DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES S

doi:10.3934/dcdss.2020212

LYAPUNOV TYPE INEQUALITY IN THE FRAME OF GENERALIZED CAPUTO DERIVATIVES

Fahd Jarad

Department of Mathematics, Çankaya University 06790, Ankara, Turkey

Yassine Adjabi

Department of Mathematics, Faculty of Sciences, University of M'hamed Bougara
UMBB, Boumerdes, 35000, Algeria

Thabet Abdeljawad*

Department of Mathematics and General Sciences, Prince Sultan University
P. O. Box 66833, 11586 Riyadh, Saudi Arabia
and
Department of Medical Research, China Medical University
40402, Taichung, Taiwan
and
Department of Computer Science and Information Engineering, Asia University
Taichung, Taiwan

Saed F. Mallak

Department of Applied mathematics, Palestine Technical University-Kadoorie
Tulkarm, West Bank, Palestine

Hussam Alrabaiah

College of Engineering, Al Ain University of Science and Technology, Al Ain, UAE
and
College of Science, Tafila Technical University, Tafila, Jordan

Abstract. In this paper, we establish the Lyapunov-type inequality for boundary value problems involving generalized Caputo fractional derivatives that unite the Caputo and Caputo-Hadamrad fractional derivatives. An application about the zeros of generalized types of Mittag-Leffler functions is given.

1. Introduction.

One of the most significant inequalities which play a critical role in acquiring qualitative properties of differential equation is the Lyapunov inequality.

The Russian mathematician A. M. Liapunov [32, 1949] proved the following:

2010 *Mathematics Subject Classification.* Primary: 26A33; Secondary: 35A23.

Key words and phrases. Lyapunov type inequality, generalized fractional integrals, generalized fractional derivatives, genralized Caputo fractional derivatives, generalized Mittag-Leffler function type, fractional boundary value problems, fractional eigenvalue problems, nontrivial solutions, zeros, Green's function.

The third author would like to thank Prince Sultan University for funding this work through research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM) group number RG-DES-2017-01-17.

* Corresponding author.

Theorem 1.1. *If the boundary value problem*

$$\begin{cases} x''(t) + p(t)x(t) = 0, \\ x(a) = 0 = x(b), \end{cases} \quad t \in (a,b), 0 < a < b < +\infty, \tag{1}$$

has a nontrivial solution, where $p : [a,b] \to \mathbb{R}$ is a continuous function, then

$$\int_a^b |p(s)|\, ds > \frac{4}{b-a}. \tag{2}$$

The number 4 in (2) can not be replaced by a larger number. This inequality was shown to have applications in many areas [36, 10, 12, 35, 40, 41]

Being considered as the generalization of the calculus of integration and differentiation, the fractional calculus is a rapidly progressing field of mathematics that has been attracting scientists working on different fields for decades because of the findings achieved when the fractional derivatives/integrals are exploited to model some phenomena [29, 37, 14]. Recently, there has been a continuous focus on fractional integrals and derivatives with nonsingular kernels. For these operators we refer to [11, 8, 18, 9, 19, 20, 43, 13, 44, 22].

For the last few years, many authors have tried to find the analogue of the Lyapunov inequality when dealing with boundary differntial equations involving fractional derivatives. Ferreirra succeeded to obtain a Lyapunov type ineaualities for boundary value problems involving Riemann-Liouville fractional derivative. The same author achieved to find a Lyapunov type inequality for boundary value problems involving Caputo fractional derivative [17] . In [33], the authors found Lyapunov-type inequalities for boundary value problems in the frame of Hadamard fractional derivatives. For other generalizations and extensions of the classical Lyapunov inequality, we refer to [25, 39, 42, 4, 5, 6, 7].

Motivated by what we mentioned above, in this work we discuss a Lyapunov type inequality for boundary value problems in the frame of a certain generalized Caputo derivative that involves the Caputo and the Caputo-Hadamarad fractional derivative in one derivative [24].

The paper is organized as follows. In section 2, we introduce notations and present the fractional differential operators that will be studied. We recover some results involving the Caputo fractional derivative in a generalized form investigate the connection of (Kilbas–Saigo) Mittag-Leffler type functions with the generalized Caputo fractional integrals and derivatives are investigated. In section 3, we discuss a Lyapunov-Type inequality for boundary value problems in the frame of generalized Caputo fractional derivatives. In section 4, we present an application and the last section is devoted to the conclusion.

2. Fractional calculus and the (Kilbas–Saigo) Mittag-Leffler type functions.

In this section, we introduce some notations, definitions and Lemmas of fractional calculus, (Kilbas–Saigo) Mittag-Leffler type functions and present preliminary results needed later.

The left-sided Riemann-Liouville fractional derivative of order $\alpha \in (n-1, n]$, of a continuous function $f : [0, \infty) \longrightarrow \mathbb{R}$ is given by [29, 37, 14]

$$\left(D_{a+}^{\alpha} f\right)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} f(s)\, ds,\ a \geq 0, \tag{3}$$

provided that the right side is pointwise defined on $\mathbb{R}^+$, where $n = [\alpha] + 1$ and $[\alpha]$ means the maximal integer not exceeding α.The corresponding left-sided Riemann-Liouville integral operator of order $\alpha > 0$, of a continuous function $f : [0, \infty) \longrightarrow \mathbb{R}$ is given by [29, 37, 14]

$$J_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds, \tag{4}$$

provided that the right side is pointwise defined on $\mathbb{R}^+$.

In [28], introduced the Hadamard fractional derivatives and their corresponding integrals were introduced as:

$$\left(\mathcal{D}_{a+}^{\alpha} f\right)(t) = \frac{1}{\Gamma(n-\alpha)} \delta^n \int_a^t \left(\ln \frac{t}{s}\right)^{n-\alpha-1} f(s) \frac{ds}{s}, \ \alpha \in (n-1, n] \tag{5}$$

and

$$\left(\mathcal{J}_{a+}^{\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s) \frac{ds}{s}, \ \alpha > 0, \tag{6}$$

where $\delta = \left(t \frac{d}{dt}\right)$ is the so-called δ-derivative.

Generalized fractional integral operator of order for $\alpha > 0$ and $t \in (a, \infty[$ is given by [26, 27]

$$\left(\mathcal{J}_{a+}^{\alpha,\rho} f\right)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (t^\rho - s^\rho)^{\alpha-1} f(s) \frac{ds}{s^{1-\rho}} \tag{7}$$

and the generalized fractional derivative [26, 27]

$$\left(\mathcal{D}_{a+}^{\alpha,\rho} f\right)(t) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \gamma^n \int_a^t (t^\rho - s^\rho)^{n-\alpha-1} f(s) \frac{ds}{s^{1-\rho}}, \ \alpha \in (n-1, n], \tag{8}$$

where $\gamma = \left(t^{1-\rho} \frac{d}{dt}\right)$.

The relation between these two fractional latter operators is as follows

$$\left(\mathcal{D}_{a+}^{\alpha,\rho} f\right)(t) = \gamma^n \left(\mathcal{J}_{a+}^{n-\alpha,\rho} f\right)(t), \ \alpha \in (n-1, n]. \tag{9}$$

The generalized operators (7) and (8) depend on extra paramater $\rho > 0$, which by taking $\rho \to 0^+$ reduces to the Hadamard fractional operator and for parameter $\rho = 1$ becomes the Riemann-Liouville fractional operator.

On the other hand, the left-sided generalized Caputo fractional derivatives of f of order α is defined by [24]

$$\left({}^{c}\mathcal{D}_{a+}^{\alpha,\rho} f\right)(t) = \mathcal{J}_{a+}^{n-\alpha,\rho} (\gamma^n f)(t), \ \alpha \in (n-1, n]. \tag{10}$$

Note that generalized Caputo derivative in (10) reduces to the Caputo-Hadamard fractional derivative introduced in [23] by taking $\rho \to 0^+$ and becomes the Caputo fractional derivative when $\rho = 1$.

Lemma 2.1. [24] *Let $\alpha \in (n-1, n]$, $\rho > 0$.*

(i) *If $f \in AC_\gamma^n [a,b]$ or $C_\gamma^n [a,b]$, then*

$$\mathcal{J}_{a+}^{n,\rho} (\gamma^n f)(t) = f(t) - \sum_{k=0}^{n-1} \frac{\left(\gamma^k f\right)(a)}{k!} \left(\frac{t^\rho - a^\rho}{\rho}\right)^k, \ \textit{for } t \in (a, b]. \tag{11}$$

(ii) *If $f \in AC_\gamma^n [a,b]$ or $C_\gamma^n [a,b]$, then*

$$\mathcal{J}_{a+}^{\alpha,\rho} \left({}^{c}\mathcal{D}_{a+}^{\alpha,\rho}\right) f(t) = f(t) - \sum_{k=0}^{n-1} \frac{\left(\gamma^k f\right)(a)}{k!} \left(\frac{t^\rho - a^\rho}{\rho}\right)^k, \ \textit{for } t \in (a, b]. \tag{12}$$

Remark 1. Let $\rho > 0$, $a \geq 0$.

(i) Let $\alpha > 0, \beta > 0$, and $n \in \mathbb{N}$ then

$$\gamma^n \left(\frac{s^\rho - a^\rho}{\rho}\right)^\beta (t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-n+1)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta-n}.$$

(ii) Let $\beta > n-1,\ n = [\alpha]+1$ then

$${}^c\mathcal{D}_{a+}^{\alpha,\rho} \left(\frac{s^\rho - a^\rho}{\rho}\right)^\beta (t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta-\alpha},$$

(iii) Let $n = [\alpha]$ then

$${}^c\mathcal{D}_{a+}^{\alpha,\rho} \left(\frac{s^\rho - a^\rho}{\rho}\right)^j (t) = 0,\ j \in \mathbb{Z}^-,\ {}^c\mathcal{D}_{a+}^{\alpha,\rho} \left(\frac{s^\rho - a^\rho}{\rho}\right)^{\alpha-j} (t) = 0,\ j \in \mathbb{N}^*,$$

(iv) Let $\beta > n-1,\ n = [\alpha]+1$ then

$$\int_s^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^\alpha \left(\frac{s^\rho - a^\rho}{\rho}\right)^\beta \frac{ds}{s^{1-\rho}} = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+1}.$$

Recently, Mittag-Leffler functions show its close relation to fractional calculus and especially to fractional problems which come from applications. This new era of research attract many scientists from different point of view (see, for example, [29, 37, 14, 17, 4, 5, 6, 7]).

