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More properties of the proportional fractional integrals and derivatives of a function with respect to another function

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Abstract

In this article, we present some new properties of the fractional proportional derivatives of a function with respect to a certain function. We use a modified Laplace transform to find the relation between the derivatives in the Riemann–Liouville setting and the one in Caputo. In addition, we provide an integration by parts formulas related to the considered operators.

Keywords: General proportional integrals; General proportional derivatives; General Caputo proportional derivative

1 Introduction

For the last 30 years or more, some scientists have shown a great deal of interest in the field of fractional calculus which addresses the derivatives and integrals with any order. As a matter of fact, this interest has sprung out by the dint of the substantial results obtained when these scientists used the tools in this calculus in order to study some models from the real world [1–6].

One of the virtues of the classical fractional calculus is that there are a variety of derivatives and integrals. Nevertheless, there has always been a need to develop this calculus more and discover some new derivatives for the sake of better understanding the universe. Some of the newly proposed fractional operators contain nonsingular kernels [7–14]. In 2011, Katugampola in [15, 16] proposed what he called generalized fractional operators for the purpose of combining the Riemann–Liouville and Hadamard fractional operators. The generalized derivatives were modified so that they cover the Caputo and the Caputo–Hadamard fractional derivatives [17]. In other respects, local derivatives permit differentiation and integration of noninteger order. In [18, 19], the authors introduced what they called conformable derivative. It should be noted that the nonlocal fractional versions of the conformable operators were discussed in [15, 16, 20]. However, more generalized forms of these operators were discussed in [21].

The main disadvantage of the conformable derivative is that contrary to the other differential operators it does not produce the function itself once the order is 0. To overcome this, the authors in [22, 23] proposed a modification of the conformable derivative so that if the order of this derivative tends to 0 it gives the function itself and if the order tends to 1 it gives the first-order derivative of the function. Later, the authors in [24] presented a new type of fractional operators generated from the above-mentioned modified conformable derivatives. In addition, more generalized forms of these fractional operators were put forward in [25], and it turned out that some of these operators coincided with some operators mentioned before in [26–28].

In this work, orientated by the above-mentioned works, we continue our study on the proportional fractional derivatives and integrals of a function with respect to another function discovered in [25]. We present the effect of the fractional integral operators on the differential operators and vice versa. In addition, we present the relation between the fractional proportional derivatives in Riemann–Liouville and Caputo settings using a modified Laplace transform.

The article is organized as follows: Sect. 2 is devoted to some essential definitions for fractional proportional derivative and integrals and their generalizations. In Sect. 3, we apply the proportional fractional integrals on fractional derivatives, discuss the Laplace transforms for the generalized fractional integrals and derivatives, and give some examples. In Sect. 4, we present an integration by parts formula, and we conclude our work in Sect. 5.

2 Essential preliminaries

In this section, we present fundamental definitions, lemmas, theorems, and corollaries needed for our findings in this article.

2.1 The proportional derivatives

Definition 2.1 (Modified conformable derivatives; [22, 23]) For $\varrho \in [0,1]$, let the functions $\varkappa_0, \varkappa_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous such that for all $t \in \mathbb{R}$ we have

$$\lim_{\varrho\to 0^+} \varkappa_1(\varrho,t) = 1, \qquad \lim_{\varrho\to 0^+} \varkappa_0(\varrho,t) = 0, \qquad \lim_{\varrho\to 1^-} \varkappa_1(\varrho,t) = 0, \qquad \lim_{\varrho\to 1^-} \varkappa_0(\varrho,t) = 1,$$

and $\varkappa_1(\varrho,t) \neq 0$, $\varrho \in [0,1)$, $\varkappa_0(\varrho,t) \neq 0$, $\varrho \in (0,1]$. Then the modified conformable differential operator of order ϱ is defined by

$$D^{\varrho} f(t) = \varkappa_1(\varrho,t) f(t) + \varkappa_0(\varrho,t) f'(t). \tag{2.1}$$

The derivative given in (2.1) is called a proportional derivative. For more details about the control theory of the proportional derivatives and their component functions $\varkappa_0$ and $\varkappa_1$, we refer the reader to [22, 23].

For the restricted case when $\varkappa_1(\varrho,t) = 1-\varrho$ and $\varkappa_0(\varrho,t) = \varrho$, the proportional derivative and its integral respectively read

$$D^{\varrho} f(t) = (1-\varrho) f(t) + \varrho f'(t) \tag{2.2}$$

and

$${}_aI^{1,\varrho} f(t) = \frac{1}{\varrho}\int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)} f(s)\,ds, \tag{2.3}$$

where ${}_aI^{0,\varrho} f(t) = f(t)$.

The nth order proportional integral has the form

$$\big({}_aI^{n,\varrho}f\big)(t)=\frac{1}{\varrho^n\Gamma(n)}\int_a^t e^{\frac{\varrho-1}{\varrho}(t-\tau)}(t-\tau)^{n-1}f(\tau)\,d\tau. \tag{2.4}$$

Definition 2.2 ([24]) For $\varrho>0$ and $\alpha\in\mathbb{C}$, $\operatorname{Re}(\alpha)>0$, the left and right proportional fractional integrals of f are respectively defined as

$$\big({}_aI^{\alpha,\varrho}f\big)(t)=\frac{1}{\varrho^\alpha\Gamma(\alpha)}\int_a^t e^{\frac{\varrho-1}{\varrho}(t-\tau)}(t-\tau)^{\alpha-1}f(\tau)\,d\tau \tag{2.5}$$

and

$$\big(I_b^{\alpha,\varrho}f\big)(t)=\frac{1}{\varrho^\alpha\Gamma(\alpha)}\int_t^b e^{\frac{\varrho-1}{\varrho}(\tau-t)}(\tau-t)^{\alpha-1}f(\tau)\,d\tau. \tag{2.6}$$

Definition 2.3 ([24]) For $\varrho>0$ and $\alpha\in\mathbb{C}$, $\operatorname{Re}(\alpha)\geq 0$, the left and right proportional derivatives of f are respectively given as

$$\begin{aligned}\big({}_aD^{\alpha,\varrho}f\big)(t)&=D^{n,\varrho}{}_aI^{n-\alpha,\varrho}f(t)\\&=\frac{D_t^{n,\varrho}}{\varrho^{n-\alpha}\Gamma(n-\alpha)}\int_a^t e^{\frac{\varrho-1}{\varrho}(t-\tau)}(t-\tau)^{n-\alpha-1}f(\tau)\,d\tau\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\big(D_b^{\alpha,\varrho}f\big)(t)&={}_\ominus D^{n,\varrho}I_b^{n-\alpha,\varrho}f(t)\\&=\frac{{}_\ominus D_t^{n,\varrho}}{\varrho^{n-\alpha}\Gamma(n-\alpha)}\int_t^b e^{\frac{\varrho-1}{\varrho}(\tau-t)}(\tau-t)^{n-\alpha-1}f(\tau)\,d\tau,\end{aligned} \tag{2.8}$$

where $n=[\operatorname{Re}(\alpha)]+1$,

$$D^{n,\varrho}=\underbrace{D^\varrho D^\varrho\ldots D^\varrho}_{n\text{ times}}$$

and

$$\big({}_\ominus D^\varrho f\big)(t):=(1-\varrho)f(t)-\varrho f'(t),\qquad {}_\ominus D^{n,\varrho}=\underbrace{{}_\ominus D^\varrho_\ominus D^\varrho\ldots{}_\ominus D^\varrho}_{n\text{ times}}.$$

