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Novel aspects of discrete dynamical type inequalities within fractional operators having generalized $\hbar$-discrete Mittag-Leffler kernels and application

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ABSTRACT

Discrete fractional calculus ($\mathcal{DFC}$) has had significant advances in the last few decades, being successfully employed in the time scale domain $\hbar\mathbb{Z}$. Understanding of $\mathcal{DFC}$ has demonstrated a valuable improvement in neural networks and modeling in other terrains. In the context of Riemann form ($\mathcal{ABR}$), we discuss the discrete fractional operator influencing discrete Atangana-Baleanu ($\mathcal{AB}$)-fractional operator having $\hbar$-discrete generalized Mittag-Leffler kernels. In the approach being presented, some new Pólya-Szegö and Chebyshev type inequalities introduced within discrete $\mathcal{AB}$-fractional operators having $\hbar$-discrete generalized Mittag-Leffler kernels. By analyzing discrete $\mathcal{AB}$-fractional operators in the time scale domain $\mathbb{Z}$, we can perform a comparison basis for notable outcomes derived from the aforesaid operators. This type of discretization generates novel outcomes for synchronous functions. The specification of this proposed strategy simply demonstrates its efficiency, precision, and accessibility in terms of the methodology of qualitative approach of discrete fractional difference equation solutions, including its stability, consistency, and continual reliance on the initial value for the solutions of many fractional difference equation initial value problems. The repercussions of the discrete $\mathcal{AB}$-fractional operators can depict new presentations for various particular cases. Finally, applications concerning bounding mappings are also illustrated.

1. Introduction

Inequalities having mappings of multiple variables are well-known for attempting to develop descriptive and analytical representations of complex and algebraic calculus. The key advantage of Pólya-Szegö and Chebyshev in terms of more comprehensive implementations is that such inequalities will provide precise bounds for multiple variables. Currently, authors are considering inventive forms of such modifications that may be beneficial in the treatment of classified differential and difference equations. A variety of Pólya-Szegö and Chebyshev type variants, which are improvements of previously reported versions and can be considered as a form of modal analysis, have been researched in [1].

Chebyshev [2] pondered the best description in 1882 as follows:

$$\frac{1}{\varphi_2-\varphi_1}\int_{\varphi_1}^{\varphi_2}\mathcal{U}(\theta)\mathcal{V}(\theta)d\theta \geq \left(\frac{1}{\varphi_2-\varphi_1}\int_{\varphi_1}^{\varphi_2}\mathcal{U}(\theta)d\theta\right)\left(\frac{1}{\varphi_2-\varphi_1}\int_{\varphi_1}^{\varphi_2}\mathcal{V}(\theta)d\theta\right), \tag{1.1}$$

for integrable functions $\mathcal{U}$ and $\mathcal{V}$ on (φ_1,φ_2) and both the mappings instantaneouslyáincrease or decrease for the same values of

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θ in (φ_1, φ_2). On the other hand, the inequality

$$\frac{1}{\varphi_2-\varphi_1}\int_{\varphi_1}^{\varphi_2}\mathcal{U}(\theta)\mathcal{V}(\theta)d\theta \le \left(\frac{1}{\varphi_2-\varphi_1}\int_{\varphi_1}^{\varphi_2}\mathcal{U}(\theta)d\theta\right)\left(\frac{1}{\varphi_2-\varphi_1}\int_{\varphi_1}^{\varphi_2}\mathcal{V}(\theta)d\theta\right). \tag{1.2}$$

Because of the same values of θ in (φ_1, φ_2), the inequality (1.2) appears to satisfy for one component to be increasing and the other to be decreasing. Ever since, persistent, differentiated forms, and advancements of certain varieties have received considerable attention in research, leading to a variety of classical inequalities, see [3–14]. Numerous esteemed versions reported in the literature are immediate consequences of diverse frameworks of nonlinear dynamics, which now address not only numerous aspects in the analysis of fractional order problems, mathematical modelling as well as certain analytical and functional science research inquiries. In this regard, Pólya-Szegö inequality is a fascinating topic that requires the most consideration. Pólya-Szegö [15] addressed this collection of variants as follows:

$$\frac{\int_{\varphi_1}^{\varphi_2}\mathcal{U}^2(\theta_1)d\theta_1\int_{\varphi_1}^{\varphi_2}\mathcal{V}^2(\theta_1)d\theta_1}{\left(\int_{\varphi_1}^{\varphi_2}\mathcal{U}(\theta_1)\mathcal{V}(\theta_1)d\theta_1\right)^2} \le \frac{1}{4}\left(\sqrt{\frac{\mathcal{R}\mathcal{T}}{r_1t_1}}+\sqrt{\frac{r_1t_1}{\mathcal{R}\mathcal{T}}}\right)^2, \tag{1.3}$$

where

$r_1 \le \mathcal{U}(\theta) \le \mathcal{R}$

and

$t_1 \le \mathcal{V}(\theta) \le \mathcal{T}$

for all $\theta \in (\varphi_1, \varphi_2)$ and for some $r_1, \mathcal{R}, t_1, \mathcal{T} \in \mathbb{R}$. In (1.3), the constant $\frac{1}{4}$ is ideally plausible, and it cannot be substituted by a smaller factor.

Stimulated by áPólya-Szegö and Chebyshev [1,3,16], Our purpose is to evaluate modified forms of Pólya-Szegö and Chebyshev type variants for discrete $\mathcal{AB}$-fractional sums in the time framework $\hbar\mathbb{Z}$.

$\mathcal{DFC}$ captivated a lot of consideration across various analysis and engineering disciplines, particularly in modelling [17], neural networks [18] and image encryption [19]. The developing approach portraying real-world problems has been exhibited to be helpful numerical devices to analyze, comprehend and predict the nature of humankind's lives [20]. While new definitions/operators help researchers to analyze and predict nature, it is significant that the expectation and understanding will be accomplished only if such techniques are illuminated with non-local effects [21]. Numerous utilities have been developed via $\mathcal{DFC}$ such as the solution of fractional difference equations and discrete boundary value problems are proposed in terms of new mathematical techniques [22].

Several diverse kinds of fractional operator have been contemplated in continuous and discrete contexts, such as Caputo, Riemann-Liouville, Hadamard, Riesz, Caputo Fabrizio and henceforth. Several researchers practiced inventing new methodologies of simulation via fractional frameworks [23–29], and to explore new strategies that can then be applied in the bulk of utilities [25], involving bioengineering [21,30], cryptography and control theory [31]. It is critical to keep evolving these innovative representations: both in terms of accumulating scientific appreciation for its specific aspects and also from the applications standpoint just mentioned [32], subsequently recovered consideration of fundamental mathematics will enable a superior analysis of the physical simulations they designate [33]. Among the computational models formulated in fractional calculus, discrete $\mathcal{AB}$-fractional operators, which is a universal operator of fractional calculus that has been traditionally employed to develop modern operators and their characterizations have been proposed in research articles [34,35]. Moreover, $\mathcal{DFC}$ has been theoretically presented by introducing and analyzing discrete forms of these fractional operators [36]. Here, we intend to find the discrete fractional inequalities analogous to fractional operators having $\hbar$-discrete Mittag-Leffler kernels, encompassing and simplifying these operators in such a manner as to recuperate certain appropriate traits such as the discrete inequalities for $\hbar$-discrete Mittag-Leffler kernels.

