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HYPERCHAOTIC DYNAMICS OF A NEW FRACTIONAL DISCRETE-TIME SYSTEM

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Abstract

In recent years, some efforts have been devoted to nonlinear dynamics of fractional discrete-time systems. A number of papers have so far discussed results related to the presence of chaos in fractional maps. However, less results have been published to date regarding the presence of hyperchaos in fractional discrete-time systems. This paper aims to bridge the gap by introducing a new three-dimensional fractional map that shows, for the first time, complex hyperchaotic behaviors. A detailed analysis of the map dynamics is conducted via computation of Lyapunov exponents, bifurcation diagrams, phase portraits, approximated entropy and C_0 complexity. Simulation results confirm the effectiveness of the approach illustrated herein.

Keywords: Chaos; Discrete Fractional Calculus; Hyperchaotic Map.

1. INTRODUCTION

Unlike fractional derivatives, which made their first appearance in 1695, discrete fractional calculus has been introduced only in 1974.[1] Namely, Diaz and Olser made a discretization of a continuous-time fractional operator and obtained the first example of a discrete fractional operator.[1] Since 1974, discrete fractional calculus has received considerable attentions. In recent years, this has led to the discovery of chaotic phenomena in discrete-time systems described by fractional difference equations.[2] These systems are nonlinear fractional-order maps that highlight unpredictable behaviors and sensitivity to initial conditions.[3] For example, in Ref. 4 the presence of chaos in the fractional delayed logistic map has been illustrated via bifurcation diagrams and phase portraits. In Ref. 5 the chaotic dynamics of two maps have been studied, i.e. the fractional sine map and the fractional standard map. In Ref. 6, the presence of chaos in three fractional chaotic maps has been discussed. Namely, in Ref. 6 the Stefanski map, the Rössler map and the Wang map have been studied via bifurcation diagrams and phase portraits. In Ref. 7, stabilization and synchronization of three chaotic maps have been illustrated, including the fractional Lorenz map, the fractional flow map and the fractional Lozi map. In Ref. 8, the chaotic dynamics of the fractional Hénon map have been analyzed, whereas in Ref. 9 chaotic attractors in the fractional generalized Hénon map have been shown. In Ref. 10, a chaotic fractional quadratic map without equilibria has been introduced, whereas in Ref. 11 bifurcation diagrams and phase portraits for a chaotic three-dimensional (3D) fractional chaotic map have been reported. In Ref. 12, dynamics and synchronization of discrete fractional complex-valued neural networks have been discussed in detail. In Ref. 13, a fractional-order higher-dimensional multicavity chaotic map, defined using the Caputo operator, has been investigated. In particular, a dynamical analysis of the multicavity map has been carried out via bifurcation diagrams and permutation entropy complexity.[13] In Ref. 14, the chaotic dynamics of a fractional-order difference Cournot duopoly model characterized by long memory have been investigated. In Ref. 15, a chaotic fractional map with hidden attractors has been studied using complexity and entropy concepts. In Ref. 16, control laws for stabilizing chaos in a fractional discrete system with hidden attractors have been developed. In Ref. 17, Poincaré plots and Julia sets have been exploited to analyze the memory effect on the dynamics of the fractional logistic map. In Ref. 18, the presence of chaos in a symmetrical fractional map, which includes only five nonlinear terms in its equations, has been analyzed in detail. In Ref. 19, it has been shown that the fractional-order Hénon–Lozi map exhibits a range of different dynamical behaviors, which include chaos and coexisting attractors. In particular, chaotic hidden attractors and transient state have been found.[20]

Referring to the literature on fractional chaotic maps, it should be noted that all the papers published to date deal with the presence of chaos, rather than hyperchaos. Namely, the dynamics of all the fractional chaotic maps illustrated in Refs. 4–7 are characterized by the presence of only one positive Lyapunov exponent. Very few papers have been published to date regarding the presence of hyperchaos in fractional maps, i.e. fractional discrete systems characterized by complex dynamics involving at least two positive Lyapunov exponents. Based on these considerations, this paper aims to bridge the gap by presenting one of the first examples of hyperchaos in fractional maps. Namely, the paper introduces a new 3D fractional chaotic map, for which a detailed analysis of its dynamics is conducted via computation of Lyapunov exponents, bifurcation diagrams, approximated entropy and complexity. The paper is organized as follows. In Sec. 2, a fractional 3D map is proposed and a stability analysis of its equilibria is conducted. In Sec. 3, the hyperchaotic behavior of the map is analyzed via the computation of bifurcation diagrams, Lyapunov exponents and phase portraits. Finally, in Sec. 4, the complexity of the dynamic behavior of the proposed hyperchaotic map is investigated by computing the C_0 algorithm and the approximate entropy.

2. A NEW 3D FRACTIONAL CHAOTIC MAP

2.1. System Equations

Let us consider the following integer-order discrete system, which has been recently proposed in Ref. 21 as an example of a 3D hyperchaotic system. Its mathematical model is described by

$$\begin{cases} x_1(n+1) = -a_1x_1(n) - a_2x_2(n) - a_3x_3(n) \\ \qquad\qquad + 1.44x_2(n) + 1.61x_1(n)x_3(n) \\ \qquad\qquad + 1.48x_2(n)x_3(n), \\ x_2(n+1) = x_1(n), \\ x_3(n+1) = x_2(n), \end{cases} \tag{1}$$

where $a_i, \forall\, i = \overline{1.3}$ three real parameters. In Ref. 21, it has been shown that the integer-order system (1) exhibits a hyperchaotic attractor when $a_1 = 2.32$, $a_2 = 1.27$ and $a_3 = 0.01$. Figure 1 illustrates the 2D and 3D phase plots of the hyperchaotic attractor.