2.2 The fractional proportional derivative of a function with respect to another function

Definition 2.4 (The proportional derivative of a function with respect to another function; [25]) For $\varrho\in[0,1]$, let the functions $\varkappa_0,\varkappa_1:[0,1]\times\mathbb{R}\to[0,\infty)$ be continuous such that for all $t\in\mathbb{R}$ we have

$$\lim_{\varrho\to 0^+}\varkappa_1(\varrho,t)=1,\qquad\lim_{\varrho\to 0^+}\varkappa_0(\varrho,t)=0,\qquad\lim_{\varrho\to 1^-}\varkappa_1(\varrho,t)=0,\qquad\lim_{\varrho\to 1^-}\varkappa_0(\varrho,t)=1,$$

and $\varkappa_1(\varrho,t)\neq 0$, $\varrho\in[0,1)$, $\varkappa_0(\varrho,t)\neq 0$, $\varrho\in(0,1]$. Let also $g(t)$ be a strictly increasing continuous function. Then the proportional differential operator of order ϱ of f with respect to g is defined by

$$D^{\varrho,g}f(t)=\varkappa_1(\varrho,t)f(t)+\varkappa_0(\varrho,t)\frac{f'(t)}{g'(t)}. \tag{2.9}$$

As in the previous subsection, for the restriction $\varkappa_1(\varrho,t)=1-\varrho$ and $\varkappa_0(\varrho,t)=\varrho$, (2.1) becomes

$$D^{\varrho,g}f(t)=(1-\varrho)f(t)+\varrho\frac{f'(t)}{g'(t)}. \tag{2.10}$$

The corresponding integral of (2.10) of order n [25]

$$\big({}_aI^{n,\varrho,g}f\big)(t)=\frac{1}{\varrho^n\Gamma(n)}\int_a^t e^{\frac{\varrho-1}{\varrho}(g(t)-g(\tau))}\big(g(t)-g(\tau)\big)^{n-1}f(\tau)g'(\tau)\,d\tau. \tag{2.11}$$

The left and right proportional integrals in their general forms are given as follows.

Definition 2.5 ([25]) For $\varrho\in(0,1]$, $\alpha\in\mathbb{C}$, $\operatorname{Re}(\alpha)>0$, we define the left and right fractional integrals of f with respect to g by

$$\big({}_aI^{\alpha,\varrho,g}f\big)(t)=\frac{1}{\varrho^\alpha\Gamma(\alpha)}\int_a^t e^{\frac{\varrho-1}{\varrho}(g(t)-g(\tau))}\big(g(t)-g(\tau)\big)^{\alpha-1}f(\tau)g'(\tau)\,d\tau. \tag{2.12}$$

The right fractional proportional integral ending at b, however, can be defined by

$$\big(I_b^{\alpha,\varrho,g}f\big)(t)=\frac{1}{\varrho^\alpha\Gamma(\alpha)}\int_t^b e^{\frac{\varrho-1}{\varrho}(g(\tau)-g(t))}\big(g(\tau)-g(t)\big)^{\alpha-1}f(\tau)g'(\tau)\,d\tau, \tag{2.13}$$

$$\big({}_aI^{\alpha,\varrho,g}f\big)(t)=\frac{1}{\varrho^\alpha\Gamma(\alpha)}\int_a^t e^{\frac{\varrho-1}{\varrho}(g(t)-g(\tau))}\big(g(t)-g(\tau)\big)^{\alpha-1}f(\tau)g'(\tau)\,d\tau, \tag{2.14}$$

and

$$\big(I_b^{\alpha,\varrho,g}f\big)(t)=\frac{1}{\varrho^\alpha\Gamma(\alpha)}\int_t^b e^{\frac{\varrho-1}{\varrho}(g(\tau)-g(t))}\big(g(\tau)-g(t)\big)^{\alpha-1}f(\tau)g'(\tau)\,d\tau. \tag{2.15}$$

Definition 2.6 ([25]) For $\varrho>0$, $\alpha\in\mathbb{C}$, $\operatorname{Re}(\alpha)\geq 0$, and $g\in C[a,b]$, where $g'(t)>0$, we define the left fractional derivative of f with respect to g as

$$\begin{aligned}\big({}_aD^{\alpha,\varrho,g}f\big)(t)&=D^{n,\varrho,g}{}_aI^{n-\alpha,\varrho,g}f(t)\\&=\frac{D_t^{n,\varrho,g}}{\varrho^{n-\alpha}\Gamma(n-\alpha)}\int_a^t e^{\frac{\varrho-1}{\varrho}(g(t)-g(\tau))}\big(g(t)-g(\tau)\big)^{n-\alpha-1}\\&\quad\times f(\tau)g'(\tau)\,d\tau\end{aligned} \tag{2.16}$$

and the right fractional derivative of f with respect to g as

$$\begin{aligned}\big(D_b^{\alpha,\varrho,g}f\big)(t)&={}_\ominus D^{n,\varrho,g}I_b^{n-\alpha,\varrho,g}f(t)\\&=\frac{{}_\ominus D_t^{n,\varrho,g}}{\varrho^{n-\alpha}\Gamma(n-\alpha)}\int_t^b e^{\frac{\varrho-1}{\varrho}(g(\tau)-g(t))}\big(g(\tau)-g(t)\big)^{n-\alpha-1}\end{aligned}$$

$$\times f(\tau)g'(\tau)\,d\tau,$$

where $n = [\mathrm{Re}(\alpha)] + 1$,

$$D^{n,\varrho,g} = \underbrace{D^{\varrho,g}D^{\varrho,g}\ldots D^{\varrho,g}}_{n \text{ times}}$$

and

$$\left({}_{\ominus}D^{\varrho,g}f\right)(t) := (1-\varrho)f(t) - \varrho\frac{f'(t)}{g'(t)}, \qquad {}_{\ominus}D^{n,\varrho,g} = \underbrace{{}_{\ominus}D^{\varrho,g}{}_{\ominus}D^{\varrho,g}\ldots{}_{\ominus}D^{\varrho,g}}_{n \text{ times}}.$$

Proposition 2.1 ([25]) *Let* $\alpha,\beta \in \mathbb{C}$ *be such that* $\mathrm{Re}(\alpha) \geq 0$ *and* $\mathrm{Re}(\beta) > 0$. *Then, for any* $\varrho > 0$, *we have*