In 1903, the Swedish mathematician G. Mittag-Leffler [34] introduced the one parametric Mittag-Leffler function $E_\alpha(z)$ defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \alpha > 0,\ z \in \mathbb{C}. \tag{13}$$

A first generalization of this function was proposed in 1905 by Wiman who defined the generalized function as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \alpha > 0, \beta \in \mathbb{R}, z \in \mathbb{C}. \tag{14}$$

When $\alpha, \beta > 0$ the series is convergent. Later, this function was rediscovered and intensively studied by R. P. Agarwal and others, This generalization is referred to as two-parameter Mittag-Leffler function. Particularly important is the case when $\beta = 1$. In this case we use notation $E_{\alpha,1}(z) = E_\alpha(z)$.

An interesting generalization of (13) is recently introduced by Kilbas and Saigo in [31, 1995], the three parametric Mittag-Leffler function defined as

$$E_{\alpha,m,\beta}(z) = \sum_{k=0}^{\infty} e_k z^k,\ e_0 = 1,\ e_k = \prod_{j=0}^{k-1} \frac{\Gamma(\alpha(jm+\beta)+1)}{\Gamma(\alpha(jm+\beta+1)+1)}, \tag{15}$$

where an empty product is to be interpreted as unity; $\alpha, \beta \in \mathbb{C}$ are complex numbers and $m \in \mathbb{R}$. When $\Re(\alpha) > 0, m > 0, \alpha(jm+\beta) \notin \mathbb{Z}^-,\ j = 0, 1, ...,$ and for $m = 1$ the above defined function reduces to a constant multiple of the Mittag-Leffler function (14), namely

$$E_{\alpha,1,\beta}(z) = \Gamma(\alpha\beta+1) E_{\alpha,\alpha\beta+1}(z) \tag{16}$$

and if further $\beta = 0, E_{\alpha,1,0}(z) = E_\alpha(z)$. Certain properties of this function associated with Riemann-Liouville fractional integrals and derivatives were obtained and exact solutions of certain integral equations of Abel-Volterra type are derived by their applications (Kilbas and Saigo, 1995, 1996).

Another generalization of the Mittag-Leffler function (13) can be found in the contemporary monographs of R. Gorenflo et al. [21, 2014].

Relations connecting the function defined by (15) and the generalized fractional integrals and the generalized Caputo derivatives are given in the form of Lemmas and Remarks.

The first statement in this paper shows the effect of $\mathcal{J}_a^{\alpha,\rho}$ on $E_{\alpha,m,\beta}(z)$.

Lemma 2.2. *Let $\rho>0,\alpha>0,\alpha\beta>-1,m>0$ and $\lambda\in\mathbb{R}^*$. Then, the following relation is valid*

$$\mathcal{J}_a^{\alpha,\rho}\left[\left(\tfrac{s^\rho-a^\rho}{\rho}\right)^{\alpha\beta}E_{\alpha,m,\beta}\left(-\lambda\left(\tfrac{s^\rho-a^\rho}{\rho}\right)^{m\alpha}\right)\right](t)=$$
$$-\frac{1}{\lambda}\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{\alpha(\beta-m+1)}\left[E_{\alpha,m,\beta}\left(-\lambda\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{m\alpha}\right)-1\right]. \tag{17}$$

Proof. In accordance with (7) and (15) we have

$$\begin{aligned}
I &= -\lambda\mathcal{J}_a^{\alpha,\rho}\left[\left(\tfrac{s^\rho-a^\rho}{\rho}\right)^{\alpha\beta}E_{\alpha,m,\beta}\left(-\lambda\left(\tfrac{s^\rho-a^\rho}{\rho}\right)^{m\alpha}\right)\right](t)\\
&= \frac{-\lambda}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}\left[\left(\tfrac{s^\rho-a^\rho}{\rho}\right)^{\alpha\beta}E_{\alpha,m,\beta}\left(-\lambda\left(\tfrac{s^\rho-a^\rho}{\rho}\right)^{m\alpha}\right)\right]\frac{ds}{s^{1-\rho}}\\
&= \frac{-\lambda}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}\left[\sum_{k=0}^{\infty}(-\lambda)^k(e_k)\left(\tfrac{s^\rho-a^\rho}{\rho}\right)^{\alpha(mk+\beta)}\right]\frac{ds}{s^{1-\rho}},
\end{aligned}$$

where (e_k) is defined in (15).

Interchanging the integration and summation and evaluating the inner integral, we find

$$I=\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{\infty}(-\lambda)^{k+1}(e_k)\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{s^\rho-a^\rho}{\rho}\right)^{\alpha(mk+\beta)}\frac{ds}{s^{1-\rho}},$$

by Remark 1-**(iii),** we have

$$\begin{aligned}
I &= \sum_{k=0}^{\infty}(-\lambda)^{k+1}\left(\prod_{j=0}^{k}\tfrac{\Gamma(\alpha(jm+\beta)+1)}{\Gamma(\alpha(jm+\beta+1)+1)}\right)\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{\alpha(mk+\beta+1)}\\
&= \left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{\alpha(\beta-m+1)}\sum_{k=1}^{\infty}(-\lambda)^k(e_k)\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{\alpha mk}\\
&= \left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{\alpha(\beta-m+1)}\left[E_{\alpha,m,\beta}\left(-\lambda\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{m\alpha}\right)-1\right],
\end{aligned}$$

where, the interchanging is guaranteed by the fact that all integrals converge from the conditions of the Lemma. □

Remark 2. for $m=1$, $\alpha>0$, $\beta>0$ and $\lambda\neq0$, there hold the formula

$$\mathcal{J}_a^{\alpha,\rho}\left[\left(\tfrac{s^\rho-a^\rho}{\rho}\right)^{\beta-1}E_{\alpha,\beta}\left(-\lambda\left(\tfrac{s^\rho-a^\rho}{\rho}\right)^{\alpha}\right)\right](t)$$
$$=-\frac{1}{\lambda}\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{\beta-1}\left[E_{\alpha,\beta}\left(-\lambda\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{\alpha}\right)-\frac{1}{\Gamma(\beta)}\right]. \tag{18}$$

In view of (16), we know that (18) can be written as

$$\begin{aligned}&\mathcal{J}_a^{\alpha,\rho}\left[\left(\tfrac{s^\rho-a^\rho}{\rho}\right)^{\beta-1}E_{\alpha,\beta}\left(-\lambda\left(\tfrac{s^\rho-a^\rho}{\rho}\right)^{\alpha}\right)\right](t)\\ &=\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{\alpha+\beta-1}E_{\alpha,\alpha+\beta}\left(-\lambda\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{\alpha}\right). \end{aligned} \tag{19}$$

- In particular, when $\beta=1$, (18) takes the form

$$\mathcal{J}_a^{\alpha,\rho}\left[E_{\alpha,1}\left(-\lambda\left(\tfrac{s^\rho-a^\rho}{\rho}\right)^{\alpha}\right)\right](t)=-\frac{1}{\lambda}\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{\alpha}E_{\alpha,\alpha+1}\left(-\lambda\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{\alpha}\right). \tag{20}$$

The application of ${}^c\mathcal{D}_a^{\alpha,\rho}$ to $E_{\alpha,m,\beta}(z)$ is given by the following statement.

Lemma 2.3. *Let $\rho>0,\alpha>0,m>0,\beta>m-1-1/\alpha$ and $\lambda\in\mathbb{R}^*$ are satisfied. Then, the following relation holds*

$$\begin{aligned}D&=\left({}^c\mathcal{D}_{a+}^{\alpha,\rho}\right)\left[\left(\tfrac{s^\rho-a^\rho}{\rho}\right)^{\alpha(\beta-m+1)}E_{\alpha,m,\beta}\left(-\lambda\left(\tfrac{s^\rho-a^\rho}{\rho}\right)^{m\alpha}\right)\right](t) \qquad (21)\\ &=\left(\tfrac{\Gamma(\alpha(\beta-m+1)+1)}{\Gamma(\alpha(\beta-m)+1)}\right)\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{\alpha(\beta-m)}-\lambda\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{\alpha\beta}E_{\alpha,m,\beta}\left(-\lambda\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{m\alpha}\right).\end{aligned}$$

- If further $\alpha(\beta-m)=-j$ for some $j=1,2,...,-[-\alpha]$

$$\begin{aligned}&\left({}^c\mathcal{D}_{a+}^{\alpha,\rho}\right)\left[\left(\tfrac{s^\rho-a^\rho}{\rho}\right)^{\alpha(\beta-m+1)}E_{\alpha,m,\beta}\left(-\lambda\left(\tfrac{s^\rho-a^\rho}{\rho}\right)^{m\alpha}\right)\right](t)=\\ &\qquad -\lambda\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{\alpha\beta}E_{\alpha,m,\beta}\left(-\lambda\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{m\alpha}\right).\end{aligned}$$