In this work, we present a new fractional chaotic map by removing and adding some terms from the integer-order system (1) and by introducing a fractional difference operator. Then, we will analyze the effects of both fractional order and system parameters on the dynamics of the map.

First, let us introduce the definition of the fractional Caputo difference operator, which has been recently used in discrete fractional systems.[22]

Definition 1. For a real order $\nu > 0$, the Caputo fractional difference operator of a function

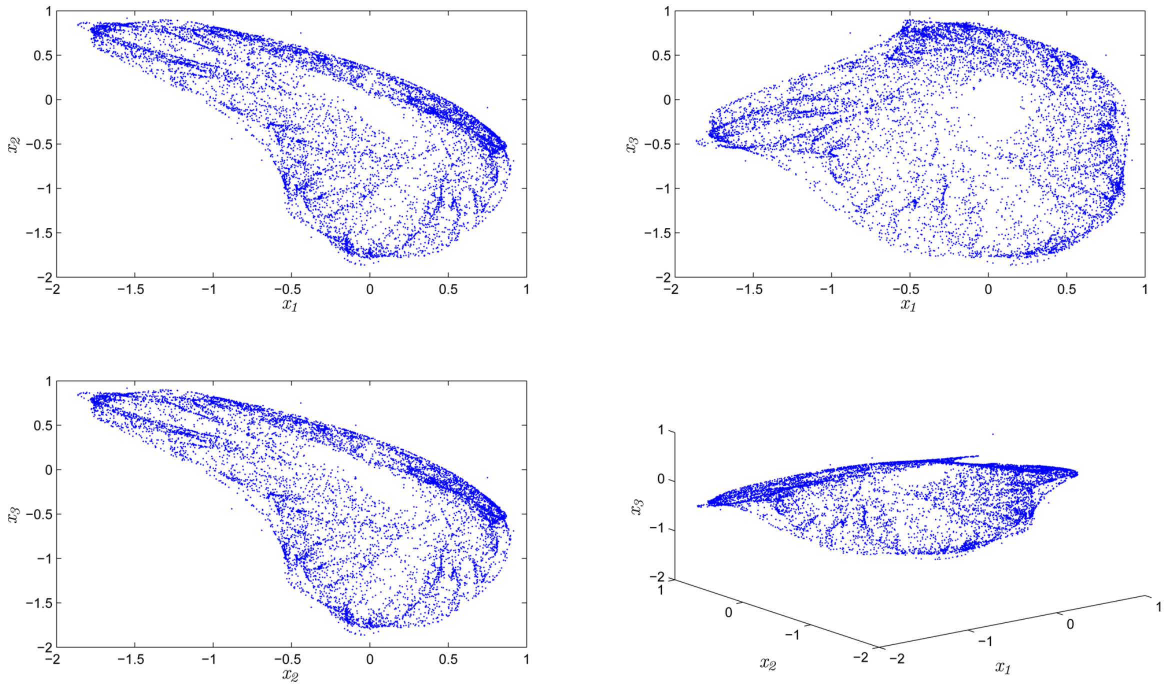

Fig. 1 Hyperchaotic attractor of the integer-order system (1).

$X : \mathbb{N}_a \to \mathbb{R}$ is defined as[22]

$$^{C}\Delta_a^{\nu} X(t) = \frac{1}{\Gamma(n-\nu)} \sum_{s=a}^{t-(n-\nu)} (t-\sigma(s))^{(n-\nu-1)} \times \Delta_s^n X(s), \quad (2)$$

where $t \in \mathbb{N}_{a+n-\nu}$, a is the starting point, $\Gamma(.)$ is the Euler's gamma function and $n = [\nu] + 1$, in which $[\nu]$ is the floor function of ν. $t^{(\nu)}$ is the generalized falling function, which is defined using Γ as follows:

$$t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}.$$

Now, the ν-Caputo of system (1) can be described as

$$\begin{cases} ^{C}\Delta_a^{\nu} x(t) = -a_1 x_1(t-1+\nu) - a_2 x_2(t+1-\nu) \\ \qquad - a_3 x_3(t+1-\nu) \\ \qquad + 1.44 x_2(t+1-\nu) \\ \qquad + 1.61 x_1(t+1-\nu) \\ \qquad + x_3(t+1-\nu) \\ \qquad + 1.48 x_2(t+1-\nu) x_3(t+1-\nu), \\ ^{C}\Delta_a^{\nu} x_2(t) = x_1(t+1-\nu) - x_2(t+1-\nu), \\ ^{C}\Delta_a^{\nu} x_3(t) = x_2(t+1-\nu) - x_3(t+1-\nu), \end{cases} \quad (3)$$

with $t \in \mathbb{N}_{a+1-\nu}$ and fractional order $0 < \nu \leq 1$.