Discrete fractional variants have been considered as fabulous tools to investigate the qualitative characterizations of difference equations. Previously, many variants have been established by several researchers, for example, Ostrowski, Hardy, Olsen, Opial, Lyapunov and Hermite-Hadamard, see [37,38]. Therefore, the most captivating and distinguished inequalities are the variants (1.2) and (1.3), respectively, which they have not studied yet for discrete $\mathcal{AB}$-fractional sums.

This paper significantly presents the implementation of the discrete $\mathcal{AB}$-fractional operator having $\hbar$-discrete Mittag-Leffler function in the kernel with a step size $0 \le \hbar < 1$. This recently generated scheme provides the discrete version of variants similar to (1.2) and (1.3), respectively, via the aforesaid operator on $\hbar\mathbb{Z}$. Hence, our proposed technique present several consequences in the discrete $\mathcal{AB}$-fractional operator. Interestingly, it is highlighted that intermingling these two approaches, $\mathcal{DFC}$ and variants might be the extreme dexterous methodology of relating inequities in fractional and time scale calculus. Finally, the proposed results are evaluated using various criteria, with the results indicating that the introduced discrete inequalities are viable for multifaceted applications in fractional difference equations and boundary value problems. However, we enhance our reference index by including intriguing literature for such implementation [39–41] from where interested readers can obtain further details.

2. Preliminaries on discrete fractional calculus

In this note, we introduce some fundamental concepts related to fractional operators, discrete generalized Mittag Leffler functions, and time scale calculus; for more information, see [36]. For the sake of convenience, we symbolize, for $\varphi_1, \varphi_2 \in \mathbb{R}$, $\hbar > 0$, $\mathbb{N}_{\varphi_1,\hbar} = \{\varphi_1, \varphi_1+\hbar, \varphi_1+2\hbar, \ldots\}$ and $\mathbb{N}_{\varphi_2,\hbar} = \{\varphi_2, \varphi_2+\hbar, \varphi_2+2\hbar, \ldots\}$.

2.1. Basics on delta and nabla $\hbar$-factorials

Definition 2.1 ([34]). The backward difference operator of a function $\mathcal{U}$ on $\hbar\mathbb{Z}$ is stated as

$$\widehat{\nabla}_\hbar\mathcal{U}(\xi) = \frac{\mathcal{U}(\xi)-\mathcal{U}(\breve{\rho}_\hbar(\xi))}{\hbar}, \tag{2.1}$$

where $\breve{\rho}_\hbar(\xi) = \xi - \hbar$ denotes the backward jump operator. Also, the forward difference operator of a function $\mathcal{U}$ on $\hbar\mathbb{Z}$ is stated as

$$\widehat{\Delta}_\hbar\mathcal{U}(\xi) = \frac{\mathcal{U}(\breve{\sigma}_\hbar(\xi))-\mathcal{U}(\xi)}{\hbar}, \tag{2.2}$$

where $\sigma_\hbar(\xi) = \xi + \hbar$ denotes the forward jump operator.

Definition 2.2 ([34]). **(i)** For any $\xi, \vartheta \in \mathbb{R}$ and $\hbar > 0$, the delta $\hbar$-factorial function is stated as

$$\xi_\hbar^{(\vartheta)} = \hbar^\vartheta\frac{\Gamma(\frac{\xi}{\hbar}+1)}{\Gamma(\frac{\xi}{\hbar}+1-\vartheta)}, \tag{2.3}$$

where Γ denotes the Euler gamma function. For $\hbar = 1$, then $\xi^{(\vartheta)} = \frac{\Gamma(\xi+1)}{\Gamma(\xi+1-\vartheta)}$. Also, a division by a pole results in zero.

(ii) For any $\xi, \vartheta \in \mathbb{R}$ and $\hbar > 0$, the nabla $\hbar$-factorial function is described as:

$$\xi_\hbar^{(\vartheta)} = \hbar^\vartheta \frac{\Gamma(\frac{\xi}{\hbar} + \vartheta)}{\Gamma(\frac{\xi}{\hbar})}. \tag{2.4}$$

For $\hbar = 1$, we observe that $\xi^{(\vartheta)} = \frac{\Gamma(\xi+\vartheta)}{\Gamma(\xi)}$.

Lemma 2.3 ([39]). *Let $\xi \in \mathbb{T} = \mathbb{N}_{\varphi_1,\hbar}$, then for all $\xi \in \mathbb{T}^\kappa$, we obtain*

$$\widehat{\nabla}_{\psi,\hbar}\left\{\frac{(\psi-\xi)_\hbar^{\overline{\kappa+1}}}{(\kappa+1)!}\right\} = \frac{(\psi-\xi)_\hbar^{\overline{\kappa}}}{\kappa!}. \tag{2.5}$$

Lemma 2.4 ([35]). *For the time scale $\mathbb{T} = \mathbb{N}_{\varphi_1,\hbar}$ then the nabla Taylor polynomial*

$$\widehat{\mathcal{H}}_\kappa(\psi,\xi) = \frac{(\psi-\xi)_\hbar^{\overline{\kappa}}}{\kappa!}, \qquad \kappa \in \mathbb{N}_0. \tag{2.6}$$

2.2. Nabla ħ-discrete Mittag-Leffler function

Now we present the idea of nabla $\hbar$-discrete Mittag-Leffler function which is introduced by Abdeljawad et al. [26].

Definition 2.5. ([26]) Let $\vartheta, \beta, \omega \in \mathbb{C}$ having $\Re(\vartheta) > 0$ such that $\lambda \in \mathbb{R}$ with $|\lambda\hbar^\vartheta| < 1$, then the nabla discrete Mittag-leffler function is stated as:

$${}_\hbar\breve{E}_{\overline{\vartheta,\beta}}(\lambda,\omega) = \sum_{\kappa=0}^{\infty}\lambda^\kappa \frac{\omega_\hbar^{\overline{\kappa\vartheta+\beta-1}}}{\Gamma(\vartheta\kappa+\beta)}, \quad |\lambda\hbar^\vartheta| < 1. \tag{2.7}$$

For $\beta = 1$, we have

$${}_\hbar\breve{E}_{\overline{\vartheta}}(\lambda,y) \triangleq {}_\hbar\breve{E}_{\overline{\vartheta,1}}(\lambda,y) = \sum_{\kappa=0}^{\infty}\lambda^\kappa \frac{y_\hbar^{\overline{\kappa\vartheta}}}{\Gamma(\vartheta\kappa+1)}, \quad |\lambda\hbar^\vartheta| < 1. \tag{2.8}$$

The following remark illustrates the strengthening properties why $\hbar\mathbb{Z}$ is important.

Remark 1. In view of $\hbar\mathbb{Z}$:

I. letting $\hbar = 1$, we attain the nabla discrete Mittag-Leffler function stated in [40,41].

II. letting $0 < \hbar < 1$, the interval of convergence to which λ lies. Observe that, when $\hbar \mapsto 0$, then $\vartheta \in (0,1)$. Moreover, when $\hbar \mapsto 1$ guarantee convergence for $\lambda = \frac{-\vartheta}{1-\vartheta}$, $\vartheta \in (0, \frac{1}{2})$.

For further investigation of the discrete Mittag-Leffler function we refer the reader to Abdeljawad and Baleanu [24].