Proof. From (10) and (15) we have

$$\begin{aligned}D&=\left({}^c\mathcal{D}_{a+}^{\alpha,\rho}\right)\left[\left(\tfrac{s^\rho-a^\rho}{\rho}\right)^{\alpha(\beta-m+1)}E_{\alpha,m,\beta}\left(-\lambda\left(\tfrac{s^\rho-a^\rho}{\rho}\right)^{m\alpha}\right)\right](t)\\ &=\left(\mathcal{J}_{a+}^{n-\alpha,\rho}\gamma^n\left[\left(\tfrac{s^\rho-a^\rho}{\rho}\right)^{\alpha(\beta-m+1)}E_{\alpha,m,\beta}\left(-\lambda\left(\tfrac{s^\rho-a^\rho}{\rho}\right)^{m\alpha}\right)\right]\right)(t)\\ &=\sum_{k=0}^{\infty}\tfrac{(-\lambda)^k(e_k)}{\Gamma(n-\alpha)}\int_a^t\left(\tfrac{t^\rho-s^\rho}{\rho}\right)^{n-\alpha-1}\left[\gamma^n\left(\tfrac{s^\rho-a^\rho}{\rho}\right)^{\alpha(mk+\beta-m+1)}\right]\frac{ds}{s^{1-\rho}},\end{aligned}$$

where (e_k) is defined in (15). By Remark 1-**(i)**, we have

$$\gamma^n\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{\alpha(mk+\beta-m+1)}(s)=\left(\tfrac{\Gamma(\alpha((k-1)m+\beta+1)+1)}{\Gamma(\alpha((k-1)m+\beta)+1-n)}\right)\left(\tfrac{s^\rho-a^\rho}{\rho}\right)^{\alpha(m(k-1)+\beta+1)-n}.$$

Then,

$$D=\sum_{k=0}^{\infty}\tfrac{(-\lambda)^k(e_k)}{\Gamma(n-\alpha)}\left(\tfrac{\Gamma(\alpha((k-1)m+\beta+1)+1)}{\Gamma(\alpha((k-1)m+\beta)+1-n)}\right)\int_a^t\left(\tfrac{t^\rho-s^\rho}{\rho}\right)^{n-\alpha-1}\left(\tfrac{s^\rho-a^\rho}{\rho}\right)^{\alpha(m(k-1)+\beta+1)-n}\frac{ds}{s^{1-\rho}}.$$

By Remark 1-**(iv)**, we have

$$\begin{aligned}
D &= \sum_{k=0}^{\infty}(-\lambda)^k(e_k)\left(\tfrac{\Gamma(\alpha((k-1)m+\beta+1)+1)}{\Gamma(\alpha((k-1)m+\beta)+1)}\right)\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{\alpha(m(k-1)+\beta)} \\
&= \sum_{k=0}^{\infty}(-\lambda)^k\prod_{j=0}^{k-2}\tfrac{\Gamma(\alpha(jm+\beta)+1)}{\Gamma(\alpha(jm+\beta+1)+1)}\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{\alpha(m(k-1)+\beta)} \\
&= \left(\tfrac{\Gamma(\alpha(\beta-m+1)+1)}{\Gamma(\alpha(\beta-m)+1)}\right)\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{\alpha(\beta-m)} \\
&\quad+\sum_{k=0}^{\infty}(-\lambda)^{k+1}\prod_{j=0}^{k-1}\tfrac{\Gamma(\alpha(jm+\beta)+1)}{\Gamma(\alpha(jm+\beta+1)+1)}\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{\alpha(km+\beta)} \\
&= \left(\tfrac{\Gamma(\alpha(\beta-m+1)+1)}{\Gamma(\alpha(\beta-m)+1)}\right)\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{\alpha(\beta-m)}-\lambda\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{\alpha\beta}E_{\alpha,m,\beta}\left(-\lambda\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{m\alpha}\right),
\end{aligned}$$

which gives (21) and thus the proof is completed. □

Remark 3. For $\alpha > 0$ and $\beta > \alpha$, the following holds

$$ {}^{c}\mathcal{D}_{a+}^{\alpha,\rho}\left[\left(\tfrac{s^\rho-a^\rho}{\rho}\right)^{\beta-1}E_{\alpha,\beta}\left(-\lambda\left(\tfrac{s^\rho-a^\rho}{\rho}\right)^{\alpha}\right)\right](t)=$$

$$\tfrac{1}{\Gamma(\beta-\alpha)}\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{\beta-\alpha-1}-\lambda\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{\beta-1}E_{\alpha,\beta}\left(-\lambda\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{\alpha}\right).$$

If further $\beta-\alpha = 0, -1, -2, ...,$ then

$$\begin{aligned}
&{}^{c}\mathcal{D}_{a+}^{\alpha,\rho}\left[\left(\tfrac{s^\rho-a^\rho}{\rho}\right)^{\beta-1}E_{\alpha,\beta}\left(-\lambda\left(\tfrac{s^\rho-a^\rho}{\rho}\right)^{\alpha}\right)\right](t)\\
&=-\lambda\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{\beta-1}E_{\alpha,\beta}\left(-\lambda\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{\alpha}\right).
\end{aligned}$$

For $\alpha > 0$ t the following is true.

$$ {}^{c}\mathcal{D}_{a+}^{\alpha,\rho}\left[E_{\alpha}\left(-\lambda\left(\tfrac{s^\rho-a^\rho}{\rho}\right)^{\alpha}\right)\right](t)=\frac{1}{\Gamma(1-\alpha)}\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{-\alpha}-\lambda E_{\alpha}\left(-\lambda\left(\tfrac{t^\rho-a^\rho}{\rho}\right)^{\alpha}\right).$$

3. **A Lyapunov-type inequality in the frame of generalized Caputo fractional derivatives.**

In this section, we consider the following fractional boundary value problem

$$\begin{cases} \left({}^{c}\mathcal{D}_{a+}^{\alpha,\rho}x\right)(t)+p(t)x(t)=0, \\ x(a)=0=x(b). \end{cases} \quad a\geq 0,\ t\in(a,b),\alpha\in(1,2]. \tag{22}$$

We begin by writing problem (22) in its equivalent integral form.

Theorem 3.1. *$x(t) \in C[a,b]$ is a solution of (22) if and only if*

$$x(t)=\int_a^b G(t,s)p(s)x(s)ds, \tag{23}$$

where $G(t,s)$ is the Green's function given by

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases} G_1(t,s), & \text{if } a\leq t\leq s\leq b, \\ G_1(t,s)-(\dfrac{t^\rho-s^\rho}{\rho})^{\alpha-1}s^{\rho-1} & \text{if } a\leq s\leq t\leq b, \end{cases} \tag{24}$$

with

$$G_1(t,s)=\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)\left(\frac{b^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}. \tag{25}$$

Proof. By applying Lemma 2.1-(**ii**), we reduce (22) to the equivalent integral equation given by

$$\begin{aligned} x(t) &= -\left(\mathcal{J}_{a+}^{\alpha,\rho}p\,(s)\,x(s)\right)(t)+c_0+c_1\left(\frac{t^\rho-a^\rho}{\rho}\right) \\ &= -\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t\left(t^\rho-s^\rho\right)^{\alpha-1}p\,(s)\,x(s)\frac{ds}{s^{1-\rho}}+c_0+c_1\left(\frac{t^\rho-a^\rho}{\rho}\right). \end{aligned}$$

From $x(a)=0$, we have $c_0=0$. Consequently the solution of (22) becomes

$$\begin{aligned} x(t) &= -\left(\mathcal{J}_{a+}^{\alpha,\rho}p\,(s)\,x(s)\right)(t)+c_1\left(\frac{t^\rho-a^\rho}{\rho}\right) \\ &= -\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t\left(t^\rho-s^\rho\right)^{\alpha-1}p\,(s)\,x(s)\frac{ds}{s^{1-\rho}}+c_1\left(\frac{t^\rho-a^\rho}{\rho}\right). \end{aligned}$$

Since

$$x(b)=-\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^b\left(b^\rho-s^\rho\right)^{\alpha-1}h\,(s)\,\frac{ds}{s^{1-\rho}}+c_1\left(\frac{b^\rho-a^\rho}{\rho}\right)$$

and $x(b)=0$, one has

$$c_1=\left(\frac{b^\rho-a^\rho}{\rho}\right)\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^b\left(b^\rho-s^\rho\right)^{\alpha-1}h\,(s)\,\frac{ds}{s^{1-\rho}}.$$

Consequently, the solution of problem (22) is

$$x(t)=-\left(\mathcal{J}_{a+}^{\alpha,\rho}p\,(s)\,x(s)\right)(t)+\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^b\left(b^\rho-s^\rho\right)^{\alpha-1}p\,(s)\,x(s)\frac{ds}{s^{1-\rho}}. \tag{26}$$

Conversely, it is easy to verify directly that (23) is the solution of (22). Thus, the unique solution $x(t)$ of problem (22) can be written as (23). The proof is finished. □

Remark 4. If we take $\rho=1$ in Theorem 3.1, then the Green function given by Theorem 3.1 reduces to the Green's function obtained in [17].

Lemma 3.2. *The function G defined in Theorem 3.1 satisfies the following property:*

$$\max\{|G(t,s)|:a\le s,t\le b\}\le G(s,s)\ \textit{for all}\ s\in[a,b],$$

and $G(s,s)$ has a unique maximum $G_{\max}$ in $[a,b]$, given by

$$G_{\max}=\begin{cases}\left(\dfrac{L-a^\rho}{b^\rho-a^\rho}\right),\left(\dfrac{b^\rho-L}{\rho}\right)^{\alpha-1}L^{\frac{\rho-1}{\rho}}, & N=0\\ \dfrac{\left((1-\alpha\rho)\,a^\rho+(2\alpha\rho-1)\,b^\rho-M\right)^{\alpha-1}\left((1-(\alpha+2)\,\rho)\,a^\rho+(2\rho-1)\,b^\rho+M\right)}{\Gamma(\alpha)\,(b^\rho-a^\rho)\,(2N)^{\frac{N}{\rho}}\left((\alpha\rho-1)\,a^\rho+(2\rho-1)\,b^\rho+M\right)^{\frac{1-\rho}{\rho}}}, & N\neq0\end{cases} \tag{27}$$

for all $s\in[a,b]$, where

$$L=\left(\frac{(\rho-1)\,a^\rho b^\rho}{(2\rho+1)\,b^\rho-a^\rho}\right)^{\frac{1}{\rho}},\ \ N=(\alpha+1)\,\rho-1, \tag{28}$$

and

$$M=\left(\left(\left(\alpha\rho-1\right)a^{\rho}+\left(2\rho-1\right)b^{\rho}\right)^{2}-4\left(1-\left(\alpha+1\right)\rho\right)\left(1-\rho\right)a^{\rho}b^{\rho}\right)^{\frac{1}{2}}. \quad (29)$$

Proof. Let us define the function

$$G_2(t,s)=G_1(t,s)-(\frac{t^{\rho}-s^{\rho}}{\rho})^{\alpha-1}s^{\rho-1},\ a\leq s\leq t\leq b, \quad (30)$$

where $G_1(t,s)$ is defined in (25). We divide the proof into two steps.