1. $\left({}_{a}I^{\alpha,\varrho,g}e^{\frac{\varrho-1}{\varrho}g(x)}\left(g(x)-g(a)\right)^{\beta-1}\right)(t) = \dfrac{\Gamma(\beta)}{\Gamma(\beta+\alpha)\varrho^{\alpha}}e^{\frac{\varrho-1}{\varrho}g(t)}\left(g(t)-g(a)\right)^{\alpha+\beta-1}$,

   $\mathrm{Re}(\alpha) > 0$.
2. $\left(I_{b}^{\alpha,\varrho,g}e^{-\frac{\varrho-1}{\varrho}g(x)}\left(g(b)-g(x)\right)^{\beta-1}\right)(t) = \dfrac{\Gamma(\beta)}{\Gamma(\beta+\alpha)\varrho^{\alpha}}e^{-\frac{\varrho-1}{\varrho}g(t)}\left(g(b)-g(t)\right)^{\alpha+\beta-1}$,

   $\mathrm{Re}(\alpha) > 0$.
3. $\left({}_{a}D^{\alpha,\varrho}e^{\frac{\varrho-1}{\varrho}g(x)}\left(g(x)-g(a)\right)^{\beta-1}\right)(t) = \dfrac{\varrho^{\alpha}\Gamma(\beta)}{\Gamma(\beta-\alpha)}e^{\frac{\varrho-1}{\varrho}g(t)}\left(g(t)-g(a)\right)^{\beta-1-\alpha}$,

   $\mathrm{Re}(\alpha) \geq 0$.
4. $\left(D_{b}^{\alpha,\varrho,g}e^{-\frac{\varrho-1}{\varrho}g(x)}\left(g(b)-g(x)\right)^{\beta-1}\right)(t) = \dfrac{\varrho^{\alpha}\Gamma(\beta)}{\Gamma(\beta-\alpha)}e^{-\frac{\varrho-1}{\varrho}g(t)}\left(g(b)-g(t)\right)^{\beta-1-\alpha}$,

   $\mathrm{Re}(\alpha) \geq 0$.

Theorem 2.1 ([25]) *If* $\varrho \in (0,1]$, $\mathrm{Re}(\alpha) > 0$, *and* $\mathrm{Re}(\beta) > 0$. *Then, for f is continuous and defined for* $t \geq a$, *we have*

$${}_{a}I^{\alpha,\varrho,g}\left({}_{a}I^{\beta,\varrho,g}f\right)(t) = {}_{a}I^{\beta,\varrho,g}\left({}_{a}I^{\alpha,\varrho,g}f\right)(t) = \left({}_{a}I^{\alpha+\beta,\varrho,g}f\right)(t), \tag{2.17}$$

$$I_{b}^{\alpha,\varrho,g}\left(I_{b}^{\beta,\varrho,g}f\right)(t) = I_{b}^{\beta,\varrho,g}\left(I_{b}^{\alpha,\varrho,g}f\right)(t) = \left(I_{b}^{\alpha+\beta,\varrho,g}f\right)(t). \tag{2.18}$$

Theorem 2.2 ([25]) *Let* $0 \leq m < [\mathrm{Re}(\alpha)] + 1$ *and f be integrable in each interval* $[a,t]$, $t > a$. *Then*

$$D^{m,\varrho,g}\left({}_{a}I^{\alpha,\varrho,g}f\right)(t) = \left({}_{a}I^{\alpha-m,\varrho,g}f\right)(t) \tag{2.19}$$

and

$${}_{\ominus}D^{m,\varrho,g}\left(I_{b}^{\alpha,\varrho,g}f\right)(t) = \left(I_{b}^{\alpha-m,\varrho,g}f\right)(t). \tag{2.20}$$

Corollary 2.1 ([25]) *Let* $0 < \mathrm{Re}(\beta) < \mathrm{Re}(\alpha)$ *and* $m-1 < \mathrm{Re}(\beta) \leq m$. *Then we have*

$${}_{a}D^{\beta,\varrho,g}{}_{a}I^{\alpha,\varrho,g}f(t) = {}_{a}I^{\alpha-\beta,\varrho,g}f(t) \tag{2.21}$$

and

$$D_b^{\beta,\varrho,g} I_b^{\alpha,\varrho,g} f(t) = I_b^{\alpha-\beta,\varrho,g} f(t). \tag{2.22}$$

Corollary 2.2 ([25]) *Let f be integrable on $t \geq a$ and* $\mathrm{Re}[\alpha] > 0$, $\varrho \in (0,1]$, $n = [\mathrm{Re}(\alpha)] + 1$. *Then we have*

$${}_aD^{\alpha,\varrho,g}\,{}_aI^{\alpha,\varrho,g} f(t) = f(t) \tag{2.23}$$

and

$$D_b^{\alpha,\varrho,g} I_b^{\alpha,\varrho,g} f(t) = f(t). \tag{2.24}$$

The generalized Caputo proportional fractional derivatives are given as follows.

Definition 2.7 ([25]) For $\varrho \in (0,1]$ and $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) \geq 0$, we define the left derivative of Caputo type starting at a by

$$\begin{aligned}\left({}_a^C D^{\alpha,\varrho,g} f\right)(t) &= {}_aI^{n-\alpha,\varrho,g}\left(D^{n,\varrho,g} f\right)(t)\\ &= \frac{1}{\varrho^{n-\alpha}\Gamma(n-\alpha)} \int_a^t e^{\frac{\varrho-1}{\varrho}(g(t)-g(\tau))}\left(g(t)-g(\tau)\right)^{n-\alpha-1}\\ &\quad \times \left(D^{n,\varrho,g} f\right)(\tau) g'(\tau)\,d\tau. \end{aligned} \tag{2.25}$$

The right derivative of Caputo type ending at b is defined by

$$\begin{aligned}\left({}^C D_b^{\alpha,\varrho} f\right)(t) &= I_b^{n-\alpha,\varrho,g}\left({}_\ominus D^{n,\varrho,g} f\right)(t)\\ &= \frac{1}{\varrho^{n-\alpha}\Gamma(n-\alpha)} \int_t^b e^{\frac{\varrho-1}{\varrho}(g(\tau)-g(t))}\left(g(\tau)-g(t)\right)^{n-\alpha-1}\\ &\quad \times \left({}_\ominus D^{n,\varrho,g} f\right)(\tau) g'(\tau)\,d\tau, \end{aligned} \tag{2.26}$$

where $n = [\mathrm{Re}(\alpha)] + 1$.

