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Research Article

Multivalued φ-Contractions on Extended b-Metric Spaces

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In this paper, we have established some fixed-point results for the class of multivalued φ-contractions in the setting of extended b-metric space. An example is furnished to show the validity of our results. The results we have obtained generalize/extend many recent results by Asl, Bota, Samreen et al., and those contained therein.

1. Introduction and Preliminaries

One of the important and pioneering results is the celebrated Banach contraction in metric fixed-point theory. Generalizations in the existence of solutions of differential, integral, and integrodifferential equations are mostly based on creating outstanding generalizations in the metric fixed-point theory. These generalizations are obtained by enriching metric structure of underlying space and/or generalizing contraction condition. Bakhtin [1] and Czerwik [2] extended first time the idea of metric space by modifying the triangle inequality and called it a b-metric space. Kamran et al. [3], in 2017, further generalized the idea of b-metric and introduced an extended b-metric (Eb-metric) space. They weakened the triangle inequality of metric and established fixed-point results for a class of contractions. Following the idea of Eb-metric space, a number of authors have published several results in this direction (see, e.g., [4, 5]). To have some insight about miscellaneous generalizations of metric, we refer the readers to a recent article [6] and for some work on b-metric, see [7–17].

In 1976, Nadler [18] extended first time the idea of Banach contraction principle for multivalued mappings. He used the set of all closed and bounded subsets of a metric space $\mathfrak{P}$ and the Hausdorff metric on it. Some of the important generalization of Nadler's result can be seen in ([19–21]). Subashi and Gjini [22] further generalized the concept of extended b-metric space to multivalued mappings by using extended Hausdorff b-metric. Unlike Nadler, they used $H(\mathfrak{P})$, the set of all compact subsets of an Eb-metric space $\mathfrak{P}$.

In this paper, we have discussed the multivalued φ-contractions on Eb-metric spaces and proved some fixed-point results. The first section of the paper consists of some essential definitions and preliminaries. The second section is dedicated to some fixed-point results for multivalued mappings where extended b-comparison function φ has been used. In the last section, some well-known theorems are mentioned which are direct consequences of our main result.

The core reason behind adding this section is to recollect some essential concepts and results which are valuable throughout this paper.

Definition 1. ([23], Czerwik) For any nonempty set $\mathfrak{P}$, a b-metric on $\mathfrak{P}$ is a function $d_b : \mathfrak{P}\times\mathfrak{P} \longrightarrow R\cup\{0\}$ such that the following axioms hold:

B_1: $d_b(p,v)=0$ if and only if $p=v : \forall p, v\in\mathfrak{P}$.

B_2: $d_b(p,v)=d_b(v,p)$: $\forall p, v\in\mathfrak{P}$.

B_3: $\exists b\geq 1$ such that $d_b(p,u)\leq b[d_b(p,v)+d_b(v,u)]$: $\forall p, v, u\in\mathfrak{P}$.

The pair $(\mathfrak{P}, d_b)$ is then termed as b-metric space with coefficient b. Evidently, we can see that the collection of b-metric spaces is a superclass of the collection of metric spaces.

A comparison function is an increasing function $\varphi : R \cup \{0\} \longrightarrow R \cup \{0\}$ such that for all $l \in R \cup \{0\}$,$\lim_{r\to\infty}\varphi^r(l) = 0$ ([24]).

A nonnegative real-valued function φ on $R \cup \{0\}$ is called a c-comparison function if it is increasing, and for every $l > 0$ and $r = 1, 2, 3, \cdots$, the series $\sum \varphi^r(l)$ converges.

It is evident from the definition that a c-comparison function is a comparison function but the converse may not be true in general (see for example [25]).

Let us consider a b-metric space $(\mathfrak{P}, d_b)$ and an increasing nonnegative function φ on $R \cup \{0\}$. We call a map φ to be a b-comparison function if for all $l \in R \cup \{0\}$, the series $\sum_{r=0}^{\infty} b^r \varphi^r(l)$ converges ([25, 26]).

The function $\varphi(l) = jl$ is an example of b-comparison function if $0 < j < 1/b$ for a b-metric space $(\mathfrak{P}, d_b)$. Note that for $b = 1$, the defined b-comparison function becomes equivalent to the definition of a comparison function.

In the following, the authors enriched the notion of b-metric space by amending the triangle inequality

Definition 2. ([3], Kamran al.) Consider a map $s : \mathfrak{P} \times \mathfrak{P} \longrightarrow [1,\infty)$ where $\mathfrak{P} \neq \phi$. An extended b-metric (Eb-metric) on $\mathfrak{P}$ is a function $d_s : \mathfrak{P} \times \mathfrak{P} \longrightarrow [0,\infty)$ which satisfies

EB_1: $d_s(p, v) = 0$ if and only if $p = v : \forall p, v \in \mathfrak{P}$.