Step (I) We start with the function G_1.

In view of the expression for the function $G_1(t,s)$, we easily find that $G_1(t,s)\geq 0,\ a\leq t\leq s\leq b$. Obviously, G_1 satisfies the following inequalities:

$$0\leq G_1(t,s)\leq G_1(s,s),\ a\leq t\leq s\leq b.$$

Differentiating $G_1(s,s)$ on (a,b), we get

$$\partial_s G_1(s,s)=\frac{\rho^{1-\alpha}s^{\rho-2}\left(b^{\rho}-s^{\rho}\right)^{\alpha-2}}{\left(b^{\rho}-a^{\rho}\right)}P\left(s\right), \quad (31)$$

where, P is a polynomial function of one variable defined by

$$P\left(s\right)=As^{2\rho}+Bs^{\rho}+C, \quad (32)$$

where

$$A=1-\left(\alpha+1\right)\rho,\ B=\left(\alpha\rho-1\right)a^{\rho}+\left(2\rho-1\right)b^{\rho}\text{ and }C=\left(1-\rho\right)a^{\rho}b^{\rho}.$$

We shall now discuss the existence and uniqueness of solutions of (32) in $[a,b]$ as follows:

When $A=0$: i.e., $\rho=\frac{1}{\alpha+1}$. Thus, we obtain

$$B=\left(\alpha\rho-1\right)a^{\rho}+\left(2\rho-1\right)b^{\rho}=\frac{1}{\alpha+1}\left(-a^{\rho}+\left(1-\alpha\right)b^{\rho}\right)<0.$$

Then,

$$\bar{s}\equiv s_0=\left[\frac{\left(\rho-1\right)a^{\rho}b^{\rho}}{\left(b^{\rho}-a^{\rho}\right)+2\rho b^{\rho}}\right]^{\frac{1}{\rho}},$$

where, s_0 is a root of the linear polynomial (32). This gives

$$\max_{s\in[a,b]}G_1(s,s)\leq G_1(s_0,s_0)=\left(\frac{L-a^{\rho}}{b^{\rho}-a^{\rho}}\right)\left(\frac{b^{\rho}-L}{\rho}\right)^{\alpha-1}L^{\frac{\rho-1}{\rho}},\text{ with }L=s_0. \quad (33)$$

When $A\neq 0$: i.e., $\rho\neq\frac{1}{\alpha+1}$, by a simple variable change, $X=s^{\rho}$ in (32), the quadratic polynomial $P\left(X\right)$ has discriminant

$$\Delta=B^2-4AC=\left(\left(\alpha\rho-1\right)a^{\rho}+\left(2\rho-1\right)b^{\rho}\right)^{2}-4\left(1-\left(\alpha+1\right)\rho\right)\left(1-\rho\right)a^{\rho}b^{\rho}.$$

Then, we have

$$\partial_{\alpha}\Delta=2\rho a^{\rho}\left(\alpha\rho a^{\rho}+b^{\rho}-a^{\rho}\right)=0\Longrightarrow\alpha=-\frac{b^{\rho}-a^{\rho}}{\rho}<0.$$

From the fact that $1<\alpha\leq 2$, it is easy to see that $\partial_{\alpha}\Delta\geq 0$. Furthermore, we get

$$\partial_{2\alpha}\Delta=2\rho^2a^{2\rho}>0,\ \partial_{\alpha}\Delta\geq 0,\ \Delta|_{\alpha=1}=\left(\left(\rho-1\right)a^{\rho}-\left(2\rho-1\right)b^{\rho}\right)^{2}>0,$$

which yields two distinct real roots of the polynomial (32)

$$X_1 = \frac{-B+\sqrt{\Delta}}{2A} \text{ and } X_2 = \frac{-B-\sqrt{\Delta}}{2A}.$$

As consequence, we have

$$\partial_s G_1(s,s) = 0 \Longleftrightarrow s \in \{0, b, s_1, s_2\}, \tag{34}$$

where,

$$s_1 = X_1^{\frac{1}{\rho}} = \begin{cases} \left(\frac{(\alpha\rho-1)a^\rho+(2\rho-1)b^\rho-M}{2N}\right)^{\frac{1}{\rho}} & \text{si} \quad \rho > 0, \\ 0 & \text{si} \quad \rho = 1, \end{cases}$$

and

$$s_2 = X_2^{\frac{1}{\rho}} = \begin{cases} \left(\frac{(\alpha\rho-1)a^\rho+(2\rho-1)b^\rho+M}{2N}\right)^{\frac{1}{\rho}} & \text{si} \quad \rho > 0, \\ \frac{(\alpha-1)a+b}{\alpha} & \text{si} \quad \rho = 1, \end{cases}$$

with

$$N = -A \text{ and } M = \Delta^{\frac{1}{2}}. \tag{35}$$

(I.1) Firstly, we prove that $s_1 \notin [a,b]$:

(a) In order to prove that

$$X_1 = \frac{-B+\sqrt{\Delta}}{2A} > b^\rho. \tag{36}$$

We consider the two cases:

(a.1) : When $A > 0$, we get $\sqrt{\Delta} > 2Ab^\rho + B$ then

If $\sqrt{\Delta} > 2Ab^\rho + B > 0$, then

$$(2Ab^\rho + B)^2 - \Delta = -4\rho(\alpha-1)Ab^\rho(b^\rho - a^\rho) < 0.$$

Thus, the inequality (36) holds.

If $\sqrt{\Delta} > 0 > 2Ab^\rho + B$, then (36) holds obviously.

(a.2) : When $A < 0$ we get $\sqrt{\Delta} < 2Ab^\rho + B$, we have

$$(2Ab^\rho + B)^2 - \Delta = -4\rho(\alpha-1)Ab^\rho(b^\rho - a^\rho) > 0. \tag{37}$$

Thus, (36) holds.

(b) Next, we show that

$$X_1 = \frac{-B+\sqrt{\Delta}}{2A} < a^\rho. \tag{38}$$

We consider also the two cases:

(b.1) : If $A > 0$, we get $\sqrt{\Delta} > 2Aa^\rho + B$, this yields to

$$(2Aa^\rho + B)^2 - \Delta = 4\rho Aa^\rho(b^\rho - a^\rho) > 0.$$

Thus the inequality (36) holds.

(b.2) : If $A < 0$, then

If $\sqrt{\Delta} > 2Aa^\rho + B > 0$, it implies that

$$(2Aa^\rho + B)^2 - \Delta = 4\rho Aa^\rho(b^\rho - a^\rho) < 0. \tag{39}$$

Thus (36) holds. If $\sqrt{\Delta} > 0 > 2Aa^\rho + B$, then (36) holds obviously.

From the above cases (**a**) and (**b**), we have $s_1 \notin [a,b]$.

(I.2) Secondly, we prove that $s_2 \in [a,b]$, by similar arguments, we can also obtain the following

$$(2Ab^\rho + B)^2 - \Delta = -4\rho(\alpha-1)Ab^\rho(b^\rho - a^\rho).$$

As a consequence, we have $X_2 = \frac{-B-\sqrt{\Delta}}{2A} \in [a^\rho, b^\rho]$. Then

$$\bar{s} \equiv s_2 = \left(\frac{(\alpha\rho-1)a^\rho + (2\rho-1)b^\rho + M}{2N}\right)^{\frac{1}{\rho}},$$

because, if $s=0$ or $s=b$, then $x=0$ is a *trivial solution,* and observe that $\partial_s G_1(s,s)$ has a unique zero in $[a,b]$, attained at the point s_2. This gives

$$\max_{s\in[a,b]} G_1(s,s) \le G_1(s_2,s_2), \tag{40}$$

where

$$G_1(s_2,s_2) = \frac{((1-\alpha\rho)a^\rho + (2\alpha\rho-1)b^\rho - M)^{\alpha-1}((1-(\alpha+2)\rho)a^\rho + (2\rho-1)b^\rho + M)}{\Gamma(\alpha)(b^\rho - a^\rho)(2N)^{\frac{N}{\rho}}((\alpha\rho-1)a^\rho + (2\rho-1)b^\rho + M)^{\frac{1-\rho}{\rho}}}, \tag{41}$$

with M and N are given by (35). Hence, we have

$$G_{\max} \equiv G_1(\bar{s},\bar{s}) = \begin{cases} G_1(s_0,s_0) & \text{if} \quad N=0, \\ G_1(s_2,s_2) & \text{if} \quad N\neq 0. \end{cases} \tag{42}$$

Step (II) Now, we turn our attention to the function G_2.

We start by differentiation $G_2(t,s)$ with respect to t for every fixed $s\in[a,b)$, we can get

$$\partial_t G_2(t,s) = \frac{t^{\rho-1}s^{\rho-1}\rho^{2-\alpha}}{(b^\rho - a^\rho)}\left[(b^\rho - s^\rho)^{\alpha-1} - (\alpha-1)(b^\rho - a^\rho)(t^\rho - s^\rho)^{\alpha-2}\right]. \tag{43}$$

We obtain

$$\partial_t G_2(t,s) = 0 \Longleftrightarrow t_s^* = \left[s^\rho + \left(\frac{(b^\rho - s^\rho)^{\alpha-1}}{(\alpha-1)(b^\rho - a^\rho)}\right)^{\frac{1}{\alpha-1}}\right]^{\frac{1}{\rho}}. \tag{44}$$

We proceed with the following two cases.

