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Research article

New iterative approach for the solutions of fractional order inhomogeneous partial differential equations

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Abstract: In this paper, the study of fractional order partial differential equations is made by using the reliable algorithm of the new iterative method (NIM). The fractional derivatives are considered in the Caputo sense whose order belongs to the closed interval [0,1]. The proposed method is directly extended to study the fractional-order Roseau-Hyman and fractional order inhomogeneous partial differential equations without any transformation to convert the given problem into integer order. The obtained results are compared with those obtained by Variational Iteration Method (VIM), Homotopy Perturbation Method (HPM), Laplace Variational Iteration Method (LVIM) and the Laplace Adominan Decomposition Method (LADM). The results obtained by NIM, show higher accuracy than HPM, LVIM and LADM. The accuracy of the proposed method improves by taking more iterations.

Keywords: fractional order Roseau-Hyman equation; fractional order inhomogeneous system; fractional calculus; new iterative method; approximate solutions
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1. Introduction

Fractional calculus plays a substantial role in different branches of physics, fluid mechanics, diffusive transport, electrical networks, electromagnetic theory, biological sciences and groundwater problems etc. [1–4]. Many researchers have modelled several physical phenomena using the fractional-order differential equations. As we know that, solving a linear differential equation is easier than that of nonlinear differential equations; therefore, numerous methods have been recommended for solving such type of equations. Some of the numerical and analytical methods for solving linear and nonlinear FDE are, Finite Element Method(FEM), time-space spectral method, compact numerical method [7,18,19], Adomian Decomposition Method (ADM) [5,6], Variational Iteration Method (VIM) [8–9], Homotopy Analysis Method (HAM) [10] and HPM [11]. Each of the above methods has its own advantages and disadvantages.

Daftardar-Gejji and Jafari presented a powerful method, namely called the new iterative method which works without any small or large parameter in the equation like other perturbation methods. The proposed method has been used in literature for the solution of different nonlinear differential equations [12,13].

In the present work, NIM has been extended to the solution of fractional order Roseau-Hyman equation and system of Inhomogeneous fractional order partial differential equations. The fractional order Roseau-Hyman equation has the following form [14–16].

$$\frac{\partial^q \Upsilon(r,t)}{\partial t^q} = \Upsilon(r,t)\frac{\partial^3 \Upsilon(r,t)}{\partial r^3} + \Upsilon(r,t)\frac{\partial \Upsilon(r,t)}{\partial r} + 3\frac{\partial \Upsilon(r,t)}{\partial r}\frac{\partial^2 \Upsilon(r,t)}{\partial r^2}, \quad t>0, \tag{1}$$

where, q is the parameter describes order the fractional derivative such that $0<q\le 1$, t time, and r represents spatial coordinate. Eq (1) has appeared in the study of the formation of patterns in liquid drops.

The system of inhomogeneous fractional order partial differential equations has the following form [17].

$$\begin{cases} \dfrac{\partial^q \Upsilon(r,t)}{\partial t^q} - \dfrac{\partial \psi(r,t)}{\partial r} - \Upsilon(r,t) + \psi(r,t) = -2, \\ \dfrac{\partial^q \psi(r,t)}{\partial t^q} + \dfrac{\partial \Upsilon(r,t)}{\partial r} - \Upsilon(r,t) + \psi(r,t) = -2. \end{cases} \tag{2}$$

The whole paper is divided into six sections. The introduction and literature survey are given in section 1, while section 2 is devoted to the basic definitions from the fractional calculus. The third section contains the fundamental theory of a new iterative method for general fractional order PDE’s. In section 4, the proposed method is tested upon fractional order Roseau-Hyman equation and system of inhomogeneous fractional order partial differential equations. In section 5, the listed results are compared with HPM, VIM, LVIM, and LADM solution, which show the precision of the planned method. The conclusions of the paper are presented in the last section.

2. Preliminaries

To investigate our problems with the help of NIM, we need some basic definitions from fractional calculus.

Definition 1. The Riemann-Liouville's (R-L) fractional integral is defined as

$$I_r^\alpha = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_0^r (r-\xi)^{\alpha-1} f(\xi)d\xi & \text{if } \alpha>0,\, t>0, \\ f(\xi) & \text{if } \alpha=0, \end{cases} \tag{3}$$

where Γ denotes the gamma function,

$$\Gamma(p) = \int_0^\infty e^{-r} r^{p-1} dr \qquad p \in \mathbb{C}.$$

Definition 2. The Riemann-Liouville fractional derivative of function $f(t)$ f with order α is defined as

$$D_r^\alpha f(r) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dr^n}\int_0^r (r-\xi)^{n-\alpha-1} f(\xi)d\xi \qquad \text{if } \alpha>0,\, t>0, \tag{4}$$

where n is a positive integer which satisfies $n-1<\alpha\le n$.

Definition 3. Fractional derivative of order α in the Caputo sense, is defined as:

For $n \in N, r > 0, t \ge -1$and $\varphi \in \mathbb{C}_t$:

$$D_t^\alpha f(r) = \begin{cases} I^{n-\alpha}\left[\frac{\partial^n}{\partial t^n} f(r)\right] & \text{if } n-1<\alpha\le n, \quad n\in \mathrm{N} \\ \frac{d^\alpha}{dt^\alpha}\left(f(r)\right) & \text{if } \alpha \in \mathrm{N}. \end{cases} \tag{5}$$

Definition 4. If $n\in \mathrm{N}$, $n-1<\alpha\le n$ and $f \in C_\alpha^\mu$, $\mu \ge -1$, then

$$I_r^\alpha\left[D_r^\alpha f(r)\right] = r(r) + \sum_{i=0}^{n-1} f^{(i)}(\xi)\frac{(r-\xi)^i}{\Gamma(i+1)}, \quad r>0. \tag{6}$$

Remarks: Basic properties of fractional integration are, when $f \in C_\lambda$, $\lambda \ge -1$, α, $\beta \ge 0$ and $\xi \ge -1$:

- $I_t^\alpha I_t^\beta f(t) = I_t^{\alpha+\beta} f(t)$,
- $\left(I_t^\alpha I_t^\beta\right) f(t) = \left(I_t^\beta I_t^\alpha\right) f(t)$,
- $I_t^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\gamma+\alpha}$.