2.3. Delta fractional sums on ħℤ

Definition 2.6 ([32]). For some $\kappa \in \mathbb{N}$, $\vartheta > 0$ and let $\varphi_2 = \varphi_1 + \kappa\hbar$. Assume that a function $\mathcal{U}$ be defined on $\mathbb{T} = \mathbb{N}_{\varphi_1,\hbar} \cap \mathbb{N}_{\varphi_2,\hbar}$. Then the delta $\hbar$-fractional sums in the left and right case are defined as follows

$$({}_{\varphi_1}\widehat{\Delta}_\hbar^{-\vartheta}\mathcal{U})(\xi) = \frac{1}{\Gamma(\vartheta)}\sum_{\kappa=\varphi_1/\hbar}^{\psi/\hbar-\vartheta}(\psi-\sigma(\kappa\hbar))_\hbar^{(\vartheta-1)}\mathcal{U}(\kappa\hbar)\hbar,$$
$$\psi \in \{\tau+\vartheta\hbar : \tau \in \mathbb{T}\}$$

and

$$({}_\hbar\widehat{\Delta}_{\varphi_2}^{-\vartheta}\mathcal{U})(\xi) = \frac{1}{\Gamma(\vartheta)}\sum_{\kappa=\psi/\hbar+\vartheta}^{\varphi_2/\hbar-\vartheta}(\kappa\hbar-\sigma(\psi))_\hbar^{(\vartheta-1)}\mathcal{U}(\kappa\hbar)\hbar,$$
$$\psi \in \{\tau-\vartheta\hbar : \tau \in \mathbb{T}\},$$

respectively.

2.4. Nabla fractional sums on ħℤ

Definition 2.7 ([26,35]). Assume that $\hbar > 0$ and the backward jump operator is $\breve{\rho}(\psi) = \psi - \hbar$. A function $\mathcal{U} : \mathbb{N}_{\varphi_1,\hbar} \mapsto \mathbb{R}$ is said to be nabla $\hbar$-fractional sum of order ϑ, if

$$({}_{\varphi_1}\widehat{\nabla}_\hbar^{-\vartheta}\mathcal{U})(\xi) = \frac{1}{\Gamma(\vartheta)}\sum_{\kappa=\varphi_1/\hbar+1}^{\psi/\hbar-\vartheta}(\psi-\breve{\rho}(\kappa\hbar))_\hbar^{(\vartheta-1)}\mathcal{U}(\kappa\hbar)\hbar,$$
$$\psi \in \mathbb{N}_{\varphi_1+\hbar,\hbar}.$$

For $\vartheta > 0$, the nabla right $\hbar$-fractional sum (ending at φ_2) for $\mathcal{U} : \mathbb{N}_{\varphi_2,\hbar} \mapsto \mathbb{R}$ is described as follows:

$$({}_\hbar\widehat{\nabla}_{\varphi_2}^{-\vartheta}\mathcal{U})(\xi) = \frac{1}{\Gamma(\vartheta)}\sum_{\kappa=\psi/\hbar}^{\varphi_2/\hbar-1}(\kappa\hbar-\breve{\rho}(\psi))_\hbar^{(\vartheta-1)}\mathcal{U}(\kappa\hbar)\hbar.$$

2.5. Nabla ħ-fractional differences depending on ħ-discrete Mittag Leffler kernels

Now, we are demonstrating some new concepts which we will be use to prove the coming results of this paper, see [24]. Also, we use the notation, $\lambda = -\frac{\vartheta}{1-\vartheta}$ and $\breve{\rho}(\psi) = \psi - \hbar$.

Definition 2.8. ([39]) For $\vartheta \in [0,1]$, $\hbar > 0$ with $|\lambda\hbar^\vartheta| < 1$ and let $\mathcal{U}$ be a function defined on $\mathbb{N}_{\varphi_1,\hbar} \cap {}_{\varphi_2,\hbar}\mathbb{N}$ with $\varphi_1 < \varphi_2$ such that $\varphi_1 \equiv \varphi_2 (mod\ \hbar)$, then the left nabla $\mathcal{ABC}$-fractional difference (in the frame of $\mathcal{AB}$) is described as

$$\left({}_{\varphi_1}^{\mathcal{ABC}}\widehat{\nabla}_\hbar^\vartheta\mathcal{U}\right)(\psi) = \mathcal{H}(\vartheta,\hbar)\frac{1-\vartheta+\vartheta\hbar}{1-\vartheta}\sum_{\kappa=\varphi_1/\hbar+1}^{\psi/\hbar} \hbar\widehat{\nabla}_\hbar\mathcal{U}(\kappa\hbar)\,{}_\hbar\breve{E}_{\overline{\vartheta}}(\lambda,\psi-\breve{\rho}(\kappa\hbar))$$

and in the left Riemann sense by

$$\left({}_{\varphi_1}^{\mathcal{ABR}}\widehat{\nabla}_\hbar^\vartheta\mathcal{U}\right)(\psi) = \mathcal{H}(\vartheta,\hbar)\frac{1-\vartheta+\vartheta\hbar}{1-\vartheta}\widehat{\nabla}_\hbar\sum_{\kappa=\varphi_1/\hbar+1}^{\psi/\hbar} \hbar\mathcal{U}(\kappa\hbar)\,{}_\hbar\breve{E}_{\overline{\vartheta}}(\lambda,\psi-\breve{\rho}(\kappa\hbar)).$$

Definition 2.9 ([39]). For $0 < \vartheta < 1$ and let the left $\hbar$-fractional sum concern to $\left({}_{\varphi_1}^{\mathcal{ABR}}\widehat{\nabla}_\hbar^\vartheta\mathcal{U}\right)(\psi)$ defined on $\mathbb{N}_{\varphi_1,\hbar}$ is stated as follows

$$\left({}_{\varphi_1}^{\mathcal{AB}}\widehat{\nabla}_\hbar^{-\vartheta}\mathcal{U}\right)(\psi) = \frac{1-\vartheta}{\mathcal{H}(\vartheta,\hbar)(1-\vartheta+\vartheta\hbar)}\mathcal{U}(\psi) + \frac{\vartheta}{\mathcal{H}(\vartheta,\hbar)(1-\vartheta+\vartheta\hbar)\Gamma(\vartheta)}\sum_{\kappa=\varphi_1/\hbar+1}^{\psi/\hbar}(\psi-\breve{\rho}(\kappa\hbar))_\hbar^{\overline{\vartheta-1}}\mathcal{U}(\kappa\hbar)\hbar. \tag{2.9}$$

The right $\hbar$-fractional sum is described on ${}_{\varphi_2,\hbar}\mathbb{N}$ by

$$\left({}_\hbar^{\mathcal{AB}}\widehat{\nabla}_{\varphi_2}^{-\vartheta}\mathcal{U}\right)(\psi) = \frac{1-\vartheta}{\mathcal{H}(\vartheta,\hbar)(1-\vartheta+\vartheta\hbar)}\mathcal{U}(\psi) + \frac{\vartheta}{\mathcal{H}(\vartheta,\hbar)(1-\vartheta+\vartheta\hbar)\Gamma(\vartheta)}\sum_{\kappa=\psi/\hbar}^{\varphi_2/\hbar-1}(\kappa\hbar-\breve{\rho}(\psi))_\hbar^{\overline{\vartheta-1}}\mathcal{U}(\kappa\hbar)\hbar. \tag{2.10}$$

3. Some discrete Pólya-Szegö and Chebyshev type inequalities

To continue, we present some new generalizations of Pólya-Szegö type variants via $\mathcal{AB}$-fractional sums within $\hbar$-discrete

Mittag-Leffler function and this is the major key part of this article.