(II-1) When $t_s^* \in [s,b]$ then $t_s^{*\rho} \le b^\rho$, i.e., as long as

$$s \le s_* \equiv ((\alpha-1)a^\rho + (2-\alpha)b^\rho)^{\frac{1}{\rho}}. \tag{45}$$

We can easily see that

$$\partial_t G_2(t,s) \begin{cases} <0 & \text{for} \quad t<t_s^*, \\ \ge 0 & \text{for} \quad t \ge t_s^*. \end{cases}$$

This together with the fact that $G_2(b,s)=0$ imply that $G_2(t_s^*,s) \le 0$. By (25), we know

$$\max|G_2(t,s)| \le \max\{\max\{G_2(t,s) : s\le t\le b\} : s\in[a,s_*]\},$$

which means

$$\max\{|G_2(t,s)| : s\le t\le b\} \le \max\left\{\max_{s\in[a,s_*]} G_2(s,s),\ \max_{s\in[a,s_*]} |G_2(t_s^*,s)|\right\}. \tag{46}$$

II-1-a: Firstly, in an entirely similar manner to Step **(I),** we deduce that

$$\max_{s\in[a,s_*]} G_2(s,s) = \begin{cases} G_2(\bar{s},\bar{s}) \equiv G_1(\bar{s},\bar{s}) & \text{if} \quad \bar{s}\in[a,s_*], \\ G_2(s_*,s_*) & \text{if} \quad \bar{s}\notin[a,s_*], \end{cases} \tag{47}$$

where

$$G_2(s_*,s_*) = (\alpha-2)\,\rho^{1-\alpha}\left((b^\rho-a^\rho)(\alpha-1)\right)^{\alpha-1}\left((\alpha-1)\,a^\rho-(\alpha-2)\,b^\rho\right)^{\frac{\rho-1}{\rho}}. \tag{48}$$

II-1-b: Secondly, by fixing an arbitrary $t\in[a,b)$ and differentiating $G_2(t,s)$ with respect to s we get

$$\partial_s G_2(t,s) = \rho^{1-\alpha}s^{\rho-1}\left[(\alpha\rho-1)\,\Theta(t,s)+(1-\rho)\,\Psi(t,s)\right], \tag{49}$$

where we denote

$$\Theta(t,s) = s^\rho\left[\varphi(t,s)-\psi(t,s)\right] \text{ and } \Psi(t,s) = \left[t^\rho\varphi(t,s)-b^\rho\psi(t,s)\right]. \tag{50}$$

with

$$\varphi(t,s) = \frac{1}{(t^\rho-s^\rho)^{2-\alpha}} \text{ and } \psi(t,s) = \frac{(t^\rho-a^\rho)}{(b^\rho-a^\rho)(b^\rho-s^\rho)^{2-\alpha}}. \tag{51}$$

From (51) we observe that

$$0<\psi(t,s)<\varphi(t,s).$$

Combining the above, we get $G_2(t,s)$ is a strictly monotonic function for all $s\in[a,s_*]$. Then $G_2(t,s_*)$ (or $G_2(t,a)$) be the maximal (or minimal) respectiely. It is now obvious that

$$\max\left\{|G_2(t,s)| : a<s\le s_*\right\} \le \max\left\{\max_{t\in[a,b)}|G_2(t,s_*)|,\ \max_{t\in[a,b)}|G_2(t,a)|\right\}. \tag{52}$$

(II-1-b-1) We consider the maximum of $|G_2(t,s_*)|$. If we differentiate $G_2(t,s_*)$ on $[a,b]$, we get

$$\partial_t G_2(t,s_*) = 0 \Longleftrightarrow \bar{t}_{s_*} = \left[s_*^\rho+\left(\frac{(b^\rho-s_*^\rho)^{\alpha-1}}{(\alpha-1)(b^\rho-a^\rho)}\right)^{\frac{1}{\alpha-1}}\right]^{\frac{1}{\rho}}. \tag{53}$$

Then, it follows from the fact that $G_2(b,s_*)=0$ that

$$\partial_t G_2(t,s_*) \begin{cases} <0 & \text{for} \quad t<\bar{t}_{s_*}, \\ \ge 0 & \text{for} \quad t\ge\bar{t}_{s_*}. \end{cases}$$

Hence, $G_2(t,s_*)$ has maximum at point $\bar{t}_{s_*}$. Since $t\in(a,b]$, we get

$$\max_{t\in[a,b]}|G_2(t,s_*)| \le |G_2(\bar{t}_{s_*},s_*)|, \tag{54}$$

where

$$G_2(\bar{t}_{s_*},s_*) = G_1(\bar{t}_{s_*},s_*) - \left(\frac{\bar{t}_{s_*}^\rho-s_*^\rho}{\rho}\right)^{\alpha-1}s_*^{\rho-1} \tag{55}$$

(II-1-b-2) Now, we consider the maximum of $|G_2(t,s)|$ which is obtained at $s=a$ For this purpose, we consider the function $G_2(t,a)$,

$$\partial_t|G_2(t,a)| = 0 \Longleftrightarrow \bar{t}_a = \left[a^\rho+(b^\rho-a^\rho)(\alpha-1)^{\frac{1}{2-\alpha}}\right]^{\frac{1}{\rho}}, \tag{56}$$

also we observe that

$$\partial_t \left|G_2(t,a)\right| \begin{cases} <0 & \text{for} \quad t<\bar{t}_a, \\ \geq 0 & \text{for} \quad t\geq \bar{t}_a. \end{cases}$$

Hence $G_2(t,a)$ has maximum at the point $\bar{t}_a$. Since $\bar{t}_a \in (a,b]$, we get

$$G_2(\bar{t}_a,a) = \rho^{1-\alpha} a^{\rho-1}\left((\alpha-1)^{\frac{1}{2-\alpha}} - (\alpha-1)^{\frac{\alpha-1}{2-\alpha}}\right)(b^\rho - a^\rho)^{\alpha-1}. \tag{57}$$

If $\alpha=2$ then $G_2(t,a)=0$. So we only consider the case that $1<\alpha<2$. Define

$$g(\alpha) = (\alpha-1)^{\frac{1}{2-\alpha}} - (\alpha-1)^{\frac{\alpha-1}{2-\alpha}},\ 0 \leq (\alpha-1)^{\frac{1}{2-\alpha}} \leq 1.$$

It is easy to see that $g(\alpha)\leq 0$ and

$$\min\left\{g(\alpha) : 1<\alpha<2\right\} = \left((\alpha-1)^{\frac{1}{2-\alpha}} - (\alpha-1)^{\frac{\alpha-1}{2-\alpha}}\right),$$

thus,

$$\min_{t,s\in[a,b]} \left|G_2(t,s)\right| = \left|G_2(\bar{t}_a,a)\right|. \tag{58}$$

Consequently, from (54) and (58) it follows that

$$\max\left\{\left|G_2(t,s)\right| : a<s\leq s_*\right\} \leq \max\left\{\left|G_2(\bar{t}_{s_*},s_*)\right|,\ \left|G_2(\bar{t}_a,a)\right|\right\}. \tag{59}$$

where $G_2(\bar{t}_{s_*},s_*)$ and $G_2(\bar{t}_a,a)$ are given by (55) and (57) respectively.

We must make a comparison among $G_2(s_*,s_*)$, $\left|G_2(\bar{t}_a,a)\right|$ to see which is the smallest. It is obvious that

$$G_2(s_*,s_*) \leq G_2(\bar{s},\bar{s}),\ 1<\alpha<2. \tag{60}$$

We now shall prove that

$$\left|G_2(\bar{t}_a,a)\right| \leq G_2(s_*,s_*), \tag{61}$$

Thus from (57) and (48), we arrive at

$$a^{\rho-1}\left((\alpha-1)^{\frac{1}{2-\alpha}} - (\alpha-1)^{\frac{\alpha-1}{2-\alpha}}\right) \leq (\alpha-2)(\alpha-1)^{\alpha-1}\left((\alpha-1)a^\rho - (\alpha-2)b^\rho\right)^{\frac{\rho-1}{\rho}},\alpha\neq 2. \tag{62}$$

Hence, we can verify that

$$\left((\alpha-1)^{\frac{1}{2-\alpha}} - (\alpha-1)^{\frac{\alpha-1}{2-\alpha}}\right) = (\alpha-1)^{\frac{\alpha-1}{2-\alpha}}(\alpha-2). \tag{63}$$

Comparing we get

$$\left((\alpha-1)^{\frac{1}{2-\alpha}} - (\alpha-1)^{\frac{\alpha-1}{2-\alpha}}\right) \leq (\alpha-2)(\alpha-1)^{\alpha-1}\left((2-\alpha)(b^\rho-a^\rho)\right)^{\frac{\rho-1}{\rho}}. \tag{64}$$

Now put $c = \left((2-\alpha)(b^\rho-a^\rho)\right)^{\frac{\rho-1}{\rho}}$.and $\sigma=\alpha-1$ Then the expression above becomes

$$\sigma^{\frac{\sigma}{\sigma-1}} \leq \sigma^\sigma c,\ \text{where } 0<\sigma=\alpha-1<1,$$

or equivalently

$$\frac{\sigma}{1-\sigma}\ln\sigma - \sigma\ln\sigma - \ln c \leq 0.$$

To prove the above inequality, it suffices to show that

$$f(\sigma) = \sigma^2\ln\sigma - (1-\sigma)\ln c \leq 0. \tag{65}$$

By differentiations of f with respect to σ we have,

$$\partial_\sigma f(\sigma) = 2\sigma\ln\sigma - \sigma + \ln c,\ \partial_{2\sigma} f(\sigma) = 2\ln\sigma - 3. \tag{66}$$

If $0 < \ln c < 1$, since $\partial_{2\sigma} f(\sigma) < 0$ the function $\partial_\sigma f(\sigma)$ is decreasing. It is easy to see that

$$\partial_\sigma f(\sigma) \longrightarrow \ln c \quad \text{as} \quad \sigma \longrightarrow 0 \quad \partial_\sigma f(\sigma) \longrightarrow \ln c - 1 \quad \text{as} \quad \sigma \longrightarrow 1$$

and there is a unique point σ_0 in $(0, 1)$ such that $\partial_\sigma f(\sigma_0) = 0$ Therefore, the function f increases from 0 to 1. Moreover, since

$$f(\sigma) \longrightarrow -\ln c \quad \text{as} \quad \sigma \longrightarrow 0 \quad f(\sigma) \longrightarrow 0 \quad \text{as} \quad \sigma \longrightarrow 1$$

we know that $f(\sigma)$ remains negative when $0 < \sigma < 1$.