3. New iterative method

The new iterative method is presented for fractional order partial differential equation as follows:
Let us consider fractional order PDE's

$$D_{\tau}^{\alpha}\Upsilon(r,\tau)=\Psi\left(\Upsilon(r,\tau)+f\right), \tag{7}$$

subject to the initial condition

$$\Upsilon(r,0)=g(m), \tag{8}$$

where Ψ denotes nonlinear functions of Υ. According to the fundamental idea of NIM, and using Eq (8) above Eq (7) takes the form

$$(r,t) = g(m) + \frac{1}{\Gamma(\alpha)}\int_0^r \Psi\big((r,t),f\big)^{\alpha-1}\, dr. \tag{9}$$

Assuming that the solution of Eq (7) has series form

$$(r,t) = \sum_{k=0}^{\infty} \,(r,t). \tag{10}$$

The decomposition of nonlinear operator $\psi\left(\Upsilon(r,t)\right)=\dfrac{1}{\Gamma(\alpha)}\int_0^r \Psi\left(\Upsilon(r,t)\right)^{\alpha-1}dr$ as follows

$$\psi\left(\sum_{k=0}^{\infty}\Upsilon(r,t)\right)=\psi\left(\Upsilon(r,t)\right)+\sum_{k=1}^{\infty}\left\{\psi\left(\sum_{i=0}^{k}\Upsilon(r,t)\right)-\psi\left(\sum_{i=0}^{k-1}\Upsilon(r,t)\right)\right\}. \tag{11}$$

Hence general equation of (7) takes the following form

$$\Upsilon_m(r,t)=\sum_{k=0}^{\infty}\Upsilon_m(r,t)=g(m)+\psi\left(\Upsilon_0(r,t)\right)+\sum_{k=1}^{\infty}\left\{\psi\left(\sum_{i=0}^{k}\Upsilon_m(r,t)\right)-\psi\left(\sum_{i=0}^{k-1}\Upsilon_{m-1}(r,t)\right)\right\}, \tag{12}$$

from the above we have

$$\begin{cases}\Upsilon_0(r,t)=g(m),\\ \Upsilon_1(r,t)=\psi\left(\Upsilon_0(r,t)\right),\\ \Upsilon_2(r,t)=\psi\left(\Upsilon_0(r,t)+\Upsilon_1(r,t)\right)-\psi\left(\Upsilon_0(r,t)\right),\\ \Upsilon_{k+1}(\chi)=\psi\left(\Upsilon_0(r,t)+\Upsilon_1(r,t)+\ldots+\Upsilon_k(r,t)\right)-\psi\left(\Upsilon_0(r,t)+\Upsilon_1(r,t)+\ldots+\Upsilon_{k-1}(r,t)\right)\\ ,k=1,2,3\ldots\end{cases} \tag{13}$$

4. Convergence

Let Υ is the series solution achieved by NIM and E the error of the solution of (8). Clearly E, satisfies (8), so, we can write

$$E(r) = f(r) + \psi\big(E(r)\big). \tag{14}$$

The recurrence relation is given as,

$$\begin{aligned} &\mathrm{E}_0 = f, \\ &\mathrm{E}_1 = N(\mathrm{E}_0), \\ &E_{n+1} = N(\mathrm{E}_0 + E_1 + E_2 + ... + E_n),\ n = 1,2,3,4.................... \end{aligned} \tag{15}$$

If $\|N(r) - N(y)\| \le k\|x - y\|, 0 < k < 1,$ then

$$\begin{aligned} &E_0 = f, \\ &\|E_1\| = \|N(E_0)\| \le k\|E_1\|, \\ &\|E_2\| = \|N(E_0 + E1) - N(E_0)\| \le k\|E_1\| \le k^2\|E_0\|, \\ \\ &\|E_{n+1}\| = \|N(E_0 + ... + E_n) - N(E_0 + ... + E_{n-1})\| \\ &\qquad \le k\|E_n\| \le k^{k+1}\|E_0\|, n = 0,1,2,3........... \end{aligned} \tag{16}$$

Thus, $E_{n+1} \to 0$, as $n \to \infty$, which proves the convergence of the new iterative method.

5. Numerical examples

To illustrate efficiency and precision of the NIM, the following fractional order differential equations are taken as test examples. All computational work has been done with the help of Mathematica 10.

