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Lie analysis, conservation laws and travelling wave structures of nonlinear Bogoyavlenskii–Kadomtsev–Petviashvili equation

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ABSTRACT

In this paper, the Bogoyavlenskii–Kadomtsev–Petviashvili (BKP) equation is taken into consideration by means of Lie symmetry analysis. Infinitesimal generators are computed under the invariance criteria of Lie groups and symmetry group for each generator is reported. Henceforth, conjugacy classes of abelian algebra are used to find the similarity reductions, which convert the considered equation into ordinary differential equations (ODEs). Further, these ODEs are taken into consideration, and travelling wave structures are computed by applying different techniques. Moreover, the discussed model is discussed by means of nonlinear selfadjointness and conservation laws are derived for each Lie symmetry generator. For specific values of the physical parameters of the equation under discussion, the graphical behaviour of some solutions is depicted.

1. Introduction

In this article, we will discuss the Bogoyavlenskii-Kadomtsev–Petviashvili (BKP) equation [1,2] of the following form;

$$Q_{\theta\theta\tau} + Q_{\theta\theta\theta\theta\zeta} + 12Q_{\theta\theta}Q_{\theta\zeta} + 8Q_{\theta}Q_{\theta\theta\zeta} + 4Q_{\theta\theta\theta}Q_{\zeta} = Q_{\zeta\zeta\zeta}, \tag{1}$$

where $Q(\theta,\zeta,\tau)$ is an real field and shows the amplitude of the relevant waves, τ is the temporal component and θ, ζ are the spatial components. Eq. (1) is used to construct the propagation of dispersive waves, where nonlinear wave envelope is described by Q. It is important to mention that after ignoring the scattering impact term $Q_{\zeta\zeta\zeta}$, the Eq. (1) converts into the Calogero–Bogoyavlenskii–Schiff (CBS) equation [3] which explains the connection of a Riemann wave proliferating along the y-axis with a long wave to the x-axis. Eq. (1) is an extension of the Bogoyavlenskii–Schiff (BS) equation and Kadomtsev–Petviashvili (KP) equation [4,5].

Differential equations (DEs) have frequently been used in literature to model [6] many physical phenomena. In economics and biology, the behaviour of complex systems can be examined by using DEs. Solutions of nonlinear DEs play a significant role in mathematics, physics, and engineering [7]. The investigation of travelling wave solutions of nonlinear equations has an important role in mathematics and other nonlinear science. Nonlinear problems are more difficult to solve than linear problems. Different techniques and their applications are accessible in literature for the calculation of accurate results of various classes of DEs emerge in many branches of science, few techniques are given in [8–15]. Some famous techniques which have been developed to study the nonlinear differential equations are Lie symmetry approach [16,17], the Hirota's bilinear method [18], and Bäcklund transformation [19], etc.

The fundamental enquiry in the theory of nonlinear partial differential equations (NLPDEs) is to investigate the existence of solutions to considered NLPDEs. For example, real-world physical problems have been described by dispersive wave equations [20]. Notably, it is amazingly hard to find the exact solution even for integrable systems. Hypothetically, it is additionally fascinating to figure out what sort of NLPDEs can be especially intriguing given solutions, for example, solitons and lumps [21,22]. In recent decades, plasma physics [23] has grown quickly in the worldwide environment, astronomical environment, and particularly in the electromagnetic spread. The constant improvement of the exploration of fractional PDEs [24] is one of

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the basic wave processes in plasmas. Countless hypothetical and trial examine have been conducted on n ion-acoustic waves for a long time. (for details see [25,26]).

The symmetry analysis technique [27,28] plays an essential role in finding the solutions of nonlinear evolution equations that appear in mathematical physics, which can explain a lot of complicated physical phenomena, such as optics, plasma physics, and fluid mechanics, etc. It is well known that investigating the exact solutions of nonlinear evolution equations is always one of the central themes in mathematical physics.

Noether discovers the other important aspect of Lie symmetry in 1918. She found a relationship between symmetry and conserved quantities. Conservation laws have their significance in the theory of DEs. Literature is full of the contributions made by many researchers in developing the different techniques to construct conservation laws. Some of them are given in [29–33]. In the recent past, a lot of efforts have been put in the theory of self-adjointness [34,35], author extends the results given in [36].

The considered equation has been described by many means in literature due to its significance in different branches of science. In the past literature, general higher-order lump-type soliton and higher-order blend solution comprising of the kink soliton and lump-type soliton solutions and Gramian determinant solutions are constructed in [1,2], and their dynamical behaviours are discussed in mention papers. According to our knowledge, Lie analysis and conservation laws of the discussed model are not reported before and examined here. The format of the paper is as follows. In Section 2, preliminaries are presented. Lie analysis of Eq. (1) and travelling wave structures are presented in Section 3. Nonlinear self-adjointness and conservation laws are discussed in Section 4. In the end, the conclusion is stated.

2. Nonlinear selfadjointness

Consider a nth order partial differential equation:

$$F(\theta, Q, Q_1, \dots, Q_n) = 0, \tag{2}$$

where Q is a dependent variable and $\theta = (\theta^1, \theta^2, \dots, \theta^m)$ represents independent variables while Q_1 and Q_n show first and nth order derivatives of Q with respect θ, respectively.

The formal Lagrangian $\mathcal{L} = vF$ is assumed so the adjoint form of Eq. (2) becomes:

$$F^* \equiv \frac{\delta}{\delta Q}(vF) = 0, \tag{3}$$

where

$$\frac{\delta}{\delta Q} = \frac{\partial}{\partial Q} + \sum_{j=1}^{\infty}(-1)^j D_{i_1}\dots D_{i_j}\frac{\partial}{\partial Q_{i_1\dots i_j}}, \tag{4}$$

is called Euler–Lagrange operator, while D_i are called total derivative operators which can be defined as:

$$D_i = \frac{\partial}{\partial\theta^1} + Q_i\frac{\partial}{\partial Q} + Q_{ij}\frac{\partial}{\partial Q_j} + \cdots . \tag{5}$$

Definition 2.1. Eq. (2) is called strictly self-adjoint If the equation gained from its adjoint equation by using the transformation $v = Q$, for some $k \in \mathcal{D}$, such that

$$F^*|_{v=Q} = k(\theta, Q, \dots)F. \tag{6}$$

Definition 2.2. Eq. (2) is called the quasi self-adjoint if the equation acquired to its adjoint equation by using the transformation $v = \Psi(\theta) \neq 0$ such that:

$$F^*|_{v=\Psi(Q)} = k(\theta, Q, \dots)F, \tag{7}$$

where $k \in \mathcal{D}$.

Definition 2.3. If the equation gained from its adjoint equation then Eq. (2) is called the weak self-adjoint by using the transformation $v = \Psi(\theta, Q) \neq 0$ for a some function Ψ such that $\Psi_Q \neq 0$ and $\Psi_{\theta^i} \neq 0$ for some θ^i such that:

$$F^*|_{v=\Psi(\theta,Q)} = k(\theta, Q, \dots)F, \tag{8}$$

where $k \in \mathcal{D}$.

Definition 2.4. If the equation acquired from its adjoint equation then Eq. (1) is called the nonlinearly self-adjoint by the transformation $v = \Psi(\theta, Q)$, with a some function such that for $\Psi(\theta, Q) \neq 0$, Eq. (1) fulfils the following condition:

$$F^*|_{v=\Psi(\theta,Q)} = k(\theta, Q, \dots)F, \tag{9}$$

where $k \in \mathcal{D}$.

The vector space of all functions of finite order which can be differentials are represented by $\mathcal{D}$ in above definitions. It is important to mention here that the concept of equations mentioned in Def.(1), (2) and (4) are reported in [37,38], while Def.(3) is taken from [39].

Theorem 2.1. *Assume Lie point, Lie–Bäcklund or non-local symmetry of Eq. (2) of the form*

$$P = \xi^i\frac{\partial}{\partial\theta^i} + \eta\frac{\partial}{\partial Q},$$

with a standard Lagrangian $\mathcal{L}$ ($\frac{\delta\mathcal{L}}{\delta Q} = 0$) then the conserved vectors for Eq. (2) can be taken as:

$$C^{\theta^i} = \xi^i L + W\left[\frac{\partial\mathcal{L}}{\partial Q_i} - D_j\frac{\partial\mathcal{L}}{\partial Q_{ij}} + D_{jkx}\frac{\partial\mathcal{L}}{\partial Q_{ijk}}\right] + D_j(W)\left[\frac{\partial\mathcal{L}}{\partial Q_{ij}} - D_k\frac{\partial\mathcal{L}}{\partial Q_{ijk}}\right] + D_jD_k(W)\frac{\partial\mathcal{L}}{\partial Q_{ijk}}, \tag{10}$$

where W is called the Lie characteristic function, it can be obtained from

$$W = \phi - \xi^i Q_i\ , \tag{11}$$

while $D_i T^{\theta^i} = 0$.

3. Lie analysis of Eq. (1)

In this section, we will compute infinitesimal generators [40–42] of Eq. (1). For this, let us assume Lie algebra of infinitesimal generators for Eq. (1) is spanned by vector field:

$$P = \xi^1(\theta,\zeta,\tau,Q)\frac{\partial}{\partial\theta} + \xi^2(\theta,\zeta,\tau,Q)\frac{\partial}{\partial\zeta} + \xi^3(\theta,\zeta,\tau,Q)\frac{\partial}{\partial\tau} + \eta(\theta,\zeta,\tau,Q)\frac{\partial}{\partial Q}\ . \tag{12}$$

The invariance condition for Eq. (1) with P becomes:

$$P^{[5]}\Big(Q_{\theta\theta\tau} + Q_{\theta\theta\theta\theta\zeta} + 12Q_{\theta\theta}Q_{\theta\zeta} + 8Q_\theta Q_{\theta\theta\zeta} + 4Q_{\theta\theta\theta}Q_\zeta - Q_{\zeta\zeta\zeta}\Big)|_{Eq.\ (1)} = 0, \tag{13}$$

where $P^{[5]}$ is the fifth prolongation of P and defined as:

$$P^{[5]} = P + \eta^\theta\frac{\partial}{\partial Q_\theta} + \eta^\zeta\frac{\partial}{\partial Q_\zeta} + \eta^{\theta\theta}\frac{\partial}{\partial Q_{\theta\theta}} + \eta^{\theta\zeta}\frac{\partial}{\partial Q_{\theta\zeta}} + \eta^{\theta\theta\theta}\frac{\partial}{\partial Q_{\theta\theta\theta}} + \eta^{\zeta\zeta\zeta}\frac{\partial}{\partial Q_{\zeta\zeta\zeta}} + \eta^{\theta\theta\tau}\frac{\partial}{\partial Q_{\theta\theta\tau}} + \eta^{\theta\theta\zeta}\frac{\partial}{\partial Q_{\theta\theta\zeta}} + \eta^{\theta\theta\theta\theta\zeta}\frac{\partial}{\partial Q_{\theta\theta\theta\theta\zeta}}\ . \tag{14}$$

Eq. (13) leads to the following five dimensional Lie algebra of Eq. (1):

$$P_1 = \frac{\partial}{\partial\tau},\ P_2 = \frac{\partial}{\partial\theta},\ P_3 = \frac{\partial}{\partial\zeta},\ P_4 = \frac{\partial}{\partial Q},\ P_5 = \frac{\theta}{4}\frac{\partial}{\partial\theta} + \frac{\zeta}{2}\frac{\partial}{\partial\zeta} + \tau\frac{\partial}{\partial\tau} - \frac{Q}{4}\frac{\partial}{\partial Q}\ . \tag{15}$$

In Table 2, $\epsilon \ll 1$ is a group parameter.