2.2. Stability of the Equilibrium Points

Now the behavior of the new system (3) is investigated by considering the stability analysis in the context of fractional-order difference equations. Let $^{C}\Delta_a^{\nu} x_1 = {}^{C}\Delta_a^{\nu} x_2 = {}^{C}\Delta_a^{\nu} x_3 = 0$. Thus, the following equation is obtained:

$$-(a_1 + a_2 + a_3) x_1 + 4.53 x_1^2 = 0. \quad (4)$$

Direct calculation shows that the fractional system (3) has two equilibrium points: $E_1 = (0, 0, 0)$ and $E_2 = (\frac{a_1+a_2+a_3}{4.53}, \frac{a_1+a_2+a_3}{4.53}, \frac{a_1+a_2+a_3}{4.53})$. By taking the same values of the system parameters as in Fig. 1, the fixed point E_2 becomes $E_2 = (\frac{3.6}{4.53}, \frac{3.6}{4.53}, \frac{3.6}{4.53})$. By computing the eigenvalues of the Jacobian matrix, we could readily get the stable conditions around the equilibrium. Namely, the Jacobian matrix of the 3D fractional system for an arbitrary point (x_1, x_2, x_3) is given by

$$J = \begin{pmatrix} A & B & C \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \quad (5)$$

where $A = -2.32 + 1.44x_2 + 1.61x_3$, $B = -1.27 + 1.44x_1 + 1.48x_3$ and $C = -0.01 + 1.61x_1 + 1.48x_2$

According to Ref. 23, an equilibrium point for the fractional difference system (3) is said to be stable if the following relationship holds

$$\lambda_i \in \left\{ z \in \mathbb{C} : |z| < \left(2\cos\frac{|\arg z| - \pi}{2-\nu}\right) \quad \text{and} \quad |\arg z| > \frac{\nu\pi}{2} \right\}. \tag{6}$$

The Jacobian matrix of the fractional system (3) computed at E_1 is given by

$$J_{E_1} = \begin{pmatrix} -2.32 & -1.27 & -0.01 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \tag{7}$$

and possesses the eigenvalues: $\lambda_1 = -1.6560 + 0.9106i$, $\lambda_2 = -1.6560 - 0.9106i$ and $\lambda_3 = -1.0079$. From Eq. (6), we can directly obtain that $|\lambda_i| = 1.8899 > (2\cos\frac{2.6388-\nu}{2-\nu})$ for every $0.499 < \nu \leq 1$, indicating that the equilibrium point E_1 is unstable in this case. Now we discuss the local stability of the equilibrium $E_2 = (\frac{3.6}{4.53}, \frac{3.6}{4.53}, \frac{3.6}{4.53})$, for which the Jacobian matrix is given by

$$J_{E_2} = \begin{pmatrix} 0.1038 & 0.0760 & 2.4456 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}. \tag{8}$$

The stability of E_2 is analyzed by evaluating the eigenvalues of J_{E_2}. The computation gives $\lambda_1 = 0.8553$, $\lambda_2 = -1.3757 + 1.0849i$, $\lambda_3 = -1.3757 - 1.0849i$. Similarly, it can be readily shown that $|\lambda_2| = 1.7520 > (2\cos\frac{2.4738-\nu}{2-\nu})$ for every $0.673 < \nu \leq 1$. Thus, according to Eq. (6), the equilibrium point E_2 is unstable.

3. ANALYSIS OF THE SYSTEM DYNAMICS

In this section, the hyperchaotic behavior of the proposed 3D fractional chaotic map (3) is analyzed via the computation of bifurcation diagrams, Lyapunov exponents and phase portraits. The effects of fractional order and system parameters on the dynamics of the map (3) are illustrated in details.

3.1. Volterra equations

At first, we briefly recall the following theorem; which is used to present the numerical discrete solution of the fractional system (3).

Theorem 2. *Given the following fractional difference equation*:

$$\begin{cases} {}^C\Delta_a^\nu X(t) = f(t+\nu-1, X(t+\nu-1)), \\ \Delta^k X(a) = X_k, \\ n = [\nu]+1, k = 0, 1, \ldots, n-1, \end{cases} \tag{9}$$

its solution can be written as a discrete equation of Volterra type[24]

$$X(t) = X(a) + \frac{1}{\Gamma(\nu)} \sum_{s=a+n-\nu}^{t-\nu} (t-s-1)^{(\nu-1)} \times f(s+\nu-1, X(s+\nu-1)),$$

where $t \in \mathbb{N}_{a+n}$.

Therefore, the discrete solution of the fractional chaotic map (3) is given as

$$\begin{cases} x_1(t) = x_1(a) + \frac{1}{\Gamma(\nu)} \sum_{s=a+1-\nu}^{t-\nu} (t-s-1)^{(\nu-1)} \\ \quad \times (-a_1 x_1(s+\nu-1) - a_2 x_2(s+\nu-1) \\ \quad - a_3 x_3(s+\nu-1) + 1.44 x_2(s+\nu-1) \\ \quad + 1.61 x_1(s+\nu-1) x_3(s+\nu-1) \\ \quad + 1.48 x_2(s+\nu-1) x_3(s+\nu-1)), \\ {}^C\Delta_a^\nu x_2(t) = x_2(a) + \frac{1}{\Gamma(\nu)} \sum_{s=a+1-\nu}^{t-\nu} \\ \quad \times (t-s-1)^{(\nu-1)} (x_1(s+\nu-1) \\ \quad - x_2(s+\nu-1)), \\ {}^C\Delta_a^\nu x_3(t) = x_3(a) + \frac{1}{\Gamma(\nu)} \sum_{s=a+1-\nu}^{t-\nu} \\ \quad \times (t-s-1)^{(\nu-1)} (x_2(s+\nu-1) \\ \quad - x_3(s+\nu-1)), \end{cases} \tag{10}$$

here $t \in \mathbb{N}_{a+n}$. Since $(t-s-1)^{(\nu-1)}/\Gamma(\nu)$ is equal to $\Gamma(t-s)/\Gamma(t-s-\nu+1)$, by taking $n = t-a$,

$a = 0$, it follows that[25]

$$\begin{cases} x_1(n) = x_1(0) + \dfrac{1}{\Gamma(\nu)} \displaystyle\sum_{j=0}^{n-1} \dfrac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} \\ \quad \times (-a_1x_1(j) - a_2x_2(j) - a_3x_3(j) \\ \quad + 1.44x_2(j) + 1.61x_1(j)x_3(j) \\ \quad + 1.48x_2(j)x_3(j)), \\ x_2(n) = x_2(0) + \dfrac{1}{\Gamma(\nu)} \displaystyle\sum_{j=0}^{n-1} \dfrac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} \\ \quad \times (x_1(j) - x_2(j)), \\ x_3(n) = x_3(0) + \dfrac{1}{\Gamma(\nu)} \dfrac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} \\ \quad \times (x_2(j) - x_3(j)), \end{cases} \tag{11}$$

where $x_1(0)$, $x_2(0)$ and $x_3(0)$ are the initial conditions. According to the numerical equation (11), the proposed fractional chaotic map (3) has memory effect; which means that the iterated solutions, $x_i, \forall\, i = 1.3$, are determined by all the previous states.