Proposition 2.2 ([25]) *Let* $\alpha, \beta \in \mathbb{C}$ *be such that* $\mathrm{Re}(\alpha) > 0$ *and* $\mathrm{Re}(\beta) > 0$. *Then, for any* $\varrho > 0$ *and* $n = [\mathrm{Re}(\alpha)] + 1$, *we have*

1. $\left({}_a^C D^{\alpha,\varrho,g} e^{\frac{\varrho-1}{\varrho}g(x)}\left(g(x)-g(a)\right)^{\beta-1}\right)(t) = \frac{\varrho^\alpha \Gamma(\beta)}{\Gamma(\beta-\alpha)} e^{\frac{\varrho-1}{\varrho}g(t)}\left(g(t)-g(a)\right)^{\beta-1-\alpha}$,

   $\mathrm{Re}(\beta) > n$.
2. $\left({}^C D_b^{\alpha,\varrho,g} e^{-\frac{\varrho-1}{\varrho}g(x)}\left(g(b)-g(x)\right)^{\beta-1}\right)(t) = \frac{\varrho^\alpha \Gamma(\beta)}{\Gamma(\beta-\alpha)} e^{\frac{\varrho-1}{\varrho}(g(b)-g(x))}\left(g(b)-g(t)\right)^{\beta-1-\alpha}$,

   $\mathrm{Re}(\beta) > n$.

For $k = 0, 1, \dots, n-1$, *we have*

$$\left({}_a^C D^{\alpha,\varrho,g} e^{\frac{\varrho-1}{\varrho}g(x)}\left(g(x)-g(a)^k\right)(t) = 0 \quad \textit{and} \quad \left({}^C D_b^{\alpha,\varrho,g} e^{-\frac{\varrho-1}{\varrho}g(x)}\left(g(b)-g(x)\right)^k\right)(t) = 0.$$

In particular, $\left({}_a^C D^{\alpha,\varrho} e^{\frac{\varrho-1}{\varrho}g(x)}\right)(t) = 0$ *and* $\left({}^C D_b^{\alpha,\varrho} e^{-\frac{\varrho-1}{\varrho}g(x)}\right)(t) = 0$.

2.3 The g-Laplace transforms

The g-Laplace transform was proposed by Jarad et al. [21].

Definition 2.8 Let $f, g : [a,\infty) \to \mathbb{R}$ be real-valued functions such that $g(t)$ is continuous and $g'(t) > 0$ on $[a,\infty)$. The generalized Laplace transform of f is defined by

$$\mathcal{L}_g\{f(t)\}(s) = \int_a^\infty e^{-s(g(t)-g(a))} f(t)g'(t)\,dt \tag{2.27}$$

for all values of s, the integral is valid.

The generalized Laplace transforms of some elementary functions were given in the following lemma.

Lemma 2.1 ([21])

1. $\mathcal{L}_g\{1\}(s) = \frac{1}{s}, \quad s > 0.$
2. $\mathcal{L}_g\{(g(t)-g(a))^\beta\}(s) = \frac{\Gamma(\beta+1)}{s^{\beta+1}}, \quad \Re(\beta) > 0, s > 0.$
3. $\mathcal{L}_g\{e^{\lambda(g(t)-g(a))}\}(s) = \frac{1}{s-\lambda}, \quad s > \lambda.$
4. $\mathcal{L}_g\{e^{\lambda(g(t)-g(a))} f(t)\}(s) = \mathcal{L}_g\{f\}(s-\lambda).$

The generalized Laplace transform of the derivative of f with respect to g is given as follows.

Theorem 2.3 ([21]) *Let the function $f(t) \in C_g[a,T]$ and of $g(t)$-exponential order such that $f^{[1]}(t)$ is piecewise continuous over every finite interval $[a,T]$. Then the generalized Laplace transform of $f^{[1]}(t) = \frac{f'}{g'}(t)$ exists and*

$$\mathcal{L}_g\{f^{[1]}(t)\}(s) = s\mathcal{L}_g\{f(t)\}(s) - f(a). \tag{2.28}$$

The generalized convolution integral is defined as follows.

Definition 2.9 ([21]) Let f and h be two functions which are piecewise continuous at each interval $[0,T]$ and of g-exponential order. The generalized convolution of f and h is defined by

$$(f *_g h)(t) = \int_a^t f(\tau)h\bigl(g^{-1}(g(t)+g(a)-g(\tau))\bigr)g'(\tau)\,d\tau. \tag{2.29}$$

The generalized convolution of two functions is commutative.

Lemma 2.2 ([21]) *Let f and h be two functions which are piecewise continuous at each interval $[a,T]$ and of exponential order. Then*

$$f *_g h = h *_g f. \tag{2.30}$$

Theorem 2.4 ([21]) *Let f and h be two functions which are piecewise continuous at each interval* $[a, T]$ *and of g-exponential order. Then*

$$\mathcal{L}_g\{f *_g h\}(s) = \mathcal{L}_g\{f\}(s)\mathcal{L}_g\{h\}(s). \tag{2.31}$$

3 The main results

Lemma 3.1 *For* $\alpha > 1$, $\varrho \in (0,1]$, *we have*

$$\left({}_aI^{\alpha,\varrho,g}D^{1,\varrho,g}f\right)(t) = \left(D^{1,\varrho,g}{}_aI^{\alpha,\varrho,g}f\right)(t) - \frac{(g(t)-g(a))^{\alpha-1}e^{\frac{\varrho-1}{\varrho}(g(t)-g(a))}}{\Gamma(\alpha)\varrho^{\alpha-1}}f(a) \tag{3.1}$$

and

$$\left(I_b^{\alpha,\varrho,g}{}_\ominus D^{1,\varrho,g}f\right)(t) = \left({}_\ominus D^{1,\varrho,g}I_b^{\alpha,\varrho,g}f\right)(t) - \frac{(g(b)-g(t))^{\alpha-1}e^{\frac{\varrho-1}{\varrho}(g(b)-g(t))}}{\Gamma(\alpha)\varrho^{\alpha-1}}f(b). \tag{3.2}$$

Proof Using the Leibniz rule, we can prove that

$$\frac{\alpha-1}{\varrho^{\alpha-1}\Gamma(\alpha)}\int_a^t e^{\frac{\varrho-1}{\varrho}(g(t)-g(\tau))}\left(g(t)-g(\tau)\right)^{\alpha-2}f(\tau)g'(\tau)\,d\tau = D^{1,\varrho,g}I_a^{\alpha,\varrho,g}f(t).$$

Now, by using Definition 2.9,

$$\begin{aligned}\left({}_aI^{\alpha,\varrho,g}D^{1,\varrho,g}f\right)(t) &= (1-\varrho)I^{\alpha,\varrho,g}f(t)\\ &\quad + \frac{\alpha-1}{\varrho^{\alpha-1}\Gamma(\alpha)}\int_a^t e^{\frac{\varrho-1}{\varrho}(g(t)-g(\tau))}\left(g(t)-g(\tau)\right)^{\alpha-1}f'(\tau)\,d\tau.\end{aligned}$$