5.1. Fractional order Roseau-Hyman equation

First, we consider time fractional Roseau-Hyman equation given as

$$\frac{\partial^q \Upsilon(r,t)}{\partial t^q} = \Upsilon(r,t)\frac{\partial^3 \Upsilon(r,t)}{\partial r^3} + \Upsilon(r,t)\frac{\partial \Upsilon(r,t)}{\partial r} + 3\frac{\partial \Upsilon(r,t)}{\partial r}\frac{\partial^2 \Upsilon(r,t)}{\partial r^2}, \tag{17}$$

Subject to initial condition

$$(r,0) = \frac{-8}{3}\varepsilon\, cos^2\left(\frac{r}{4}\right). \tag{18}$$

For special case $q = 1$, exact solution for Eq (17) can be found in [14] and ε is an arbitrary constant

$$\Upsilon(r,t)=\frac{-8}{3}\varepsilon\cos^2\left(\frac{r-\varepsilon t}{4}\right). \tag{19}$$

By applying I^q to both sides Eq (17), we get the equivalent integral form of (16) is

$$\Upsilon(r,t)=\Upsilon_0(r,t)+I^q\left(\Upsilon(r,t)\frac{\partial^3\Upsilon(r,t)}{\partial r^3}+\Upsilon(r,t)\frac{\partial\Upsilon(r,t)}{\partial r}+3\frac{\partial\Upsilon(r,t)}{\partial r}\frac{\partial^2\Upsilon(r,t)}{\partial r^2}\right). \tag{20}$$

The nonlinear term is

$$N(\Upsilon(r,t))=I^q\left(\Upsilon(r,t)\frac{\partial^3\Upsilon(r,t)}{\partial r^3}+\Upsilon(r,t)\frac{\partial\Upsilon(r,t)}{\partial r}+3\frac{\partial\Upsilon(r,t)}{\partial r}\frac{\partial^2\Upsilon(r,t)}{\partial r^2}\right). \tag{21}$$

Using NIM formulation discussed in section 3, we get the approximations as

$$\begin{aligned}
&\Upsilon_0(r,t)=\frac{-8}{3}\varepsilon\cos^2\left(\frac{r}{4}\right), && \Upsilon_1(r,t)=-\frac{2\varepsilon^2t^q\sin\left(\frac{r}{2}\right)}{3\Gamma(1+q)},\\
&\Upsilon_2(r,t)=\frac{\varepsilon^3t^{2q}\cos\left(\frac{r}{2}\right)}{3\Gamma(1+2q)}, && \Upsilon_3(r,t)=\frac{\varepsilon^4t^{3q}\sin\left(\frac{r}{2}\right)}{6\Gamma(1+3q)},\\
&\Upsilon_4(r,t)=-\frac{\varepsilon^5t^{4q}\cos\left(\frac{r}{2}\right)}{12\Gamma(1+4q)}, && \Upsilon_5(r,t)=-\frac{\varepsilon^6t^{5q}\sin\left(\frac{r}{2}\right)}{24\Gamma(1+5q)},\\
&\Upsilon_6(r,t)=\frac{\varepsilon^7t^{6q}\cos\left(\frac{r}{2}\right)}{48\Gamma(1+6q)}, && \Upsilon_7(r,t)=\frac{\varepsilon^8t^{7q}\sin\left(\frac{r}{2}\right)}{96\Gamma(1+7q)}....
\end{aligned} \tag{22}$$

The expression for the solution of $\Upsilon(r,t)$ is given as

$$\tilde{\Upsilon}(r,t)=\Upsilon_0(r,t)+\sum_{i=1}^{n=7}\Upsilon_i(r,t).\quad i=1,2,3,... \tag{23}$$

For $q=1.0$, the seventh order NIM solution for Roseau-Hyman equation is

$$\begin{aligned}
\tilde{\Upsilon}(r,t)=&-\frac{8}{3}\varepsilon\cos^2\left(\frac{r}{4}\right)+\frac{1}{6}\varepsilon^3t^2\cos\left(\frac{r}{4}\right)-\frac{1}{288}\varepsilon^5t^4\cos\left(\frac{r}{4}\right)+\frac{1}{34560}\left(\varepsilon^7t^6\cos\left(\frac{r}{4}\right)\right)-\frac{2}{3}\varepsilon^2t\sin\left(\frac{r}{4}\right)\\
&+\frac{1}{36}\varepsilon^4t^3\sin\left(\frac{r}{4}\right)-\frac{1}{2880}\left(\varepsilon^6t^5\sin\left(\frac{r}{4}\right)\right)+\frac{1}{483840}\left(\varepsilon^8t^7\sin\left(\frac{r}{4}\right)\right).
\end{aligned}$$

Similarly for $q=0.7$, the seventh order NIM solution is

$$\tilde{\Upsilon}(r,t) = -\frac{8}{3}\varepsilon\cos^2\left(\frac{r}{4}\right) + 0.26834773761572084\varepsilon^3 t^{1.4}\cos\left(\frac{r}{4}\right) - 0.017752501224054806\varepsilon^5 t^{2.8}\cos\left(\frac{r}{4}\right) +$$
$$0.0006394889775371471c^7 t^{4.199999999999999}\cos\left(\frac{x}{4}\right) - 0.7336982703491104\varepsilon^2 t^{0.7}\sin\left(\frac{r}{4}\right) +$$
$$0.07583961083057564\varepsilon^4 t^{2.0999999999999996}\sin\left(\frac{r}{4}\right) - 0.0035821560860175\varepsilon^6 t^{3.5}\sin\left(\frac{r}{4}\right) +$$
$$0.00010286014612875396\varepsilon^8 t^{4.8999999999999995}\sin\left(\frac{r}{4}\right).$$

5.2. Fractional Order inhomogeneous system

Consider time fractional inhomogeneous system given as [17]

$$\begin{aligned} &\frac{\partial^q \Upsilon(r,t)}{\partial t^q} - \frac{\partial\psi(r,t)}{\partial r} - \Upsilon(r,t) + \psi(r,t) = -2, \\ &\frac{\partial^q \psi(r,t)}{\partial t^q} + \frac{\partial\Upsilon(r,t)}{\partial r} - \Upsilon(r,t) + \psi(r,t) = -2. \end{aligned} \tag{24}$$

With initial condition as

$$\begin{cases} \Upsilon(r,0) = 1 + e^r, \\ \psi(r,0) = -1 + e^r. \end{cases} \tag{25}$$