EB_2: $d_s(p, v) = d_s(v, p) \forall p, v \in \mathfrak{P}$.

EB_3: $d_s(p, u) \le s(p, u)[d_s(p, t) + d_s(t, v)]$: $\forall p, t, v \in \mathfrak{P}$.

The pair $(\mathfrak{P}, d_s)$ is then termed as an extended b-metric (Eb-metric) space.

If $s(p_1, p_2) = b$ for some $b \ge 1$, then Definition 2 reduces to the definition of b-metric space with coefficient b.

Definition 3. ([3], Kamran al.) Let us consider an Eb-metric space$(\mathfrak{P}, d_s)$. A sequence $\{\varpi_r\}$ in $\mathfrak{P}$ is said to be

(i) convergent which converges to ϖ in $\mathfrak{P}$ if and only if $d_s(\varpi_r, \varpi) \longrightarrow 0$ as $r \longrightarrow \infty$; we write $\lim_{r\to\infty}\varpi_r = \varpi$

(ii) a Cauchy sequence if $d_s(\varpi_r, \varpi_k) \longrightarrow 0$ as $r, k \longrightarrow \infty$

We say that an Eb-metric space $(\mathfrak{P}, d_s)$ is complete if every Cauchy sequence in $\mathfrak{P}$ converges in $\mathfrak{P}$. We note that the extended b-metric d_s is not a continuous functional in general and every convergent sequence converges to a single point.

Next, we define the concept of γ-orbital lower semicontinuity (lsc in short) in the case of Eb-metric space which we will use

Definition 4. [27]. Let $\gamma : D \subset \mathfrak{P} \longrightarrow \mathfrak{P}$, $\varpi_0 \in D$, and the orbit of $\varpi_0 \in D$, $\mathcal{O}(\varpi_0) = \{\varpi_0, \gamma(\varpi_0), \gamma^2\varpi_0, \cdots\} \subset D$. A real-valued function G on D is said to be a γ-orbitally lsc at $p \in D$ if $\varpi_r \longrightarrow p$ and $(\varpi_r) \subset \mathcal{O}(\varpi_0)$ implies $G(s) \le \lim_{r\to\infty} \inf G(\varpi_r)$. In case if $\gamma : D \subset \mathfrak{P} \longrightarrow P(\mathfrak{P})$ is multivalued, then the orbit of γ at ϖ_0 is given as $\mathcal{O}(\varpi_0) = \{\varpi_r : \varpi_r \in \gamma(\varpi_{r-1})\}$.

2. Main Results

For some technical reasons, Samreen et al., introduced another class of comparison functions for Eb-metric spaces given as follows

Definition 5. Let $(\mathfrak{P}, d_s)$ be an Eb-metric space. A nonnegative increasing real-valued function φ on $R \cup \{0\}$ is called an extended b-comparison function if there exists a mapping $\gamma : D \subset \mathfrak{P} \longrightarrow \mathfrak{P}$ such that for some $\varpi_0 \in D$, $\mathcal{O}(\varpi_0) \subset D$ and the infinite series $\sum_{r=0}^{\infty} \varphi^r(l) \prod_{i=1}^{r} s(\varpi_i, \varpi_k)$ converges for all $l \in R \cup \{0\}$ and for every $k \in N$. Here, $\varpi_r = \gamma^r \varpi_0$ for $r = 1, 2, \cdots$. We say that φ is an extended b-comparison function for γ at ϖ_0.

Remark 6. It can be easily seen that by taking $s(p_1, p_2) = b \ge 1$ (a constant), Definition 5 coincides with the definition of a b-comparison function for an arbitrary self-map γ on $\mathfrak{P}$. Every extended b-comparison function is also a comparison function for some b; i.e., if $s(p_1, p_2) \ge 1$ for every $p_1, p_2 \in \mathfrak{P}$, then by setting $b = \inf_{p_1,p_2\in\mathfrak{P}} s(p_1, p_2)$, we have

$$\sum_{r=0}^{\infty} b^r \varphi^r(l) \le \sum_{r=0}^{\infty} \varphi^r(l) \left(\prod_{i=1}^{r} s(\varpi_i, \varpi_k) \right). \tag{1}$$