3.2. Bifurcation Diagrams, Lyapunov Exponents and Phase Portraits

It is always useful to examine the bifurcation diagram corresponding to a specific critical parameter, with the aim to gain a comprehensive understanding of the system dynamics. In the following, we fix the values $a_1 = 2.32, a_2 = 1.27$ and $a_3 = -0.01$ and then we choose the fractional order $0 < \nu \leq 1$ as bifurcation parameter. The corresponding bifurcation diagram is plotted in Fig. 2 with respect to $0.96 \leq \nu \leq 1$. From Fig. 2, it can be noted that system (3) is chaotic over most of the range $0.964 \leq \nu \leq 1$. In particular, when $\nu = 0.9678$, the chaotic motion suddenly disappears and a periodic motion appears. The lowest order to generate chaos is 0.9645.

For better observation, the Lyapunov exponent is considered. Lyapunov exponent is an effective method generally used to characterize the exponential rate of separation of two initially very close trajectories. It is recognized citation that a nonlinear system is hyperchaotic when two Lyapunov exponents are positive, whereas it is chaotic when only one Lyapunov exponent is positive. Herein,

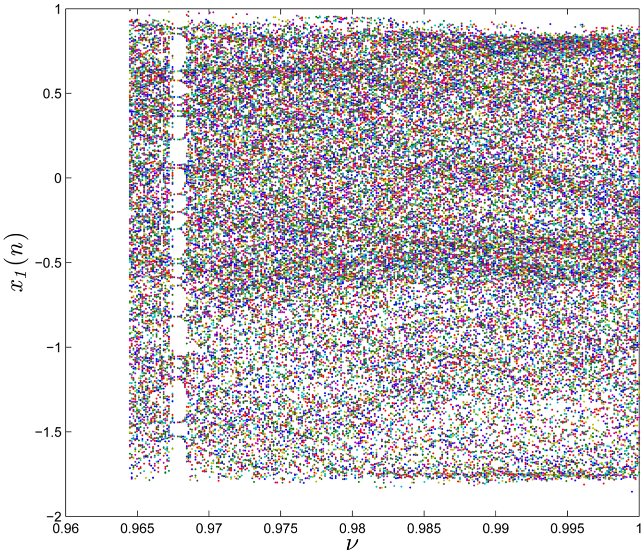

Fig. 2 Bifurcation diagram of the new fractional discrete system (3) with respect to μ for $a_3 = -0.01$.

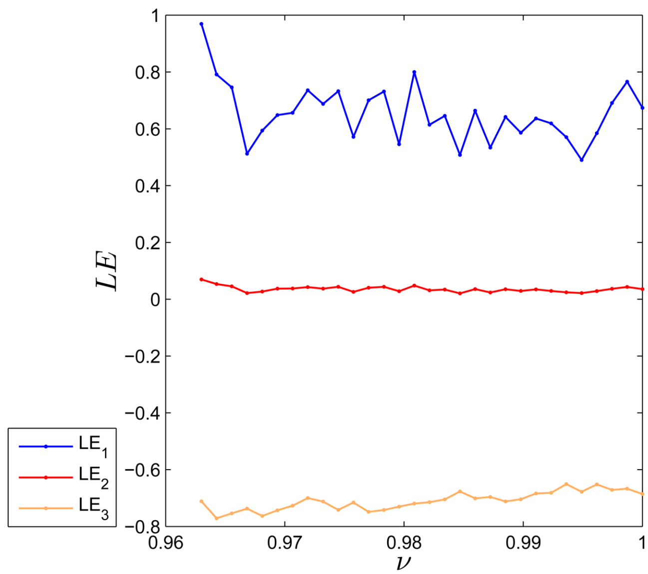

Fig. 3 Lyapunov exponents of the new fractional discrete system (3) with respect to μ for $a_3 = -0.01$.

the Lyapunov exponents are calculated by applying the QR decomposition method.[20] In particular, the Lyapunov exponents of system (3) have been obtained in MATLAB with initial conditions $x_1 = 0.75$, $x_2 = 0.09$, $x_3 = 0.32$ (see Fig. 3). In particular, regarding hyperchaotic and chaotic behaviors, from Fig. 3 it can be deduced that the fractional map (3) is hyperchaotic over the two ranges $0.964 \leq \nu < 0.9668$ and $0.9949 < \nu \leq 1$, whereas it is chaotic over the rest of the interval. In