For $q = 1$, exact solution for system (23) is [17]

$$\begin{cases} \Upsilon(r,t) = 1 + e^{r+t}, \\ \psi(r,t) = -1 + e^{r-t} \end{cases} \tag{26}$$

By applying I^q to both sides of Eq (24), we get the equivalent integral form of (23) given as

$$\begin{cases} \Upsilon(r,t) = \Upsilon_0(r,t) + I^q\left(\dfrac{\partial\psi(r,t)}{\partial r} + \Upsilon(r,t) - \psi(r,t) - 2\right), \\ \psi(r,t) = \psi_0(r,t) + I^q\left(\dfrac{\partial\Upsilon(r,t)}{\partial r} + \Upsilon(r,t) - \psi(r,t) - 2\right). \end{cases} \tag{27}$$

Here nonlinear terms are in Eq (25)

$$\begin{cases} N(\Upsilon(r,t)) = I^q\left(\dfrac{\partial\psi(r,t)}{\partial r} + \Upsilon(r,t) - \psi(r,t) - 2\right), \\ N(\psi(r,t)) = I^q\left(\dfrac{\partial\Upsilon(r,t)}{\partial r} + \Upsilon(r,t) - \psi(r,t) - 2\right). \end{cases} \tag{28}$$

Using NIM formulation discussed in section 3, we get the approximations as

$$\begin{aligned}
&\Upsilon_0(r,t)=1+e^r, && \Upsilon_3(r,t)=\frac{e^r t^{3q}}{\Gamma(1+3q)},\\
&\psi_0(r,t)=-1+e^r, && \psi_3(r,t)=-\frac{e^r t^{3q}}{\Gamma(1+3q)},\\
&\Upsilon_1(r,t)=\frac{e^r t^{q}}{\Gamma(1+q)}, && \Upsilon_4(r,t)=\frac{e^r t^{4q}}{\Gamma(1+4q)},\\
&\psi_1(r,t)=-\frac{e^r t^{q}}{\Gamma(1+q)}, && \psi_4(r,t)=\frac{e^r t^{4q}}{\Gamma(1+4q)},\\
&\Upsilon_2(r,t)=\frac{e^r t^{2q}}{\Gamma(1+2q)}, && \Upsilon_5(r,t)=\frac{e^r t^{5q}}{\Gamma(1+5q)},\\
&\psi_2(r,t)=\frac{e^r t^{2q}}{\Gamma(1+2q)}, && \psi_5(r,t)=-\frac{e^r t^{5q}}{\Gamma(1+5q)}.
\end{aligned} \tag{29}$$

The expression for the solution of $\Upsilon(r,t)$ and $\psi(r,t)$ is given as

$$\begin{cases}\Upsilon(r,t)=\sum_{i=1}^{n=5}\Upsilon_i(r,t)\\ \tilde{\psi}(r,t)=\sum_{i=1}^{n=5}\psi_i(r,t).\end{cases} \quad i=1,2,3,\ldots. \tag{30}$$

$$\tilde{\Upsilon}(r,t)=1+e^r+\frac{e^r t^{q}}{\Gamma(1+q)}+\frac{e^r t^{2q}}{\Gamma(1+2q)}+\frac{e^r t^{3q}}{\Gamma(1+3q)}+\frac{e^r t^{4q}}{\Gamma(1+4q)}+\frac{e^r t^{5q}}{\Gamma(1+5q)},$$
$$\tilde{\psi}(r,t)=-1+e^r-\frac{e^r t^{q}}{\Gamma(1+q)}+\frac{e^r t^{2q}}{\Gamma(1+2q)}-\frac{e^r t^{3q}}{\Gamma(1+3q)}+\frac{e^r t^{4q}}{\Gamma(1+4q)}-\frac{e^r t^{5q}}{\Gamma(1+5q)}.$$

6. Results and discussion

We implemented NIM for finding the approximate solutions of Roseau-Hyman and fractional order inhomogeneous partial differential equations. The results obtained by NIM for Roseau-Hyman and fractional order inhomogeneous partial differential equations with VIM, HPM and LADM in the form of tables and figures in section 3.

Table 1 shows the approximate solution obtain by NIM for Roseau-Hyman equation at different values of q Table 2 shows the comparison of absolute errors of NIM with VIM and HPM for Roseau-Hyman equation. Table 3 shows the numerical solution obtained by the proposed method for $\gamma(r,t)$ and $\gamma(r,t)$ inhomogeneous partial differential equations at q=0.5. Similarly Table 4 shows the residuals obtain by the NIM $\gamma(r,t)$ and $\psi(r,t)$ inhomogeneous partial differential equations at q=0.5. It is clear that for the fractional order inhomogeneous system, NIM has the same solution like LADM and LVIM, mention in [17]. That is why, we listed only NIM solution as well as residual obtain by NIM in Tables 3 and 4.

Table 1. NIM Solution for Eq (17) at different values of q at $\varepsilon = 1.0$.

r	t	Solution q=0.7	Solution q=0.9	Solution q=1.0	Exact q=1
	0.2	−1.302026382	−1.296886512	−1.294567416	−1.294567416
$\pi/4$	0.4	−1.316024885	−1.315094415	−1.313795239	−1.313795239
	1.0	−1.320505052	−1.325168954	−1.326557167	−1.326557168
	0.2	−1.177509671	−1.165865484	−1.161042461	−1.161042461
$\pi/2$	0.4	−1.216132042	−1.208017039	−1.203223634	−1.203223634
	1.0	−1.241129500	−1.241413136	−1.240043707	−1.240043707
	0.2	−0.975221742	−0.958845962	−0.952253273	−0.952253273
$3\pi/4$	0.4	−1.032588076	−1.018523975	−1.010966095	−1.010966095
	1.0	−1.074297190	−1.070157378	−1.066238790	−1.066238790
	0.2	−0.725959101	−0.707344790	−0.699986113	−0.699986113
π	0.4	−0.793335893	−0.775463826	−0.766292088	−0.766292088
	1.0	−0.845406828	−0.837473817	−0.831602639	−0.831602639

Table 2. Comparison of absolute errors obtained by NIM with HPM and VIM for, q=1.0 at $\varepsilon = 1.0$ for Eq (17).