If $\ln c > 1$, the function is decreasing. It is easy to see that

$$\partial_\sigma f(\sigma) \longrightarrow \ln c \quad \text{as} \quad \sigma \longrightarrow 0 \quad \partial_\sigma f(\sigma) \longrightarrow \ln c - 1 \quad \text{as} \quad \sigma \longrightarrow 1$$

Therefore, the function f decreases first from 0 to σ_0, and then increases from σ_0 to 1. Moreover, since

$$f(\sigma) \longrightarrow -\ln c \quad \text{as} \quad \sigma \longrightarrow 0 \quad f(\sigma) \longrightarrow 0 \quad \text{as} \quad \sigma \longrightarrow 1$$

we know that $f(\sigma)$ remains negative when $0 < \sigma < 1$. Considering the above cases, we have, inequality (64) is shown to be true.

From (60), (61) and (64), we conclude that

$$|G_2(\bar{t}_a, a)| \leq G_1(s_*, s_*) \leq G_2(\bar{s}, \bar{s}).$$

As consequence, we have

$$|G_2(\bar{t}_{s_*}, s_*)| \leq G_1(s_*, s_*) \leq G_2(\bar{s}, \bar{s}).$$

then, we conclude that

$$\max_{t,s\in[a,b]} |G_2(t,s)| = \begin{cases} G_2(\bar{s}, \bar{s}), & \text{if} \quad \bar{s} \in [a, s_*], \\ G_2(s_*, s_*) & \text{if} \quad \bar{s} \notin [a, s_*]. \end{cases} \tag{67}$$

(II-2) When $t_s^* \notin [a, b]$ then $s_* < s \leq t \leq b$. Hence, $G_2(t, s)$ is strictly decreasing as a function of t and, since $G_2(b, s) = 0$, we conclude that

$$\max_{t\in[s,b]} |G_2(t,s)| = G_2(s,s) \equiv G_1(s,s), \;\; s \in (s_*, b].$$

In summary, for each $s \in (s_*, b]$, we conclude that,

$$\max\{G_2(s,s) : s_* < s \leq b\} \leq \begin{cases} G_2(\bar{s}, \bar{s}), & \text{for} \quad \bar{s} \in (s_*, b], \\ G_2(s_*, s_*) & \text{for} \quad \bar{s} \notin (s_*, b], \end{cases} \tag{68}$$

where $\bar{s}$ and s_* are given in step (**I**) and (45). From the above discussion, thus (27) holds.

From (67) and (68) we have

$$\max_{t,s\in[a,b]} |G_2(t,s)| \leq |G_2(\bar{s}, \bar{s})|. \tag{69}$$

From the steps (**I**) and (**II**), the maximum value of $G(t, s)$ is

$$\max\{|G(t,s)| : a \leq s, t \leq b\} \leq \max_{s\in[a,b]} G(s,s) = G_{\max}.$$

This completes the proof of Lemma. □

Theorem 3.3. *If a nontrivial continuous solution of the problem (22) exists, then*

$$\int_a^b |p(s)|\, ds > G_{\max}, \tag{70}$$

where $G_{\max}$ is defined in (27).

Proof. By Lemma 3.2 and from (23), it follows that if x is a nontrivial continuous solution of the (22), then

$$|x(t)| \leq \int_a^b |G(t,s)p(s)|\, |x(s)|\, ds. \tag{71}$$

Let $\mathcal{B} = C[a,b]$ be a Banach space endowed a norm

$$\|x\|_\infty = \max_{t\in[a,b]} |x(t)|, x \in \mathcal{B}. \tag{72}$$

Hence, from (71) and (72), we get

$$\|x\|_\infty \leq \max_{t\in[a,b]} \left| \int_a^b G(t,s)p(s)\, ds \right| \|x\|_\infty ,$$

or equivalently,

$$\max_{t\in[a,b]} \int_a^b |G(t,s)p(s)|\, ds \geq 1. \tag{73}$$

Using the properties of Green's function $G(t,s)$ particularly, $G_{\max}$ in Lemma 3.2 gives the inequality

$$\int_a^b |p(s)|\, ds \geq \frac{1}{G_{\max}}, \tag{74}$$

called the Lyapunov-type inequality for (22), where $G_{\max}$ is defined in (27). □

Particular cases. In the case $\rho = 1$ we have

$$\bar{s} = \frac{(\alpha-1)\,a + b}{\alpha},\ \ M = (\alpha-1)\,a + b,\ \ N = \alpha.$$

The corresponding maximum Green's function $G_{\max}$ can be written as

$$G_{\max} = \frac{\alpha^{-\alpha}}{\Gamma(\alpha)} \left((\alpha-1)(b-a)\right)^{\alpha-1}.$$

Thus, our results matches the results of Theorem 1 in [17].

When $\rho = 1, \alpha = 2$

$$\bar{s} = \frac{a+b}{2},\ \ M = a+b,\ \ N = 2.$$

The corresponding maximum Green's function $G_{\max}$ can be written as

$$G_{\max} = \frac{b-a}{4}.$$

Thus, we obtain Theorem 1.1.

4. **Applications.** This section can be considered as the applied aspect of this paper. Relying on the Lyapunov inequalitie (70), we are going to establish nontrivial solutions of fractional boundary value problems (22) . Also, considering corresponding fractional eigenvalue problems we find spreading interval of the eigenvalues. The eigenvalues and eigenfunctions are characterized in terms of the Mittag–Leffler functions.

4.1. **Lyapunov-type inequality for fractional boundary value prolems.** In this subsection, we obtain a Lyapunov-type inequality for fractional boundary value prolems having the form

$$\begin{cases} \left({}^{c}\mathcal{D}_{a+}^{\alpha,\rho}\right)\left(\left(\frac{s^{\rho}-a^{\rho}}{\rho}\right)^{\alpha(1-m)}y\left(s\right)\right)(t)+q(t)y(t)=0, \\ y\left(a\right)=0=y\left(b\right),\ m>0,t\in\left(a,b\right),\alpha\in(1,2],\rho>0, \end{cases} \tag{75}$$

where $y(t)\in C_{\rho,\alpha(1-m)}$ of the functions $g\left(t\right)\in\mathcal{B}$ such that $\left(\frac{s^{\rho}-a^{\rho}}{\rho}\right)^{\alpha(1-m)}g\left(t\right)\in\mathcal{B}$ and $q:[a,b]\to\mathbb{R}$ is a continuous function.

The fractional boundary value prolems (75) can be reduced to (22) by a change of

$$y\left(t\right)=\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha(m-1)}x\left(t\right)\ \text{ and } q(t)=\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha(1-m)}p(t). \tag{76}$$

For $x\left(t\right)$ and $p(t)$ in (76), Theorem 3.3, yields to the following Corollary.

Corollary 1. *If a nontrivial continuous solution of the problem (75) exists, then*

$$\int_{a}^{b}\left|\left(\frac{s^{\rho}-a^{\rho}}{\rho}\right)^{\alpha(m-1)}q(s)\right|ds>G_{\max}, \tag{77}$$

where $G_{\max}$ is defined in (27).

4.2. **The real zeroes of the generalized Mittag–Leffler functions $E_{\alpha,m,\beta}\left(z\right)$.** The zeros of $E_{\alpha,m,\beta}\left(z\right)$, which play a significant role in the dynamic solutions, are of intrinsic interest, we will use Lyapunov-type inequalities (70) to obtain intervals where certain generalized Mittag-Leffler functions have no real zeros.

Firstly, we present explicit solutions to fractional differential equations

$$\left({}^{c}\mathcal{D}_{a+}^{\alpha,\rho}\right)\left(\left(\frac{s^{\rho}-a^{\rho}}{\rho}\right)^{\alpha(1-m)}y\left(s\right)\right)(t)=-\lambda y\left(t\right),\ \alpha>0,\ m>0,\ \lambda\neq0. \tag{78}$$

Theorem 4.1. *Let $\rho>0,\ m>0,\lambda\in\mathbb{R}^{*}$.*

(i) *If $\alpha\in(0,1]$ the equation (78) has the solution*

$$y\left(t\right)=\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha m-1}E_{\alpha,m,m-1/\alpha}\left(-\lambda\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{m\alpha}\right),\ \text{for } t>a>0. \tag{79}$$

(ii) *If $\alpha>1$ and $\alpha m\left(i-1\right)\neq1,2,...,-\left[-\alpha\right]-1,\ i=0,1,2,...,$ the equation (78) has $\left(-\left[-\alpha\right]\right)$ linearly independent solutions*

$$y_{j}\left(t\right)=\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha\left(m-\frac{j}{\alpha}\right)}E_{\alpha,m,m-j/\alpha}\left(-\lambda\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{m\alpha}\right),\ \text{for } j=1,2,...,-\left[-\alpha\right]. \tag{80}$$

Proof. Applying the relation (21) and (80) we have

$$\begin{aligned} &{}^{c}\mathcal{D}_{a+}^{\alpha,\rho}\left(\left(\frac{s^{\rho}-a^{\rho}}{\rho}\right)^{\alpha(1-m)}y_{j}\left(s\right)\right)(t)= \\ &{}^{c}\mathcal{D}_{a+}^{\alpha,\rho}\left(\left(\frac{s^{\rho}-a^{\rho}}{\rho}\right)^{\alpha\left(m-\frac{j}{\alpha}+1-m\right)}E_{\alpha,m,m-j/\alpha}\left(-\lambda\left(\frac{s^{\rho}-a^{\rho}}{\rho}\right)^{m\alpha}\right)\right)(t) \\ &=-\lambda\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha\left(m-\frac{j}{\alpha}\right)}E_{\alpha,m,m-j/\alpha}\left(-\lambda\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{m\alpha}\right)=-\lambda y_{j}\left(t\right), \end{aligned}$$

which gives (78), for $j=1,2,...,-\left[-\alpha\right]$. ∎

Corollary 2. *Let $\rho > 0,\ m > (-[-\alpha]-1)/\alpha,\ \lambda \in \mathbb{R}^*$, then the equation (78) has $(-[-\alpha])$ linearly independent solutions given by (80).*