Theorem 3.1. *For $0<\vartheta<1$ and let two positive functions $\mathcal{U}$ and $\mathcal{V}$ defined on $\mathbb{N}_{\varphi_1,\hbar}$. Suppose that there exist four positive functions $\Lambda_1,\Lambda_2,\Theta_1$ and Θ_2 on $\mathbb{N}_{\varphi_1,\hbar}$ such that:*

$$(\mathcal{A}_1)\quad 0<\Lambda_1(\lambda)\le\mathcal{U}(\lambda)\le\Lambda_2(\lambda),\qquad 0<\Theta_1(\lambda)\le\mathcal{V}(\lambda)\le\Theta_2(\lambda),\quad(\lambda\in\mathbb{N}_{\varphi_1,\hbar}).\tag{3.1}$$

Then, for all $\psi\in\{\varphi_1,\varphi_1+\hbar,\varphi_1+2\hbar,\ldots\}$, the following inequality holds:

$$\frac{{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[\Theta_1\Theta_2\mathcal{U}^2\big](\psi)\,{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[\Lambda_1\Lambda_2\mathcal{V}^2\big](\psi)}{\big[{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[(\Theta_1\Lambda_1+\Theta_2\Lambda_2)\mathcal{U}\mathcal{V}\big](\psi)\big]^2}\le\frac{1}{4}.\tag{3.2}$$

Proof. By means of condition $(\mathcal{A}_1)$, for $\lambda\in\mathbb{N}_{\varphi_1,\hbar}$, we have

$$\left(\frac{\Lambda_2(\lambda)}{\Theta_1(\lambda)}-\frac{\mathcal{U}(\lambda)}{\mathcal{V}(\lambda)}\right)\ge 0.\tag{3.3}$$

Analogously, we have

$$\left(\frac{\mathcal{U}(\lambda)}{\mathcal{V}(\lambda)}-\frac{\Lambda_1(\lambda)}{\Theta_2(\lambda)}\right)\ge 0.\tag{3.4}$$

Multiplying (3.3) and (3.4), it follows that

$$\left(\frac{\mathcal{U}(\lambda)}{\mathcal{V}(\lambda)}-\frac{\Lambda_1(\lambda)}{\Theta_2(\lambda)}\right)\left(\frac{\Lambda_2(\lambda)}{\Theta_1(\lambda)}-\frac{\mathcal{U}(\lambda)}{\mathcal{V}(\lambda)}\right)\ge 0.$$

The above inequality can be expressed as

$$\big(\Lambda_1(\lambda)\Theta_1(\lambda)+\Lambda_2(\lambda)\Theta_2(\lambda)\big)\mathcal{U}(\lambda)\mathcal{V}(\lambda)\ge\Theta_1(\lambda)\Theta_2(\lambda)\mathcal{U}^2(\lambda)+\Lambda_1(\lambda)\Lambda_2(\lambda)\mathcal{V}^2(\lambda).\tag{3.5}$$

Taking product both sides of (3.5) by $\frac{1-\vartheta}{\mathcal{H}(\vartheta,\hbar)(1-\vartheta+\vartheta\hbar)}$, we get

$$\frac{(1-\vartheta)\mathcal{U}(\lambda)\mathcal{V}(\lambda)}{\mathcal{H}(\vartheta,\hbar)(1-\vartheta+\vartheta\hbar)}\big(\Lambda_1(\lambda)\Theta_1(\lambda)+\Lambda_2(\lambda)\Theta_2(\lambda)\big)\ge\frac{(1-\vartheta)\Theta_1(\lambda)\Theta_2(\lambda)\mathcal{U}^2(\lambda)}{\mathcal{H}(\vartheta,\hbar)(1-\vartheta+\vartheta\hbar)}+\frac{(1-\vartheta)\Lambda_1(\lambda)\Lambda_2(\lambda)\mathcal{V}^2(\lambda)}{\mathcal{H}(\vartheta,\hbar)(1-\vartheta+\vartheta\hbar)}.\tag{3.6}$$

Moreover, interchanging λ by ξ in (3.5) and conducting product both sides by $\frac{\vartheta(\psi-\breve{\rho}(\xi))_{\hbar}^{\overline{\vartheta-1}}}{\mathcal{H}(\vartheta,\hbar)\Gamma(\vartheta)}$, we have

$$\frac{\vartheta(\psi-\breve{\rho}(\xi))_{\hbar}^{\overline{\vartheta-1}}}{\mathcal{H}(\vartheta,\hbar)\Gamma(\vartheta)}\big(\Lambda_1(\xi)\Theta_1(\xi)+\Lambda_2(\xi)\Theta_2(\xi)\big)\mathcal{U}(\xi)\mathcal{V}(\xi)\ge\frac{\vartheta(\psi-\breve{\rho}(\xi))_{\hbar}^{\overline{\vartheta-1}}}{\mathcal{H}(\vartheta,\hbar)\Gamma(\vartheta)}\Theta_1(\xi)\Theta_2(\xi)\mathcal{U}^2(\xi)+\frac{\vartheta(\psi-\breve{\rho}(\xi))_{\hbar}^{\overline{\vartheta-1}}}{\mathcal{H}(\vartheta,\hbar)\Gamma(\vartheta)}\Lambda_1(\xi)\Lambda_2(\xi)\mathcal{V}^2(\xi).\tag{3.7}$$

Summing both sides for $\xi\in\{\varphi_1,\varphi_1+\hbar,\varphi_1+2\hbar,\ldots\}$, we get

$$\sum_{\kappa=\varphi_1/\hbar+1}^{\psi/\hbar}\frac{\vartheta(\psi-\breve{\rho}(\kappa\hbar))_{\hbar}^{\overline{\vartheta-1}}}{\mathcal{H}(\vartheta,\hbar)\Gamma(\vartheta)}\big(\Lambda_1(\kappa\hbar)\hbar\Theta_1(\kappa\hbar)\hbar+\Lambda_2(\kappa\hbar)\hbar\Theta_2(\kappa\hbar)\hbar\big)\mathcal{U}(\kappa\hbar)\hbar.\mathcal{V}(\kappa\hbar)\hbar$$
$$\ge\sum_{\kappa=\varphi_1/\hbar+1}^{\psi/\hbar}\frac{\vartheta(\psi-\breve{\rho}(\kappa\hbar))_{\hbar}^{\overline{\vartheta-1}}}{\mathcal{H}(\vartheta,\hbar)\Gamma(\vartheta)}\Theta_1(\kappa\hbar)\hbar\Theta_2(\kappa\hbar)\hbar\mathcal{U}^2(\kappa\hbar)\hbar$$
$$+\sum_{\kappa=\varphi_1/\hbar+1}^{\psi/\hbar}\frac{\vartheta(\psi-\breve{\rho}(\kappa\hbar))_{\hbar}^{\overline{\vartheta-1}}}{\mathcal{H}(\vartheta,\hbar)\Gamma(\vartheta)}\Lambda_1(\kappa\hbar)\hbar\Lambda_2(\kappa\hbar)\hbar\mathcal{V}^2(\kappa\hbar)\hbar.\tag{3.8}$$