Now, using the integration by parts formula, we obtain

$$\begin{aligned}&\left({}_aI^{\alpha,\varrho,g}D^{1,\varrho,g}f\right)(t)\\ &\quad= (1-\varrho)I^{\alpha,\varrho,g}f(t) - \frac{(g(t)-g(a))^{\alpha-1}e^{\frac{\varrho-1}{\varrho}(g(t)-g(a))}}{\Gamma(\alpha)\varrho^{\alpha-1}}f(a)\\ &\qquad + (\varrho-1)I^{\alpha,\varrho,g}f(t) + \frac{\alpha-1}{\varrho^{\alpha}\Gamma(\alpha)}\int_a^t e^{\frac{\varrho-1}{\varrho}(g(t)-g(\tau))}\left(g(t)-g(\tau)\right)^{\alpha-2}f(\tau)g'(\tau)\,d\tau\\ &\quad= D^{1,\varrho,g}I_a^{\alpha,\varrho,g}f(t) - \frac{(g(t)-g(a))^{\alpha-1}e^{\frac{\varrho-1}{\varrho}(g(t)-g(a))}}{\Gamma(\alpha)\varrho^{\alpha-1}}f(a).\end{aligned}$$

(3.2) can be proved similarly. □

We can generalize Lemma 3.1 as follows.

Corollary 3.1 *For* $\alpha > m$, $\varrho \in (0,1]$, *and m is a positive integer, we have*

$$\begin{aligned}\left({}_aI^{\alpha,\varrho,g}D^{m,\varrho,g}f\right)(t) &= \left(D^{m,\varrho,g}{}_aI^{\alpha,\varrho,g}f\right)(t)\\ &\quad - \sum_{k=0}^{m-1}\frac{(g(t)-g(a))^{\alpha-m+k}e^{\frac{\varrho-1}{\varrho}(g(t)-g(a))}}{\Gamma(\alpha+k-m+1)\varrho^{\alpha-m+k}}\left(D^{k,\varrho,g}f\right)(a),\end{aligned} \tag{3.3}$$

$$\begin{aligned}\left(I_b^{\alpha,\varrho,g}{}_{\ominus}D^{m,\varrho,g}f\right)(t) &= \left({}_{\ominus}D^{m,\varrho,g}I_b^{\alpha,\varrho,g}f\right)(t)\\ &\quad-\sum_{k=0}^{m-1}\frac{(-1)^k(g(t)-g(a))^{\alpha-m+k}e^{\frac{\varrho-1}{\varrho}(g(t)-g(a))}}{\Gamma(\alpha+k-m+1)\varrho^{\alpha-m+k}}\\ &\quad\times\left({}_{\ominus}D^{k,\varrho,g}f\right)(b).\end{aligned}\tag{3.4}$$

Proof The proof can be done by mathematical induction. □

In the following theorems we present the impact of the fractional integral on the fractional derivative of the same order.

Theorem 3.1 *Let* $\mathrm{Re}(\alpha)>0$, $n=-[-\mathrm{Re}(\alpha)]$, $f\in L_1(a,b)$, *and* $({}_aI^{\alpha,\varrho,g}f)(t),(I_b^{\alpha,\varrho,g}f)(t)\in AC^n[a,b]$. *Then*

$$\begin{aligned}({}_aI^{\alpha,\varrho,g}{}_aD^{\alpha,\varrho,g}f(t) &= f(t)\\ &\quad-e^{\frac{\varrho-1}{\varrho}(g(t)-g(a))}\sum_{j=1}^{n}\left({}_aI^{j-\alpha,\varrho,g}f\right)\left(a^+\right)\frac{(g(t)-g(a))^{\alpha-j}}{\varrho^{\alpha-j}\Gamma(\alpha+1-j)}\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}(I_b^{\alpha,\varrho,g}{}_{\ominus}D_b^{\alpha,\varrho,g}f(t) &= f(t)\\ &\quad-e^{\frac{\varrho-1}{\varrho}(g(b)-g(t))}\sum_{j=1}^{n}(-1)^j\left(I_b^{j-\alpha,\varrho,g}f\right)\left(b^-\right)\frac{(g(b)-g(t))^{\alpha-j}}{\varrho^{\alpha-j}\Gamma(\alpha+1-j)}.\end{aligned}\tag{3.6}$$

Proof By applying Corollary 3.1 and Theorem 2.1, we can observe that

$$\begin{aligned}({}_aI^{\alpha,\varrho}{}_aD^{\alpha,\varrho}f(t) &= {}_aI^{\alpha,\varrho}D^{n,\varrho}{}_aI^{n-\alpha,\varrho}f(t)\\ &= D^{n,\varrho}{}_aI^{\alpha,\varrho}{}_aI^{n-\alpha,\varrho}f(t)\\ &\quad-\sum_{k=0}^{n-1}\frac{(t-a)^{\alpha-n+k}e^{\frac{\varrho-1}{\varrho}(t-a)}}{\Gamma(\alpha+k-n+1)\varrho^{\alpha-n+k}}\left(D^{k,\varrho}{}_aI^{n-\alpha,\varrho}f\right)(a)\\ &= f(t)-\sum_{k=0}^{n-1}\frac{(t-a)^{\alpha-n+k}e^{\frac{\varrho-1}{\varrho}(t-a)}}{\Gamma(\alpha+k-n+1)\varrho^{\alpha-n+k}}\left({}_aI^{n-\alpha-k,\varrho}f\right)\left(a^+\right)\\ &= f(t)-e^{\frac{\varrho-1}{\varrho}(t-a)}\sum_{j=1}^{n}\left({}_aI^{j-\alpha,\varrho}f\right)\left(a^+\right)\frac{(t-a)^{\alpha-j}}{\varrho^{\alpha-j}\Gamma(\alpha+1-j)},\end{aligned}$$

where the change of variable $j=n-k$ has been used. Equation (3.6) can be analogously proved. □

Theorem 3.2 *For* $\varrho>0$ *and* $n=[\mathrm{Re}(\alpha)]+1$, *we have*

$${}_aI^{\alpha,\varrho,g}\left({}_a^CD^{\alpha,\varrho,g}f\right)(t)=f(t)-\sum_{k=0}^{n-1}\frac{(D^{k,\varrho,g}f)(a)}{\varrho^kk!}\left(g(t)-g(a)\right)^ke^{\frac{\varrho-1}{\varrho}(g(t)-g(a))}\tag{3.7}$$

and

$$I_b^{\alpha,\varrho,g}\left({}^C D_b^{\alpha,\varrho,g} f\right)(t) = f(t) - \sum_{k=0}^{n-1} \frac{(-1)^k ({}_\ominus D^{k,\varrho,g} f)(b)}{\varrho^k k!} \big(g(b) - g(t)\big)^k e^{\frac{\varrho-1}{\varrho}(g(b)-g(t))}. \tag{3.8}$$