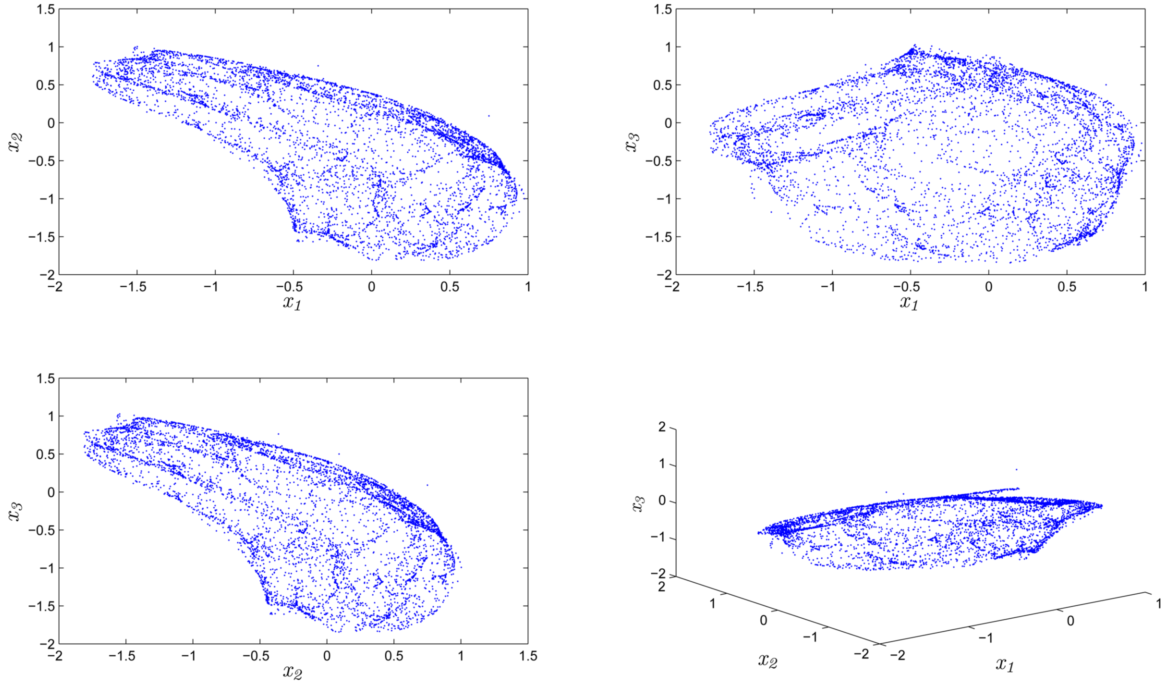

Fig. 4 Chaotic attractor of the fractional discrete-time system (3) with $a_3 = -0.01$ and $\mu = 0.97$.

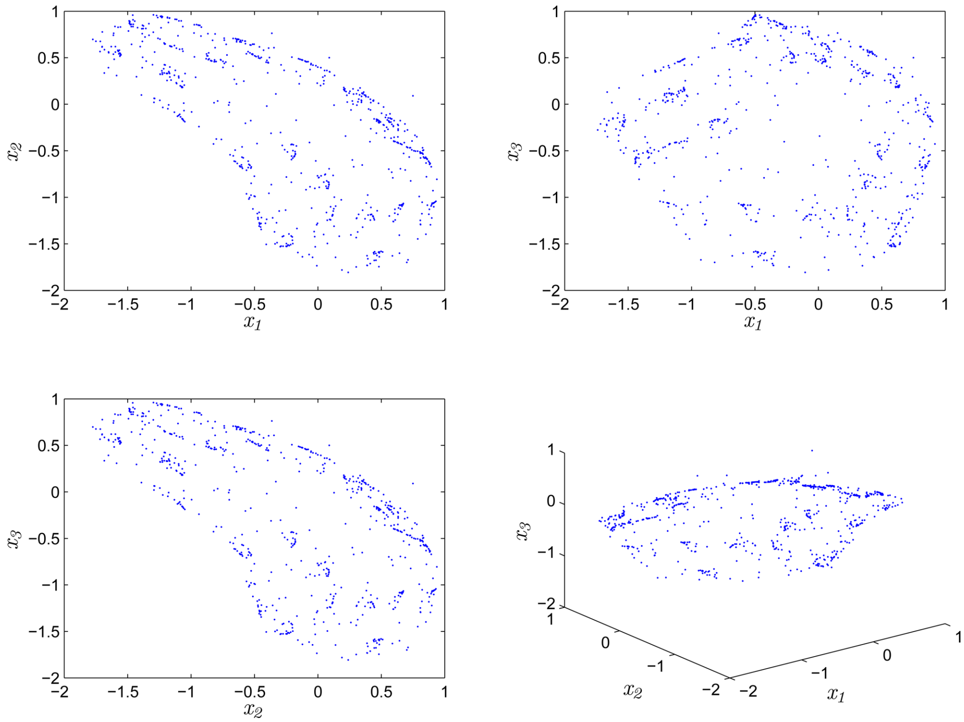

Fig. 5 Periodic attractor of the fractional discrete-time system (3) with $a_3 = -0.01$ and $\mu = 0.9678$.

order to show the different dynamic behaviors, some phase portraits are reported for three different values of ν. When $\nu = 0.97$, the maximum LE is positive indicating that the proposed fractional map (3) is chaotic. The corresponding attractor is plotted in Fig. 4, while, when $\nu = 0.9678$ two negative LEs are observed indicating that the proposed fractional chaotic map (3) is periodic (see Fig. 5). On the other hand, when we choose the parameter $\mu = 0.9655$, LEs has two positive values, indicating that the