Adding (3.6) and (3.8), we have

$$\frac{(1-\vartheta)\mathcal{U}(\lambda)\mathcal{V}(\lambda)}{\mathcal{H}(\vartheta,\hbar)(1-\vartheta+\vartheta\hbar)}\big(\Lambda_1(\lambda)\Theta_1(\lambda)+\Lambda_2(\lambda)\Theta_2(\lambda)\big)+\sum_{\kappa=\varphi_1/\hbar+1}^{\psi/\hbar}\frac{\vartheta(\psi-\breve{\rho}(\kappa\hbar))_{\hbar}^{\overline{\vartheta-1}}}{\mathcal{H}(\vartheta,\hbar)\Gamma(\vartheta)}\big(\Lambda_1(\kappa\hbar)\hbar\Theta_1(\kappa\hbar)\hbar+\Lambda_2(\kappa\hbar)\hbar\Theta_2(\kappa\hbar)\hbar\big)\mathcal{U}(\kappa\hbar)\hbar.\mathcal{V}(\kappa\hbar)\hbar$$
$$\ge\frac{(1-\vartheta)\Theta_1(\lambda)\Theta_2(\lambda)\mathcal{U}^2(\lambda)}{\mathcal{H}(\vartheta,\hbar)(1-\vartheta+\vartheta\hbar)}+\sum_{\kappa=\varphi_1/\hbar+1}^{\psi/\hbar}\frac{\vartheta(\psi-\breve{\rho}(\kappa\hbar))_{\hbar}^{\overline{\vartheta-1}}}{\mathcal{H}(\vartheta,\hbar)\Gamma(\vartheta)}\Theta_1(\kappa\hbar)\hbar\Theta_2(\kappa\hbar)\hbar\mathcal{U}^2(\kappa\hbar)\hbar$$
$$+\frac{(1-\vartheta)\Lambda_1(\lambda)\Lambda_2(\lambda)\mathcal{V}^2(\lambda)}{\mathcal{H}(\vartheta,\hbar)(1-\vartheta+\vartheta\hbar)}+\sum_{\kappa=\varphi_1/\hbar+1}^{\psi/\hbar}\frac{\vartheta(\psi-\breve{\rho}(\kappa\hbar))_{\hbar}^{\overline{\vartheta-1}}}{\mathcal{H}(\vartheta,\hbar)\Gamma(\vartheta)}\Lambda_1(\kappa\hbar)\hbar\Lambda_2(\kappa\hbar)\hbar\mathcal{V}^2(\kappa\hbar)\hbar.$$

Consequently, we have

$${}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[\big(\Lambda_1\Theta_1+\Lambda_2\Theta_2\big)\mathcal{U}\mathcal{V}\big](\psi)\ge{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[\Theta_1\Theta_2\mathcal{U}^2\big](\psi)+{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[\Lambda_1\Lambda_2\mathcal{V}^2\big](\psi).$$

Besides, by $\mathcal{AM}-\mathcal{GM}$ inequality, that is, $c_1+c_2\ge 2\sqrt{c_1c_2}$, $c_1,c_2\in\mathbb{R}^+$, we obtain

$${}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[\big(\Lambda_1\Theta_1+\Lambda_2\Theta_2\big)\mathcal{U}\mathcal{V}\big](\psi)\ge 2\sqrt{{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[\Theta_1\Theta_2\mathcal{U}^2\big](\psi)+{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[\Lambda_1\Lambda_2\mathcal{V}^2\big](\psi)}$$

and it follows straightforward the statement (3.1). □

Corollary 3.2. *For $0<\vartheta<1$ and let two positive functions $\mathcal{U}$ and $\mathcal{V}$ defined on $\mathbb{N}_{\varphi_1,\hbar}$ satisfying*

$$(\mathcal{A}_2)\quad 0<r_1\le\mathcal{U}(\lambda)\le\mathcal{R},\quad 0<t_1\le\mathcal{V}(\lambda)\le\mathcal{T},\quad(\lambda\in\mathbb{N}_{\varphi_1,\hbar}).$$

Then, for all $\psi\in\{\varphi_1,\varphi_1+\hbar,\varphi_1+2\hbar,\ldots\}$, the following inequality holds:

$$\frac{{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[\mathcal{U}^2\big](\psi)\,{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[\mathcal{V}^2\big](\psi)}{\big[{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[\mathcal{U}\mathcal{V}\big](\psi)\big]^2}\le\left(\sqrt{\frac{r_1t_1}{\mathcal{R}\mathcal{T}}}+\sqrt{\frac{\mathcal{R}\mathcal{T}}{r_1t_1}}\right)^2.$$

As a special case of Theorem 3.1 with the assumption of $\hbar=1$, we get the following result.

Corollary 3.3. *For $0<\vartheta<1$ and let two positive functions $\mathcal{U}$ and $\mathcal{V}$ defined on $\mathbb{N}_{\varphi_1}$. Suppose that there exist four positive functions $\Lambda_1,\Lambda_2,\Theta_1$ and Θ_2 on $\mathbb{N}_{\varphi_1}$ such that:*

$$(\mathcal{A}_1)\quad 0<\Lambda_1(\lambda)\le\mathcal{U}(\lambda)\le\Lambda_2(\lambda),\qquad 0<\Theta_1(\lambda)\le\mathcal{V}(\lambda)\le\Theta_2(\lambda),\quad(\lambda\in\mathbb{N}_{\varphi_1}).$$

Then, for all $\psi\in\{\varphi_1,\varphi_1+1,\varphi_1+2,\ldots\}$, the following inequality holds:

$$\frac{{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}\big[\Theta_1\Theta_2\mathcal{U}^2\big](\psi)\,{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}\big[\Lambda_1\Lambda_2\mathcal{V}^2\big](\psi)}{\big[{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}\big[(\Theta_1\Lambda_1+\Theta_2\Lambda_2)\mathcal{U}\mathcal{V}\big](\psi)\big]^2}\le\frac{1}{4}.$$

Theorem 3.4. *For $0<\vartheta_1,\vartheta_2<1$ and let two positive functions $\mathcal{U}$ and $\mathcal{V}$ defined on $\mathbb{N}_{\varphi_1,\hbar}$. Suppose that there exist four positive functions $\Lambda_1,\Lambda_2,\Theta_1$ and Θ_2 on $\mathbb{N}_{\varphi_1,\hbar}$ satisfying $(\mathcal{A}_1)$ on $\mathbb{N}_{\varphi_1,\hbar}$. Then, for all $\psi\in\{\varphi_1,\varphi_1+\hbar,\varphi_1+2\hbar,\ldots\}$, the following inequality holds:*

$$\frac{{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}_{\hbar}\big[\mathcal{U}^2(\psi)\big]\,{}^{\mathcal{AB}}_{\varsigma_1}\widehat{\nabla}^{-\vartheta_2}_{\hbar}\big[\Theta_1(\psi)\Theta_2(\psi)\big]+{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}_{\hbar}\big[\Lambda_1(\psi)\Lambda_2(\psi)\big]\,{}^{\mathcal{AB}}_{\varsigma_1}\widehat{\nabla}^{-\vartheta_2}_{\hbar}\big[\mathcal{V}^2(\psi)\big]}{{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}_{\hbar}\big[\Lambda_1(\psi)\mathcal{U}(\psi)\big]\,{}^{\mathcal{AB}}_{\varsigma_1}\widehat{\nabla}^{-\vartheta_2}_{\hbar}\big[\Theta_1(\psi)\mathcal{V}(\psi)\big]+{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}_{\hbar}\big[\Lambda_2(\psi)\mathcal{U}(\psi)\big]\,{}^{\mathcal{AB}}_{\varsigma_1}\widehat{\nabla}^{-\vartheta_2}_{\hbar}\big[\Theta_2(\psi)\mathcal{V}(\psi)\big]}\le\frac{1}{4}.\tag{3.9}$$

Proof. By means of assumption $(\mathcal{A}_1)$, it is clear that

$$\left(\frac{\Lambda_2(\lambda)}{\Theta_1(\omega)}-\frac{\mathcal{U}(\lambda)}{\mathcal{V}(\omega)}\right)\ge 0$$

and

$$\left(\frac{\mathcal{U}(\lambda)}{\mathcal{V}(\omega)}-\frac{\Lambda_1(\lambda)}{\Theta_2(\omega)}\right)\geq 0.$$