Table 1
Commutator table.

$[P_i, P_j]$	P_1	P_2	P_3	P_4	P_5
P_1	0	0	0	0	$4P_1$
P_2	0	0	0	0	P_2
P_3	0	0	0	0	$2P_3$
P_4	0	0	0	0	$-P_4$
P_5	$-4P_1$	$-P_2$	$-2P_3$	P_4	0

Table 2
Adjoint representation.

Ad_g	P_1	P_2	P_3	P_4	P_5
P_1	P_1	P_2	P_3	P_4	$P_5 - \varepsilon P_1$
P_2	P_1	P_2	P_3	P_4	$2P_5 - \varepsilon P_3$
P_3	P_1	P_2	P_3	P_4	$4P_5 - \varepsilon P_2$
P_4	P_1	P_2	P_3	P_4	$4P_5 - \varepsilon P_4$
P_5	$P_1 e^{\frac{3}{2}\varepsilon}$	$P_2 e^{-\frac{1}{4}\varepsilon}$	$P_3 e^{\frac{1}{4}\varepsilon}$	P_4	P_5

3.1. Symmetry group of $(2+1)$-dimensional BKP equation

In this section, we obtain some new exact solutions from known ones, for this we compute the Lie symmetry groups from the corresponding symmetries.

The one parameter group is defined as:

$$H_i : (\theta, \zeta, \tau, Q) \to (\bar{\theta}, \bar{\zeta}, \bar{\tau}, \bar{Q}) \ , \tag{16}$$

which is produced by the generators of infinitesimal transformations P_i for $1 \le i \le 5$ after solving the following system of ordinary differential equations:

$$\frac{d}{d\varepsilon}(\bar{\theta}, \bar{\zeta}, \bar{\tau}, \bar{Q}) = \mu(\bar{\theta}, \bar{\zeta}, \bar{\tau}, \bar{Q}), \ with \ \ (\bar{\theta}, \bar{\zeta}, \bar{\tau}, \bar{Q})|_{\varepsilon=0} = (\theta, \zeta, \tau, Q) \ , \tag{17}$$

where ε is an discretionary real parameter and

$$\mu = \xi^1 Q_\theta + \xi^2 Q_\zeta + \xi^3 Q_\tau + \eta Q. \tag{18}$$

By using the infinitesimal generators ξ^1, ξ^2, ξ^3 and η, we have the following groups:

$$\begin{aligned} &H_1 : (\theta, \zeta, \tau, Q) \to (\theta, \zeta, \tau + \varepsilon, Q), \\ &H_2 : (\theta, \zeta, \tau, Q) \to (\theta + \varepsilon, \zeta, \tau, Q), \\ &H_3 : (\theta, \zeta, \tau, Q) \to (\theta, \zeta + \varepsilon, \tau, Q), \\ &H_4 : (\theta, \zeta, \tau, Q) \to (\theta, \zeta, \tau, Qe^{\varepsilon}), \\ &H_5 : (\theta, \zeta, \tau, Q) \to (\theta e^{\varepsilon}, \zeta e^{2\varepsilon}, \tau e^{4\varepsilon}, Qe^{-\varepsilon}) - A. \end{aligned} \tag{19}$$

It is significant to mention here, that the symmetry group H_1 is a time interpretation and H_2, H_3 illustrate the space invariance of Eq. (1). Further, corresponding new solutions can be obtained by using H_i, $1 \le i \le 5$. For example, if $Q = f(\theta, \zeta, \tau)$ is a known solution of Eq. (1), then by utilizing H_i, $1 \le i \le 5$ the new solutions Q_i, $1 \le i \le 5$ are acquired as follow:

$$\begin{aligned} Q_1 &= f_2(\theta, \zeta, \tau - \varepsilon), \\ Q_2 &= f_4(\theta - \varepsilon, \zeta, \tau), \\ Q_3 &= f_3(\theta, \zeta - \varepsilon, \tau), \\ Q_4 &= e^{-\varepsilon} f_5(\theta, \zeta, \tau), \\ Q_5 &= e^{\varepsilon} f_1(\theta e^{-\varepsilon}, \zeta e^{-2\varepsilon}, \tau e^{-4\varepsilon}). \end{aligned} \tag{20}$$

3.2. Optimal system and similarity reduction of Eq. (1)

It can be seen from Table 1 that $P = \{P_1, P_2, P_3, P_4\}$ forms an abelian subalgebra. Thus the one dimensional optimal system for P [43] is:

$$\begin{aligned} \pounds_1 &= \langle P_1 \rangle, \\ \pounds_2 &= \langle P_1 + aP_2 \rangle, \\ \pounds_3 &= \langle P_1 + aP_2 + bP_3 \rangle, \\ \pounds_4 &= \langle P_1 + aP_2 + bP_3 + cP_4 \rangle. \end{aligned} \tag{21}$$

Next task is to calculate the similarity variables for (21), which are further used to compute the all possible similarity reductions for Eq. (1).

3.2.1. $\pounds_1 = \langle P_1 \rangle$

In this case, we have

$$\varrho = \tau, \quad Q = U(\varrho),$$

which leads to a constant solution.

3.2.2. $\pounds_2 = \langle P_1 + aP_2 \rangle$

In this case, one can obtain

$$\varrho = \tau - a\theta, \quad Q = U(\varrho), \tag{22}$$

by putting the similarity variables (22) into Eq. (1) and we get the following solution:

$$Q(\theta, \zeta, \tau) = \frac{\alpha_1}{2}(\tau - a\theta)^2 + \alpha_2(\tau - a\theta) + \alpha_3. \tag{23}$$

3.2.3. $\pounds_3 = \langle P_1 + aP_2 + bP_3 \rangle$

For this class, one can easily get

$$\varrho = \tau - a\theta - b\zeta, \quad Q = U(\varrho). \tag{24}$$

By putting the similarity variables (24) into Eq. (1), we get the following ODE:

$$a^2 U''' - a^4 b U''''' + 12a^3 b (U'')^2 + 12a^3 b U' U''' + b^3 U''' = 0. \tag{25}$$

3.2.4. $\pounds_4 = \langle P_1 + aP_2 + bP_3 + cP_4 \rangle$

For this class, one can easily get

$$\varrho = b\theta - a\zeta, \quad Q(\theta, \zeta, \tau) = c\tau - U(\varrho). \tag{26}$$

By putting the similarity variables (26) into Eq. (1), we get the following ODE:

$$ab^4 U''''' - 12ab^3 U''^2 - 12ab^3 U' U''' - a^3 U''' = 0. \tag{27}$$

3.3. Travelling wave structure of Eq. (1)

In this section, we will compute the travelling wave structures of Eq. (1) by using Eqs. (25) and (27) by practicing two different techniques.

3.3.1. Travelling wave solutions from Eq. (25)

In this section, we will compute the travelling wave structures of Eq. (1) from Eq. (25) with the help of new extended direct algebraic method. Equating the linear term U''''' and nonlinear term $U'U'''$ in Eq. (25), gives the solution of the following form (for details see Appendix A section)

$$U(\varrho) = a_0 + a_1 g(\varrho). \tag{28}$$

Let us suppose $g(\varrho)$ is the solution of the following equation

$$g'(\varrho) = ln(\vartheta)(\alpha + \varsigma g(\varrho) + \gamma g^2(\varrho)), \tag{29}$$

putting Eqs. (28) and (29) into Eq. (25) and comparing the coefficients of powers of $g(\varrho)$, we get a system of algebraic equations. After solving the obtained system with the help of Maple for a_0, a_1 and a, we get the following sets of solutions:

$$a_0 = m_1, \quad a_1 = \gamma \ln(\vartheta) a, \quad a = \sqrt{\frac{(4\varsigma^2 b^4 ln^2(\vartheta) - 16\alpha\gamma b^4 ln^2(\vartheta))^{\frac{1}{2}} - 1}{8\alpha\gamma b \ln^2(\vartheta) - 2\varsigma^2 b \ln^2(\vartheta)}}, \tag{30}$$

where m_1 is an real constant.

Following the routine calculation as mentioned in preliminaries, we obtain the following travelling wave solutions of BKP equation:

Case 1. When $\Delta < 0$ and $\gamma \neq 0$, then
$Q_1(\theta,\zeta,\tau) = m_1 + \frac{1}{2}a\ln(\vartheta)[-\varsigma + \sqrt{-\Delta}\tan_\vartheta(\frac{\sqrt{-\Delta}}{2}\varrho)]$,
$Q_2(\theta,\zeta,\tau) = m_1 + \frac{1}{2}a\ln(\vartheta)[-\varsigma - \sqrt{-\Delta}\cot_\vartheta(\frac{\sqrt{-\Delta}}{2}\varrho)]$,
$Q_3(\theta,\zeta,\tau) = m_1 + \frac{1}{2}a\ln(\vartheta)[-\varsigma + \sqrt{-\Delta}(\tan_\vartheta(\sqrt{-\Delta}\varrho) \pm \sqrt{rs}\sec_\vartheta(\sqrt{-\Delta}\varrho))]$,
$Q_4(\theta,\zeta,\tau) = m_1 + \frac{1}{2}a\ln(\vartheta)[-\varsigma - \sqrt{-\Delta}(\cot_\vartheta(\sqrt{-\Delta}\varrho) \pm \sqrt{rs}\csc_\vartheta(\sqrt{-\Delta}\varrho))]$,
$Q_5(\theta,\zeta,\tau) = m_1 + \frac{1}{2}a\ln(\vartheta)[-\varsigma + \frac{\sqrt{-\Delta}}{2}(\tan_\vartheta(\frac{\sqrt{-\Delta}}{4}\varrho) - \cot_\vartheta(\frac{\sqrt{-\Delta}}{4}\varrho))]$.
Case 2. When $\Delta > 0$ and $\gamma \neq 0$, then
$Q_6(\theta,\zeta,\tau) = m_1 + \frac{1}{2}a\ln(\vartheta)[-\varsigma - \sqrt{\Delta}\tanh_\vartheta(\frac{\sqrt{\Delta}}{2}\varrho)]$,
$Q_7(\theta,\zeta,\tau) = m_1 + \frac{1}{2}a\ln(\vartheta)[-\varsigma - \sqrt{\Delta}\coth_\vartheta(\frac{\sqrt{\Delta}}{2}\varrho)]$,
$Q_8(\theta,\zeta,\tau) = m_1 + \frac{1}{2}a\ln(\vartheta)[-\varsigma - \sqrt{\Delta}(\tanh_\vartheta(\sqrt{\Delta}\varrho) \pm \imath\sqrt{rs}\,\mathrm{sech}_\vartheta(\sqrt{\Delta}\varrho))]$,
$Q_9(\theta,\zeta,\tau) = m_1 + \frac{1}{2}a\ln(\vartheta)[-\varsigma - \sqrt{\Delta}(\coth_\vartheta(\sqrt{\Delta}\varrho) \pm \sqrt{rs}\,\mathrm{csch}_\vartheta(\sqrt{\Delta}\varrho))]$,
$Q_{10}(\theta,\zeta,\tau) = m_1 + \frac{1}{2}a\ln(\vartheta)[-\varsigma - \frac{\sqrt{\Delta}}{2}(\tanh_\vartheta(\frac{\sqrt{\Delta}}{4}\varrho) + \coth_\vartheta(\frac{\sqrt{\Delta}}{4}\varrho))]$.
Case 3. When $\gamma\alpha > 0$ and $\varsigma = 0$, then
$Q_{11}(\theta,\zeta,\tau) = m_1 + \sqrt{\gamma\alpha}a\ln(\vartheta)\tan_\vartheta(\sqrt{\gamma\alpha}\varrho)$,
$Q_{12}(\theta,\zeta,\tau) = m_1 - \sqrt{\gamma\alpha}a\ln(\vartheta)\cot_\vartheta(\sqrt{\gamma\alpha}\varrho)$,
$Q_{13}(\theta,\zeta,\tau) = m_1 + \sqrt{\gamma\alpha}a\ln(\vartheta)[\tan_\vartheta(2\sqrt{\gamma\alpha}\varrho) \pm \sqrt{rs}\sec_\vartheta(2\sqrt{\gamma\alpha}\varrho)]$,
$Q_{14}(\theta,\zeta,\tau) = m_1 - \sqrt{\gamma\alpha}a\ln(\vartheta)[\cot_\vartheta(2\sqrt{\gamma\alpha}\varrho) \pm \sqrt{rs}\csc_\vartheta(2\sqrt{\gamma\alpha}\varrho)]$,
$Q_{15}(\theta,\zeta,\tau) = m_1 + \frac{1}{2}\sqrt{\gamma\alpha}a\ln(\vartheta)[\tan_\vartheta(\frac{\sqrt{\gamma\alpha}}{2}\varrho) - \cot_\vartheta(\frac{\sqrt{\gamma\alpha}}{2}\varrho)]$.
Case 4. When $\gamma\alpha < 0$ and $\varsigma = 0$, then
$Q_{16}(\theta,\zeta,\tau) = m_1 - \sqrt{-\gamma\alpha}a\ln(\vartheta)\tanh_\vartheta(\sqrt{-\gamma\alpha}\varrho)$,
$Q_{17}(\theta,\zeta,\tau) = m_1 - \sqrt{-\gamma\alpha}a\ln(\vartheta)\coth_\vartheta(\sqrt{-\gamma\alpha}\varrho)$,
$Q_{18}(\theta,\zeta,\tau) = m_1 - \sqrt{-\gamma\alpha}a\ln(\vartheta)[\tanh_\vartheta(2\sqrt{-\gamma\alpha}\varrho) \pm \imath\sqrt{rs}\,\mathrm{sech}_\vartheta(2\sqrt{-\gamma\alpha}\varrho)]$,
$Q_{19}(\theta,\zeta,\tau) = m_1 - \sqrt{-\gamma\alpha}a\ln(\vartheta)[\coth_\vartheta(2\sqrt{-\gamma\alpha}\varrho) \pm \sqrt{rs}\,\mathrm{csch}_\vartheta(2\sqrt{-\gamma\alpha}\varrho)]$,
$Q_{20}(\theta,\zeta,\tau) = m_1 - \frac{1}{2}\sqrt{-\gamma\alpha}a\ln(\vartheta)[\tanh_\vartheta(\frac{\sqrt{-\gamma\alpha}}{2}\varrho) + \coth_\vartheta(\frac{\sqrt{-\gamma\alpha}}{2}\varrho)]$.
Case 5. When $\varsigma = 0$ and $\gamma = \alpha$, then
$Q_{21}(\theta,\zeta,\tau) = m_1 + \gamma a\ln(\vartheta)\tan_\vartheta(\alpha\varrho)$,
$Q_{22}(\theta,\zeta,\tau) = m_1 - \gamma a\ln(\vartheta)\cot_\vartheta(\alpha\varrho)$,
$Q_{23}(\theta,\zeta,\tau) = m_1 + \gamma a\ln(\vartheta)[\tan_\vartheta(2\alpha\varrho) \pm \sqrt{rs}\sec_\vartheta(2\alpha\varrho)]$,
$Q_{24}(\theta,\zeta,\tau) = m_1 - \gamma a\ln(\vartheta)[\cot_\vartheta(2\alpha\varrho) \pm \sqrt{rs}\csc_\vartheta(2\alpha\varrho)]$,
$Q_{25}(\theta,\zeta,\tau) = m_1 + \frac{1}{2}\gamma a\ln(\vartheta)[\tan_\vartheta(\frac{\alpha}{2}\varrho) - \cot_\vartheta(\frac{\alpha}{2}\varrho)]$.
Case 6. When $\varsigma = 0$ and $\gamma = -\alpha$, then
$Q_{26}(\theta,\zeta,\tau) = m_1 - \gamma a\ln(\vartheta)\tanh_\vartheta(\alpha\varrho)$,
$Q_{27}(\theta,\zeta,\tau) = m_1 - \gamma a\ln(\vartheta)\coth_\vartheta(\alpha\varrho)$,
$Q_{28}(\theta,\zeta,\tau) = m_1 - \gamma a\ln(\vartheta)[\tanh_\vartheta(2\alpha\varrho) \pm \imath\sqrt{rs}\,\mathrm{sech}_\vartheta(2\alpha\varrho)]$,
$Q_{29}(\theta,\zeta,\tau) = m_1 - \gamma a\ln(\vartheta)[\coth_\vartheta(2\alpha\varrho) \pm \sqrt{rs}\,\mathrm{csch}_\vartheta(2\alpha\varrho)]$,
$Q_{30}(\theta,\zeta,\tau) = m_1 - \frac{1}{2}\gamma a\ln(\vartheta)[\tanh_\vartheta(\frac{\alpha}{2}\varrho) + \coth_\vartheta(\frac{\alpha}{2}\varrho)]$.
Case 7. When $\varsigma^2 = 4\gamma\alpha$ then
$Q_{31}(\theta,\zeta,\tau) = m_1 - \frac{2\gamma a d(\varsigma\varrho\ln(\vartheta)+2)}{\varsigma^2\varrho}$.
Case 8. When $\varsigma = \lambda$, $\alpha = m\lambda$ $(m \neq 0)$ and $\gamma = 0$, then
$Q_{32}(\theta,\zeta,\tau) = m_1$.
we get trivial solution.
Case 9. When $\varsigma = \gamma = 0$, then
$Q_{33}(\theta,\zeta,\tau) = m_1$.
we get trivial solution.
Case 10. When $\varsigma = \alpha = 0$, then
$Q_{34}(\theta,\zeta,\tau) = m_1 - \frac{d}{\varrho}$.
Case 11. When $\alpha = 0$ and $\varsigma \neq 0$, then
$Q_{35}(\theta,\zeta,\tau) = m_1 - \frac{r\varsigma a\ln(\vartheta)}{[\cosh_\vartheta(\varsigma\varrho)-\sinh_\vartheta(\varsigma\varrho)+r]}$.
$Q_{36}(\theta,\zeta,\tau) = m_1 - \frac{\varsigma a\ln(\vartheta)[\cosh_\vartheta(\varsigma\varrho)+\sinh_\vartheta(\varsigma\varrho)]}{[\cosh_\vartheta(\varsigma\varrho)+\sinh_\vartheta(\varsigma\varrho)+s]}$.
Case 12. When $\varsigma = \lambda$, $\gamma = m\lambda$ $(m \neq 0)$ and $\alpha = 0$, then
$Q_{37}(\theta,\zeta,\tau) = m_1 + \gamma a\ln(\vartheta)\frac{r\vartheta^{\lambda\varrho}}{s-mr\vartheta^{\lambda\varrho}}$.

3.3.2. Exact explicit solutions from Eq. (27)

In this section, we will compute the exact explicit solutions of Eq. (1) from Eq. (27) with the help of tanh technique.

Suppose the general solution of Eq. (1) is of the form (for details see Appendix B section)

$$U(\varrho) = a_0 + \sum_{n=1}^{N} a_n Y^n, \tag{31}$$

Equating the linear term U''''' and nonlinear term $U'U'''$ in Eq. (27), we get $N = 1$, using in Eq. (31) and we get:

$$U(\varrho) = a_0 + a_1 Y, \tag{32}$$

After doing routine calculations as mentioned in the description, we get the following sets of solutions for Eq. (1):

Set 1 :
$a_0 = d_1, \quad a_1 = -b, \quad a = 2b^2.$
$Q(\theta,\zeta,\tau) = c\tau + b\tanh(b\theta - 2b^2\zeta) - d_1$,
where d_1 is an arbitrary constant.
Set 2 :
$a_0 = d_2, \quad a_1 = -b, \quad a = -2b^2.$
$Q(\theta,\zeta,\tau) = c\tau + b\tanh(b\theta + 2b^2\zeta) - d_2$,
where d_2 is an arbitrary constant.
Set 3 :
$a_0 = d_3, \quad a_1 = d_4, \quad a = 0.$
$Q(\theta,\zeta,\tau) = c\tau - d_4\tanh(b\theta) - d_3$,
where d_3 and d_4 are arbitrary constants.

3.3.3. Graphical interpretation of travelling wave structures

In this section, we will interpret some solutions graphically. By taking the different values of involving parameters, we have represented the different $2D$ and $3D$ graphical behaviour of travelling wave solutions. Different graphical behaviour of $Q_6(\theta,\zeta,\tau)$ for $\alpha = -1$, $\varsigma = 0$, $\gamma = 1$, $b = -1$, $m_1 = 1$, $\tau = 1$, and $\vartheta = e$. in Figs. 1(a) and 1(b). Furthermore, we have represented the different $2D$ and $3D$ graphical behaviour of $Q_8(\theta,\zeta,\tau)$ for $\alpha = -1$, $\varsigma = 0$, $\gamma = 1$, $b = -1$, $m_1 = 1$, $\tau = 1$, $r = -1$, $s = 1$, and $\vartheta = e$ is presented in Figs. 2(a) and 2(b). we have showed the different $2D$ and $3D$ graphical structures of $Q_{10}(\theta,\zeta,\tau)$ for $\alpha = -1$, $\varsigma = 0$, $\gamma = 1$, $b = -1$, $m_1 = 1$, $\tau = 1$, $r = -1$, $s = 1$ and $\vartheta = e$ in Figs. 3(a) and 3(b). By choosing the different values of involving parameters and we represent the $2D$ and $3D$ graphical behaviour of the solution $Q_{11}(\theta,\zeta,\tau)$ for $\alpha = 1$, $\gamma = 1$, $a = 1$, $b = 1$, $m_1 = 1$, $\tau = 1$, $\vartheta = e$ in Figs. 4(a) and 4(b). Different graphical behaviour of $Q_{26}(\theta,\zeta,\tau)$ for $\alpha = 1, \varsigma = 0, \gamma = 1, b = -1, m_1 = 1, \tau = 1$ and $\vartheta = e$ in Figs. 5(a) and 5(b). In Figs. 6(a) and 6(b), we have showed the graphical representation of $Q_{35}(\theta,\zeta,\tau)$ for $\varsigma = 1$, $d = 1$, $r = 0.5$, $a = 1$, $b = 1$, $m_1 = 1$, $\tau = 5$ and $\vartheta = e$.

We have showed the different $2D$ and $3D$ graphical behaviour of travelling wave solutions. Different graphical structures of Set 1 for $b = 1$, $c = 1$, $d_1 = 0.5$ and $\tau = 2$ in Figs. 7(a) and 7(b). By taking the different values of involving parameters and we represent the graphical behaviour of Set 2 for $b = 1$, $c = 1$, $d_2 = 0.5$ and $\tau = 2$ in Figs. 8(a) and 8(b) and graphical representation of Set 3 for $b = 1$, $c = 1$, $d_3 = 0.5$ and $d_4 = 1$. are represented in Figs. 9(a) and 9(b).

