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# Laplace Transform Method for Economic Models with Constant Proportional Caputo Derivative

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**Abstract:** In this study, we solved the economic models based on market equilibrium with constant proportional Caputo derivative using the Laplace transform. We proved the accuracy and efficiency of the method. We constructed the relations between the solutions of the problems and bivariate Mittag–Leffler functions.

**Keywords:** Laplace transforms; constant proportional Caputo derivative; modeling

**MSC:** 44A10; 26A33

## 1. Introduction

Fractional differential equations have taken much interest in the recent years. Ghanbari et al. [1,2] investigated the new application of fractional Atangana–Baleanu derivatives and abundant new analytical and approximate solutions to the generalized Schamel equation. Allahviranloo et al. [3] worked the fuzzy fractional differential equation with Atangana–Baleanu fractional derivative approach. Salari et al. [4] searched existence and multiplicity for some boundary value problems involving Caputo and Atangana–Baleanu fractional derivatives. Akgül et al. [5–7] investigated the solutions of new type fractional differential equations occurring in the electrohydrodynamic flow, analysis and dynamical behavior of fractional-order cancer model with vaccine strategy, and the solutions of fractional order telegraph partial differential equation by Crank–Nicholson finite difference method.

We consider the following economic models:

$$q_d(x) = d_0 - d_1 h(x), \quad q_s(x) = -s_0 + s_1 h(x), \quad (1)$$

where  $h$  the price of goods,  $d_0, s_0, d_1, s_1$  are positive constants. For  $q_d(x) = q_s(x)$ , when the demanded quantity and the supplied quantity are equal, the equilibrium price is found as;  $h^* = \frac{d_0 + s_0}{d_1 + s_1}$ . Then, the price is disposed to stay stable. Now, considering [8]:

$$h'(x) = k(q_d - q_s). \quad (2)$$

where  $k > 0$ . Then, we get

$$h'(x) + k(d_1 + s_1)h(x) = k(d_0 + s_0). \quad (3)$$

When we solved the first order ordinary differential equation, the solution can be found:

$$h(x) = \frac{d_0 + s_0}{d_1 + s_1} - \left[ h(0) + \frac{d_0 + s_0}{d_1 + s_1} \right] \exp(-k(d_1 + s_1)x), \quad (4)$$

where  $h(0)$  is the price at the time  $x = 0$  and in here we do not interest to the expectation of agents in market.

In the above equation, we define the  $h(0)$  as the expense at the time  $x = 0$ . If we take into consideration the prospects of agents, the request and provision functions containing supplement elements  $q_d$  and  $q_s$  alters as:

$$q_d(x) = d_0 - d_1 h(x) + d_2 h'(x), \quad q_s(x) = -s_0 + s_1 h(x) - s_2 h'(x). \quad (5)$$

We equalize  $q_d(x)$ ,  $q_s(x)$  and obtain:

$$h'(x) - \frac{d_1 + s_1}{d_2 + s_2} h(x) = -\frac{d_0 + s_0}{d_2 + s_2}. \quad (6)$$

and when the linear differential Equation (6) is solved, the solution is acquired as:

$$h(x) = \frac{d_0 + s_0}{d_1 + s_1} - \left[ h(0) + \frac{d_0 + s_0}{d_1 + s_1} \right] \exp\left(\frac{d_1 + s_1}{d_2 + s_2} x\right). \quad (7)$$

We consider the above economic models with constant proportional Caputo derivative that has been presented very recently in [9]. We construct the Laplace transform method (LTM) to solve the economic models. The Laplace transform is one of the best integral transform used by many researchers.

Gupta et al. [10] investigated analytical solutions of convection–diffusion problems by combining Laplace transform method and homotopy perturbation method. Anjum et al. [11] worked Laplace transform making the variational iteration method easier. Convergence of iterative Laplace transform methods for a system of fractional partial differential equations and partial integro-differential equations arising in option pricing has been searched by Zhou [12]. Bashir et al. [13] studied solution of non-homogeneous differential equations using Faddeev–Leverrier method together with Laplace transform. Convergence analysis of iterative Laplace transform methods for the coupled partial differential equations from regime-switching option pricing was investigated by Jingtang [14]. Eljaoui et al. [15] researched Aumann fuzzy improper integral and its application to solve fuzzy integro-differential equations by LTM. Zhou et al. [16] have worked fast LTM for free-boundary problems of fractional diffusion equations. The transform method for the ulam stability of linear fractional differential equations with constant coefficient was studied by Yonghong [17]. Fatoorehchi et al. [18] investigated series solution of nonlinear differential equations by a novel extension of the LTM. Jacobs [19] searched high-order compact finite difference and Laplace transform method for the solution of time-fractional heat equations with Dirichlet and Neumann boundary conditions. See these reference for more details [20,21].