Implies that

$$\left(\frac{\Lambda_1(\lambda)}{\Theta_2(\omega)}-\frac{\Lambda_2(\lambda)}{\Theta_1(\omega)}\right)\frac{\mathcal{U}(\lambda)}{\mathcal{V}(\omega)}\geq\frac{\mathcal{U}^2(\lambda)}{\mathcal{V}^2(\omega)}+\frac{\Lambda_1(\lambda)\Lambda_2(\lambda)}{\Theta_1(\omega)\Theta_2(\omega)}. \tag{3.10}$$

Besides, multiplying both sides of (3.10) by $\Theta_1(\omega)\Theta_2(\omega)\mathcal{V}^2(\omega)$, we get

$$\begin{aligned}&\Lambda_1(\lambda)\mathcal{U}(\lambda)\Theta_1(\omega)\mathcal{V}(\omega)+\Lambda_2(\lambda)\mathcal{U}(\lambda)\Theta_2(\omega)\mathcal{V}(\omega)\\&\geq\Theta_1(\omega)\Theta_2(\omega)\mathcal{U}^2(\lambda)+\Lambda_1(\lambda)\Lambda_2(\lambda)\mathcal{V}^2(\omega).\end{aligned} \tag{3.11}$$

Taking product both sides of (3.11) by $\frac{1-\vartheta_1}{\mathcal{H}(\vartheta_1,\hbar)(1-\vartheta_1+\vartheta_1\hbar)}$, we get

$$\begin{aligned}&\frac{1-\vartheta_1}{\mathcal{H}(\vartheta_1,\hbar)(1-\vartheta_1+\vartheta_1\hbar)}\Lambda_1(\lambda)\mathcal{U}(\lambda)\Theta_1(\omega)\mathcal{V}(\omega)\\&\quad+\frac{1-\vartheta_1}{\mathcal{H}(\vartheta_1,\hbar)(1-\vartheta_1+\vartheta_1\hbar)}\Lambda_2(\lambda)\mathcal{U}(\lambda)\Theta_2(\omega)\mathcal{V}(\omega)\\&\geq\frac{1-\vartheta_1}{\mathcal{H}(\vartheta_1,\hbar)(1-\vartheta_1+\vartheta_1\hbar)}\Theta_1(\omega)\Theta_2(\omega)\mathcal{U}^2(\lambda)\\&\quad+\frac{1-\vartheta_1}{\mathcal{H}(\vartheta_1,\hbar)(1-\vartheta_1+\vartheta_1\hbar)}\Lambda_1(\lambda)\Lambda_2(\lambda)\mathcal{V}^2(\omega).\end{aligned} \tag{3.12}$$

Moreover, interchanging λ by ξ_1 in (3.12) and conducting product both sides by $\frac{\vartheta_1(\psi-\breve{\rho}(\xi_1))_\hbar^{\overline{\vartheta_1-1}}}{\mathcal{H}(\vartheta_1,\hbar)\Gamma(\vartheta_1)}$, we have

$$\begin{aligned}&\frac{\vartheta_1(\psi-\breve{\rho}(\xi_1))_\hbar^{\overline{\vartheta_1-1}}}{\mathcal{H}(\vartheta_1,\hbar)\Gamma(\vartheta_1)}\Lambda_1(\xi_1)\mathcal{U}(\xi_1)\Theta_1(\omega)\mathcal{V}(\omega)\\&\quad+\frac{1-\vartheta}{\mathcal{H}(\vartheta,\hbar)(1-\vartheta+\vartheta\hbar)}\Lambda_2(\xi_1)\mathcal{U}(\xi_1)\Theta_2(\omega)\mathcal{V}(\omega)\\&\geq\frac{\vartheta_1(\psi-\breve{\rho}(\xi_1))_\hbar^{\overline{\vartheta_1-1}}}{\mathcal{H}(\vartheta_1,\hbar)\Gamma(\vartheta_1)}\Theta_1(\omega)\Theta_2(\omega)\mathcal{U}^2(\xi_1)\\&\quad+\frac{\vartheta_1(\psi-\breve{\rho}(\xi_1))_\hbar^{\overline{\vartheta_1-1}}}{\mathcal{H}(\vartheta_1,\hbar)\Gamma(\vartheta_1)}\Lambda_1(\xi_1)\Lambda_2(\xi_1)\mathcal{V}^2(\omega).\end{aligned} \tag{3.13}$$

Summing both sides for $\xi_1\in\{\varphi_1,\varphi_1+\hbar,\varphi_1+2\hbar,\ldots\}$, we get

$$\begin{aligned}&\sum_{\kappa=\varphi_1/\hbar+1}^{\psi/\hbar}\frac{\vartheta_1(\psi-\breve{\rho}(\kappa\hbar))_\hbar^{\overline{\vartheta_1-1}}}{\mathcal{H}(\vartheta_1,\hbar)\Gamma(\vartheta_1)}\Lambda_1(\kappa\hbar)\hbar\mathcal{U}(\kappa\hbar)\hbar\Theta_1(\omega)\mathcal{V}(\omega)\\&\quad+\sum_{\kappa=\varphi_1/\hbar+1}^{\psi/\hbar}\frac{\vartheta_1(\psi-\breve{\rho}(\kappa\hbar))_\hbar^{\overline{\vartheta_1-1}}}{\mathcal{H}(\vartheta_1,\hbar)\Gamma(\vartheta_1)}\Lambda_2(\kappa\hbar)\hbar\mathcal{U}(\kappa\hbar)\hbar\Theta_2(\omega)\mathcal{V}(\omega)\\&\geq\sum_{\kappa=\varphi_1/\hbar+1}^{\psi/\hbar}\frac{\vartheta_1(\psi-\breve{\rho}(\kappa\hbar))_\hbar^{\overline{\vartheta_1-1}}}{\mathcal{H}(\vartheta_1,\hbar)\Gamma(\vartheta_1)}\Theta_1(\omega)\Theta_2(\omega)\mathcal{U}^2(\kappa\hbar)\hbar\\&\quad+\sum_{\kappa=\varphi_1/\hbar+1}^{\psi/\hbar}\frac{\vartheta_1(\psi-\breve{\rho}(\kappa\hbar))_\hbar^{\overline{\vartheta_1-1}}}{\mathcal{H}(\vartheta_1,\hbar)\Gamma(\vartheta_1)}\Lambda_1(\kappa\hbar)\hbar\Lambda_2(\kappa\hbar)\hbar\mathcal{V}^2(\omega)\end{aligned} \tag{3.14}$$