Example 7. Let $(\mathfrak{P}, d_s)$ be an Eb-metric space, γ a self-map on $\mathfrak{P}$, and $\varpi_0 \in \mathfrak{P}$, $\lim_{r,k\to\infty} s(\varpi_r, \varpi_k)$ exists for $\varpi_r = \gamma^r \varpi_0$. Define $\varphi : [0,\infty) \longrightarrow [0,\infty)$ as

$$\varphi(l) = jl, \text{such that } \lim_{r,k\to\infty} s(\varpi_r, \varpi_k) < 1/j. \tag{2}$$

Then, by using ratio test, one can easily see that the series $\sum_{r=1}^{\infty} \varphi^r(l) \prod_{i=1}^{r} s(\varpi_i, \varpi_k)$ converges.where $d_s(u, Z) = \inf \{d_s(u, z): z \in Z\}$ is a distance from a point $u \in \mathfrak{P}$ to a set Z and $s(W, Z) = \sup \{s(w, z): w \in W, z \in Z\}$.

Definition 8. [22] Let $(\mathfrak{P}, d_s)$ be an Eb-metric space and $A, B \in H(\mathfrak{P})$. An extended Pompeiu-Hausdorff metric induced by d_s is a function $\mathcal{H}_s : H(\mathfrak{P}) \times H(\mathfrak{P}) \longrightarrow R \cup \{0\}$ defined as:

$$\mathcal{H}_s(W, Z) = \max \left\{ \sup_{w\in W} d_s(w, Z), \sup_{z\in Z} d_s(W, z) \right\}, \tag{3}$$

Theorem 9. *[22] Let $(\mathfrak{P}, d_s)$ be a complete Eb-metric space. Then, $H(\mathfrak{P})$ is a complete Eb-metric space with respect to the metric $\mathcal{H}_s$*

The following lemma is trivial.

Lemma 10. *Let $(\mathfrak{P}, d_s)$ be an Eb-metric space and $W, Z \in H(\mathfrak{P})$. Then, for any $\beta > 0$ and for every $z \in Z$, there exist $w \in W$ such that*

$$d_s(w, z) \le \mathcal{H}_s(W, Z) + \beta. \tag{4}$$

Now we are able to state our main result.

Theorem 11. *Let d_s be a continuous functional on $\mathfrak{P}$ such that $(\mathfrak{P}, d_s)$ is an Eb-metric space. Let D be a closed subset of $\mathfrak{P}$ and $\gamma : D \longrightarrow H(\mathfrak{P})$ be such that $\mathcal{O}(\varpi_0) \subset D$. Assume that for all $p \in \mathcal{O}(\varpi_0)$ and $t \in \gamma(p)$;*

$$\mathscr{H}_s(\gamma(p), \gamma(t)) \leq \varphi(d_s(p, t)). \tag{5}$$

Moreover, the inequality (1) strictly holds if and only if $p \neq t$ and φ is an extended b-comparison function for γ at $\varpi_0 \in D$. Then, there exists ϖ in $\mathfrak{P}$ such that $\varpi_r \longrightarrow \varpi$, where $\varpi_r \in T(\varpi_{r-1})$. Furthermore, $\varpi \in \mathfrak{P}$ is a point fixed under the map γ if and only if the map $G(l) = d_s(l, \gamma(l))$ is γ-orbitally lsc at ϖ.

Proof. Let $\varpi_0 \in D$ and $\varpi_1 \in \gamma(\varpi_0)$. Then, $\varpi_0 \neq \varpi_1$ because if it is equal, then ϖ_0 is a fixed point of γ. By using (1) for $\gamma(\varpi_0)$, $\gamma(\varpi_1) \in H(\mathfrak{P})$, we obtain

$$\mathscr{H}_s(\gamma(\varpi_0), \gamma(\varpi_1)) < \varphi(d_s(\varpi_0, \varpi_1)). \tag{6}$$

Choose $\varepsilon_1 > 0$ such that

$$\mathscr{H}_s(\gamma(\varpi_0), \gamma(\varpi_1)) + \varepsilon_1 \leq \varphi(d_s(\varpi_0, \varpi_1)). \tag{7}$$

Now, $\varpi_1 \in \gamma(\varpi_0)$ and $\varepsilon_1 > 0$; then, by Lemma 2, there exists $\varpi_2 \in \gamma(\varpi_1)$ such that

$$d_s(\varpi_1, \varpi_2) \leq \mathscr{H}_s(\gamma(\varpi_0), \gamma(\varpi_1)) + \varepsilon_1 \leq \varphi(d_s(\varpi_0, \varpi_1)). \tag{8}$$

Again, $\varpi_1 \neq \varpi_2$; otherwise, ϖ_1 is fixed under the map γ. By using (1), we obtain

$$\mathscr{H}_s(\gamma(\varpi_1), \gamma(\varpi_2)) < \varphi(d_s(\varpi_1, \varpi_2)). \tag{9}$$