4. Nonlinear self-adjointness and conservation laws

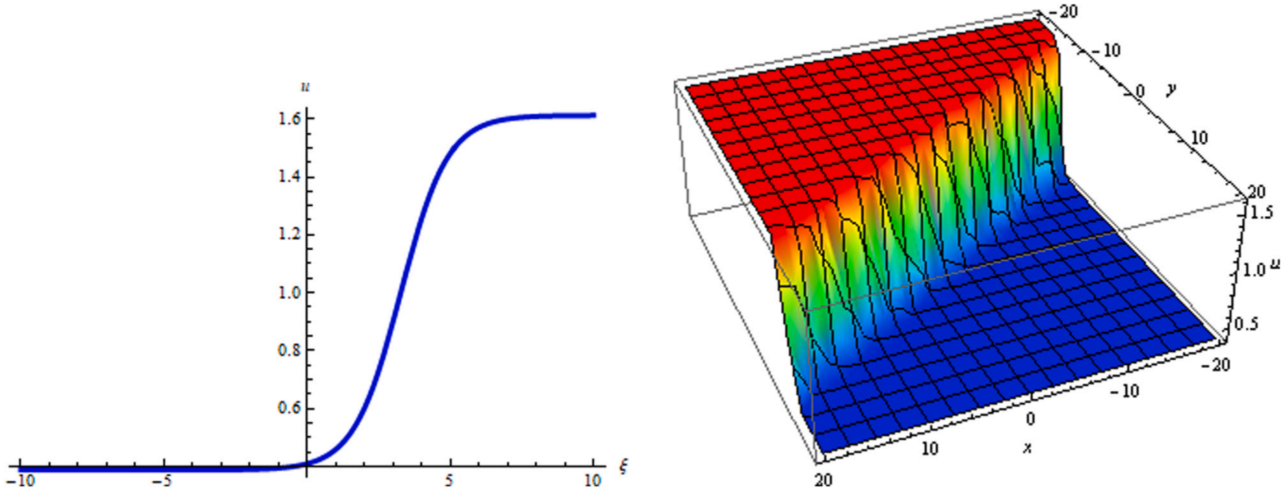
4.1. Nonlinear self-adjointness classification

In this section, we will present classification of Eq. (1) via theory of nonlinear self-adjointness. For this let us suppose formal Lagrangian $\mathcal{L}$ of the form:

$$\mathcal{L} = \psi(\theta,\zeta,\tau)\Big(Q_{\theta\theta\tau} + Q_{\theta\theta\theta\theta\zeta} + 12Q_{\theta\theta}Q_{\theta\zeta} + 8Q_\theta Q_{\theta\theta\zeta} + 4Q_{\theta\theta\theta}Q_\zeta - Q_{\zeta\zeta\zeta}\Big), \tag{33}$$

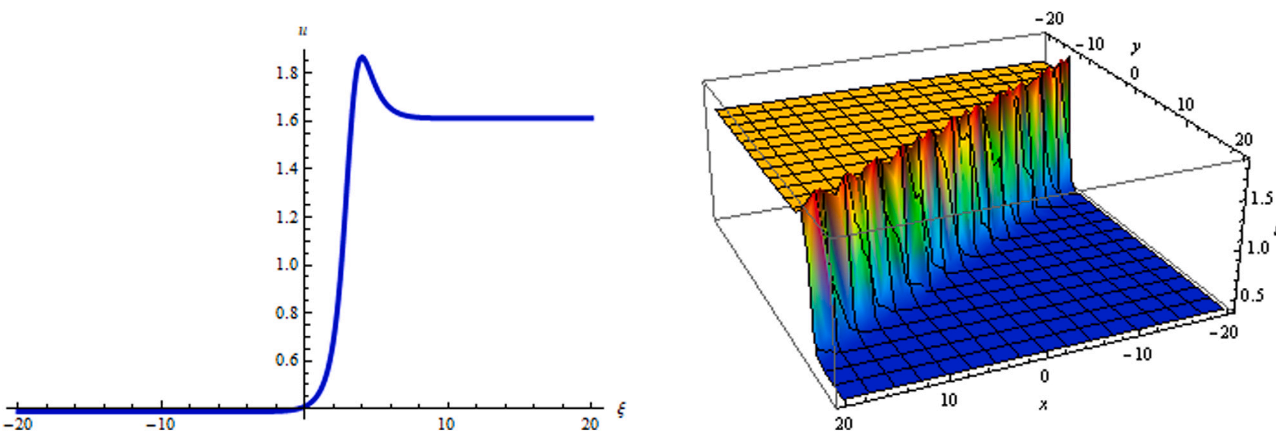
where $\psi(\theta,\zeta,\tau)$ is the new dependent variable based on Eq. (33), now we define action integral which can be written as:

$$\int_0^\tau\int_{\Omega_1}\int_{\Omega_2}\mathcal{L}(\theta,\zeta,\tau,Q,Q_\theta,Q_\zeta,Q_{\theta\theta},Q_{\theta\zeta},Q_{\theta\theta\tau},Q_{\theta\theta\zeta},Q_{\theta\theta\theta},Q_{\zeta\zeta\zeta},Q_{\theta\theta\theta\theta\zeta})d\theta d\zeta d\tau. \tag{34}$$



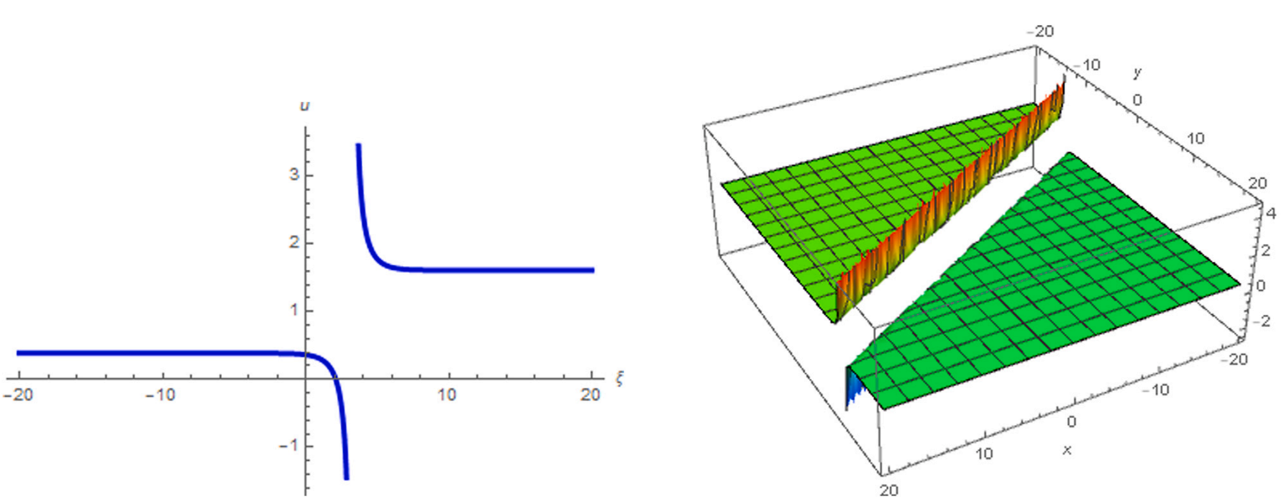
2D Graphics 3D Graphics

Fig. 1. Graphical representation of $Q_6(\theta,\zeta,\tau)$ for $\alpha=-1,\varsigma=0,\gamma=1,b=-1,m_1=1,\tau=1,$ and $\vartheta=e$.



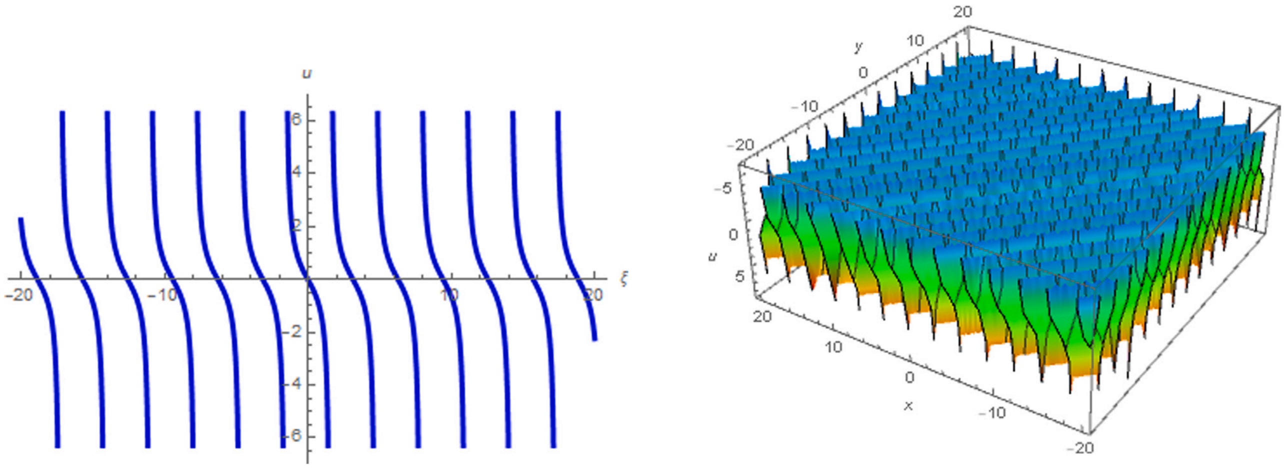
2D Graphics 3D Graphics

Fig. 2. Graphical representation of $Q_8(\theta,\zeta,\tau)$ for $\alpha=-1,\varsigma=0,\gamma=1,b=-1,m_1=1,\tau=1,r=-1,s=1,$and $\vartheta=e$.



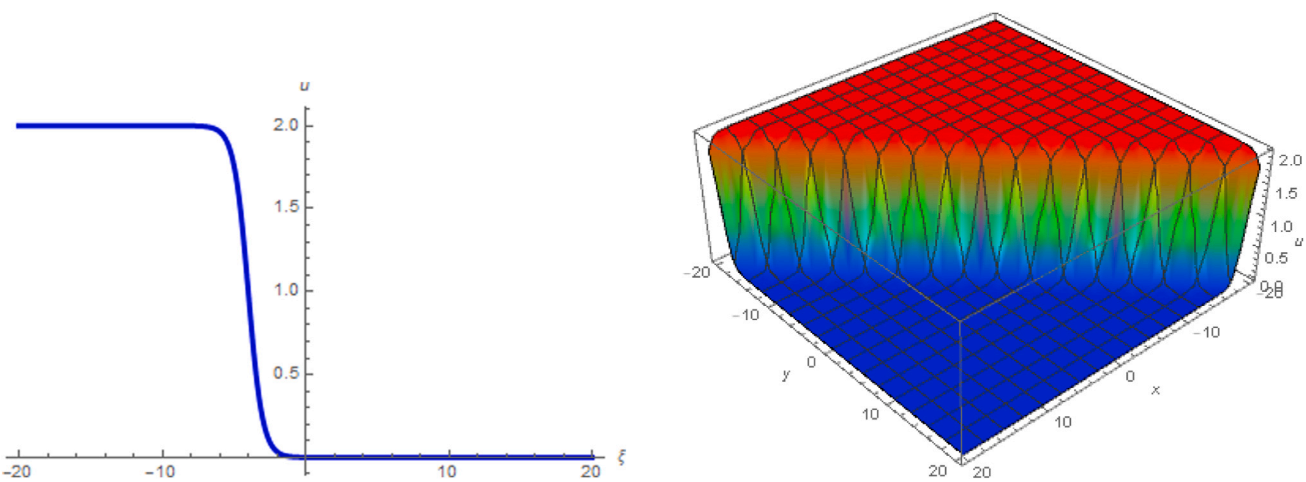
2D Graphics 3D Graphics

Fig. 3. Graphical representation of $Q_{10}(\theta,\zeta,\tau)$ for $\alpha=-1,\ \varsigma=0,\ \gamma=1,\ b=-1,\ m_1=1,\ \tau=1,\ r=-1,\ s=1$ and $\vartheta=e$.