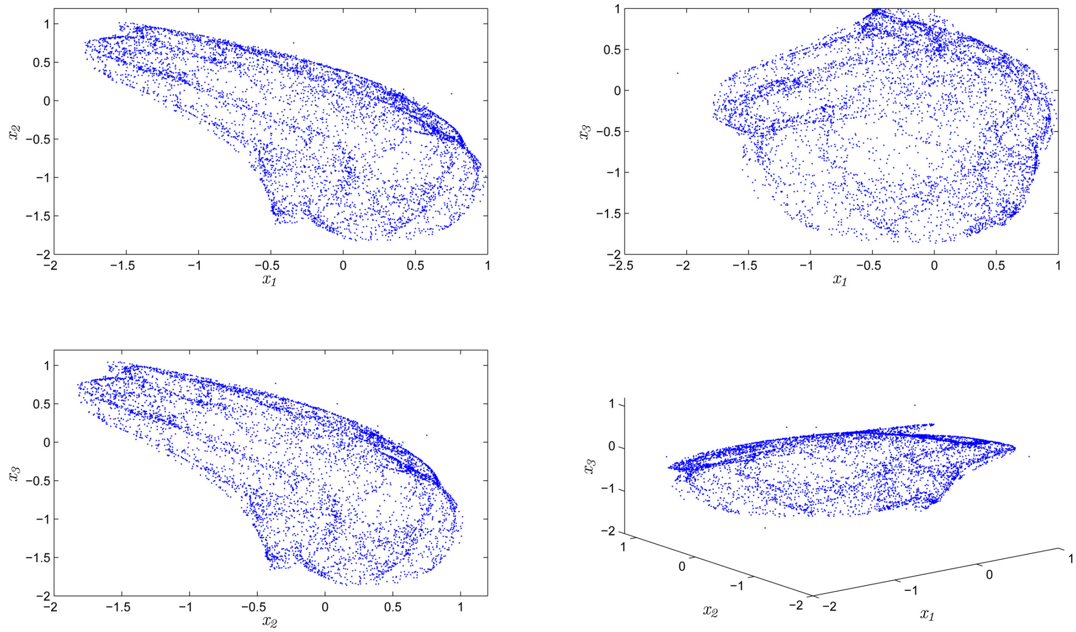

Fig. 6 Hyperchaotic attractor of the fractional discrete system (3) with $a_3 = -0.01$ and $\mu = 0.9655$.

fractional chaotic map is in hyperchaos, as shown in Fig. 6.

In order to further analyze the properties of the fractional chaotic map (3), we plot the bifurcation diagrams for different values of ν (i.e. $\nu = 1, \nu = 0.995$, and $\nu = 0.985$) by considering a_3 as a bifurcation parameter and by taking the values of the remaining parameters as $a_1 = 2.32$, $a_2 = 1.27$, and $a_3 = 0.01$, results are reported in Fig. 7. Various dynamical behavior can be observed by changing the value of a_3 at the interval $]-0.01, 0.25]$. In particular, we observe that decreasing the value of the fractional order leads to the disappearance of some chaotic regions. Moreover, when we decrease the value of ν, the bifurcation diagram shrinks. Finally, a flip bifurcation is observed when we consider the integer-order system (i.e. $\nu = 1$).

4. ANALYSIS OF THE SYSTEM COMPLEXITY

In this section, the complexity of the new hyperchaotic system (3) is investigated by exploiting the C_0 algorithm and the approximate entropy. Note that the C_0 algorithm[10] is a statistical measure that can quantify the irregularity of a time series, whereas the approximate entropy[25] estimates the complexity of a series of data from a multidimensional perspective. Both the C_0 algorithm and the approximate entropy have been computed by varying the parameter a_3 and the fractional order ν.

4.1. C_0 Complexity

At first, we briefly present the C_0 algorithm. Suppose that $\{x(n), n = 0, 1, \ldots, N-1\}$ is a time series with a length of N and $F(k) = \sum_{k=0}^{n-1} x(k) e^{(-2j\pi k/n)}$, $k = \overline{0, n-1}$ is the corresponding discrete Fourier transform. Then, we construct the following function[10]

$$\bar{F}(k) = \begin{cases} F(k) & \text{if } |F(k)|^2 > rG, \\ 0 & \text{if } |F(k)|^2 \le rG, \end{cases} \tag{12}$$

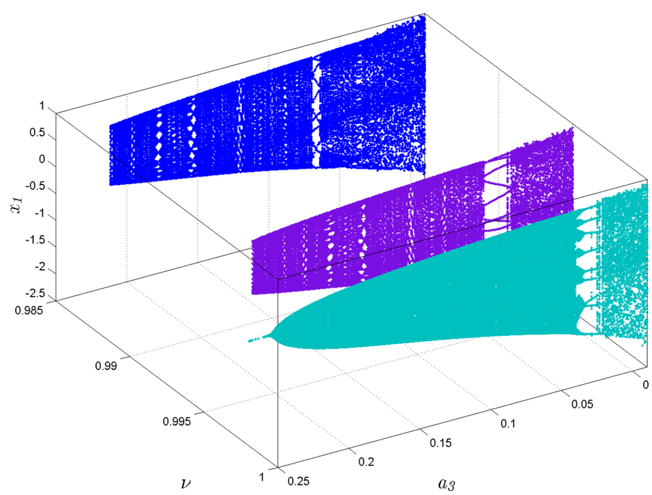

Fig. 7 Bifurcation diagrams of the fractional discrete system (3) for different fractional-order values.

where r is a control parameter and G is the mean square value of suite of data $\{F(k)\}$. Herein, the value $r = 15$ is selected. By denoting with $\bar{x}$ the inverse Fourier transform of $\bar{F}$, the C_0 complexity is defined as[10]

$$C_0(n) = \frac{\sum_{i=0}^{n-1} |x(i) - \bar{x}(i)|^2}{\sum_{i=0}^{n-1} |x(i)|^2}. \quad (13)$$

By considering the system parameters $a_1 = 2.32$, $a_2 = 1.27$, $a_3 = 0.01$, Fig. 8 displays the C_0 complexity of the $a_1 = 2.32$, $a_2 = 1.27$, and $a_3 = 0.01$. Corresponding to the fractional-order value $\nu = 0.995$ (blue diagram) and the value $\nu = 0.985$ (red diagram). From Fig. 8, it can be deduced that the complexity of the system is higher for smallest values of a_3. Moreover, the complexity of the system takes the highest value when the value of order ν increases. Based on these considerations, it can be concluded that the fractional chaotic map shows more complexity (3) when a_3 takes smallest values and for order $\nu \in [0.964, 0.9668[\cup]0.9949, 1]$.