Mittag–Leffler functions are very important in the fractional calculus. There are many useful works related to the Mittag–Leffler functions in the literature. Özarslan et al. [22,23] studied on a singular integral equation including a set of multivariate polynomials suggested by Laguerre polynomials and on a certain bivariate Mittag–Leffler function analyzed from a fractional-calculus point of view. For more details [24–29].

Acay et al. [30] found the solutions of these models with three different derivatives which are Caputo, Caputo–Fabrizio and Atangana–Baleanu. They presented the efficiency of the Laplace transform method for these models. For more details see [31–33].

We organize the paper as follows: we give the main definitions and lemmas related to the constant proportional Caputo derivative in Section 2. We present the applications of the economic models by Laplace transform method in Section 3. We give some conclusions remarks in the last section.

## 2. Preliminaries

**Definition 1.** The Caputo fractional derivative is defined as [34]:

$${}^C_a D^\sigma h(x) = \frac{1}{\Gamma(1-\sigma)} \int_a^x (x-\tau)^{-\sigma} h'(\tau) d\tau, \quad (8)$$

where  $0 < \sigma \leq 1$ .

**Definition 2.** The Riemann–Liouville integral is given as [34]:

$${}^{RL}_a I^\sigma h(x) = \frac{1}{\Gamma(\sigma)} \int_a^x (x-\tau)^{\sigma-1} h(\tau) d\tau. \quad (9)$$

in here  $h$  is an integrable function and  $\sigma > 0$ .

From above definitons,

$${}^C_a D^\sigma h(x) = {}^{RL}_a I^{1-\sigma} h'(x). \quad (10)$$

is written.

**Definition 3.** The constant proportional Caputo (CPC) derivative is defined by [9]

$${}^{CPC}_0 D_x^\alpha h(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x [k_1(\alpha)h(\tau) + k_0(\alpha)h'(\tau)] (x-\tau)^{-\alpha} d\tau. \quad (11)$$

**Definition 4.** Let  $\xi, \eta : [0, \infty) \rightarrow \mathfrak{R}$ , then the convolution of  $\xi, \eta$  is

$$(\xi * \eta) = \int_0^x \xi(x-u)\eta(u) du. \quad (12)$$

and assume that  $\xi, \eta : [0, \infty) \rightarrow \mathfrak{R}$ , then we write:

$$L\{(\xi * \eta)(x)\} = L\{\xi(x)\}L\{\eta(x)\}. \quad (13)$$

**Definition 5.** We define  $E_\alpha(v)$  by:

$$E_\alpha(v) = \sum_{l=0}^{\infty} \frac{v^l}{\Gamma(\alpha l + 1)} \quad (v \in \mathbb{C}, \operatorname{Re}(\alpha) > 0). \quad (14)$$

The Mittag–Leffler function which has two parameters is given as:

$$E_{\alpha,\beta}(v) = \sum_{l=0}^{\infty} \frac{v^l}{\Gamma(\alpha l + \beta)} \quad (v, \beta \in \mathbb{C}, \operatorname{Re}(\alpha) > 0), \quad (15)$$

it is worth,  $E_{\alpha,\beta}(v)$  corresponds to the Mittag–Leffler function (14) when  $\beta = 1$ .

**Lemma 1.** The Laplace transform of constant proportional Caputo (CPC) derivative is found as [9], for Laplace transform of the derivatives see Table 1:

$$L\{{}^{CPC}_0 D_x^\alpha h(x)\} = \left[ \frac{k_1(\alpha)}{s} + k_0(\alpha) \right] s^\alpha L\{h(x)\} - k_0(\alpha) s^{\alpha-1} h(0). \quad (16)$$

**Table 1.** Laplace transform of the Caputo (C), constant proportional Caputo (CPC) derivatives and the Riemann–Liouville (RL) integral.