Proof By the help of Theorem 3.1 with $\alpha = n$, we have

$$\begin{aligned} {}_aI^{\alpha,\varrho,g}\left({}_a^C D^{\alpha,\varrho,g} f\right)(t) &= {}_aI^{\alpha,\varrho,g}\left({}_aI^{n-\alpha,\varrho,g} D^{n,\varrho,g} f\right)(t) = \left({}_aI^{n,\varrho,g} D^{n,\varrho,g} f\right)(t) \\ &= f(t) \\ &\quad - e^{\frac{\varrho-1}{\varrho}(g(t)-g(a))} \sum_{j=1}^{n} \frac{({}_aI^{j-n,\varrho,g} f)(a^+)(g(t)-g(a))^{n-j}}{\varrho^{n-j}\Gamma(n-j+1)} \\ &= f(t) - \sum_{k=0}^{n-1} \frac{(D^{k,\varrho,g} f)(a)}{\varrho^k k!} \big(g(t)-g(a)\big)^k e^{\frac{\varrho-1}{\varrho}(g(t)-g(a))}. \end{aligned}$$

Equation (3.8) can be proved similarly. □

Before we find the g-Laplace transform, we need to find the g-Laplace transform of the n^th order derivative of a function with respect to another function.

Theorem 3.3 *Let the function $f \in C_g[a,T]$ and of g-exponential order such that $D^{\varrho,g} f$ is piecewise continuous over every finite interval $[a,T]$. Then the generalized Laplace transform of $D^{\varrho,g} f$ exists and*

$$\mathcal{L}_g\left\{D^{\varrho,g} f(t)\right\}(s) = (\varrho s + 1 - \varrho)\mathcal{L}_g\left\{f(t)\right\}(s) - \varrho f(a). \tag{3.9}$$

Proof Using (2.10) and Theorem 2.3, we have

$$\begin{aligned} \mathcal{L}_g\left\{D^{\varrho,g} f(t)\right\}(s) &= (1-\varrho)\mathcal{L}_g\left\{f(t)\right\} + \varrho\mathcal{L}_g\left\{f^{[1]}(t)\right\}(s) \\ &= (1-\varrho)\mathcal{L}_g\left\{f(t)\right\} + \varrho\left(s\mathcal{L}_g\left\{f(t)\right\}(s) - f(a)\right) \\ &= (\varrho s + 1 - \varrho)\mathcal{L}_g\left\{f(t)\right\}(s) - \varrho f(a). \end{aligned}$$

□

Using induction on n, we can prove the following corollary.

Corollary 3.2 *Let $f \in C^{n-1}[a,\infty)$ be such that $f^{[i]}$, $i = 1,2,\dots,n-1$, are of exponential order e^{ct} on each subinterval $[a,T]$. Then*

$$\begin{aligned} \mathcal{L}_g\left\{\left(D^{n,\varrho,g} f\right)(t)\right\}(s) &= (\varrho s + 1 - \varrho)^n \mathcal{L}_g\left\{f(t)\right\}(s) \\ &\quad - \varrho \sum_{k=0}^{n-1} (\varrho s + 1 - \varrho)^{n-1-k}\left(D^{k,\varrho,g} f\right)(a). \end{aligned} \tag{3.10}$$

Theorem 3.4 *Let $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$ and $\varrho \in (0,]$, $n = [\mathrm{Re}(\alpha)] + 1$. Assume that f is of g-exponential order. Then*

$$\mathcal{L}_g\left\{\left({}_aI^{\alpha,\varrho,g} f\right)(t)\right\}(s) = \frac{1}{(\varrho s + 1 - \varrho)^\alpha}\mathcal{L}_g\left\{f(t)\right\}(s), \quad s > c. \tag{3.11}$$

Proof From the convolution formula we have

$$\begin{aligned}\mathcal{L}_g\big\{\big({}_aI^{\alpha,\varrho,g}f\big)(t)\big\}(s) &= \frac{1}{\varrho^\alpha\Gamma(\alpha)}\mathcal{L}_g\big\{e^{\frac{\varrho-1}{\varrho}(g(t)-g(a))}\big(g(t)-g(a)\big)^{\alpha-1} *_g f(t)\big\}(s)\\ &= \frac{1}{\varrho^\alpha\Gamma(\alpha)}\frac{\Gamma(\alpha)}{(s-\frac{\varrho-1}{\varrho})^\alpha}\mathcal{L}_a\big\{f(t)\big\}(s)\\ &= \frac{1}{(\varrho s+1-\varrho)^\alpha}\mathcal{L}_a\big\{f(t)\big\}(s).\end{aligned}$$

□

Theorem 3.5 *For any* $\alpha\in\mathbb{C}$ *with* $\operatorname{Re}(\alpha)>0$ *and* $\varrho\in(0,1]$, $n=[\operatorname{Re}(\alpha)]+1$, *we have*

$$\begin{aligned}\mathcal{L}_g\big\{\big({}_aD^{\alpha,\varrho,g}f\big)(t)\big\}(s) &= (\varrho s+1-\varrho)^\alpha F_g(s)\\ &\quad -\varrho\sum_{k=0}^{n-1}(\varrho s+1-\varrho)^{n-k-1}\big(I^{n-\alpha-k,\varrho,g}f\big)\big(a^+\big),\end{aligned}\tag{3.12}$$

where $F_g(s)=\mathcal{L}_g\{f(t)\}(s)$. *In particular, if* f *is continuous at* a, *then*

$$\mathcal{L}_g\big\{\big({}_aD^{\alpha,\varrho,g}f\big)(t)\big\}(s) = (\varrho s+1-\varrho)^\alpha F_g(s).$$

Proof By applying Corollary 3.2 and Theorem 3.4, we have

$$\begin{aligned}\mathcal{L}_a\big\{\big({}_aD^{\alpha,\varrho}f\big)(t)\big\}(s) &= \mathcal{L}_g\big\{{}_aD^{n,\varrho,g}{}_aI^{n-\alpha,\varrho,g}f)(t)\big\}(s)\\ &= (\varrho s-1+\varrho)^n\mathcal{L}_g\big\{{}_aI^{n-\alpha,\varrho,g}f)(t)\big\}(s)\\ &\quad -\varrho\sum_{k=0}^{n-1}(\varrho s-1+\varrho)^{n-1-k}\big(D^{k,\varrho,g}{}_aI^{n-\alpha,\varrho,g}f\big)\big(a^+\big)\\ &= (\varrho s+1-\varrho)^n(\varrho s+1-\varrho)^{\alpha-n}F_g(s)\\ &\quad -\varrho\sum_{k=0}^{n-1}(\varrho s-1+\varrho)^{n-1-k}\big({}_aI^{n-\alpha-k,\varrho,g}f\big)\big(a^+\big).\end{aligned}$$

The last part follows by noting that $({}_aI^{n-\alpha-k,\varrho}f)(a^+)$ vanishes for a continuous function f on $[a,b]$. □