Corollary 3. *Let $m = 1, \rho > 0, \alpha > 0$, then the equation (78) has $(-[-\alpha])$ linearly independent solutions*

$$y_j(t) = \Gamma(\alpha - j + 1)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha - j} E_{\alpha,\alpha-j+1}\left(-\lambda\left(\frac{t^\rho - a^\rho}{\rho}\right)^\alpha\right),\ \text{for } j = 1, 2, ..., -[-\alpha]. \tag{81}$$

Remark 5. In particular, if we take $1 < \alpha < 2$ in Corollary 3, then the equation (78) has the general solution

$$y(t) = c_2\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-2} E_{\alpha,\alpha-1}\left(-\lambda\left(\frac{t^\rho - a^\rho}{\rho}\right)^\alpha\right) + c_1\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho - a^\rho}{\rho}\right)^\alpha\right). \tag{82}$$

where c_1 and c_2 are the constants.

When $\alpha = 2$ in Corollary 3, the equation (78), has the general solution

$$y(t) = c_2 E_{2,1}\left(-\lambda\left(\frac{t^\rho - a^\rho}{\rho}\right)^2\right) + c_1\left(\frac{t^\rho - a^\rho}{\rho}\right) E_{2,2}\left(-\lambda\left(\frac{t^\rho - a^\rho}{\rho}\right)^2\right). \tag{83}$$

When $\rho = 1$ in Corollary 3, the equation (78), has the well-known general solution

$$y(t) = c_2 E_{\alpha,1}(-\lambda(t-a)^\alpha) + c_1(t-a) E_{\alpha,2}(-\lambda(t-a)^\alpha). \tag{84}$$

If $\lambda = 0$ the general solution (78) degenerates to

$$y(t) = c_2 + c_1(t-a). \tag{85}$$

Secondly, we consider the particular case of the following fractional eigenvalue problem (75)

$$\begin{cases} \left({}^c\mathcal{D}_{0+}^{\alpha,\rho}\right)\left(\left(\dfrac{s^\rho}{\rho}\right)^{\alpha(1-m)} y(s)\right)(t) = -\lambda y(t), \\ y(0) = 0 = y(1),\ t \in [0,1], \alpha \in (1,2], m > 0,\ \lambda \neq 0, \end{cases} \tag{86}$$

Let $z \in \mathbb{R}$ and consider the real zeros of the generalized Mittag-Leffler functions $E_{\alpha,m,\beta}(z)$.

Obviously $E_{\alpha,m,\beta}(z) > 0$ for all $z \geq 0$. Hence, the real zeros of $E_{\alpha,m,\beta}(z)$ if they exist, must be negative real numbers. The values of α, m and β determine if the function $E_{\alpha,m,\beta}(z)$ has real zeroes.

Theorem 4.2. *The fractional eigenvalue problem (86) has an infinite number of eigenvalues, and they are roots of the generalized Mittag-Leffler type equation*

$$E_{\alpha,m,m-1/\alpha}\left(-\lambda\left(\frac{1}{\rho}\right)^{m\alpha}\right) = 0,$$

and the corresponding eigenfunctions are given by

$$y(t) = \left(\frac{t^\rho}{\rho}\right)^{\alpha\left(m-\frac{1}{\alpha}\right)} E_{\alpha,m,m-1/\alpha}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^{m\alpha}\right). \tag{87}$$

Proof. Using Theorem 4.1-ii, the general solution of (86) can be obtained as

$$\begin{aligned} y(t) &= c_2\left(\frac{t^\rho}{\rho}\right)^{\alpha\left(m-\frac{2}{\alpha}\right)} E_{\alpha,m,m-2/\alpha}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^{m\alpha}\right) \\ &+ c_1\left(\frac{t^\rho}{\rho}\right)^{\alpha\left(m-\frac{1}{\alpha}\right)} E_{\alpha,m,m-1/\alpha}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^{m\alpha}\right). \end{aligned}$$

If $\lambda \leq 0$ then the problem (86) only has zero solution.

If $\lambda > 0$ with $y(0) = 0$ we have $c_2 = 0$ and

$$y(1) = c_1 \left(\frac{1}{\rho}\right)^{\alpha m-1} E_{\alpha,m,m-1/\alpha}\left(-\lambda \left(\frac{1}{\rho}\right)^{m\alpha}\right) = 0,$$

where c_1 is an arbitrary constant, we get

$$E_{\alpha,m,m-1/\alpha}\left(-\lambda \left(\frac{1}{\rho}\right)^{m\alpha}\right) = 0.$$

The eigenfunctions of the problem (86) then has the form

$$y(t) = \left(\frac{t^\rho}{\rho}\right)^{\alpha m-1} E_{\alpha,m,m-1/\alpha}\left(-\lambda \left(\frac{t^\rho}{\rho}\right)^{m\alpha}\right),$$

where $-\lambda \left(\frac{t^\rho}{\rho}\right)^{m\alpha}$ are zeros of the generalized Mittag-Leffler function. □

Corollary 4. *In particular, if $m = 1$, the fractional eigenvalue problem (86) has an infinite number of eigenvalues, and they are roots of the Mittag-Leffler type equation*

$$E_{\alpha,1,1-1/\alpha}\left(-\lambda \left(\frac{1}{\rho}\right)^{\alpha}\right) = \Gamma(\alpha) E_{\alpha,\alpha}\left(-\lambda \left(\frac{1}{\rho}\right)^{\alpha}\right) = 0$$

and the corresponding eigenfunctions are given by

$$y(t) = \left(\frac{t^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,1,1-1/\alpha}\left(-\lambda \left(\frac{t^\rho}{\rho}\right)^{\alpha}\right),\ t \in [0,1].$$

Finally in this section, inequality (70) can be used to determine intervals for the real zeros of the Mittag-Leffler function $E_{\alpha,m,\beta}(z)$.

Let us consider the fractional eigenvalue problem (22) (with $[a,b] = [0,1]$ and $q(t) = -\lambda$) Theorem 3.3, yields to the following Corollary.

Corollary 5. *Let λ be the smallest eigenvalue of (86). Then the eigenvalues λ are indeed real zeros of the generalized Mittag-Leffler function $E_{\alpha,m,\beta}(z)$ provided that*

$$\int_0^1 \left|\lambda \left(\frac{1}{\rho}\right)^{\alpha(m-1)}\right| ds > \frac{\Gamma(\alpha)(2N)^{\frac{N}{\rho}} (M + 2\rho - 1)^{\frac{1-\rho}{\rho}}}{(2\alpha\rho - 1 - M)^{\alpha-1}(M - \alpha\rho - 1)},\ N \neq 0. \tag{88}$$

Equivalently,

$$|\lambda| > (\rho)^{\alpha(m-1)} \frac{\Gamma(\alpha)((\alpha+1)\rho - 1)^{\frac{((\alpha+1)\rho-1)}{\rho}}}{(2\rho - 1)^{\frac{2\rho-1}{\rho}}(\alpha - 1)^{\alpha-1}}. \tag{89}$$

Hence, it follows that for each

$$\lambda \in (\rho)^{\alpha(m-1)} \left[-\frac{\Gamma(\alpha)((\alpha+1)\rho - 1)^{\frac{((\alpha+1)\rho-1)}{\rho}}}{(2\rho - 1)^{\frac{2\rho-1}{\rho}}(\alpha - 1)^{\alpha-1}}, \frac{\Gamma(\alpha)((\alpha+1)\rho - 1)^{\frac{((\alpha+1)\rho-1)}{\rho}}}{(2\rho - 1)^{\frac{2\rho-1}{\rho}}(\alpha - 1)^{\alpha-1}}\right]$$

λ is not an eigenvalue of the fractional eigenvalue problem (75). Also,

$$\mathrm{LB}_{\text{eigenvalue}} := (\rho)^{\alpha(m-1)} \frac{\Gamma(\alpha)((\alpha+1)\rho - 1)^{\frac{((\alpha+1)\rho-1)}{\rho}}}{(2\rho - 1)^{\frac{2\rho-1}{\rho}}(\alpha - 1)^{\alpha-1}},$$

can be considered as a lower bound for the positive eigenvalues of the eigenvalue problem (86).

Corollary 6. *If (88) is does not hold then the eigenfunctions*

$$y(t) = \left(\frac{t^\rho}{\rho}\right)^{\alpha m-1} E_{\alpha,m,m-1/\alpha}\left(-\lambda \left(\frac{t^\rho}{\rho}\right)^{m\alpha}\right),\ t \in [0,1], \tag{90}$$

of the eigenvalue problem (86) has no real zeros.

Corollary 7. *If (88) is does not hold then the the problem (86) has no nontrivial solutions in the class of real functions.*

Corollary 8. *The generalized Mittag-Leffler function $E_{\alpha,m,\beta}(z)$ has no real zeros for*

$$|z| \leq \frac{\Gamma(\alpha)\left((\alpha+1)\rho-1\right)^{\frac{((\alpha+1)\rho-1)}{\rho}}}{(2\rho-1)^{\frac{2\rho-1}{\rho}}(\alpha-1)^{\alpha-1}}. \tag{91}$$

Remark 6. When $\rho = 1$, the result in (91), coincides with the result found in [2], where a Lyapunov type inequality was obtained by considering boundary value problems involving different fractional derivatives.