4.2. Approximate Entropy (ApEn)

The approximate entropy (ApEn)[25] is a technique used to quantify the amount of regularity and the unpredictability of fluctuations over time-series data. Its computation returns a non-negative number, where higher values indicate higher complexity. The detailed steps for computing the ApEn are now illustrated.[25] We consider a set of points $x(1), \ldots, x(N)$ that are obtained from the discrete formula $(x_1(n))$. The value of the approximate entropy depends on two important parameters m and r, where m is embedding dimension and r is the similar tolerance. Now, we reconstruct a subsequence of x such that $\chi(i) = [x(i), \ldots, x(i+m-1)]$, where m presents the points from $x(i)$ to $x(i+m-1)$. Let K be the number of $\chi(i)$ such that the maximum absolute difference of two vectors $\chi(i)$ and $\chi(j)$ is lower or equal to the tolerance τ. The relative frequency of $\chi(i)$ being similar to $\chi(j)$, and it has the form: $C_i^m(\tau) = \frac{K}{N-m+1}$. From C_i^m, we calculate the logarithm and then define the average for all i as follows:

$$\phi^m(r) = \frac{1}{N-m-1} \sum_{i=1}^{N-m+1} \log C_i^m(r). \quad (14)$$

Thus, the approximate entropy of order m is setting as

$$\mathrm{ApEn} = \phi^m(r) - \phi^{m+1}(r). \quad (15)$$

The complexity of the fractional chaotic map (3) is investigated using the approximate entropy as a function of the control parameter a_3 and the fractional order ν. The behavior of the ApEn in the 3D space is plotted in Fig. 9. It can be deduced that the complexity of system (3) changes according to the variations of ν and a_3. Namely, the complexity of system (3) goes to zero when the fractional order assumes small values. Moreover, the ApEn decreases with the increase of fractional order ν and system parameter a_3, indicating that the system with smaller order is more complex. We conclude our analysis by observing that the approximate entropy results are in good agreement with

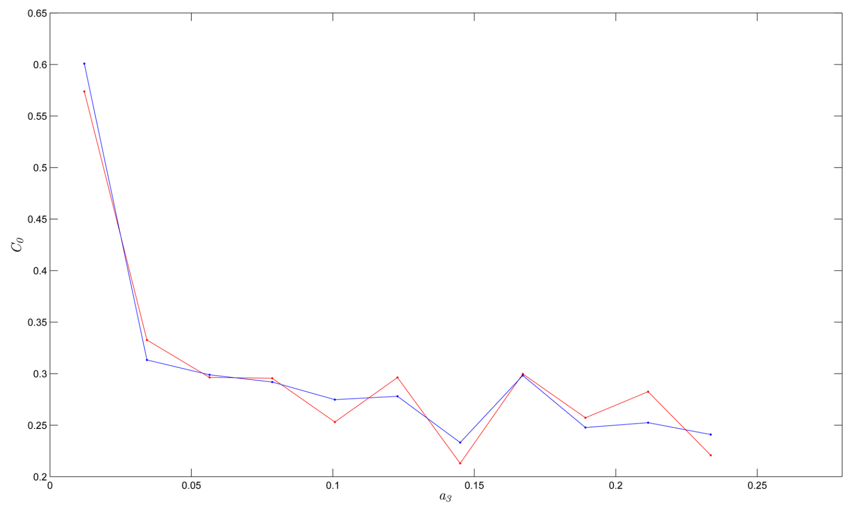

Fig. 8 (Color online) Complexity C_0 as a function of the parameter a_3 ($\mu = 0.995$ for the blue line and $\mu = 0.985$ for the red line).

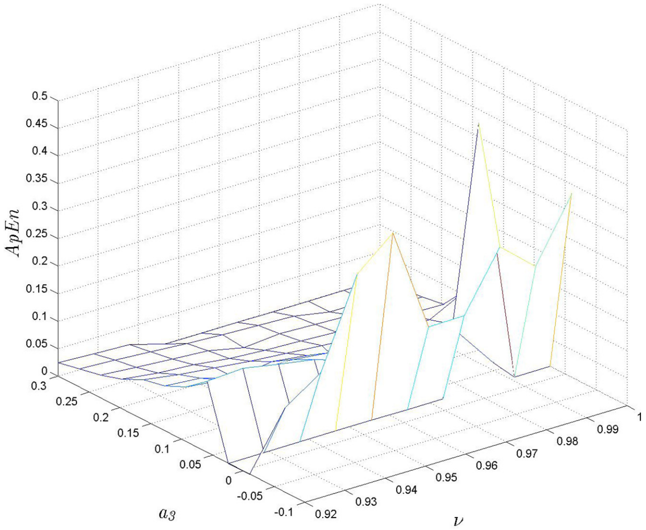

Fig. 9 Approximate entropy of the fractional system (3) as a function of the order μ and of system parameter a_3.

the remaining results obtained through the paper, clearly indicating that the fractional chaotic map (3) is more complex than its integer-order counterpart.