	$r = \pi$			$r = 3\pi/2$		
t	HPM **[14]**	VIM**[14]**	NIM	HPM **[14]**	VIM**[14]**	NIM
0.1	1.0000×10^{-11}	5.0000×10^{-10}	2.2204×10^{-16}	1.0000×10^{-11}	2.0000×10^{-10}	5.5511×10^{-17}
0.2	1.7360×10^{-9}	5.0000×10^{-10}	4.2188×10^{-16}	1.2378×10^{-9}	3.0000×10^{-10}	4.7185×10^{-16}
0.3	1.3182×10^{-8}	5.0000×10^{-10}	1.4099×10^{-16}	9.4375×10^{-9}	3.0000×10^{-10}	1.1491×10^{-14}
0.4	5.5542×10^{-8}	1.0000×10^{-10}	1.8807×10^{-15}	3.9925×10^{-8}	9.0000×10^{-10}	1.1549×10^{-13}
0.5	1.6948×10^{-7}	4.0000×10^{-10}	1.4008×10^{-14}	1.2233×10^{-7}	1.9000×10^{-9}	6.8706×10^{-13}
0.6	4.2165×10^{-7}	7.0000×10^{-10}	7.226×10^{-14}	3.0561×10^{-7}	6.1000×10^{-9}	2.9458×10^{-12}
0.7	9.1117×10^{-7}	1.2000×10^{-9}	2.8266×10^{-13}	6.6309×10^{-7}	1.8300×10^{-8}	1.0081×10^{-11}
0.8	1.7761×10^{-6}	2.1000×10^{-9}	9.4025×10^{-13}	1.2978×10^{-6}	3.9300×10^{-8}	2.9252×10^{-11}
0.9	3.1998×10^{-6}	4.0000×10^{-9}	2.7141×10^{-12}	2.3474×10^{-6}	8.2200×10^{-8}	7.4833×10^{-11}
1.0	5.4173×10^{-6}	8.6000×10^{-9}	7.0042×10^{-12}	3.9903×10^{-6}	1.5230×10^{-7}	1.7332×10^{-10}

Table 3. The approximate solution obtained by the NIM, for Eq (24) at q=0.5.

		q=0.5		q=0.5	
t	r	NIM Solution $\gamma(r,t)$	NIM Solution $\psi(r,t)$	Residual $\gamma(r,t)$	Residual $\psi(r,t)$
	−4	1.0240330988	−0.9855237049	-3.08085×10^{-6}	-3.08085×10^{-6}
	−3	1.0655328735	−0.9606493501	-8.37462×10^{-6}	-8.37462×10^{-6}
	−2	1.1775819155	−0.8930338435	-2.27646×10^{-5}	-2.27646×10^{-5}
0.05	−1	1.4827176940	−0.7092355840	-6.18806×10^{-5}	-6.18806×10^{-5}
	0	2.3121627361	−0.2096210694	-1.68209×10^{-4}	-1.68209×10^{-4}
	1	4.5668281215	1.14847226845	-4.57239×10^{-4}	-4.57239×10^{-4}
	2	10.695644068	4.84015425729	-1.24290×10^{-3}	-1.24290×10^{-3}
	−4	1.0272304344	−0.9867467269	-1.74279×10^{-5}	-1.74279×10^{-5}
0.1	−3	1.0740199950	−0.9639739783	-4.73740×10^{-5}	-4.73740×10^{-5}
	−2	1.2012072075	−0.9020711198	-1.28776×10^{-4}	-1.28776×10^{-4}

		q=0.5		q=0.5	
t	r	NIM Solution $\gamma(r,t)$	NIM Solution $\psi(r,t)$	Residual $\gamma(r,t)$	Residual $\psi(r,t)$
	−1	1.5469378961	−0.7338017046	-3.50049×10^{-4}	-3.50049×10^{-4}
0.1	0	2.4867313443	−0.2763980109	-9.51533×10^{-4}	-9.51533×10^{-4}
	1	5.0413547970	0.9669564137	-2.58653×10^{-3}	-2.58653×10^{-3}
	2	11.985541307	4.3467356902	-7.03093×10^{-3}	-7.03093×10^{-3}

Table 4. Absolute errors obtained by 5th order approximate solution of NIM for Eq (24) at q=1.0.