Choose $\varepsilon_2 > 0$ such that

$$\begin{aligned} \mathscr{H}_s(\gamma(\varpi_1), \gamma(\varpi_2)) + \varepsilon_2 &\leq \varphi(d_s(\varpi_1, \varpi_2)) \\ &\leq \varphi(\varphi(d_s(\varpi_0, \varpi_1))) \\ &= \varphi^2(d_s(\varpi_0, \varpi_1)), \end{aligned} \tag{10}$$

while the second inequality is due to (4). By Lemma 10, for $\varpi_2 \in \gamma(\varpi_1)$ and $\varepsilon_2 > 0$, $\exists \varpi_3 \in \gamma(\varpi_2)$ such that

$$d_s(\varpi_2, \varpi_3) \leq \mathscr{H}_s(\gamma(\varpi_1), \gamma(\varpi_2)) + \varepsilon_2 \leq \varphi^2(d_s(\varpi_0, \varpi_1)). \tag{11}$$

Continuing in the same way, we get

$$d_s(\varpi_r, \varpi_{r+1}) \leq \varphi^r(d_s(\varpi_0, \varpi_1)). \tag{12}$$

If $k > r$, then by using (6) and the triangle inequality in Eb-metric, we obtain,

$$\begin{aligned} d_s(\varpi_r, \varpi_k) &\leq s(\varpi_r, \varpi_k) d_s(\varpi_r, \varpi_{r+1}) \\ &\quad + s(\varpi_r, \varpi_k) s(\varpi_{r+1}, \varpi_k) d_s(\varpi_{r+1}, \varpi_{r+2}) + \cdots \\ &\quad + s(\varpi_r, \varpi_k) s(\varpi_{r+1}, \varpi_k) \cdots s(\varpi_{k-1}, \varpi_k) d_s(\varpi_{k-1}, \varpi_k) \\ &\leq d_s(\varpi_r, \varpi_{r+1}) \prod_{i=1}^{r} s(\varpi_i, \varpi_k) \\ &\quad + d_s(\varpi_{r+1}, \varpi_{r+2}) \prod_{i=1}^{r+1} s(\varpi_i, \varpi_k) + \cdots \\ &\quad + d_s(\varpi_{k-1}, \varpi_k) \prod_{i=1}^{k-1} s(\varpi_i, \varpi_k) \\ &\leq \varphi^r(d_s(\varpi_0, \varpi_1)) \prod_{i=1}^{r} s(\varpi_i, \varpi_k) \\ &\quad + \varphi^{r+1}(d_s(\varpi_0, \varpi_1)) \prod_{i=1}^{r+1} s(\varpi_i, \varpi_k) + \cdots \\ &\quad + \varphi^{k-1}(d_s(\varpi_0, \varpi_1)) \prod_{i=1}^{k-1} s(\varpi_i, \varpi_k). \end{aligned} \tag{13}$$

But φ is an extended b-comparison function, so the series $\sum_{j=1}^{\infty} \varphi^j(d_s(\varpi_0, \varpi_1)) \prod_{i=1}^{j} s(\varpi_i, \varpi_k)$ converges. Let S be the sum of the series. By setting $S_n = \sum_{j=1}^{n} \varphi^j(d_s(\varpi_0, \varpi_1)) \prod_{i=1}^{j} s(\varpi_i, \varpi_k)$, from inequality (7), we obtain

$$d_s(\varpi_r, \varpi_k) \leq (S_{k-1} - S_{r-1}), \tag{14}$$

which further implies that $\lim_{r,k\to\infty} d_s(\varpi_r, \varpi_k) \longrightarrow 0$. Hence, $\{\varpi_r\}$ is a Cauchy sequence in D. But D is a closed subset of complete space $\mathfrak{P}$ so there exists $\varpi \in D$ such that $\varpi_r \longrightarrow \varpi$.

Using the definition of an extended Hausdorff b-metric $\mathscr{H}_s$ and (1), we have

$$\begin{aligned} d_s(\varpi_r, \varpi_{r+1}) &\leq \mathscr{H}_s(\gamma(\varpi_{r-1}), \gamma(\varpi_r)) \\ &\leq \varphi(d_s(\varpi_{r-1}, \varpi_r)) < d_s(\varpi_{r-1}, \varpi_r). \end{aligned} \tag{15}$$

But $\varpi_r \longrightarrow \varpi$ as $r \longrightarrow \infty$ which infers that $\lim_{r\to\infty} d_s(\varpi_r , \gamma(\varpi_r)) = 0$.