| C, CPC and RL                | Convolution   | Laplace Transform   |
|------------------------------|---|---|
| ${}_0^C D_t^\alpha f(t)$     | $\frac{df(t)}{dt} * \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$   | $(sL\{f(t)\} - f(0))s^{\alpha-1}$   |
| ${}_0^{RL} I_t^\alpha f(t)$  | $f(t) * \frac{t^{\alpha-1}}{\Gamma(\alpha)}$  | $s^{-\alpha}L\{f(t)\}$  |
| ${}_0^{CPC} D_t^\alpha f(t)$ | $f(t) * \frac{k_1(\alpha)t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{df(t)}{dt} * \frac{k_0(\alpha)t^{-\alpha}}{\Gamma(1-\alpha)}$ | $k_1(\alpha)s^{\alpha-1}L\{f(t)\} + (sL\{f(t)\} - f(0))s^{\alpha-1}k_0(\alpha)$ |

### 3. Applications of the Economic Model

Let us take into consideration the first model with the constant proportional Caputo derivative as:

$${}_0^{CPC} D_x^\alpha h(x) + k(d_1 + s_1)h(x) = k(d_0 + s_0). \tag{17}$$

Applying the Laplace transform to the equation, we acquire

$$L\{{}_0^{CPC} D_x^\alpha h(x)\} + k(d_1 + s_1)L\{h(x)\} = L\{k(d_0 + s_0)\}. \tag{18}$$

Using the expression Lemma 1, then we obtain:

$$\left[ \frac{k_1(\alpha)}{s} + k_0(\alpha) \right] s^\alpha L\{h(x)\} - k_0(\alpha)s^{\alpha-1}h(0) + k(d_1 + s_1)L\{h(x)\} = \frac{k(d_0 + s_0)}{s}, \tag{19}$$

and

$$\begin{aligned} L\{h(x)\} &= \frac{k(d_0 + s_0)}{k_1(\alpha)s^\alpha + k_0(\alpha)s^{\alpha+1} + k(d_1 + s_1)s} \\ &\quad + \frac{k_0(\alpha)h(0)s^\alpha}{k_1(\alpha)s^\alpha + k_0(\alpha)s^{\alpha+1} + k(d_1 + s_1)s} \\ &= \frac{d_0 + s_0}{d_1 + s_1} s^{-1} \left[ 1 - \frac{-k_1(\alpha)s^{\alpha-1} - k_0\alpha s^\alpha}{k(d_0 + s_0)} \right]^{-1} \\ &\quad + h(0)s^{-1} \left[ 1 - \frac{-k_1(\alpha)s^{-1} - k(d_1 + s_1)s^{-\alpha}}{k_0(\alpha)} \right]^{-1} \\ &= \frac{d_0 + s_0}{d_1 + s_1} s^{-1} \sum_{j=0}^{\infty} \left[ \frac{-k_1(\alpha)s^{\alpha-1} - k_0(\alpha)s^\alpha}{k(d_0 + s_0)} \right]^j \\ &\quad + h(0)s^{-1} \sum_{j=0}^{\infty} \left[ \frac{-k_1(\alpha)s^{-1} - k(d_1 + s_1)s^{-\alpha}}{k_0(\alpha)} \right]^j \\ &= \frac{d_0 + s_0}{d_1 + s_1} s^{-1} \sum_{j=0}^{\infty} \frac{1}{k^j(d_0 + s_0)^j} \sum_{r=0}^j \binom{j}{r} [-k_1(\alpha)s^{\alpha-1}]^{j-r} [-k_0(\alpha)s^\alpha]^r \\ &\quad + h(0)s^{-1} \sum_{j=0}^{\infty} \frac{1}{k_0(\alpha)^j} \sum_{r=0}^j \binom{j}{r} [-k_1(\alpha)s^{-1}]^{j-r} [-k(d_1 + s_1)s^{-\alpha}]^r \\ &= \frac{d_0 + s_0}{d_1 + s_1} \sum_{j=0}^{\infty} \sum_{r=0}^j (-1)^j \frac{k_1(\alpha)^{j-r} k_0(\alpha)^r}{k^j(d_0 + s_0)^j} \binom{j}{r} s^{(\alpha-1)(j-r) + \alpha r - 1} \\ &\quad + h(0) \sum_{j=0}^{\infty} \sum_{r=0}^j (-1)^j \frac{k_1(\alpha)^{j-r} k^r (d_1 + s_1)^r}{k_0(\alpha)^j} \binom{j}{r} s^{j-r-\alpha r - 1}. \end{aligned}$$