t	r	NIM Solution $\gamma(r,t)$	NIM Solution $\psi(r,t)$	Absolute Error $\gamma(r,t)$	Absolute Error $\psi(r,t)$
	−4	1.0192547017	−0.9825776253	2.886×10^{-15}	2.664×10^{-15}
	−3	1.0522339705	−0.9526410756	7.771×10^{-15}	7.660×10^{-15}
	−2	1.1422740715	−0.8712655064	2.131×10^{-14}	2.098×10^{-14}
0.005	−1	1.3867410234	−0.6500622508	1.563×10^{-14}	5.662×10^{-14}
	0	2.0512710963	−0.0487705754	4.241×10^{-13}	1.540×10^{-13}
	1	3.8576511180	1.5857096593	1.151×10^{-13}	4.187×10^{-13}
	2	8.7679011063	6.0286887558	1.138×10^{-12}	1.139×10^{-12}
	−4	1.0202419114	−0.9834273245	3.679×10^{-13}	3.588×10^{-13}
	−3	1.0552322005	−0.9549507976	1.000×10^{-12}	9.755×10^{-13}
	−2	1.1495686192	−0.8775435717	2.718×10^{-12}	2.651×10^{-12}
0.1	−1	1.4065696597	−0.6671289162	7.391×10^{-12}	7.209×10^{-12}
	0	2.1051709180	−0.0951625819	2.009×10^{-11}	1.959×10^{-11}
	1	4.0041660238	1.4596031112	5.461×10^{-11}	5.326×10^{-11}
	2	9.1661699124	5.6858944442	1.484×10^{-10}	1.447×10^{-10}

Figures 1–4 show the 3D plots obtain by NIM for $\gamma(r,t)$ and exact solution for $\gamma(r,t)$ of Roseau-Hyman equation respectively. Figures 5 and 6 shows the 2D plot of exact verses NIM solution and convergence of NIM for different values of q for $\psi(r,t)$ respectively. Similarly, Figures 7 and 8 show, 2D graph for residual obtained by the NIM for Roseau-Hyman equation for q=0.5 and 0.7 respectively.

Figures 9–12 show the 3D plots obtain by NIM for $\gamma(r,t)$ and exact solution of inhomogeneous partial differential equations respectively at different values of q. Figures 13 and 14 show the 2D plot of exact verses NIM solution and convergence of NIM for different values of q for $\gamma(r,t)$ part of inhomogeneous partial differential equation respectively.

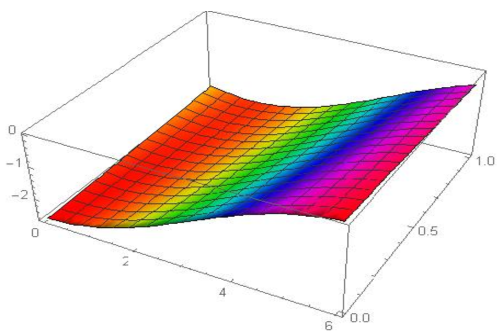

Figure 1. NIM solution for Eq (17) at q =0.5.

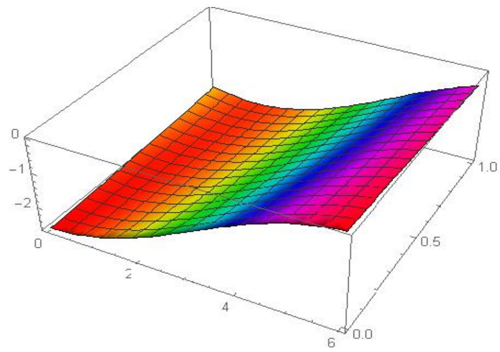

Figure 2. NIM solution for Eq (17) at q =0.9.

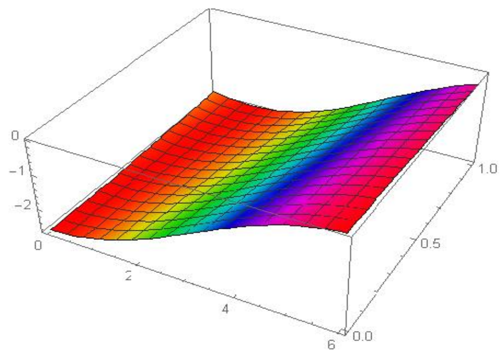

Figure 3. NIM solution for Eq (17) at q =1.0.

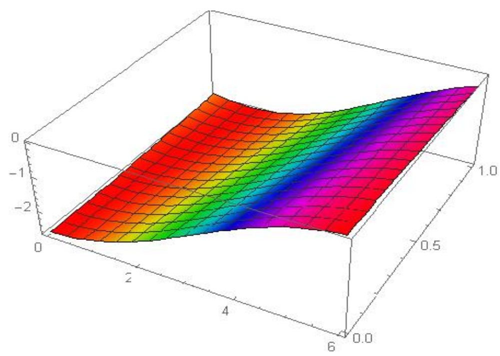

Figure 4. NIM solution for Eq (17) at q= 1.0.

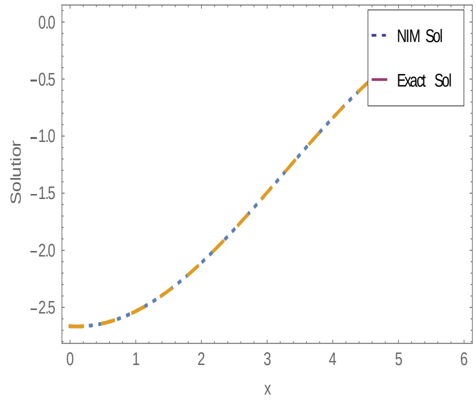

Figure 5. NIM solution verses exact solution for Eq (17) at t=0.1.

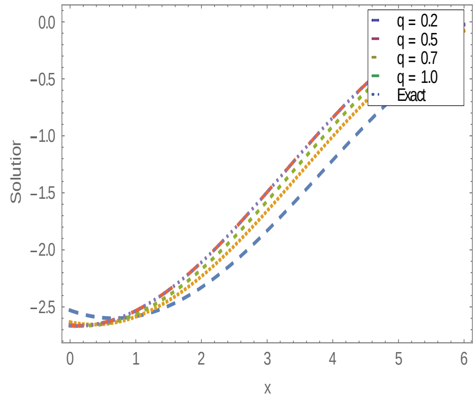

Figure 6. NIM solution for Eq (17) for different values of q at t=0.1.