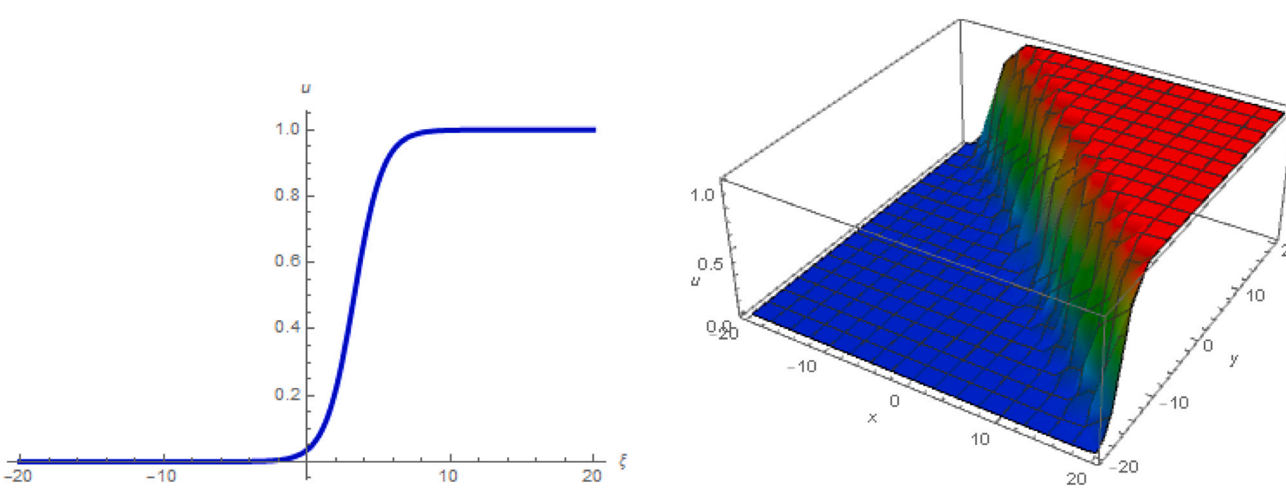
2D Graphics 3D Graphics

Fig. 4. Graphical representation of $Q_{11}(\theta, \zeta, \tau)$ for $\alpha = 1,\ \gamma = 1,\ a = 1,\ b = 1,\ m_1 = 1,\ \tau = 1,\ \vartheta = e$.



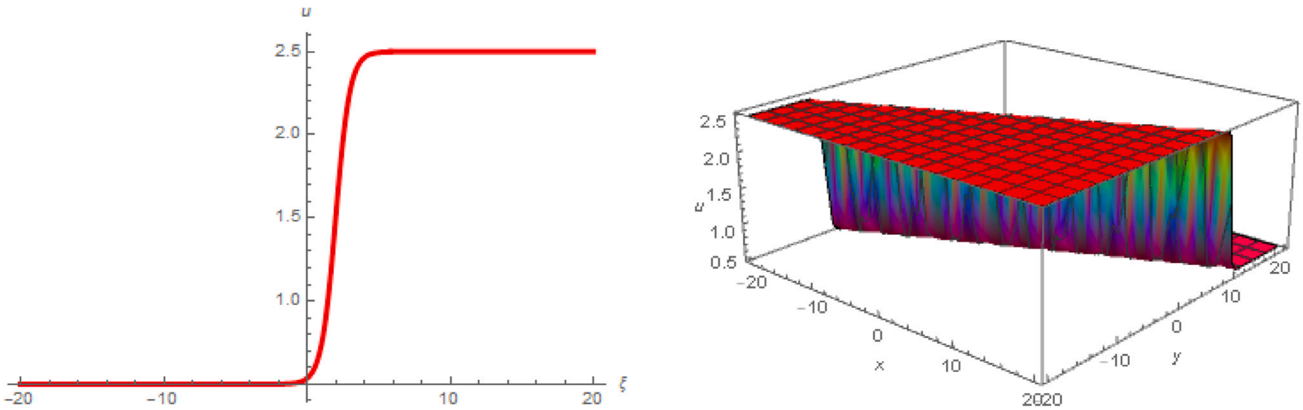
2D Graphics 3D Graphics

Fig. 5. Graphical representation of $Q_{26}(\theta, \zeta, \tau)$ for $\alpha = 1,\ \varsigma = 0,\ \gamma = 1,\ b = -1,\ m_1 = 1,\ \tau = 1$ and $\vartheta = e$.



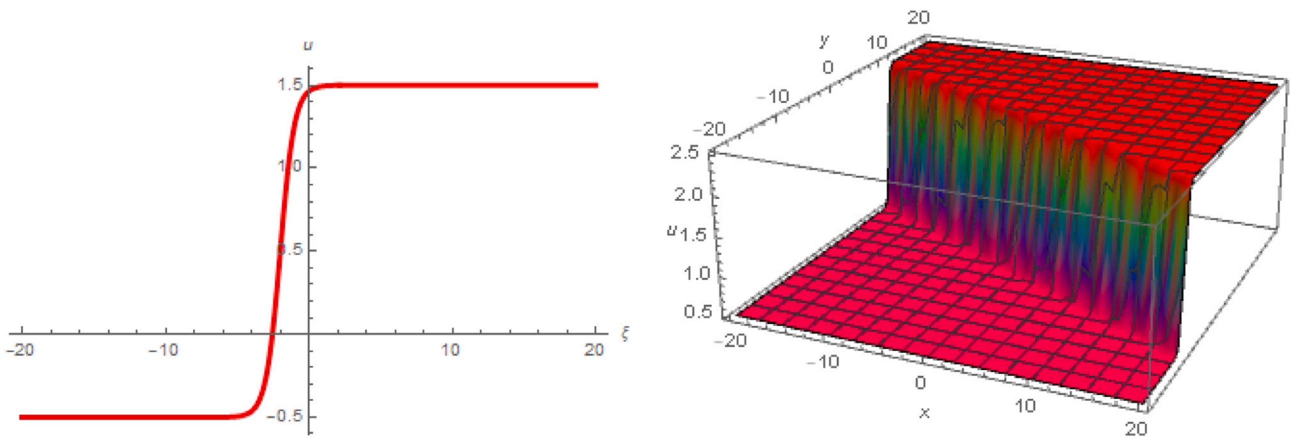
2D Graphics 3D Graphics

Fig. 6. Graphical representation of $Q_{35}(\theta, \zeta, \tau)$ for $\varsigma = 1,\ d = 1,\ r = 0.5,\ a = 1,\ b = 1,\ m_1 = 1,\ \tau = 5$ and $\vartheta = e$.



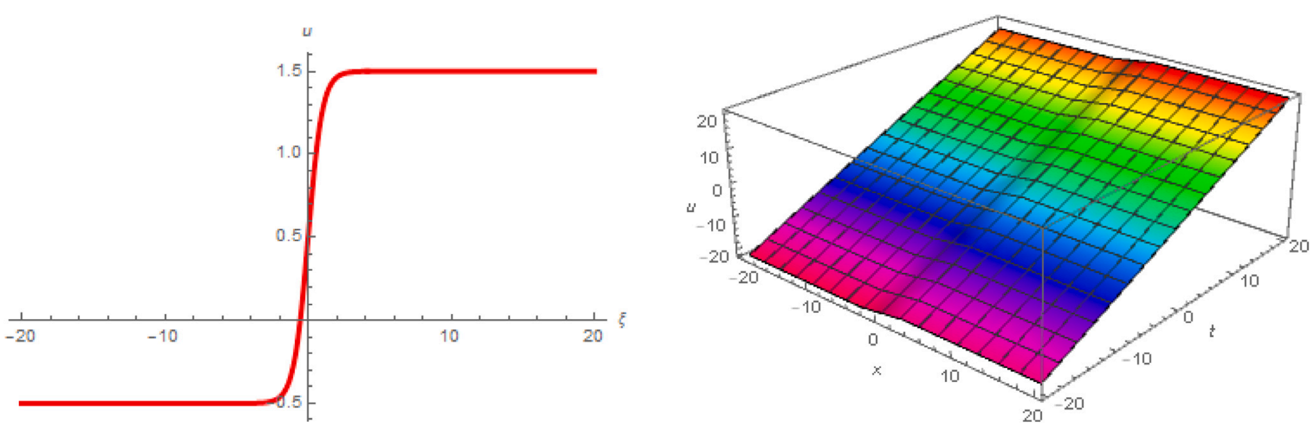
2D Graphics   3D Graphics

Fig. 7. Graphical representation of Set 1 for $b = 1$, $c = 1$, $d_1 = 0.5$ and $\tau = 2$.



2D Graphics   3D Graphics

Fig. 8. Graphical representation of Set 2 for $b = 1$, $c = 1$, $d_2 = 0.5$ and $\tau = 2$.



2D Graphics   3D Graphics

Fig. 9. Graphical representation of Set 3 for $b = 1$, $c = 1$, $d_3 = 0.5$ and $d_4 = 1$.

After applying Euler–Lagrange operator on Lagrangian (33), we get:

$$D_\zeta^3[\psi] - D_\theta^4 D_\zeta[\psi] - D_\theta^2 D_\tau[\psi] + 12 D_\theta D_\zeta[\psi Q_{\theta\theta}] - 8 D_\theta^2 D_\zeta[\psi Q_\theta] + 12 D_\theta^2[\psi Q_{\theta\zeta}] - 4 D_\theta^3[\psi Q_\zeta] - 8 D_\theta[\psi Q_{\theta\theta\zeta}] - 4 D_\zeta[\psi Q_{\theta\theta\theta}] = 0. \tag{35}$$

where $\psi = \psi(\theta, \zeta, \tau)$ and $Q = Q(\theta, \zeta, \tau)$. Further computing D_θ, D_ζ and D_τ in Eq. (35) and we get

$$\psi_{\theta\theta\theta}(1 - 4Q_\zeta) - \psi_{\theta\theta\theta\theta\zeta} - 8\psi_{\theta\theta} Q_{\theta\zeta} - 4\psi_{\theta\zeta} Q_{\theta\theta} - 8\psi_{\theta\theta\zeta} Q_\theta - \psi_{\theta\theta\tau} = 0. \tag{36}$$

Now by using the Definitions 2.1–2.4 and doing the routine calculations, we can state the following theorem:

Theorem 4.1. *Eq. (1) is neither nonlinear self-adjoint or quasi self-adjoint nor weak self-adjoint, however Eq. (1) is strictly self-adjoint for $\upsilon = \psi$, where ψ can be taken as:*

$$\psi(\theta,\zeta,\tau) = \theta f(\tau) + h(\zeta,\tau), \tag{37}$$

where $f(\tau)$ and $h(\zeta,\tau)$ are arbitrary functions.