Then, implementing the inverse LTM gives:

$$h(x) = \frac{d_0 + s_0}{d_1 + s_1} \sum_{j=0}^{\infty} \sum_{r=0}^j (-1)^j \frac{k_1(\alpha)^{j-r} k_0(\alpha)^r}{k^j (d_0 + s_0)^j} \binom{j}{r} \frac{x^{(1-\alpha)j-r}}{\Gamma((1-\alpha)j-r+1)}$$

$$+ h(0) \sum_{j=0}^{\infty} \sum_{r=0}^j (-1)^j \frac{k_1(\alpha)^{j-r} k^r (d_1 + s_1)^r}{k_0(\alpha)^j} \binom{j}{r} \frac{x^{j+(\alpha-1)r}}{\Gamma(j+(\alpha-1)r+1)},$$

When we take  $p = j - r$ , we will get:

$$h(x) = \frac{d_0 + s_0}{d_1 + s_1} \sum_{r=0}^{\infty} \sum_{p=0}^{\infty} \frac{(r+p)!}{r!p!} \frac{(-k_1(\alpha))^p (-k_0(\alpha))^r}{k^{p+r} (d_0 + s_0)^{p+r}} \frac{x^{(1-\alpha)p-\alpha r}}{\Gamma((1-\alpha)p-\alpha r+1)}$$

$$+ h(0) \sum_{r=0}^{\infty} \sum_{p=0}^{\infty} \frac{(r+p)!}{r!p!} \frac{(-k_1(\alpha))^p (-k(d_1 + s_1))^r}{k_0(\alpha)^{p+r}} \frac{x^{p+\alpha r}}{\Gamma(p+\alpha r+1)},$$

$$h(x) = \frac{d_0 + s_0}{d_1 + s_1} \sum_{r=0}^{\infty} \sum_{p=0}^{\infty} \frac{(r+p)!}{r!p!} \left[ \frac{-k_0(\alpha)}{k(d_0 + s_0)} x^{-\alpha} \right]^r \left[ \frac{-k_1(\alpha)}{k(d_0 + s_0)} x^{1-\alpha} \right]^p \frac{1}{\Gamma((1-\alpha)p-\alpha r+1)}$$

$$+ h(0) \sum_{r=0}^{\infty} \sum_{p=0}^{\infty} \frac{(r+p)!}{r!p!} \left[ \frac{-k(d_1 + s_1)}{k_0(\alpha)} x^{\alpha} \right]^r \left[ \frac{-k_1(\alpha)}{k_0(\alpha)} x \right]^p \frac{1}{\Gamma(p+\alpha r+1)}.$$

We can write geometric series as [35]:

$$h(x) = \frac{d_0 + s_0}{d_1 + s_1} E_{1-\alpha, -\alpha, 1}^1 \left( \frac{-k_1(\alpha)}{k(d_0 + s_0)} x^{1-\alpha}, \frac{-k_0(\alpha)}{k(d_0 + s_0)} x^{-\alpha} \right)$$

$$+ h(0) E_{1, \alpha, 1}^1 \left( \frac{-k_1(\alpha)}{k_0(\alpha)} x, \frac{-k(d_1 + s_1)}{k_0(\alpha)} x^{\alpha} \right).$$

Let us take into consideration the second model with the constant proportional Caputo derivative as:

$${}_0^{CPC} D_x^{\alpha} h(x) - \frac{d_1 + s_1}{d_2 + s_2} h(x) = -\frac{d_0 + s_0}{d_2 + s_2}. \tag{20}$$

If we implement the LTM, we will get

$$L\{{}_0^{CPC} D_x^{\alpha} h(x)\} + k(d_1 + s_1)L\{h(x)\} = L\{k(d_0 + s_0)\}. \tag{21}$$