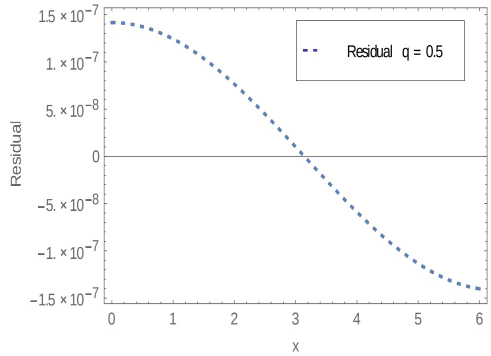

Figure 7. Residual obtained by NIM for Eq (17) t=0.1.

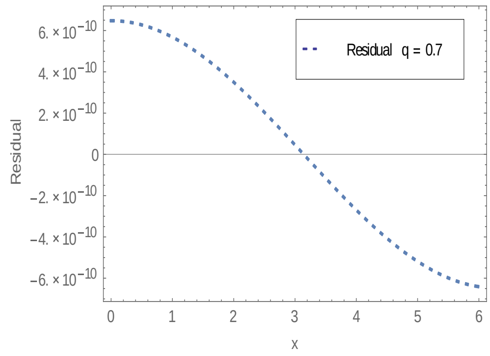

Figure 8. Residual obtained by NIM for Eq (17) *t*=0.1.

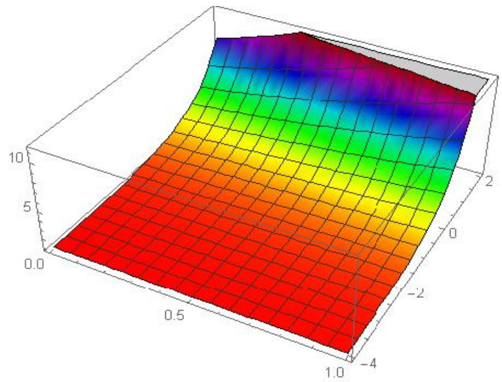

Figure 9. NIM Solution of γ(r,t) for Eq (24) when q= 0.5.

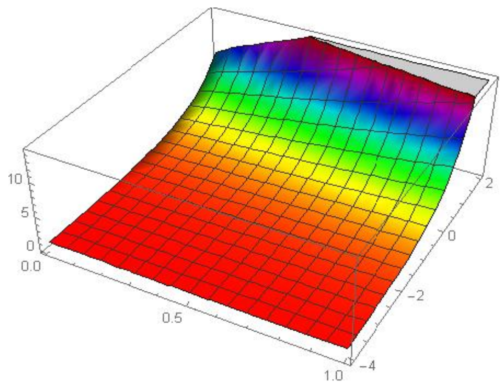

Figure 10. NIM Solution of γ(r,t) for Eq (24) when q= 0.7.

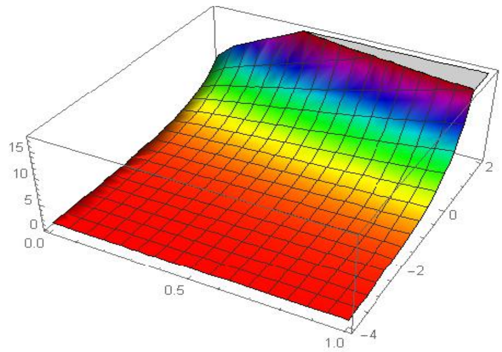

Figure 11. NIM Solution of γ(r,t) for Eq. (24)

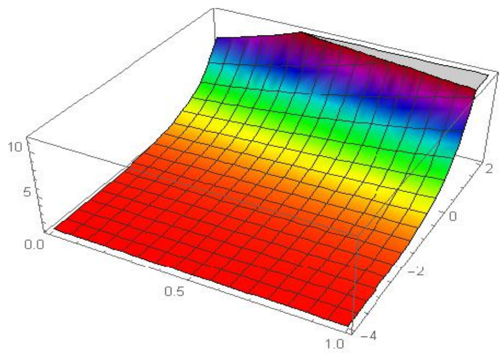

Figure 12. Exact Solution of γ(r,t) for Eq (24).

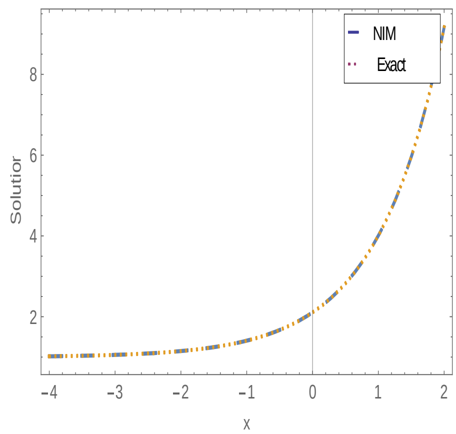

Figure 13. NIM solution of γ(r,t) verses exact solution for Eq (24) at t=0.1.

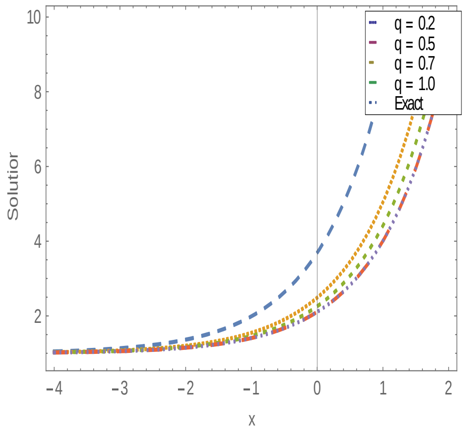

Figure 14. NIM solution of γ(r,t) for Eq (24) for different values of q at t=0.1.