Assume that $G(\varpi) = d_s(\varpi, \gamma\varpi)$ is γ-orbitally lsc at ϖ. Then,

$$d_s(\varpi, \gamma(\varpi)) = G(\varpi) \leq \liminf_{r\to\infty} G(\varpi_r) = \liminf_{r\to\infty} d_s(\varpi_r, \gamma(\varpi_r)) = 0. \tag{16}$$

Hence, $\varpi \in \gamma(\varpi)$. But $\gamma(\varpi)$ is closed, so $\varpi \in \gamma(\varpi)$ and thus, ϖ is fixed under the map γ. Conversely, if ϖ is a point fixed under the map γ, then $G(\varpi) = 0 \leq \liminf_{r\to\infty} G(\varpi_r)$.

Remark 12. Note that Theorem 11 extends/generalizes the main result by Samreen et al. (, Theorem 15.9) to the case of multivalued mappings. Moreover, Theorem 11 includes main results such as by Czerwik (Theorem 9 [2]) and Samreen *et al.* (Theorem 3.10 (6) [28]) as special cases when the self-mapping is taken on a b-metric space. It also invokes some of the results by Proinov [29] and Hicks and Rhoades [30] in the case of metric space.

Example 13. Let $\mathfrak{P} = [0, 1/4]$ and $d_s : \mathfrak{P} \times \mathfrak{P} \longrightarrow R$ be defined as $d_s(l, m) = (l - m)^2$. Then, $(\mathfrak{P}, d_s)$ is an Eb-metric space with $s(p, q) = p + q + 2$. Define $\gamma : \mathfrak{P} \longrightarrow H(\mathfrak{P})$ by $\gamma(p) = [0, l^2]$; then, for each $\varpi_0 \in \mathfrak{P}$ and $\varpi_r \in \gamma(\varpi_{r-1})$, we have $\lim_{r,k\to\infty} s(\varpi_r, \varpi_k) = \lim_{r,k\to\infty}(\varpi_r + \varpi_k + 2) = 2 < 4$. For every $l \in \mathfrak{P}$ and $m \in T(l)$, we obtain

$$\begin{aligned}\mathscr{H}_s(\gamma l, \gamma m) &= \mathscr{H}_s\left([0, l^2], [0, m^2]\right) = \left(l^2 - m^2\right)^2 \\ &= (l + m)^2 (l - m)^2 \leq \frac{1}{4}(l - m)^2.\end{aligned} \tag{17}$$

If we define $\varphi : [0,\infty) \longrightarrow [0,\infty)$ by $\varphi(j) = j/4$, then γ fulfilled all the conditions present in our main Theorem 11. So $\exists \varpi$ in $\mathfrak{P}$ such that $\varpi \in \gamma$ as we can see here that $\varpi = 0 \in \gamma 0$.

3. Consequences

In this section, we will discuss an important consequence of Theorem 11 which involves $\beta_* - \varphi$ multivalued contractions on Eb-metric spaces. The obtained result generalizes some results by Asl et al. (Theorem 2.1 [31]) and Bota *et al.* (Theorem 9 [32]).where φ is an extended b-comparison function for γ at ϖ_0. Then, $\exists \varpi$ in $\mathfrak{P}$ such that $\gamma^r \varpi_0 \longrightarrow \varpi($ *as* $r \longrightarrow \infty)$. Additionally, ϖ is a point in $\mathfrak{P}$ fixed under the map γ if and only if the map $G(l) = d_s(l, \gamma l)$ is γ-orbitally lsc at ϖ .for every $p \in \mathcal{O}(\varpi_0)$. Then, $\gamma^r \varpi_0 \longrightarrow \varpi \in \mathfrak{P}$ as $r \longrightarrow \infty$. Additionally, ϖ is a point fixed under the map γ if and only if the map $G(l) = d_s(l, \gamma(l))$ is γ-orbitally lsc at ϖ.for all $m, u \in \mathfrak{P}$. Here, Φ_{Eb} denotes the class of all extended b-comparison functions.Theorem 4. Let d_s be a continuous functional on $\mathfrak{P}$ such that $(\mathfrak{P}, d_s)$ is a complete Eb-metric space. Suppose $\gamma : \mathfrak{P} \longrightarrow H(\mathfrak{P})$ is a $\beta_* - \varphi$ contractive multivalued operator of type (Eb) satisfies the following:

(i) γ is β_*-admissible

(ii) There exist $\varpi_0 \in \mathfrak{P}$ and $\varpi_1 \in \gamma(\varpi_0)$ such that $\beta(\varpi_0, \varpi_1) \geq 1$

Corollary 14. *(Theorem 3.9) Let d_s be a continuous functional on $\mathfrak{P}$ such that $(\mathfrak{P}, d_s)$ is a complete Eb-metric space. Let $\gamma : D \subset \mathfrak{P} \longrightarrow \mathfrak{P}$ be a map such that $\mathcal{O}(\varpi_0) \subseteq D$. Assume that for every $q \in \mathcal{O}(\varpi_0)$*

$$d_s\left(\gamma q, \gamma^2(q)\right) \leq \varphi(d_s(q, \gamma(q))), \tag{18}$$

Proof. The assertion simply follows by taking γ a self-map and then using Theorem 11.