Adding (3.12) and (3.14), we obtain

$$\begin{aligned}&\frac{1-\vartheta_1}{\mathcal{H}(\vartheta_1,\hbar)(1-\vartheta_1+\vartheta_1\hbar)}\Lambda_1(\lambda)\mathcal{U}(\lambda)\Theta_1(\omega)\mathcal{V}(\omega)\\&\quad+\sum_{\kappa=\varphi_1/\hbar+1}^{\psi/\hbar}\frac{\vartheta_1(\psi-\breve{\rho}(\kappa\hbar))_\hbar^{\overline{\vartheta_1-1}}}{\mathcal{H}(\vartheta_1,\hbar)\Gamma(\vartheta_1)}\Lambda_1(\kappa\hbar)\hbar\mathcal{U}(\kappa\hbar)\hbar\Theta_1(\omega)\mathcal{V}(\omega)\\&\quad+\frac{1-\vartheta}{\mathcal{H}(\vartheta,\hbar)(1-\vartheta+\vartheta\hbar)}\Lambda_2(\lambda)\mathcal{U}(\lambda)\Theta_2(\omega)\mathcal{V}(\omega)\\&\quad+\sum_{\kappa=\varphi_1/\hbar+1}^{\psi/\hbar}\frac{\vartheta_1(\psi-\breve{\rho}(\kappa\hbar))_\hbar^{\overline{\vartheta_1-1}}}{\mathcal{H}(\vartheta_1,\hbar)\Gamma(\vartheta_1)}\Lambda_2(\kappa\hbar)\hbar\mathcal{U}(\kappa\hbar)\hbar\Theta_2(\omega)\mathcal{V}(\omega)\\&\geq\frac{1-\vartheta_1}{\mathcal{H}(\vartheta_1,\hbar)(1-\vartheta_1+\vartheta_1\hbar)}\Theta_1(\omega)\Theta_2(\omega)\mathcal{U}^2(\lambda)\\&\quad+\sum_{\kappa=\varphi_1/\hbar+1}^{\psi/\hbar}\frac{\vartheta_1(\psi-\breve{\rho}(\kappa\hbar))_\hbar^{\overline{\vartheta_1-1}}}{\mathcal{H}(\vartheta_1,\hbar)\Gamma(\vartheta_1)}\Theta_1(\omega)\Theta_2(\omega)\mathcal{U}^2(\kappa\hbar)\hbar\\&\quad+\frac{\vartheta_1(\psi-\breve{\rho}(\xi_1))_\hbar^{\overline{\vartheta_1-1}}}{\mathcal{H}(\vartheta_1,\hbar)\Gamma(\vartheta_1)}\Lambda_1(\lambda)\Lambda_2(\lambda)\mathcal{V}^2(\omega)\\&\quad+\sum_{\kappa=\varphi_1/\hbar+1}^{\psi/\hbar}\frac{\vartheta_1(\psi-\breve{\rho}(\kappa\hbar))_\hbar^{\overline{\vartheta_1-1}}}{\mathcal{H}(\vartheta_1,\hbar)\Gamma(\vartheta_1)}\Lambda_1(\kappa\hbar)\hbar\Lambda_2(\kappa\hbar)\hbar\mathcal{V}^2(\omega).\end{aligned}$$

Consequently, we have

$$\begin{aligned}&{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_\hbar^{-\vartheta_1}\big[\Lambda_1(\psi)\mathcal{U}(\psi)\big]\Theta_1(\omega)\mathcal{V}(\omega)\\&\quad+{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_\hbar^{-\vartheta_1}\big[\Lambda_2(\psi)\mathcal{U}(\psi)\big]\Theta_2(\omega)\mathcal{V}(\omega)\\&\geq{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_\hbar^{-\vartheta_1}\big[\mathcal{U}^2(\psi)\big]\Theta_1(\omega)\Theta_2(\omega)\\&\quad+{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_\hbar^{-\vartheta_1}\big[\Lambda_1(\psi)\Lambda_2(\psi)\big]\mathcal{V}^2(\omega).\end{aligned} \tag{3.15}$$

Taking product both sides of (3.15) by $\frac{1-\vartheta_2}{\mathcal{H}(\vartheta_2,\hbar)(1-\vartheta_2+\vartheta_2\hbar)}$, we get

$$\begin{aligned}&\frac{(1-\vartheta_2)\,{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_\hbar^{-\vartheta_1}\big[\Lambda_1(\psi)\mathcal{U}(\psi)\big]}{\mathcal{H}(\vartheta_2,\hbar)(1-\vartheta_2+\vartheta_2\hbar)}\Theta_1(\omega)\mathcal{V}(\omega)\\&\quad+\frac{(1-\vartheta_2)\,{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_\hbar^{-\vartheta_1}\big[\Lambda_2(\psi)\mathcal{U}(\psi)\big]}{\mathcal{H}(\vartheta_2,\hbar)(1-\vartheta_2+\vartheta_2\hbar)}\Theta_2(\omega)\mathcal{V}(\omega)\\&\geq\frac{(1-\vartheta_2)\,{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_\hbar^{-\vartheta_1}\big[\mathcal{U}^2(\psi)\big]}{\mathcal{H}(\vartheta_2,\hbar)(1-\vartheta_2+\vartheta_2\hbar)}\Theta_1(\omega)\Theta_2(\omega)\\&\quad+\frac{(1-\vartheta_2)\,{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_\hbar^{-\vartheta_1}\big[\Lambda_1(\psi)\Lambda_2(\psi)\big]}{\mathcal{H}(\vartheta_2,\hbar)(1-\vartheta_2+\vartheta_2\hbar)}\mathcal{V}^2(\omega).\end{aligned} \tag{3.16}$$

Again, interchanging ω by ξ_2 in (3.15) and conducting product both sides by $\frac{\vartheta_2(\psi-\breve{\rho}(\xi_2))_\hbar^{\overline{\vartheta_2-1}}}{\mathcal{H}(\vartheta_2,\hbar)\Gamma(\vartheta_2)}$. Also, summing both sides for $\xi_2\in\{\varsigma_1,\varsigma_1+\hbar,\varsigma_1+2\hbar,\ldots\}$, we get

$$\begin{aligned}&\sum_{\kappa=\varsigma_1/\hbar+1}^{\psi/\hbar}{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_\hbar^{-\vartheta_1}\big[\Lambda_1(\psi)\mathcal{U}(\psi)\big]\Theta_1(\hbar\kappa)\kappa\mathcal{V}(\hbar\kappa)\kappa\\&\quad+\sum_{\kappa=\varsigma_1/\hbar+1}^{\psi/\hbar}{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_\hbar^{-\vartheta_1}\big[\Lambda_2(\psi)\mathcal{U}(\psi)\big]\Theta_2(\hbar\kappa)\kappa\mathcal{V}(\hbar\kappa)\kappa\\&\geq\sum_{\kappa=\varsigma_1/\hbar+1}^{\psi/\hbar}{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_\hbar^{-\vartheta_1}\big[\mathcal{U}^2(\psi)\big]\Theta_1(\hbar\kappa)\kappa\Theta_2(\hbar\kappa)\kappa\\&\quad+\sum_{\kappa=\varsigma_1/\hbar+1}^{\psi/\hbar}{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_\hbar^{-\vartheta_1}\big[\Lambda_1(\psi)\Lambda_2(\psi)\big]\mathcal{V}^2(\hbar\kappa)\kappa.\end{aligned} \tag{3.17}$$