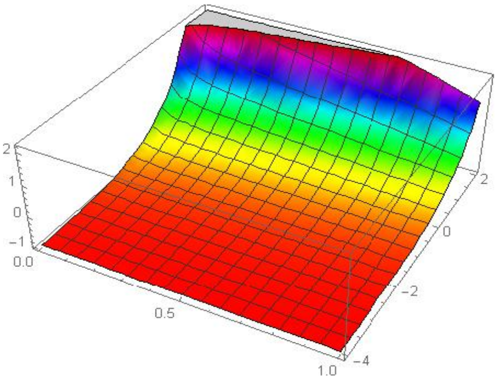

Figure 15. NIM Solution of ψ(r,t) for Eq (24) when q= 0.5.

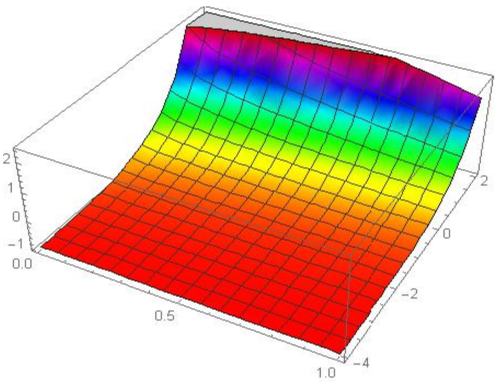

Figure 16. NIM Solution of ψ(r,t) for Eq (24) when q= 0.5.

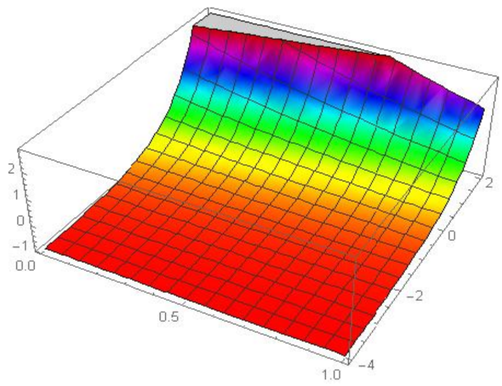

Figure 17. NIM Solution of ψ(r,t) for Eq (24) when q= 1.0.

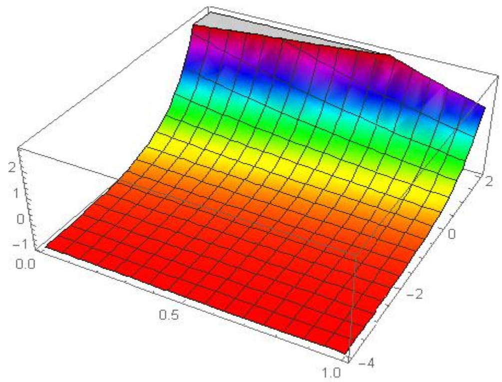

Figure 18. Exact Solution of ψ(r,t) when q= 1.0.

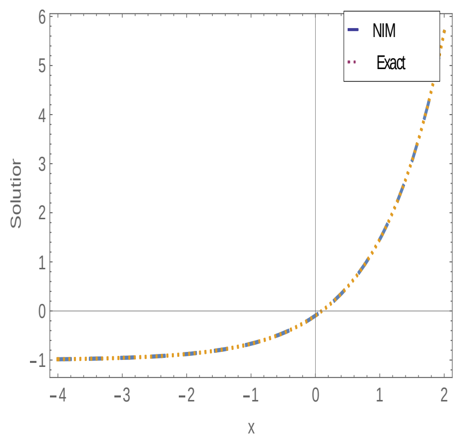

Figure 19. NIM solution of ψ(r,t) verses exact solution for Eq (24) at *t*=0.1.

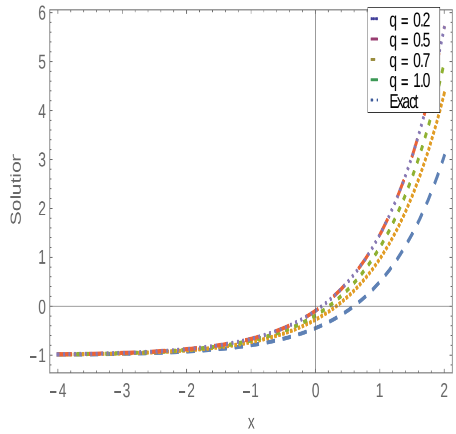

Figure 20. NIM solution of ψ(r,t) for Eq (24) for different values of q, at t=0.1.

Figures 15–18 show the 3D plots obtain by NIM for $\psi(r,t)$ and exact solution of inhomogeneous partial differential equations respectively at different values of q. Figures. 19 and 20 shows the 2D plot of exact verses NIM solution and convergence of NIM for different values of q for $\psi(r,t)$ part of inhomogeneous partial differential equation respectively.

From the numerical values and graphs, it is clear that NIM is very powerful tool for solution of coupled fractional order system of partial differential equations. The accuracy of the NIM can further be increased by taking higher order approximations.

7. Conclusion

In the current article, we presented some FDE's arising in modern sciences. A novel and classy technique which is identified as NIM, is applied for fractional order problems. For pertinence and unwavering quality of the proposed method, the fractional order Roseau-Hyman equation and the system of fractional order non-homogeneous equations. It has been explored through graphical and tabulated results that the current method gives a precise and meriting investigation about the physical occurring of the problems. Also, the current method is favored when contrasted with other technique in light of its better pace of convergence. This course rouses the scientists towards the execution of the present method for other non-linear FDE's.

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Conflict of interest

The authors declare no conflict of interest.

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