Theorem 3.6 *Let* $\alpha\in\mathbb{C}$ *with* $\operatorname{Re}(\alpha)>0$ *and* $\varrho>0$, $n=[\operatorname{Re}(\alpha)]+1$. *If* $F_g(s)=\mathcal{L}\{f(t)\}(s)$, *then*

$$\mathcal{L}_a\big\{\big({}^C_aD^{\alpha,\varrho}f\big)(t)\big\}(s) = (\varrho s+1-\varrho)^\alpha F_a(s)-\varrho\sum_{k=0}^{n-1}(\varrho s+1-\varrho)^{\alpha-1-k}\big(D^{k,\varrho}f\big)(a).\tag{3.13}$$

Proof

$$\begin{aligned}\mathcal{L}_g\big\{\big({}^C_aD^{\alpha,\varrho,g}f\big)(t)\big\}(s) &= \mathcal{L}_g\big\{\big({}_aI^{n-\alpha,\varrho,g}D^{n,\varrho,g}f\big)(t)\big\}(s)\\ &= (\varrho s+1-\varrho)^{\alpha-n}\mathcal{L}_g\big\{\big(D^{\alpha,\varrho,g}f\big)(t)\big\}(s)\\ &= (\varrho s+1-\varrho)^{\alpha-n}\\ &\quad\times\Big[(\varrho s+1-\varrho)^n F_g(s)\end{aligned}$$

$$- \varrho \sum_{k=0}^{n-1} (\varrho s + 1 - \varrho)^{n-1-k} \bigl(D^{k,\varrho,g} f\bigr)(a) \Biggr]$$
$$= (\varrho s + 1 - \varrho)^{\alpha} F_g(s) - \varrho \sum_{k=0}^{n-1} (\varrho s + 1 - \varrho)^{\alpha-1-k} \bigl(D^{k,\varrho,g} f\bigr)(a). \qquad \square$$

Using the Laplace transform of the generalized proportional derivatives in Riemann–Liouville settings in Theorem 3.5 and the Laplace transform in the Caputo in Theorem 3.6, we can state the following relation that links the Caputo and Riemann–Liouville general proportional fractional derivatives.

Corollary 3.3 *For any* $\alpha \in \mathbb{C}$ *with* $\mathrm{Re}(\alpha) > 0$ *and* $\varrho > 0$, $n = [\mathrm{Re}(\alpha)] + 1$, *we have*

$$\bigl({}_a^C D^{\alpha,\varrho} f\bigr)(t) = \bigl({}_a D^{\alpha,\varrho,g} f\bigr)(t) - \sum_{k=0}^{n-1} \frac{\varrho^{\alpha-k}}{\Gamma(k+1-\alpha)} \bigl(g(t) - g(a)\bigr)^{k-\alpha} e^{\frac{\varrho-1}{\varrho}(g(t)-g(a))} \bigl(D^{k,\varrho,g} f\bigr)(a). \tag{3.14}$$

Example 3.1 Consider the linear proportional fractional initial value problem

$${}_a D^{\alpha,\varrho,g} y(t) - \varrho^{\alpha} \lambda y(t) = f(t), \qquad \bigl({}_a I^{1-\alpha,\varrho,g} y\bigr)\bigl(a^+\bigr) = y_a, \quad 0 < \alpha \le 1. \tag{3.15}$$

Then $y(t)$ is a solution of (3.15) if and only if it satisfies the integral equation

$$\begin{aligned} y(t) &= y_a \varrho^{1-\alpha} e^{\frac{\varrho-1}{\varrho}(g(t)-g(a))} \bigl(g(t) - g(a)\bigr)^{\alpha-1} E_{\alpha,\alpha}\bigl(\lambda \bigl(g(t) - g(a)\bigr)^{\alpha}\bigr) \\ &\quad + \varrho^{-\alpha} \int_a^t E_{\alpha,\alpha}\bigl(\lambda \bigl(g(t) - g(s)\bigr)^{\alpha}\bigr) e^{\frac{\varrho-1}{\varrho}(g(t)-g(s))} \bigl(g(t) - g(s)\bigr)^{\alpha-1} \\ &\quad \times f(s) g'(s)\, ds, \end{aligned} \tag{3.16}$$

where $E_{\alpha,\beta}$ is the Mittag-Leffler function of two parameters [29]. Actually, applying $\mathcal{L}_g$ to (3.15) and making use of Theorem 3.5 with $n = 1$, we obtain

$$\bigl((\varrho s + 1 - \varrho)^{\alpha} - \lambda \varrho^{\alpha}\bigr) Y_g(s) = \varrho y_a + F_a(s).$$

Hence,

$$Y_a(s) = \frac{\varrho^{1-\alpha} y_a}{(s - \frac{\varrho-1}{\varrho})^{\alpha} - \lambda} + \frac{\varrho^{-\alpha} F_g(s)}{(s - \frac{\varrho-1}{\varrho})^{\alpha} - \lambda}.$$

Applying the inverse of $\mathcal{L}_a$ and using the fact that [21]

$$\mathcal{L}_g \bigl\{ \bigl(g(t) - g(a)\bigr)^{\alpha-1} E_{\alpha,\alpha}\bigl(\lambda \bigl(g(t) - g(a)\bigr)^{\alpha}\bigr) \bigr\} = \frac{1}{s^{\alpha} - \lambda},$$

together with Theorem 2.4, we reach (3.16). Conversely, if $y(t)$ has the representation (3.16), then by the help of Proposition 2.1 it satisfies (3.15).

Example 3.2 Consider the linear Caputo proportional fractional initial value problem

$$ {}_{a}^{C}D^{\alpha,\varrho,g}y(t) - \varrho^{\alpha}\lambda y(t) = f(t), \quad y(a) = y_a, 0 < \alpha \leq 1. \tag{3.17}$$

Then $y(t)$ is a solution of (3.17) if and only if it satisfies the integral equation

$$\begin{aligned} y(t) &= y_a e^{\frac{\varrho-1}{\varrho}(g(t)-g(a))}\big(g(t)-g(a)\big)^{\alpha-1}E_{\alpha}\big(\lambda\big(g(t)-g(a)\big)^{\alpha}\big) \\ &\quad + \varrho^{-\alpha}\int_a^t E_{\alpha,\alpha}\big(\lambda\big(g(t)-g(s)\big)^{\alpha}\big)e^{\frac{\varrho-1}{\varrho}(g(t)-g(s))}\big(g(t)-g(s)\big)^{\alpha-1} \\ &\quad \times f(s)g'(s)\,ds, \end{aligned} \tag{3.18}$$

where E_α is the Mittag-Leffler function of one parameter [29]. Applying $\mathcal{L}_a$ to (3.17) and making use of Theorem 3.6 with $n = 1$, we have

$$\big((\varrho s + 1 - \varrho)^{\alpha} - \lambda\varrho^{\alpha}\big)Y_g(s) = \varrho y_a(\varrho s + 1 - \varrho)^{\alpha-1} + F_g(s).$$