5. CONCLUSIONS

This paper has presented the first example of hyperchaos in fractional chaotic maps. The objective has been achieved by introducing a new 3D map based on the Caputo difference operator. At first, a stability analysis of the system equilibria has been conducted. Successively, the hyperchaotic behavior of the map has been analyzed via the computation of bifurcation diagrams, Lyapunov exponents and phase portraits. The conducted analysis has shown that the proposed map is hyperchaotic when the fractional order ν belongs to the two ranges $0.964 \leq \nu < 0.9668$ and $0.9949 < \nu \leq 1$. Finally, the complexity of the dynamic behavior of the hyperchaotic map has been investigated by computing the C_0 algorithm and the approximate entropy.

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REFERENCES

1. J. B. Diaz and T. J. Osler, Differences of fractional order, *Math. Comput.* **28**(125) (1974) 185–202.
2. G. C. Wu and D. Baleanu, Discrete fractional logistic map and its chaos, *Nonlinear Dyn.* **75**(1) (2014) 283–287.
3. P. Ostalczyk, *Discrete Fractional Calculus: Applications in Control and Image Processing*, Vol. 4 (World Scientific, 2015).
4. G. C. Wu and D. Baleanu, Discrete chaos in fractional delayed logistic maps, *Nonlinear Dyn.* **80**(4) (2015) 1697–1703.
5. G. C. Wu, D. Baleanu and S. D. Zeng, Discrete chaos in fractional sine and standard maps, *Phys. Lett. A* **378**(5–6) (2014) 484–487.
6. A. A. Khennaoui, A. Ouannas, S. Bendoukha, G. Grassi, X. Wang, V. T. Pham and F. E. Alsaadi, Chaos, control, and synchronization in some fractional-order difference equations, *Adv. Differ. Equ.* **2019** (2019) 1–23.
7. A. A. Khennaoui, A. Ouannas, S. Bendoukha, G. Grassi, R. P. Lozi and V. T. Pham, On fractional–order discrete-time systems: Chaos, stabilization and synchronization, *Chaos Solitons Fractals* **119** (2019) 150–162.
8. T. Hu, Discrete chaos in fractional Hénon map, *Appl. Math.* **18** (2014) 170–175.
9. M. K. Shukla and B. B. Sharma, Investigation of chaos in fractional order generalized hyperchaotic Hénon map, *AEU-Int. J. Electron. Commun.* **78** (2017) 265–273.
10. A. Ouannas, A. A. Khennaoui, S. Momani, G. Grassi, V. T. Pham, R. El-Khazali and D. Vo Hoang, A quadratic fractional map without equilibria: Bifurcation, 0-1 Test, complexity, entropy, and control, *Electronics* **9**(5) (2020) 748.
11. A. Ouannas, A. A. Khennaoui, S. Momani, G. Grassi and V. T. Pham, Chaos and control of a three-dimensional fractional order discrete-time system with no equilibrium and its synchronization, *AIP Adv.* **10**(4) (2020) 045310.
12. Y. Gu, H. Wang and Y. Yu, Synchronization for fractional-order discrete-time neural networks with time delays, *Appl. Math. Comput.* **372** (2020) 124995.
13. L. Wang, K. Sun, Y. Peng and S. He, Chaos and complexity in a fractional-order higher-dimensional multicavity chaotic map, *Chaos Solitons Fractals* **131** (2020) 109488.
14. B. Xin, W. Peng and Y. Kwon, A fractional-order difference Cournot duopoly game with long memory, preprint (2019), arXiv:1903.0430.
15. A. Ouannas, A. A. Khennaoui, S. Momani, V. T. Pham and R. El-Khazali, Hidden attractors in a new fractional-order discrete system: Chaos, complexity,

entropy, and control, *Chin. Phys. B* **29**(5) (2020) 050504.
16. A. A. Khennaoui, A. Ouannas, S. Boulaaras, V. T. Pham and A. Taher Azar, A fractional map with hidden attractors: Chaos and control, *Eur. Phys. J. Spec. Top.* **229** (2020) 1083–1093.
17. Y. Wang, S. Liu and H. Li, On fractional difference logistic maps: Dynamic analysis and synchronous control, *Nonlinear Dyn.* **102** (2020) 579–588.
18. V. P. Thoai, M. S. Kahkeshi, V. V. Huynh, A. Ouannas and V. T. Pham, A nonlinear five-term system: Symmetry, chaos, and prediction, *Symmetry* **12**(5) (2020) 865.
19. A. Ouannas, A. A. Khennaoui, X. Wang, V. T. Pham, S. Boulaaras and S. Momani, Bifurcation and chaos in the fractional form of Hénon–Lozi type map, *Eur. Phys. J. Spec. Top.* **229**(12) (2020) 2261–2273.
20. S. Jafari and T. Kapitaniak, Special chaotic systems, *Eur. Phys. J. Spec. Top.* **229** (2020) 877–886.
21. W. Li, W. Yan, R. Zhang, C. Wang and Q. Ding, A new 3D discrete hyperchaotic system and its application in secure transmission, *Int. J. Bifurcation Chaos* **29**(14) (2019) 1950206.
22. T. Abdeljawad, On Riemann and Caputo fractional differences, *Comput. Math. Appl.* **62**(3) (2011) 1602–1611.
23. J. Cermak, I. Gyori and L. Nechvatal, On explicit stability conditions for a linear fractional difference system, *Fract. Calc. Appl. Anal.* **18**(3) (2015) 651–672.
24. G. A. Anastassiou, Principles of delta fractional calculus on time scales and inequalities, *Math. Comput. Model.* **52**(3–4) (2010) 556–566.
25. S. M. Pincus, Approximate entropy as a measure of system complexity, *Proc. Natl. Acad. Sci.* **88**(6) (1991) 2297–2301.