Theorem 15. *Let d_s be a continuous functional on $\mathfrak{P}$ such that $(\mathfrak{P}, d_s)$ is a complete Eb-metric space. Let $\gamma : D \subset \mathfrak{P} \longrightarrow \mathfrak{P}$ be such that the orbit of ϖ_0, $\mathcal{O}(\varpi_0)$ is a subset of D. Suppose that* $\lim_{r,k\to\infty} s(\varpi_r, \varpi_k)$ *exists and j is a constant so that* $\lim_{r,k\to\infty} s(\varpi_r, \varpi_k) < 1/j$ *for all $\varpi_r, \varpi_k \in \mathcal{O}(\varpi_0)$. Assume that*

$$d_s\left(\gamma(p), \gamma^2 p\right) \leq j(d_s(p, \gamma(p))), \tag{19}$$

Proof. Define $\varphi : R \cup \{0\} \longrightarrow R \cup \{0\}$ by $\varphi(l) = jl$. By taking γ a self-map, Example 7 invokes that φ is an extended b-comparison function for γ at ϖ_0. Hence, the result follows from Theorem 11.

Remark 16. Note that Theorem 15 generalizes Theorem 9 [30] for multivalued mappings in the case of Eb-metric spaces.

Definition 17. Let $s : \mathfrak{P} \times \mathfrak{P} \longrightarrow [1, \infty)$ be a map such that $\mathfrak{P}$ is an Eb-metric space. A multivalued mapping $\gamma : \mathfrak{P} \longrightarrow P(\mathfrak{P})$ is said to be a β_*-admissible map if there exists a real-valued mapping β on $\mathfrak{P} \times \mathfrak{P}$ which is nonnegative and $\beta(p, q) \geq 1$ implies that $\beta_*(\gamma(p), \gamma(q)) \geq 1$ for all $p, q \in \mathfrak{P}$. Note that $\beta_* : P(\mathfrak{P}) \times P(\mathfrak{P}) \longrightarrow R \cup \{0\}$ is defined by

$$\beta_*(W, Z) = \inf \{\beta(p, q) : p \in W, q \in Z\}. \tag{20}$$

Definition 18. [32] Let $(\mathfrak{P}, d_s)$ be an Eb-metric space. A multivalued mapping $\gamma : \mathfrak{P} \longrightarrow P(\mathfrak{P})$ is said to be a $\beta_* - \varphi$-contractive multivalued operator of type (Eb) if there exist two functions $\beta : \mathfrak{P} \times \mathfrak{P} \longrightarrow R \cup \{0\}$ and $\varphi \in \Phi_{\mathrm{Eb}}$ such that [33]

$$\varphi(d_s(m, u)) \geq \beta_*(\gamma(m), \gamma(u)) \mathscr{H}_s(\gamma(m), \gamma(u)), \tag{21}$$

Then, $\exists \varpi \in \mathfrak{P}$ such that $\varpi_r \longrightarrow \varpi$ as $rt\infty$ where $\varpi_r \in \gamma(\varpi_{r-1})$. Furthermore, the point ϖ is fixed under the map γ if and only if the function $G(l) = d_s(l, \gamma l)$ is γ-orbitally lsc at ϖ.

Proof. Since γ is β_*-admissible and $\beta(\varpi_0, \varpi_1) \geq 1$ for $\varpi_1 \in \gamma(\varpi_0)$, so $\beta_*(\gamma(\varpi_0), \gamma(\varpi_1)) \geq 1$. By using infimum property, for $\varpi_1 \in \gamma(\varpi_0)$ and $\varpi_2 \in \gamma(\varpi_1)$,

$$\beta(\varpi_1, \varpi_2) \geq \beta_*(\gamma(\varpi_0), \gamma(\varpi_1)). \tag{22}$$