It is important to mention that due to arbitrariness of $f(\tau)$ and $h(\zeta,\tau)$, Eq. (1) contains infinite many conservation laws, which are reported in next section.

4.2. Conservation laws

In this portion, we are represent the conserved vectors for Eq. (1) which fulfils the following condition

$$[D_\tau(C^\tau) + D_\theta(C^\theta) + D_\zeta(C^\zeta)]_{Eq.\ (1)} = 0\ , \tag{38}$$

where C^τ, C^θ and C^ζ are the conserved vectors. Now supposing, the case if θ, ζ and τ are independent variables and $Q(\theta,\zeta,\tau)$ is dependent variable, then we have

$$\bar{P} + D_\tau(\xi^1)I + D_\theta(\xi^2)I + D_\zeta(\xi^3)I = W\frac{\delta}{\delta Q} + D_\tau(C^\tau) + D_\theta(C^\theta) + D_\zeta(C^\zeta)\ , \tag{39}$$

where I is identity operator and $\frac{\delta}{\delta Q}$ is EL-generator, C^τ, C^θ and C^ζ are the conserved vectors.

$\bar{P}$ is written as

$$\bar{P} = \xi^1\frac{\partial}{\partial\tau} + \xi^2\frac{\partial}{\partial\theta} + \xi^3\frac{\partial}{\partial\zeta} + \eta\frac{\partial}{\partial Q} + \eta^\theta\frac{\partial}{\partial Q_\theta} + \eta^\zeta\frac{\partial}{\partial Q_\zeta} + \eta^{\theta\theta}\frac{\partial}{\partial Q_{\theta\theta}} + \eta^{\theta\zeta}\frac{\partial}{\partial Q_{\theta\zeta}} + \eta^{\theta\theta\tau}\frac{\partial}{\partial Q_{\theta\theta\tau}} + \eta^{\theta\theta\zeta}\frac{\partial}{\partial Q_{\theta\theta\zeta}} + \eta^{\zeta\zeta\zeta}\frac{\partial}{\partial Q_{\zeta\zeta\zeta}} + \eta^{\theta\theta\theta\zeta}\frac{\partial}{\partial Q_{\theta\theta\theta\zeta}}, \tag{40}$$

and W is the Lie characteristic function which can be written as

$$W = \eta - \xi^1 u_\tau - \xi^3 Q_\theta - \xi^1 Q_\zeta.$$

In this case, C^i for three independent variables θ, ζ and τ can be written as

$$\begin{aligned} C^i =& \xi^i\mathcal{L} + W_\alpha\left[\frac{\partial\mathcal{L}}{\partial Q^\alpha_i} - D_j\left(\frac{\partial\mathcal{L}}{\partial Q^\alpha_{ij}}\right) + D_jD_k\left(\frac{\partial\mathcal{L}}{\partial Q^\alpha_{ijk}}\right) - \cdots\right] + D_j(W_\alpha)\left[\frac{\partial\mathcal{L}}{\partial Q^\alpha_{ij}}\right. \\ &\left. - D_k\left(\frac{\partial\mathcal{L}}{\partial Q^\alpha_{ijk}}\right) + \cdots\right] + D_jD_k(W_\alpha)\left[\frac{\partial\mathcal{L}}{\partial Q^\alpha_{ijk}} - \cdots\right] + \cdots\ , \end{aligned} \tag{41}$$

where

$$\xi^\tau = \xi^1,\ \xi^\theta = \xi^2,\ \xi^\zeta = \xi^3 \ \textit{and}\ \alpha = 1,2,3\ldots\ .$$

For the operators $P_i (i = 1,2\ldots,5)$ in (12), the corresponding characteristic functions have the following form

$$W_1 = -\frac{Q}{4} - \tau Q_\tau - \frac{\theta}{4}Q_\theta - \frac{\zeta}{2}Q_\zeta, W_2 = -Q_\tau, W_3 = -Q_\zeta, W_4 = -Q_\theta, W_5 = 1. \tag{42}$$

Putting (33) and (42) into (41), one can obtain

$$\begin{aligned} C^\tau =& \xi^\tau\mathcal{L} + W_i\left[D_\theta D_\tau(\frac{\partial\mathcal{L}}{\partial Q_{\theta\theta\tau}})\right] + D_\tau(W_i)\left[-D_\theta\frac{\partial\mathcal{L}}{\partial Q_{\theta\theta\tau}}\right] + D_\theta(W_i)\left[-D_\tau\frac{\partial\mathcal{L}}{\partial Q_{\theta\theta\tau}}\right] \\ &+ D_\theta D_\tau(W_i)\left[-D_\tau(\frac{\partial\mathcal{L}}{\partial Q_{\theta\theta\tau}})\right], \quad \textit{where} \quad i = 1,2,3,4,5\ . \end{aligned} \tag{43}$$

$$\begin{aligned} C^\theta =& \xi^\theta\mathcal{L} + W_i\left[\frac{\partial\mathcal{L}}{\partial Q_\theta} + D_\theta D_\tau(\frac{\partial\mathcal{L}}{\partial Q_{\theta\theta\tau}}) + D^3_\theta D_\zeta(\frac{\partial\mathcal{L}}{\partial Q_{\theta\theta\theta\theta\zeta}})\right. \\ &- D_\theta(\frac{\partial\mathcal{L}}{\partial Q_{\theta\theta}}) - D_\zeta(\frac{\partial\mathcal{L}}{\partial Q_{\theta\zeta}}) \\ &\left.+ D_\theta D_\zeta(\frac{\partial\mathcal{L}}{\partial Q_{\theta\theta\zeta}}) + D^2_\theta(\frac{\partial\mathcal{L}}{\partial Q_{\theta\theta\theta}})\right] + D_\tau(W_i) \\ &\times\left[-D_\theta\frac{\partial\mathcal{L}}{\partial Q_{\theta\theta\tau}}\right] + D_\theta(W_i)\left[\frac{\partial\mathcal{L}}{\partial Q_{\theta\theta}}\right. \\ &- D_\tau\frac{\partial\mathcal{L}}{\partial Q_{\theta\theta\tau}} + D_\theta D_\zeta(\frac{\partial\mathcal{L}}{\partial Q_{\theta\theta\theta\theta\zeta}}) - D_\zeta(\frac{\partial\mathcal{L}}{\partial Q_{\theta\theta\zeta}}) \\ &\left.- D_\theta(\frac{\partial\mathcal{L}}{\partial Q_{\theta\theta\theta}})\right] + D_\zeta(W_i)\left[\frac{\partial\mathcal{L}}{\partial Q_{\theta\zeta}}\right. \\ &\left.- D_\theta(\frac{\partial\mathcal{L}}{\partial Q_{\theta\theta\zeta}}) - D^3_\theta(\frac{\partial\mathcal{L}}{\partial Q_\theta})\right] + D_\theta D_\tau(W_i)\left[\frac{\partial\mathcal{L}}{\partial Q_{\theta\theta\tau}}\right] \\ &+ D_\theta D_\zeta(W_i)\left[\frac{\partial\mathcal{L}}{\partial Q_{\theta\theta\zeta}}\right] \\ &+ D_\theta D_\theta(W_1)\left[\frac{\partial\mathcal{L}}{\partial Q_{\theta\theta\theta}}\right] + D^2_\theta D_\zeta(W_i)\left[-D_\theta\frac{\partial\mathcal{L}}{\partial Q_{\theta\theta\theta\theta\zeta}}\right] \\ &+ D^3_\theta D_\zeta(W_i)\left[\frac{\partial\mathcal{L}}{\partial Q_{\theta\theta\theta\theta\zeta}}\right], \end{aligned} \tag{44}$$

where $i = 1,2,3,4,5$.

$$\begin{aligned} C^\zeta =& \xi^\zeta\mathcal{L} + W_i\left[\frac{\partial\mathcal{L}}{\partial Q_\zeta} - D_\zeta(\frac{\partial\mathcal{L}}{\partial Q_{\theta\zeta}}) + D^3_\theta D_\zeta(\frac{\partial\mathcal{L}}{\partial Q_{\theta\theta\theta\theta\zeta}})\right. \\ &\left.+ D_\theta D_\zeta(\frac{\partial\mathcal{L}}{\partial Q_{\theta\theta\zeta}}) + D^2_\zeta(\frac{\partial\mathcal{L}}{\partial Q_{\zeta\zeta\zeta}})\right] \\ &+ D_\theta(W_i)\left[-D^2_\theta D_\zeta(\frac{\partial\mathcal{L}}{\partial Q_{\theta\theta\theta\zeta}}) - D_\zeta(\frac{\partial\mathcal{L}}{\partial Q_{\theta\theta\zeta}})\right. \\ &\left.- D_\zeta(\frac{\partial\mathcal{L}}{\partial Q_{\zeta\zeta\zeta}})\right] + D^2_\zeta(W_i)\left[\frac{\partial\mathcal{L}}{\partial Q_{\zeta\zeta\zeta}}\right] \\ &+ D_\theta D_\zeta(W_i)\left[\frac{\partial\mathcal{L}}{\partial Q_{\theta\theta\zeta}}\right] + D^2_\theta D_\zeta(W_i) \\ &\times\left[-D_\theta\frac{\partial\mathcal{L}}{\partial Q_{\theta\theta\theta\theta\zeta}}\right] + D^3_\theta D_\zeta(W_i)\left[\frac{\partial\mathcal{L}}{\partial Q_{\theta\theta\theta\theta\zeta}}\right] \\ &+ D_\zeta(W_i)\left[-D^3_\theta(\frac{\partial\mathcal{L}}{\partial Q_{\theta\theta\theta\theta\zeta}}) + \frac{\partial\mathcal{L}}{\partial Q_{\theta\zeta}} - D_\theta(\frac{\partial\mathcal{L}}{\partial Q_{\theta\theta\zeta}}) - D_\zeta(\frac{\partial\mathcal{L}}{\partial Q_{\zeta\zeta\zeta}})\right], \end{aligned} \tag{45}$$

where $i = 1,2,3,4,5$.