Using the expression Lemma 1, then we obtain:

$$\left[ \frac{k_1(\alpha)}{s} + k_0(\alpha) \right] s^{\alpha} L\{h(x)\} - k_0(\alpha) s^{\alpha-1} h(0) - \frac{d_1 + s_1}{d_2 + s_2} L\{h(x)\} = -\frac{d_0 + s_0}{d_2 + s_2 s}, \tag{22}$$

and

$$\begin{aligned}
 L\{h(x)\} &= \frac{d_0 + s_0}{d_2 + s_2 k_1(\alpha) s^\alpha + k_0(\alpha)(d_2 + s_2) s^{\alpha+1} - (d_1 + s_1) s} \\
 &\quad + \frac{k_0(\alpha) h(0) s^\alpha (d_2 + s_2)}{(d_2 + s_2) k_1(\alpha) s^\alpha + k_0(\alpha)(d_2 + s_2) s^{\alpha+1} - (d_1 + s_1) s} \\
 &= \frac{d_0 + s_0}{d_1 + s_1} s^{-1} \left[ 1 - \frac{(d_2 + s_2) s^{\alpha-1} + k_0 \alpha s^\alpha}{(d_1 + s_1)} \right]^{-1} \\
 &\quad + h(0) s^{-1} \left[ 1 - \frac{-k_1(\alpha)(d_2 + s_2) s^{-1} + (d_1 + s_1) s^{-\alpha}}{k_0(\alpha)(d_2 + s_2)} \right]^{-1} \\
 &= \frac{d_0 + s_0}{d_1 + s_1} s^{-1} \sum_{j=0}^{\infty} \left[ \frac{(d_2 + s_2) s^{\alpha-1} + k_0(\alpha)(d_2 + s_2) s^\alpha}{(d_1 + s_1)} \right]^j \\
 &\quad + h(0) s^{-1} \sum_{j=0}^{\infty} \left[ \frac{-k_1(\alpha)(d_2 + s_2) s^{-1} + (d_1 + s_1) s^{-\alpha}}{k_0(\alpha)(d_2 + s_2)} \right]^j \\
 &= \frac{d_0 + s_0}{d_1 + s_1} s^{-1} \sum_{j=0}^{\infty} \frac{1}{(d_1 + s_1)^j} \sum_{r=0}^j \binom{j}{r} [(d_2 + s_2) s^{\alpha-1}]^{j-r} [k_0(\alpha)(d_2 + s_2) s^\alpha]^r \\
 &\quad + h(0) s^{-1} \sum_{j=0}^{\infty} \frac{1}{k_0(\alpha)^j (d_2 + s_2)^j} \sum_{r=0}^j \binom{j}{r} [-k_1(\alpha)(d_2 + s_2) s^{-1}]^{j-r} [(d_1 + s_1) s^{-\alpha}]^r \\
 &= \frac{d_0 + s_0}{d_1 + s_1} \sum_{j=0}^{\infty} \sum_{r=0}^j \frac{(d_2 + s_2)^{j-r} k_0(\alpha)^r (d_2 + s_2)^r}{(d_1 + s_1)^j} \binom{j}{r} s^{(\alpha-1)(j-r) + \alpha r - 1} \\
 &\quad + h(0) \sum_{j=0}^{\infty} \sum_{r=0}^j \frac{(-k_1(\alpha))^{j-r} (d_2 + s_2)^{j-r} (d_1 + s_1)^r}{k_0(\alpha)^j (d_2 + s_2)^j} \binom{j}{r} s^{r - j - \alpha r - 1}.
 \end{aligned}$$

Applying the inverse Laplace transform, we have:

$$\begin{aligned}
 h(x) &= \frac{d_0 + s_0}{d_1 + s_1} \sum_{j=0}^{\infty} \sum_{r=0}^j \frac{(d_2 + s_2)^{j-r} k_0(\alpha)^r (d_2 + s_2)^r}{(d_1 + s_1)^j} \binom{j}{r} \frac{x^{(1-\alpha)(j-r) - \alpha r}}{\Gamma((1-\alpha)(j-r) - \alpha r + 1)} \\
 &\quad + h(0) \sum_{j=0}^{\infty} \sum_{r=0}^j \frac{(-k_1(\alpha))^{j-r} (d_2 + s_2)^{j-r} (d_1 + s_1)^r}{k_0(\alpha)^j (d_2 + s_2)^j} \binom{j}{r} \frac{x^{j-r + \alpha r}}{\Gamma((j-r + \alpha r + 1))}.
 \end{aligned}$$