Remark 7. We stress that, when $m = 1$ and $\rho = 1$, the result stated in Corollary 8 coincides with that of Theorem 2.2 in [17].

$$|z| \leq \frac{\Gamma(\alpha)\alpha^{\alpha}}{(\alpha-1)^{\alpha-1}}. \tag{92}$$

5. **Conlusion.** In this article, we obtained Lyapunov type inequalities for certain classes of fractional boundary value problems involving generalized Caputo fractional derivatives. In all cases it was demonstrated that the results previously obtained in the literature are just special cases of our results. We think it is worth to mention that because of the complexity to obtain the maximum of the Green's functions discussed we were forced to use symbolic manipulation program Maple. Because of the fact that the fractional integrals considered in this paper combine the Riemann-Liouville and the Hadamard fractional integrals (derivative), it is appreciated if the researchers consider them, although difficult, to obtain new inequalities that help in the development of the qualitative properties of the fractional differential equations that contain these operators. In addition, researchers can use the newly discovered Atangana-Baleanu fractional operators in order to establish mathematical inequalities. This will contribute in pushing the theory of the fractional calculus in the frame of these operators forward.

REFERENCES

[1] T. Abdeljawad, B. Benli and D. Baleanu, A generalized q-Mittag-Leffler function by q-Captuo fractional linear equations, *Abstract and Appl. Anal.*, **2012** (2012), Article ID 546062, 11pp.

[2] T. Abdeljawad, F. Jarad, S. F. Mallak and J. Alzabut, Lyapunov type inequalities via fractional proportional derivatives and application on the free zero disc of Kilbas-Saigo generalized Mittag-Leffler functions, *Eur. Phys. J. Plus*, **134** (2019), 247.

[3] T. Abdeljawad, J. Alzabut and F. Jarad, A generalized Lyapunov-type inequality in the frame of conformable derivatives, *Adv. Differ. Equ.*, **2017** (2017), Paper No. 321, 10 pp.

[4] T. Abdeljawad, A Lyapunov type inequality for fractional operators with nonsingular Mittag-Leffler kernel, *J. Ineq. Appl.*, **2017** (2017), Paper No. 130, 11 pp.

[5] T. Abdeljawad, Fractional operators with exponential kernels and a Lyapunov type inequality, *Adv. Differ. Equ.*, **2017** (2017), Paper No. 313, 11 pp.

[6] T. Abdeljawad, Q. M. Al-Mdallal and M. A. Hajji, Arbitrary order fractional difference operators with discrete exponential kernels and applications, *Discrete Dyn. Nat. Soc.*, **2017** (2017), Art. ID 4149320, 8 pp.

[7] T. Abdeljawad and F. Madjidi, A Lyaponuv inequality for fractional difference operators with discrete Mittag-Leffler kernel of order $2 \leq \alpha < 5/2$, *Eur. Phys. J. Spec. Top.*, **226** (2017), 3355–3368.

[8] A. Atangana and D. Baleanu, New fractional derivatives with non-local and non-singular kernel: Theory and application to heat transfer model, *Thermal Sci.*, **20** (2016), 763–769.

[9] A. Atangana and J. F. Gomez-Aguilar, Hyperchaotic behaviour obtained via a nonlocal operator with exponential decay and Mittag-Leffler laws, *Chaos Soliton Fractals*, **102** (2017), 285–294.

[10] D. Çakmak, Lyapunov-type integral inequalities for certain higher order differential equations, *Appl. Math. Comp.*, **216** (2010), 368–373.

[11] M. Caputo and M. Fabrizio, A new definition of fractional derivative without singular kerne, *Prog. Frac. Diff. Appl.*, **1** (2015), 73-85.

[12] S. Clark and D. B. Hinton, A Lyapunov-type inequality for linear Hamiltonian systems, *Math. Ineq. Appl.*, **1** (2010), 201–209.

[13] B. Cuahutenango-Barro, M. A. Taneco-Hernandez and J. F. Gomez-Aguilar, On the solutions of fractional-time wave equation with memory effect involving operators with regular kernel, *Chaos Solitons Fractals*, **115** (2018), 283–299.

[14] K. Diethelm, *The Analysis of Fractional Differential Equations,* Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2010.

[15] R. A. C. Ferreira, A Lyapunov-type inequality for a fractional initial value problem, *Fract. Calc. Appl. Anal.*, **16** (2013), 978–984.

[16] R. A. C. Ferreira, Lyapunov-type inequalities for some sequential fractional initial value problems, *Adv. Dyn. Syst. Appl.*, **11** (2016), 33–43.

[17] R. A. C. Ferreira, On a Lyapunov-type inequality and the zeros of a certain Mittag-Leffler function, *J. Math. Anal. Appl.*, **412** (2014), 1058–1063.

[18] J. F. Gomez-Aguilar, A. Atangana and V. F. Morales-Delgado, Electrical circuits RC, LC and RL described by Atangana-Baleanu fractional derivatives, *Int. J. Circ. theor. Appl.*, **45** (2017), 1514–1533.

[19] J. F. Gomez-Aguilar and A. Atangana, New insight in fractional differentiation: Power, exponential decay and Mittag-Leffler laws and applications, *Eur. Phys. J. Plus*, **2017** (2017), 13pp.

[20] J. F. Gomez-Aguilar, H. Yepez-Martinez, R. F. Escobar-Jimenez, C. M. Astorga-Zaragoza and J. Reyes-Reyes, Analytical and numerical solutions of electrical circuits described by fractional derivatives, *Appl. Math. Model.*, **40** (2016), 9079–9094.

[21] R. Gorenflo, A. A. Kilbas, F. Mainardi and S. V. Rogosin, *Mittag-Leffler Functions, Related Topics and Applications*, Springer Monographs in Mathematics, Springer Heidelberg New York Dordrecht London, 2014.

[22] F. Jarad, T. Abdeljawad and Z. Hammouch, On a class of ordinary differential equations in the frame of Atagana-Baleanu derivative, *Chaos Solitons Fractals*, **117** (2018), 16–20.

[23] F. Jarad, T. Abdeljawad and D. Baleanu, Caputo-type modification of the Hadamard fractional derivative, *Adv. Differ. Equ.*, **2012** (2012), 8pp.

[24] F. Jarad, T. Abdeljawad and D. Baleanu, On the generalized fractional derivatives and their Caputo modification, *J. Nonlinear Sci. Appl.*, **10** (2017), 2607–2619.

[25] M. Jleli and B. Samet, Lyapunov-type inequalities for fractional boundary value problems equation with fractional initial conditions, *Electron. J. Diff. Equ.*, **2015** (2015), 1–11.

[26] U. N. Katugampola, New approach to a generalized fractional integral, *Appl. Math. Comput.*, **218** (2011), 860–865.

[27] U. N. Katugampola, A new approach to generalized fractional derivatives, *Bull. Math. Anal. App.*, **6** (2014), 1–15.

[28] A. A. Kilbas, Hadamard type fractional calculus, *J. Korean Math. Soc.*, **38** (2001), 1191–1204.

[29] A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations,* Elsevier, Amsterdam, 2006.

[30] A. A. Kilbas and M. Saigo, Fractional integrals and derivatives of Mittag-Leffler type function (Russian), *Dokl. Akad. Nauk Belarusi*, **39** (1995), 22–26, 123.

[31] A. A. Kilbas and M. Saigo, On solutions of integral equations of Abel-Volterra type, *Diff. Integral Equ.*, **8** (1995), 993–1011.

[32] A. M. Liapunov, Problème général de la stabilitie du mouvement, *Ann. of Math. Stud.*, **17**, Princeton Univ. Press, Princeton, N. J., 1949.

[33] Q. Ma, C. Ma and J. Wang, A Lyapunov-type inequality for a fractional differential equation with Hadamard derivative, *J. Math. Ineq.*, **11** (2017), 135–141.

[34] G. M. Mittag-Leffler, Sur la nouvelle fonction $E_\alpha(z)$, *C. R. Acad. Sci. Paris*, **137** (1903), 554–558.

[35] N. Parhi and S. Panigrahi, A Lyapunov-type integral inequality for higher order differential equations, *Math. Slovaca*, **52** (2002), 31–46.

[36] J. P. Pinasco, *Lyapunov-type Inequalities*, Springer Briefs in Mathematics, Springer, New York, 2013.

[37] I. Podlubny, *Fractional Differential Equations,* Academic Press, San Diego, California, 1999.

[38] T. R. Prabhakar, A singular integral equation with a generalised Mittag-Leffler function in the kernel, *Yokohama Math. J.*, **19** (1971), 7–15.

[39] J. Rongand and C. Bai, Lyapunov-type inequality for afractional differential equation with fractional initial conditions, *Adv. Differ. Equ.*, **2015** (2015), 10pp.

[40] X. Yang, On Lyapunov-type inequality for certain higher-order differential equations, *Appl. Math. Comp.*, **134** (2003), 307–317.

[41] X. Yang and K. Lo, Lyapunov-type inequality for a class of even-order differential equations, *Appl. Math. Comp.*, **215** (2010), 3884–3890.

[42] H. Ye, J. Gao and Y. Ding, A generalized Lyapunov inequality and its application to a fractional differential equation, *J. Math. Anal. Appl.*, **328** (2007), 1075–1081.

[43] H. Yepez-Martinez and J. F. Gomez-Aguilar, A new modified definition of Caputo-Fabrizio fractional-order derivative and their applications to the Multi Step Homotopy Analysis Method (MHAM), *J. Comput. Appl. Math.*, **346** (2019), 247–260.

[44] H. Yepez-Martinez, J. F. Gomez-Aguilar, I. O. Sosa, J. M. Reyes and J. Torres-Jimenez, The Feng's first integral method applied to the nonlinear mKdV space-time fractional partial differential equation, *Revista Mexicana Fisica*, **62** (2016), 310–316.

Received April 2019; revised May 2019.

E-mail address: fahd@cankaya.edu.tr
E-mail address: adjabiy@yahoo.fr
E-mail address: tabdeljawad@psu.edu.sa
E-mail address: s.mallak@ptuk.edu.ps
E-mail address: hussam.alrabaiah@aau.ac.ae