Case:1 For P_1, we have $W = -Q_\tau$ and $\xi^\tau = 1$. Substituting these values in Eqs. (43)–(45), we find

$$\begin{aligned} C^\tau =& \mathcal{L} - Q_\tau\psi_{\theta\tau} + Q_{\tau\tau}\psi_\theta + Q_{\theta\tau}\psi_\tau + Q_{\theta\tau\tau}\psi_\tau\ , \\ C^\theta =& -Q_\tau[-4Q_{\theta\theta\zeta}\psi + \psi_{\theta\tau} + \psi_{\theta\theta\theta\zeta} - 4Q_{\theta\theta}\psi_\zeta + 8Q_\theta\psi_{\theta\zeta} + 4\psi_{\theta\theta}Q_\zeta] + Q_{\tau\tau}\psi_\theta \\ &- Q_{\theta\tau}[13\psi_{\theta\zeta} - \psi_\tau - 8\psi_\zeta Q_\theta - 12\psi Q_{\theta\zeta} - 4\psi_\theta Q_\zeta] \\ &- Q_{\zeta\tau}[4\psi Q_{\theta\theta} - 8\psi_\theta Q_\theta - 8(\psi_{\theta\theta\theta}Q_{\theta\theta\zeta} + 3\psi_{\theta\theta}Q_{\theta\theta\theta\zeta} + 3\psi_\theta Q_{\theta\theta\theta\theta\zeta} + \psi_\theta Q_{\theta\theta\theta\theta\theta\zeta})] \\ &- \psi Q_{\theta\tau\tau} - 8\psi Q_\theta Q_{\theta\zeta\tau} - 4\psi_\theta Q_\zeta Q_{\theta\theta\tau} + \psi_\theta Q_{\theta\theta\zeta\tau} - \psi_\theta Q_{\theta\theta\theta\zeta\tau}\ , \\ C^\zeta =& -Q_\tau[4Q_{\theta\theta\theta}\psi - 12\psi_\zeta Q_{\theta\theta} - 12\psi Q_{\theta\theta\zeta} + \psi_{\theta\theta\theta\zeta} + 8Q_{\theta\theta}\psi_\zeta + 8Q_\theta Q_{\theta\zeta} + 8Q_{\theta\zeta}\psi_\theta \\ &+ \psi Q_{\theta\theta\zeta} - \psi_{\zeta\zeta}] - Q_{\theta\tau}[-\psi_{\theta\theta\zeta} - 8\psi Q_{\theta\zeta} - 8\psi_\zeta Q_\theta + \psi_\zeta] - Q_{t\zeta}[-\psi_{\theta\theta\theta} \\ &+ 12\psi Q_{\theta\theta} - 8\psi Q_{\theta\theta} - 8\psi_\theta Q_\theta + \psi_\zeta] - 8\psi Q_\theta Q_{\theta\zeta\tau} - \psi_{\zeta\zeta} - \psi_\theta Q_{\theta\theta\zeta\tau} - \psi Q_{\theta\theta\theta\zeta\tau}\ , \end{aligned}$$

where $\psi(\theta,\zeta,\tau) = \theta f(\tau) + h(\zeta,\tau)$.

Case:2 For P_2, we have $W = -Q_\theta$ and $\xi^\theta = 1$, we find

$$\begin{aligned} C^\tau =& 2Q_{\theta\theta}\psi_\tau + Q_{\theta\tau}\psi_\theta - Q_\theta\psi_{\theta\tau} + Q_{\theta\theta\tau}\psi_\tau\ , \\ C^\theta =& \mathcal{L} - Q_\theta[-4Q_{\theta\theta\zeta}\psi + \psi_{\theta\tau} + \psi_{\theta\theta\theta\zeta} - 4Q_{\theta\theta}\psi_\zeta + 8Q_\theta\psi_{\theta\zeta} + 4\psi_{\theta\theta}Q_\zeta] + Q_{\theta\tau}\psi_\theta \\ &- Q_{\theta\theta}[13\psi_{\theta\zeta} - \psi_\tau - 8\psi_\zeta Q_\theta - 12\psi Q_{\theta\zeta} - 4\psi_\theta Q_\zeta] \\ &- Q_{\theta\zeta}[4\psi Q_{\theta\theta} - 8\psi_\theta Q_\theta - 8(\psi_{\theta\theta\theta}Q_{\theta\theta\zeta} + 3\psi_{\theta\theta}Q_{\theta\theta\theta\zeta} + 3\psi_\theta Q_{\theta\theta\theta\theta\zeta} + \psi_\theta Q_{\theta\theta\theta\theta\theta\zeta})] \\ &- \psi Q_{\theta\theta\tau} - 8\psi Q_\theta Q_{\theta\theta\zeta} - 4\psi Q_\zeta Q_{\theta\theta\theta} + \psi_\theta Q_{\theta\theta\theta\zeta} - \psi Q_{\theta\theta\theta\theta\zeta}\ , \\ C^\zeta =& -Q_\theta[4Q_{\theta\theta\theta}\psi - 12\psi_\zeta Q_{\theta\theta} - 12\psi Q_{\theta\theta\zeta} + \psi_{\theta\theta\theta\zeta} + 8Q_{\theta\theta}\psi_\zeta + 8Q_\theta Q_{\theta\zeta} + 8Q_{\theta\zeta}\psi_\theta \\ &+ \psi Q_{\theta\theta\zeta} - \psi_{\zeta\zeta}] - Q_{\theta\theta}[-\psi_{\theta\theta\zeta} - 8\psi Q_{\theta\zeta} - 8\psi_\zeta Q_\theta + \psi_\zeta] - Q_{\theta\zeta}[-\psi_{\theta\theta\theta} \\ &+ 12\psi Q_{\theta\theta} - 8\psi Q_{\theta\theta} - 8\psi_\theta Q_\theta + \psi_\zeta] - 8\psi Q_\theta Q_{\theta\theta\zeta} - \psi_{\zeta\zeta} - \psi_\theta Q_{\theta\theta\theta\zeta} - \psi Q_{\theta\theta\theta\theta\zeta}\ , \end{aligned}$$

where $\psi(\theta,\zeta,\tau) = \theta f(\tau) + h(\zeta,\tau)$.

Case:3 For P_3, we have $W = -Q_\zeta$ and $\xi^\zeta = 1$, we find

$$
\begin{aligned}
C^\tau =& Q_{\zeta\tau}\psi_\theta + Q_{\theta\zeta}\psi_\tau - Q_\zeta\psi_{\theta\tau} + Q_{\theta\tau\zeta}\psi_\tau\ ,\\
C^\theta =& -Q_\zeta[-4Q_{\theta\theta\zeta}\psi + \psi_{\theta\tau} + \psi_{\theta\theta\theta\zeta} - 4Q_{\theta\theta}\psi_\zeta + 8Q_\theta\psi_{\theta\zeta} + 4\psi_{\theta\theta}Q_\zeta] + Q_{\tau\zeta}\psi_\theta\\
& - Q_{\theta\zeta}[13\psi_{\theta\zeta} - \psi_\tau - 8\psi_\zeta Q_\theta - 12\psi Q_{\theta\zeta} - 4\psi_\theta Q_\zeta]\\
& - Q_{\zeta\zeta}[4\psi Q_{\theta\theta} - 8\psi_\theta Q_\theta - 8(\psi_{\theta\theta\theta}Q_{\theta\theta\zeta} + 3\psi_{\theta\theta}Q_{\theta\theta\theta\zeta} + 3\psi_\theta Q_{\theta\theta\theta\theta\zeta} + \psi_\theta Q_{\theta\theta\theta\theta\theta\zeta})]\\
& - \psi Q_{\theta\zeta\tau} - 8\psi Q_\theta Q_{\theta\zeta\zeta} - 4\psi Q_\zeta Q_{\theta\theta\zeta} + \psi_\theta Q_{\theta\theta\zeta\zeta} - \psi Q_{\theta\theta\theta\zeta\zeta}\ ,\\
C^\zeta =& \mathcal{L} - Q_\zeta[4Q_{\theta\theta\theta}\psi - 12\psi_\zeta Q_{\theta\theta} - 12\psi Q_{\theta\theta\zeta} + \psi_{\theta\theta\theta\zeta} + 8Q_{\theta\theta}\psi_\zeta + 8Q_\theta Q_{\theta\zeta} + 8Q_{\theta\zeta}\psi_\theta\\
& + \psi Q_{\theta\theta\zeta} - \psi_{\zeta\zeta}] - Q_{\theta\zeta}[-\psi_{\theta\theta\zeta} - 8\psi Q_{\theta\zeta} - 8\psi_\zeta Q_\theta + \psi_\zeta] - Q_{\zeta\zeta}[-\psi_{\theta\theta\theta}\\
& + 12\psi Q_{\theta\theta} - 8\psi Q_{\theta\theta} - 8\psi_\theta Q_\theta + \psi_\zeta] - 8\psi Q_\theta Q_{\theta\zeta\zeta} - \psi_{\zeta\zeta} - \psi_\theta Q_{\theta\theta\zeta\zeta} - \psi Q_{\theta\theta\theta\zeta\zeta}\ ,
\end{aligned}
$$

where $\psi(\theta,\zeta,\tau) = \theta f(\tau) + h(\zeta,\tau)$.

Note: Here we have calculated the conservation laws for W_1, W_2 and W_3. The conservation laws for other W_4 and W_5 can also be calculated, which have been excluded here.

5. Conclusions

In this article, BKP equation was discussed by means of Lie analysis. Lie point symmetries were computed, and one-dimensional conjugacy classes were reported for the abelian algebra of the Lie group. These classes were further utilized to find the reductions of the discussed model via similarity variables. The reduced differential equations were solved by using different techniques to find the new solitary wave solutions and exact explicit solutions of the BKP equation. Different kinds of explicit exact solutions were calculated, which contain trigonometric, rational and hyperbolic functions. The considered equation was classified by using nonlinear-selfadjointness theory, and conservation laws were computed.

CRediT authorship contribution statement

Adil Jhangeer: Conceptualization, Funding acquisition, Resources, Validation, Writing - original draft. **Amjad Hussain:** Conceptualization, Funding acquisition, Project administration, Resources, Supervision, Validation, Writing - original draft. **M. Junaid-U-Rehman:** Formal analysis, Investigation, Methodology, Software, Visualization, Writing - review & editing. **Ilyas Khan:** Formal analysis, Investigation, Methodology, Project administration, Software, Supervision, Visualization, Writing - review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A

A.1. Description of methods

A.1.1. The new extended direct algebraic method

In this section, the general procedure of the new extended direct algebraic method [44,45] is discussed. We will follow given below steps to practice the said methods.

Step1 : By using the substitution

$$Q(\theta,\zeta,\tau) = U(\varrho)$$

where ϱ is a linear combination of independent variables, Eq. (2) can be converted into following nonlinear ODE of the form:

$$\Theta(U, U', U'', \ldots) = 0. \tag{A.1}$$

Step 2 : Assuming the general solution of Eq. (A.1) is of the form:

$$U(\varrho) = \sum_{i=0}^{N} a_i g^i(\varrho), \tag{A.2}$$

where $a_i (0 < i \leq n)$ are the coefficients which can be decided later and $g(\varrho)$ is the solution of the following equation:

$$g'(\varrho) = ln(\vartheta)(\alpha + \varsigma g(\varrho) + \gamma g^2(\varrho)), \tag{A.3}$$

where $\vartheta \neq 0, 1$ and α, ς, γ are the constants.

After assuming $\Delta = \varsigma^2 - 4\gamma\alpha$, the solutions of Eq. (A.3) can be taken as:

(1): If $\Delta < 0$ and $\gamma \neq 0$, then

$g_1(\varrho) = -\frac{\varsigma}{2\gamma} + \frac{\sqrt{-\Delta}}{2\gamma}\tan_\vartheta(\frac{\sqrt{-\Delta}}{2}\varrho)$,

$g_2(\varrho) = -\frac{\varsigma}{2\gamma} - \frac{\sqrt{-\Delta}}{2\gamma}\cot_\vartheta(\frac{\sqrt{-\Delta}}{2}\varrho)$,

$g_3(\varrho) = -\frac{\varsigma}{2\gamma} + \frac{\sqrt{-\Delta}}{2\gamma}(\tan_\vartheta(\sqrt{-\Delta}\varrho) \pm \sqrt{rs}\sec_\vartheta(\sqrt{-\Delta}\varrho))$,

$g_4(\varrho) = -\frac{\varsigma}{2\gamma} - \frac{\sqrt{-\Delta}}{2\gamma}(\cot_\vartheta(\sqrt{-\Delta}\varrho) \pm \sqrt{rs}\csc_\vartheta(\sqrt{-\Delta}\varrho))$,

$g_5(\varrho) = -\frac{\varsigma}{2\gamma} + \frac{\sqrt{-\Delta}}{4\gamma}(\tan_\vartheta(\frac{\sqrt{-\Delta}}{4}\varrho) - \cot_\vartheta(\frac{\sqrt{-\Delta}}{4}\varrho))$.