When we take  $p = j - r$ , we will get:

$$\begin{aligned}
 h(x) &= \frac{d_0 + s_0}{d_1 + s_1} \sum_{r=0}^{\infty} \sum_{p=0}^{\infty} \frac{(r+p)!}{r! p!} \frac{((d_2 + s_2))^{p+m} (k_0(\alpha))^r}{(d_1 + s_1)^{p+r}} \frac{x^{(1-\alpha)p - \alpha r}}{\Gamma((1-\alpha)p - \alpha r + 1)} \\
 &\quad + h(0) \sum_{r=0}^{\infty} \sum_{p=0}^{\infty} \frac{(r+p)!}{r! p!} \frac{(-k_1(\alpha))^p ((d_1 + s_1))^r}{k_0(\alpha)^{p+r}} (d_2 + s_2)^r \frac{x^{p + \alpha r}}{\Gamma(p + \alpha r + 1)},
 \end{aligned}$$

$$\begin{aligned}
 h(x) &= \frac{d_0 + s_0}{d_1 + s_1} \sum_{r=0}^{\infty} \sum_{p=0}^{\infty} \frac{(r+p)!}{r! p!} \left[ \frac{(d_2 + s_2) k_0(\alpha)}{(d_1 + s_1)} x^{-\alpha} \right]^r \left[ \frac{d_2 + s_2}{d_1 + s_1} x^{1-\alpha} \right]^p \frac{1}{\Gamma((1-\alpha)p - \alpha r + 1)} \\
 &\quad + h(0) \sum_{r=0}^{\infty} \sum_{p=0}^{\infty} \frac{(r+p)!}{r! p!} \left[ \frac{(d_1 + s_1)}{k_0(\alpha)(d_2 + s_2)} x^\alpha \right]^r \left[ \frac{-k_1(\alpha)}{k_0(\alpha)} x \right]^p \frac{1}{\Gamma(p + \alpha r + 1)}.
 \end{aligned}$$

We can write geometric series as [35]:

$$h(x) = \frac{d_0 + s_0}{d_1 + s_1} E_{1-\alpha, -\alpha, 1}^1 \left( \frac{d_2 + s_2}{d_1 + s_1} x^{1-\alpha}, \frac{(d_2 + s_2)k_0(\alpha)}{(d_1 + s_1)} x^{-\alpha} \right) + h(0) E_{1, \alpha, 1}^1 \left( \frac{-k_1(\alpha)}{k_0(\alpha)} x, \frac{(d_1 + s_1)}{k_0(\alpha)(d_2 + s_2)} x^\alpha \right).$$

### 4. Comparison

In this section, we give the results that were found for the first economic model with Caputo fractional derivative [30], Caputo–Fabrizio derivative in Caputo sense [30], Atangana–Baleanu derivative [30] and constant proportional Caputo derivative.

$$h(x) = \frac{(d_0 + s_0)}{(d_1 + s_1)} [1 - E_\alpha(-k(d_1 + s_1)x^\alpha)] + h(0) E_\alpha(-k(d_1 + s_1)x^\alpha),$$

$$h(x) = \frac{M(\alpha)h(0) \exp\left(\frac{ak(d_1+s_1)x}{-M(\alpha)+(\alpha-1)k(d_1+s_1)}\right)}{M(\alpha) - (\alpha - 1)k(d_1 + s_1)} - \frac{M(\alpha)(d_0 + s_0)(-1 + \exp\left(\frac{ak(d_1+s_1)x}{-M(\alpha)+(\alpha-1)k(d_1+s_1)}\right)) + (\alpha - 1)k(d_1 + s_1)}{(d_1 + s_1)(-M(\alpha) + (\alpha - 1)k(d_1 + s_1))}$$

$$h(x) = \frac{AB(\alpha)h(0)}{AB(\alpha) + (1 - \alpha)k(d_1 + s_1)} E_\alpha\left(-\frac{ak(d_1 + s_1)x^\alpha}{AB(\alpha) + (1 - \alpha)k(d_1 + s_1)}\right) + \frac{(1 - \alpha)k(d_0 + s_0)}{AB(\alpha) + (1 - \alpha)k(d_1 + s_1)} E_\alpha\left(-\frac{ak(d_1 + s_1)x^\alpha}{AB(\alpha) + (1 - \alpha)k(d_1 + s_1)}\right) + \frac{(d_0 + s_0)}{(d_1 + s_1)} \left[1 - E_\alpha\left(-\frac{ak(d_1 + s_1)x^\alpha}{AB(\alpha) + (1 - \alpha)k(d_1 + s_1)}\right)\right]$$

$$h(x) = \frac{(d_0 + s_0)}{(d_1 + s_1)} E_{1-\alpha, -\alpha, 1}^1 \left( \frac{-k_1(\alpha)}{k(d_0 + s_0)} x^{1-\alpha}, \frac{-k_0(\alpha)}{k(d_0 + s_0)} x^{-\alpha} \right) + h(0) E_{1, \alpha, 1}^1 \left( \frac{-k_1(\alpha)}{k_0(\alpha)} x, \frac{-k(d_1 + s_1)}{k_0(\alpha)} x^\alpha \right).$$