Thus, $\beta(\varpi_1, \varpi_2) \geq 1$ which further implies that $\beta_*(\gamma(\varpi_1), \gamma(\varpi_2)) \geq 1$. Again, by using the same property, for $\varpi_2 \in \gamma(\varpi_1)$ and $\varpi_3 \in T\varpi_2$, $\beta(\varpi_2, \varpi_3) \geq \beta_*(\gamma(\varpi_1), \gamma(\varpi_2)) \geq 1$. Continue the similar process to obtain

$$\beta_*(\gamma(\varpi_r), \gamma\varpi_{r+1}) \geq 1, \quad r = 1, 2, 3, \cdots. \tag{23}$$

The contractive condition (8) thus implies

$$\begin{aligned}\mathscr{H}_s(\gamma(\varpi_r), \gamma(\varpi_{r+1})) &\leq \beta_*(\gamma(\varpi_r), \gamma(\varpi_{r+1})) \mathscr{H}_s(\gamma(\varpi_r), \gamma(\varpi_{r+1})) \\ &\leq \varphi\left(d_s\left(\gamma^{r-1}(\varpi_0), \gamma^r(\varpi_0)\right)\right),\end{aligned} \tag{24}$$

which becomes equivalent to the following condition:

$$\mathscr{H}_s(\gamma p_1, \gamma p_2) \leq \varphi(d_s(p_1, p_2)), \tag{25}$$

for every $p_1 \in \mathcal{O}(\varpi_0)$ and $p_2 \in \gamma p_1$. Thus, all the conditions of Theorem 11 are satisfied and so the assertions follow.

Remark 19. 1. Note that Theorem 4.2 in becomes a special case of Theorem 4 for a self-map. Also, for a selfmap γ and $s(p_1, p_2) = 1$, Theorem 4 reduces to Theorem 2, 1 [33].

Data Availability

No data is used.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

M. S., W. U., and E. K. contributed in writing, reviewing, and editing the manuscript. All authors contributed equally and significantly in writing this article. All authors have read and agreed to the published version of the manuscript.

References

[1] I. A. Bakhtin, "The contraction mapping in almost metric spaces, Funct," *Funct. Ana. Gos. Ped. Inst. Unianowsk*, vol. 30, pp. 26–37, 1989.

[2] S. Czerwik, "Contraction mappings in *b*-metric spaces," *Acta Mathematica et Informatica Universitatis Ostraviensis*, vol. 1, no. 1, pp. 5–11, 1993.

[3] T. Kamran, M. Samreen, and Q. U. L. Ain, "A generalization of b-Metric space and some fixed point theorems," *Mathematics*, vol. 5, no. 2, p. 19, 2017.

[4] B. Alqahtani, E. Karapinar, and A. Ozturk, "On (β-φ)-K-contractions in the *Eb*-metric space," *Univerzitet u Nišu*, vol. 32, no. 15, pp. 5337–5345, 2018.

[5] B. Alqahtani, A. Fulga, and E. Karapinar, "Common fixed point results on an extended b-metric space," *Journal of Inequalities and Applications*, vol. 2018, no. 1, Article ID 158, 2018.

[6] T. Van An, N. Van Dung, Z. Kadelburg, and S. Radenović, "Various generalizations of metric spaces and fixed point theorems," *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas*, vol. 109, no. 1, pp. 175–198, 2015.

[7] Q. Mahmood, A. Shoaib, T. Rasham, and M. Arshad, "Fixed point results for the family of multivalued F-contractive mappings on closed ball in complete dislocated b-Metric spaces," *Mathematics*, vol. 7, no. 1, p. 56, 2019.

[8] T. Rasham, A. Shoaib, N. Hussain, B. A. S. Alamri, and M. Arshad, "Multivalued fixed point results in dislocated b-Metric spaces with application to the system of nonlinear integral equations," *Symmetry*, vol. 11, no. 1, p. 40, 2019.

[9] M. Samreen, T. Kamran, and N. Shahzad, "Some Fixed Point Theorems in -Metric Space Endowed with Graph," *Abstract and Applied Analysis*, vol. 2013, Article ID 967132, 9 pages, 2013.

[10] M. Samreen, K. Waheed, and Q. Kiran, "Multivalued φ-contractions and fixed points," *Univerzitet u Nišu*, vol. 32, no. 4, pp. 1209–1220, 2018.

[11] U. Aksoy, E. Karapinar, and I. M. Erhan, "Fixed points of generalized alpha-admissible contractions on b-metric spaces with an application to boundary value problems," *Journal of Nonlinear and Convex Analysis*, vol. 17, no. 6, pp. 1095–1108, 2016.

[12] H. H. Alsulami, S. Gülyaz, E. Karapınar, and İ. M. Erhan, "An Ulam stability result on quasi-b-metric-like spaces," *Open Mathematics*, vol. 14, no. 1, pp. 1087–1103, 2016.