Hence,

$$Y_g(s) = \frac{(s - \frac{\varrho-1}{\varrho})^{\alpha-1}y_a}{(s - \frac{\varrho-1}{\varrho})^{\alpha} - \lambda} + \frac{\varrho^{-\alpha}F_g(s)}{(\frac{\varrho-1}{\varrho} - s)^{\alpha} - \lambda}.$$

Applying the inverse of $\mathcal{L}_a$ and using the facts that

$$\mathcal{L}_g\big\{(g\big((t) - g(a)\big)^{\alpha-1}E_{\alpha,\alpha}\big(\lambda\big(g(t) - g(a)\big)^{\alpha}\big)\big\} = \frac{1}{s^{\alpha} - \lambda}$$

and

$$\mathcal{L}_g\big\{E_{\alpha}\big(\lambda\big(g(t) - g(a)\big)^{\alpha}\big)\big\} = \frac{s^{\alpha-1}}{s^{\alpha} - \lambda},$$

together with the convolution formula, we reach (3.18). Conversely, if $y(t)$ has the representation (3.18), then by the help of Proposition 2.2 one can easily show that it satisfies (3.17).

4 Some integration by parts formulas

In this section, we present some integration by parts formulas.

Theorem 4.1 *Let* $f, h, g \in C^n[a, b]$, $g'(t) > 0$, $\alpha \in \mathbb{R}$, $n = [\alpha] + 1$, *and* $\varrho \in (0, 1]$. *Then*

$$\begin{aligned} \int_a^b f(t)\big({}_{a}^{C}D^{\alpha,\varrho,g}h\big)(t)g'(t)\,dt &= \int_a^b h(t)\big({}^{C}D_b^{\alpha,\varrho,g}f\big)(t)g'(t)\,dt \\ &\quad + \varrho\left[\sum_{k=0}^{n-1}\big(D^{k,\varrho,g}h\big)(t)\big(I_b^{k-\alpha,\varrho,g}f\big)(t)\right]\Bigg|_a^b. \end{aligned} \tag{4.1}$$

Proof

$$\int_a^b f(t)\big({}_{a}^{C}D^{\alpha,\varrho,g}h\big)(t)g'(t)\,dt$$

$$= \int_a^b f(t) \frac{1}{\varrho^{n-\alpha}\Gamma(n-\alpha)} \int_a^t e^{\frac{\varrho-1}{\varrho}(g(t)-g(s))} \big(g(t)-g(s)\big)^{n-\alpha-1}$$
$$\times \big(D^{n,\varrho,g}h\big)(s)g'(s)\,ds g'(t)\,dt$$
$$= \int_a^b \big(D^{n,\varrho,g}h\big)(s) \frac{1}{\varrho^{n-\alpha}\Gamma(n-\alpha)} \int_s^b e^{\frac{\varrho-1}{\varrho}(g(t)-g(s))} \big(g(t)-g(s)\big)^{n-\alpha-1}$$
$$\times f(t)g'(t)\,dt g'(s)\,ds$$
$$= \int_a^b \big(D^{n,\varrho,g}h\big)(s)\big(I_b^{\alpha,\varrho,g}f\big)(s)g'(s)\,ds$$
$$= \int_a^b (1-\varrho)\big(D^{n-1,\varrho,g}h\big)(s)\big(I_b^{\alpha,\varrho,g}f\big)(s)g'(s)$$
$$+ \varrho\left(\frac{d}{ds}\big(D^{n-1,\varrho,g}h\big)\right)(s)(I_b^{\alpha,\varrho,g}f(s)\,ds$$
$$= \int_a^b (1-\varrho)\big(D^{n-1,\varrho,g}h\big)(s)g'(s) - \varrho\big(D^{n-1,\varrho,g}h\big)(s)\frac{d}{ds}\big(\big(I_b^{\alpha,\varrho,g}f\big)(s)\big)$$
$$+ \varrho\big[\big(D^{n-1,\varrho,g}h\big)(s)\big(I_b^{\alpha,\varrho,g}f\big)(s)\big]\Big|_a^b.$$

The result is obtained by repeating the above procedure $n-1$ times. □

Using Theorem 4.1 and the relation between the Caputo and Riemann–Liouville type derivatives, we can conclude the following.

Corollary 4.1

$$\int_a^b f(t)\big({}_aD^{\alpha,\varrho,g}h\big)(t)g'(t)\,dt = \int_a^b h(t)\big(D_b^{\alpha,\varrho,g}f\big)(t)g'(t)\,dt$$
$$+ \varrho\sum_{k=0}^{n-1}\big(D^{k,\varrho,g}h\big)(b)\big(I_b^{k+1-\alpha,\varrho,g}f\big)(b). \tag{4.2}$$

Analogously, the following theorem and corollary can be proved.

Theorem 4.2 *Let $f,h,g \in C^n[a,b]$, $g'(t) > 0$, $\alpha \in \mathbb{R}$, $n = [\alpha]+1$, and $\varrho \in (0,1]$. Then*

$$\int_a^b f(t)\big({}^CD_b^{\alpha,\varrho,g}h\big)(t)g'(t)\,dt = \int_a^b h(t)\big({}_aD^{\alpha,\varrho,g}f\big)(t)g'(t)\,dt$$
$$- \varrho\left[\sum_{k=0}^{n-1}\big(D^{k,\varrho,g}h\big)(t)\big({}_aI^{k-\alpha,\varrho,g}f\big)(t)\right]\Bigg|_a^b.$$

Corollary 4.2

$$\int_a^b f(t)\big(D_b^{\alpha,\varrho,g}h\big)(t)g'(t)\,dt = \int_a^b h(t)\big({}_aD^{\alpha,\varrho,g}f\big)(t)g'(t)\,dt$$
$$+ \varrho\sum_{k=0}^{n-1}\big({}_{\ominus}D^{k,\varrho,g}h\big)(a)\big({}_aI^{k-\alpha+1,\varrho,g}f\big)(a). \tag{4.3}$$

5 Conclusions

We presented the activities of the general fractional proportional integrals on the general fractional proportional derivatives. These actions are indispensable for the study of the qualitative aspects of differential and integral equations in the frame of the considered operators. In addition, we heralded the suitable integral transforms of these operators that we believe will help in discussing the stability of systems involving such fractional operators.

The proportional derivatives D^ϱ and the proportional derivative are defined only when $0 \leq \varrho \leq 1$, and higher-order derivatives are defined when we have a sequential of these derivatives. This may cause some obstacles in dealing with equations containing such derivatives. For this reason, we present the following definition of the proportional derivative of any order $n \leq \varrho \leq n+1$, $n = 0, 1, \ldots$:

$$D^\varrho f(t) = (n+1-\varrho) f^{(n)}(t) + (\varrho - n) f^{(n+1)}(t),$$

which is equivalent to Definition 5.5 in [30]. It can be clearly observed that this derivative is an interpolation between the nth derivative of the function and its derivative of order $n+1$. We believe that it would be interesting to work on such a definition.

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