We give the results that were found for the second economic model with Caputo fractional derivative [30], Caputo–Fabrizio derivative in Caputo sense [30], Atangana–Baleanu derivative [30] and constant proportional Caputo derivative respectively as:

$$h(x) = -\frac{(d_0 + s_0)}{(d_1 + s_1)} \left[1 - E_\alpha\left(\frac{(d_1 + s_1)}{(d_2 + s_2)} x^\alpha\right)\right] + h(0) E_\alpha\left(\frac{(d_1 + s_1)}{(d_2 + s_2)} x^\alpha\right),$$

$$h(x) = \frac{M(\alpha)h(0)(d_2 + s_2)}{M(\alpha)(d_2 + s_2) + (\alpha - 1)d_1 + (\alpha - 1)s_1} \exp\left(\frac{\alpha(d_1 + s_1)x}{M(\alpha)(d_2 + s_2) + (\alpha - 1)d_1 + (\alpha - 1)s_1}\right) + \frac{(d_0 + s_0)((\alpha - 1)d_1 + (\alpha - 1)s_1 - M(\alpha)(-1 + \exp\left(\frac{ak(d_1+s_1)x}{M(\alpha)(d_2+s_2)+(\alpha-1)d_1+(\alpha-1)s_1}\right))(d_2 + s_2))}{(d_1 + s_1)(M(\alpha)(d_2 + s_2) + (\alpha - 1)d_1 + (\alpha - 1)s_1)}$$

$$\begin{aligned}
 h(x) &= \frac{AB(\alpha)h(0)(d_2 + s_2)}{AB(\alpha)(d_2 + s_2) + (\alpha - 1)d_1 + (\alpha - 1)s_1} E_\alpha \left( \frac{\alpha(d_1 + s_1)x^\alpha}{AB(\alpha)(d_2 + s_2) + (\alpha - 1)d_1 + (\alpha - 1)s_1} \right) \\
 &\quad - \frac{(\alpha - 1)(d_0 + s_0)}{AB(\alpha)(d_2 + s_2) + (\alpha - 1)d_1 + (\alpha - 1)s_1} E_\alpha \left( \frac{\alpha(d_1 + s_1)x^\alpha}{AB(\alpha)(d_2 + s_2) + (\alpha - 1)d_1 + (\alpha - 1)s_1} \right) \\
 &\quad + \frac{(\alpha - 1)(d_0 + s_0)}{\alpha(d_1 + s_1)} \left[ 1 - E_\alpha \left( \frac{\alpha(d_1 + s_1)x^\alpha}{AB(\alpha)(d_2 + s_2) + (\alpha - 1)d_1 + (\alpha - 1)s_1} \right) \right] \\
 h(x) &= \frac{(d_0 + s_0)}{(d_1 + s_1)} E_{1-\alpha, -\alpha, 1}^1 \left( \frac{(d_2 + s_2)}{(d_1 + s_1)} x^{1-\alpha}, \frac{(d_2 + s_2)k_0(\alpha)}{(d_1 + s_1)} x^{-\alpha} \right) \\
 &\quad + h(0) E_{1, \alpha, 1}^1 \left( \frac{-k_1(\alpha)}{k_0(\alpha)} x, \frac{(d_1 + s_1)}{k_0(\alpha)(d_2 + s_2)} x^\alpha \right).
 \end{aligned}$$

## 5. Conclusions

In this study, we gave some details of the Laplace transform. Then, we constructed the economic models and we solved these models by the Laplace transform. We concluded that the Laplace transform is very effective for solving such problems. We presented an application of the economic models by using the newly introduced constant proportional Caputo derivative. The obtained results will be useful for researchers who interest the economic models and the integral transforms.

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