[13] H. Aydi, M. F. Bota, E. Karapinar, and S. Moradi, "A common fixed point for weak-φ-contractions on b-metric spaces," *Fixed Point Theory*, vol. 13, no. 2, pp. 337–346, 2012.

[14] M.-F. Bota, E. Karapinar, and O. Mlesnite, "Ulam-hyers stability results for fixed point problems via \- -contractive mapping in $(\alpha_* - \psi)$-metric space," *Abstract and Applied Analysis*, vol. 2013, Article ID 825293, 6 pages, 2013.

[15] M.-F. Bota and E. Karapınar, "A note on "Some results on multi-valued weakly Jungck mappings in b-metric space"," *Open Mathematics*, vol. 11, no. 9, pp. 1711-1712, 2013.

[16] S. Gülyaz-Özyurt, "On some alpha-admissible contraction mappings on Branciari b-metric spaces," *Advances in the Theory of Nonlinear Analysis and its Application*, vol. 1, no. 1, pp. 1–13, 2017.

[17] B. Samet, C. Vetro, and P. Vetro, "Fixed point theorems for α-ψ-contractive type mappings," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 75, no. 4, pp. 2154–2165, 2012.

[18] M. A. Alghamdi, S. Gulyaz-Ozyurt, and E. Karapınar, "A note on extended Z-contraction," *Mathematics*, vol. 8, no. 2, p. 195, 2020.

[19] S. B. Nadler, "Multi-valued contraction mappings," *Pacific Journal of Mathematics*, vol. 30, no. 2, pp. 475–488, 1969.

[20] H. Afshari, H. Aydi, and E. Karapinar, "Existence of fixed points of set-valued mappings in b-METRIC spaces," *East Asian mathematical journal*, vol. 32, no. 3, pp. 319–332, 2016.

[21] M. Boriceanu, M. Bota, and A. Petrusel, "Multivalued fractals in b-metric spaces," *Central European Journal of Mathematics*, vol. 8, no. 2, pp. 367–377, 2010.

[22] M. F. Bota, C. Chifu, and E. Karapinar, "Fixed point theorems for generalized $(\alpha * -\psi)$-Ćirić-type contractive multivalued operators in b-metric spaces," *Journal of Nonlinear Sciences and Applications*, vol. 9, no. 3, pp. 1165–1177, 2016.

[23] L. Subashi and N. Gjini, "Some results on extended b-metric spaces and Pompeiu-Hausdorff metric," *Journal of Progressive Research in Mathematics*, vol. 12, no. 4, pp. 2021–2029, 2017.

[24] S. Czerwik, "Nonlinear set-valued contraction mappings in *b* -metric spaces," *Atti del Seminario Matematico e Fisico dell'Università di Modena*, vol. 46, pp. 263–276, 1998.

[25] J. Matkowski, *Integrable Solutions of Functional Equations*, Instytut Matematyczny Polskiej Akademi Nauk, Warszawa, 1975, http://eudml.org/doc/268343.

[26] I. A. Rus, *Generalized Contractions and Applications*, Cluj University Press, 2001.

[27] V. Berinde, "Generalized contractions in quasimetric spaces," *Seminar on Fixed Point Theory at Babes-Bolyai University*, vol. 3, no. 9, 1993.

[28] M. Samreen, T. Kamran, and M. Postolache, "Extended *b*-metric space, Extended *b*-comparison function and nonlinear contractions," *University of Bucharest Scientific Bulletin Series A Applied Mathematics and Physics*, vol. 4, pp. 21–28, 2018.

[29] M. Samreen, Q. Kiran, and T. Kamran, "Fixed point theorems for φ-contractions," *Journal of Inequalities and Applications*, vol. 2014, no. 1, Article ID 266, 2014.

[30] P. D. Proinov, "A generalization of the Banach contraction principle with high order of convergence of successive approximations," *Nonlinear Analysis*, vol. 67, no. 8, pp. 2361–2369, 2007.

[31] T. L. Hicks and B. E. Rhoades, "A Banach type fixed point theorem," *Mathematica Japonica*, vol. 24, pp. 327–330, 1979.

[32] J. Asl, S. Rezapour, and N. Shahzad, "On fixed points of α-ψ-contractive multifunctions," *Fixed Point Theory and Applications*, vol. 2012, no. 1, Article ID 212, 2012.

[33] M. Bota, V. Llea, E. Karapinar, and O. Mlesnite, "On $\beta-\varphi$-contractive multi-valued operators in b-metric spaces and applications," *Applied Mathematics Information Sciences*, vol. 5, no. 9, pp. 2611–2620, 2015.