(2): If $\Delta > 0$ and $\gamma \neq 0$, then

$g_6(\varrho) = -\frac{\varsigma}{2\gamma} - \frac{\sqrt{\Delta}}{2\gamma}\tanh_\vartheta(\frac{\sqrt{\Delta}}{2}\varrho)$,

$g_7(\varrho) = -\frac{\varsigma}{2\gamma} - \frac{\sqrt{\Delta}}{2\gamma}\coth_\vartheta(\frac{\sqrt{\Delta}}{2}\varrho)$,

$g_8(\varrho) = -\frac{\varsigma}{2\gamma} - \frac{\sqrt{\Delta}}{2\gamma}(\tanh_\vartheta(\sqrt{\Delta}\varrho) \pm \imath\sqrt{rs}\ \mathrm{sech}_\vartheta(\sqrt{\Delta}\varrho))$,

$g_9(\varrho) = -\frac{\varsigma}{2\gamma} - \frac{\sqrt{\Delta}}{2\gamma}(\coth_\vartheta(\sqrt{\Delta}\varrho) \pm \sqrt{rs}\ \mathrm{csch}_\vartheta(\sqrt{\Delta}\varrho))$,

$g_{10}(\varrho) = -\frac{\varsigma}{2\gamma} - \frac{\sqrt{\Delta}}{4\gamma}(\tanh_\vartheta(\frac{\sqrt{\Delta}}{4}\varrho) + \coth_\vartheta(\frac{\sqrt{\Delta}}{4}\varrho))$.

(3): If $\gamma\alpha > 0$ and $\varsigma = 0$, then

$g_{11}(\varrho) = \sqrt{\frac{\alpha}{\gamma}}\tan_\vartheta(\sqrt{\gamma\alpha}\varrho)$,

$g_{12}(\varrho) = -\sqrt{\frac{\alpha}{\gamma}}\cot_\vartheta(\sqrt{\gamma\alpha}\varrho)$,

$g_{13}(\varrho) = \sqrt{\frac{\alpha}{\gamma}}(\tan_\vartheta(2\sqrt{\gamma\alpha}\varrho) \pm \sqrt{rs}\sec_\vartheta(2\sqrt{\gamma\alpha}\varrho))$,

$g_{14}(\varrho) = \sqrt{\frac{\alpha}{\gamma}}(-\cot_\vartheta(2\sqrt{\gamma\alpha}\varrho) \pm \sqrt{rs}\csc_\vartheta(2\sqrt{\gamma\alpha}\varrho))$,

$g_{15}(\varrho) = \frac{1}{2}\sqrt{\frac{\alpha}{\gamma}}(\tan_\vartheta(\frac{\sqrt{\gamma\alpha}}{2}\varrho) - \cot_\vartheta(\frac{\sqrt{\gamma\alpha}}{2}\varrho))$.

(4): If $\gamma\alpha < 0$ and $\varsigma = 0$, then

$g_{16}(\varrho) = -\sqrt{-\frac{\alpha}{\gamma}}\tanh_\vartheta(\sqrt{-\gamma\alpha}\varrho)$,

$g_{17}(\varrho) = -\sqrt{-\frac{\alpha}{\gamma}}\coth_\vartheta(\sqrt{-\gamma\alpha}\varrho)$,

$g_{18}(\varrho) = -\sqrt{-\frac{\alpha}{\gamma}}(\tanh_\vartheta(2\sqrt{-\gamma\alpha}\varrho) \pm \imath\sqrt{rs}\ \mathrm{sech}_\vartheta(2\sqrt{-\gamma\alpha}\varrho))$,

$g_{19}(\varrho) = -\sqrt{-\frac{\alpha}{\gamma}}(\coth_\vartheta(2\sqrt{-\gamma\alpha}\varrho) \pm \sqrt{rs}\ \mathrm{csch}_\vartheta(2\sqrt{-\gamma\alpha}\varrho))$,

$g_{20}(\varrho) = -\frac{1}{2}\sqrt{-\frac{\alpha}{\gamma}}(\tanh_\vartheta(\frac{\sqrt{-\gamma\alpha}}{2}\varrho) + \coth_\vartheta(\frac{\sqrt{-\gamma\alpha}}{2}\varrho))$.

(5): If $\varsigma = 0$ and $\gamma = \alpha$, then

$g_{21}(\varrho) = \tan_\vartheta(\alpha\varrho)$,

$g_{22}(\varrho) = -\cot_\vartheta(\alpha\varrho)$,

$g_{23}(\varrho) = \tan_\vartheta(2\alpha\varrho) \pm \sqrt{rs}\sec_\vartheta(2\alpha\varrho)$,

$g_{24}(\varrho) = -\cot_\vartheta(2\alpha\varrho) \pm \sqrt{rs}\csc_\vartheta(2\alpha\varrho)$,

$g_{25}(\varrho) = \frac{1}{2}(\tan_\vartheta(\frac{\alpha}{2}\varrho) - \cot_\vartheta(\frac{\alpha}{2}\varrho))$.

(6): If $\varsigma = 0$ and $\gamma = -\alpha$, then

$g_{26}(\varrho) = -\tanh_\vartheta(\alpha\varrho)$,

$g_{27}(\varrho) = -\coth_\vartheta(\alpha\varrho)$,

$g_{28}(\varrho) = -\tanh_\vartheta(2\alpha\varrho) \pm \imath\sqrt{rs}\ \mathrm{sech}_\vartheta(2\alpha\varrho)$,

$g_{29}(\varrho) = -\coth_\vartheta(2\alpha\varrho) \pm \sqrt{rs}\ \mathrm{csch}_\vartheta(2\alpha\varrho)$,

$g_{30}(\varrho) = -\frac{1}{2}(\tanh_\vartheta(\frac{\alpha}{2}\varrho) + \coth_\vartheta(\frac{\alpha}{2}\varrho))$.

(7): If $\varsigma^2 = 4\gamma\alpha$, then

$g_{31}(\varrho) = \frac{-2\alpha(\varsigma\varrho\ln(\vartheta)+2)}{\varsigma^2\varrho\ln(\vartheta)}$.

(8): If $\varsigma = \lambda$, $\alpha = p\lambda (p \neq 0)$ and $\gamma = 0$, then

$g_{32}(\varrho) = \vartheta^{\lambda\varrho} - p$.

(9): If $\varsigma = \gamma = 0$, then

$g_{33}(\varrho) = \alpha\varrho\ln(\vartheta)$.

(10): If $\varsigma = \alpha = 0$, then

$g_{34}(\varrho) = \frac{-1}{\gamma\varrho\ln(\vartheta)}$.

(11): If $\alpha = 0$ and $\varsigma \neq 0$, then

$g_{35}(\varrho) = -\frac{r\varsigma}{\gamma(\cosh_\vartheta(\varsigma\varrho) - \sinh_\vartheta(\varsigma\varrho) + r)}$,

$g_{36}(\varrho) = -\frac{\varsigma(\sinh_\vartheta(\varsigma\varrho)+\cosh_\vartheta(\varsigma\varrho))}{\gamma(\sinh_\vartheta(\varsigma\varrho)+\cosh_\vartheta(\varsigma\varrho)+s)}$.

(**12**): If $\varsigma = \lambda$, $\gamma = p\lambda(p \neq 0)$ and $\alpha = 0$, then

$g_{37}(\varrho) = \frac{r\vartheta^{\lambda\varrho}}{s-pr\vartheta^{\lambda\varrho}}$.

Here we define the hyperbolic and trigonometric functions as follows:

$\sinh_\vartheta(\varrho) = \frac{r\vartheta^{\varrho}-s\vartheta^{-\varrho}}{2}, \quad \cosh_\vartheta(\varrho) = \frac{r\vartheta^{\varrho}+s\vartheta^{-\varrho}}{2},$

$\tanh_\vartheta(\varrho) = \frac{r\vartheta^{\varrho}-s\vartheta^{-\varrho}}{r\vartheta^{\varrho}+s\vartheta^{-\varrho}}, \quad \coth_\vartheta(\varrho) = \frac{r\vartheta^{\varrho}+s\vartheta^{-\varrho}}{r\vartheta^{\varrho}-s\vartheta^{-\varrho}},$

$\mathrm{csch}_\vartheta(\varrho) = \frac{2}{r\vartheta^{\varrho}-s\vartheta^{-\varrho}}, \quad \mathrm{sech}_\vartheta(\varrho) = \frac{2}{r\vartheta^{\varrho}+s\vartheta^{-\varrho}},$

$\sin_\vartheta(\varrho) = \frac{r\vartheta^{\imath\varrho}-s\vartheta^{-\imath\varrho}}{2\imath}, \quad \cos_\vartheta(\varrho) = \frac{r\vartheta^{\imath\varrho}+s\vartheta^{-\imath\varrho}}{2},$

$\tan_\vartheta(\varrho) = -\imath\frac{r\vartheta^{\imath\varrho}-s\vartheta^{-\imath\varrho}}{r\vartheta^{\imath\varrho}+s\vartheta^{-\imath\varrho}}, \quad \cot_\vartheta(\varrho) = \imath\frac{r\vartheta^{\imath\varrho}+s\vartheta^{-\imath\varrho}}{r\vartheta^{\imath\varrho}-s\vartheta^{-\imath\varrho}},$

$\csc_\vartheta(\varrho) = \frac{2\imath}{r\vartheta^{\imath\varrho}-s\vartheta^{-\imath\varrho}}, \quad \mathrm{sech}_\vartheta(\varrho) = \frac{2}{r\vartheta^{\imath\varrho}+s\vartheta^{-\imath\varrho}},$

where r and s are constants and also called the deformation parameters.

Step 3 : Where N can be determined by equating the highest order linear term in Eq. (A.1) with the nonlinear terms of highest order.

Step 4 : After substituting Eqs. (A.3) and (B.1) into Eq. (A.1) and comparing all coefficients of $g(\varrho)$, an algebraic system of equations is obtained. Obtained system can be further solved by using Maple.

Appendix B

B.1. The tanh method

In this section, the general procedure of the tanh technique [40,46] is presented below;

Step 1 : Assuming the general solution of Eq. (A.1) is of the form;

$$U(\varrho) = a_0 + \sum_{n=1}^{N} a_n Y^n, \tag{B.1}$$

where N is an positive integer and computed as defined in previous method.

Step 2 : Let us take a new independent variable $Y = tanh(\varrho)$ then $U'(\varrho)$ and $U''(\varrho)$ can be represented as follow:

$$\begin{aligned} \frac{dU}{d\varrho} &= (1-Y^2)\frac{dU}{dY}, \\ \frac{d^2U}{d\varrho^2} &= (1-Y^2)\left(-2Y\frac{dU}{dY} + (1-Y^2)\frac{d^2U}{dY^2}\right), \end{aligned} \tag{B.2}$$

similarly we can find other derivatives.

Step3 : After substituting (B.1) and (B.2) in Eq. (A.1) and taking the coefficients of Y^n $(n = 0, 1, 2, \ldots)$ equal to zero we get system of algebraic equations which can be further solved by using Maple.

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