Adding (3.16) and (3.17), we have

$$\begin{aligned}&\frac{(1-\vartheta_2)\,{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_\hbar^{-\vartheta_1}\big[\Lambda_1(\psi)\mathcal{U}(\psi)\big]}{\mathcal{H}(\vartheta_2,\hbar)(1-\vartheta_2+\vartheta_2\hbar)}\Theta_1(\omega)\mathcal{V}(\omega)\\&\quad+\sum_{\kappa=\varsigma_1/\hbar+1}^{\psi/\hbar}{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_\hbar^{-\vartheta_1}\big[\Lambda_1(\psi)\mathcal{U}(\psi)\big]\Theta_1(\hbar\kappa)\kappa\mathcal{V}(\hbar\kappa)\kappa\\&\quad+\frac{(1-\vartheta_2)\,{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_\hbar^{-\vartheta_1}\big[\Lambda_2(\psi)\mathcal{U}(\psi)\big]}{\mathcal{H}(\vartheta_2,\hbar)(1-\vartheta_2+\vartheta_2\hbar)}\Theta_2(\omega)\mathcal{V}(\omega)\\&\quad+\sum_{\kappa=\varsigma_1/\hbar+1}^{\psi/\hbar}{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_\hbar^{-\vartheta_1}\big[\Lambda_2(\psi)\mathcal{U}(\psi)\big]\Theta_2(\hbar\kappa)\kappa\mathcal{V}(\hbar\kappa)\kappa\\&\geq\frac{(1-\vartheta_2)\,{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_\hbar^{-\vartheta_1}\big[\mathcal{U}^2(\psi)\big]}{\mathcal{H}(\vartheta_2,\hbar)(1-\vartheta_2+\vartheta_2\hbar)}\Theta_1(\omega)\Theta_2(\omega)\end{aligned}$$

$$+\sum_{\kappa=\varsigma_1/\hbar+1}^{\psi/\hbar} {}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_{\hbar}^{-\vartheta_1}\big[\mathcal{U}^2(\psi)\big]\Theta_1(\hbar\kappa)\kappa\Theta_2(\hbar\kappa)\kappa$$
$$+\frac{(1-\vartheta_2)\,{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_{\hbar}^{-\vartheta_1}\big[\Lambda_1(\psi)\Lambda_2(\psi)\big]}{\mathcal{H}(\vartheta_2,\hbar)(1-\vartheta_2+\vartheta_2\hbar)}\mathcal{V}^2(\omega)$$
$$+\sum_{\kappa=\varsigma_1/\hbar+1}^{\psi/\hbar} {}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_{\hbar}^{-\vartheta_1}\big[\Lambda_1(\psi)\Lambda_2(\psi)\big]\mathcal{V}^2(\hbar\kappa)\kappa. \tag{3.18}$$

Therefore, we have

$$\begin{aligned}&{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_{\hbar}^{-\vartheta_1}\big[\Lambda_1(\psi)\mathcal{U}(\psi)\big]\,{}^{\mathcal{AB}}_{\varsigma_1}\widehat{\nabla}_{\hbar}^{-\vartheta_2}\big[\Theta_1(\psi)\mathcal{V}(\psi)\big]\\&\quad+{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_{\hbar}^{-\vartheta_1}\big[\Lambda_2(\psi)\mathcal{U}(\psi)\big]\,{}^{\mathcal{AB}}_{\varsigma_1}\widehat{\nabla}_{\hbar}^{-\vartheta_2}\big[\Theta_2(\psi)\mathcal{V}(\psi)\big]\\&\geq{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_{\hbar}^{-\vartheta_1}\big[\mathcal{U}^2(\psi)\big]\,{}^{\mathcal{AB}}_{\varsigma_1}\widehat{\nabla}_{\hbar}^{-\vartheta_2}\big[\Theta_1(\psi)\Theta_2(\psi)\big]\\&\quad+{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_{\hbar}^{-\vartheta_1}\big[\Lambda_1(\psi)\Lambda_2(\psi)\big]\,{}^{\mathcal{AB}}_{\varsigma_1}\widehat{\nabla}_{\hbar}^{-\vartheta_2}\big[\mathcal{V}^2(\psi)\big].\end{aligned} \tag{3.19}$$

Consequently, applying the $\mathcal{AM}-\mathcal{GM}$ inequality to the last inequality, we come to (3.9). □

With the assumption of $\hbar=1$, we get the following result as a particular case of Theorem 3.4.

Corollary 3.5. *For $0<\vartheta_1,\vartheta_2<1$ and let two positive functions $\mathcal{U}$ and $\mathcal{V}$ defined on $\mathbb{N}_{\varphi_1}$. Suppose that there exist four positive functions $\Lambda_1,\Lambda_2,\Theta_1$ and Θ_2 on $\mathbb{N}_{\varphi_1}$ satisfying $(\mathcal{A}_1)$ on $\mathbb{N}_{\varphi_1}$. Then, for all $\psi\in\{\varphi_1,\varphi_1+1,\varphi_1+2,\ldots\}$, the following inequality holds:*

$$\frac{{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}\big[\mathcal{U}^2(\psi)\big]\,{}^{\mathcal{AB}}_{\varsigma_1}\widehat{\nabla}^{-\vartheta_2}\big[\Theta_1(\psi)\Theta_2(\psi)\big]+{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}\big[\Lambda_1(\psi)\Lambda_2(\psi)\big]\,{}^{\mathcal{AB}}_{\varsigma_1}\widehat{\nabla}^{-\vartheta_2}\big[\mathcal{V}^2(\psi)\big]}{{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}\big[\Lambda_1(\psi)\mathcal{U}(\psi)\big]\,{}^{\mathcal{AB}}_{\varsigma_1}\widehat{\nabla}^{-\vartheta_2}\big[\Theta_1(\psi)\mathcal{V}(\psi)\big]+{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}\big[\Lambda_2(\psi)\mathcal{U}(\psi)\big]\,{}^{\mathcal{AB}}_{\varsigma_1}\widehat{\nabla}^{-\vartheta_2}\big[\Theta_2(\psi)\mathcal{V}(\psi)\big]}\leq\frac{1}{4}.$$

Theorem 3.6. *Suppose that all assumptions of Theorem 3.4 are satisfied. Then, for all $\psi\in\{\varphi_1,\varphi_1+\hbar,\varphi_1+2\hbar,\ldots\}$, the following inequality holds:*

$$\begin{aligned}&{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_{\hbar}^{-\vartheta_1}\big[\mathcal{U}^2(\psi)\big]\,{}^{\mathcal{AB}}_{\varsigma_1}\widehat{\nabla}_{\hbar}^{-\vartheta_2}\big[\mathcal{V}^2(\psi)\big]\leq{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_{\hbar}^{-\vartheta_1}\Big[\frac{\Lambda_2\mathcal{UV}}{\Theta_1}(\psi)\Big]\\&{}^{\mathcal{AB}}_{\varsigma_1}\widehat{\nabla}_{\hbar}^{-\vartheta_2}\Big[\frac{\Theta_2\mathcal{UV}}{\Lambda_1}(\psi)\Big].\end{aligned} \tag{3.20}$$

Proof. By means of assumption $(\mathcal{A}_1)$, it is clear that

$$\mathcal{U}^2(\lambda)\leq\frac{\Lambda_2(\lambda)}{\Theta_1(\lambda)}\mathcal{U}(\lambda)\mathcal{V}(\lambda). \tag{3.21}$$

Taking product both sides of (3.11) by $\frac{1-\vartheta_1}{\mathcal{H}(\vartheta_1,\hbar)(1-\vartheta_1+\vartheta_1\hbar)}$, we get

$$\begin{aligned}&\frac{1-\vartheta_1}{\mathcal{H}(\vartheta_1,\hbar)(1-\vartheta_1+\vartheta_1\hbar)}\mathcal{U}^2(\lambda)\leq\\&\frac{1-\vartheta_1}{\mathcal{H}(\vartheta_1,\hbar)(1-\vartheta_1+\vartheta_1\hbar)}\frac{\Lambda_2(\lambda)}{\Theta_1(\lambda)}\mathcal{U}(\lambda)\mathcal{V}(\lambda).\end{aligned} \tag{3.22}$$

Moreover, interchanging λ by ξ_1 in (3.21) and conducting product both sides by $\frac{\vartheta_1(\psi-\breve{\rho}(\xi_1))_{\hbar}^{\overline{\vartheta_1-1}}}{\mathcal{H}(\vartheta_1,\hbar)\Gamma(\vartheta_1)}$. Also, summing both sides for $\xi_1\in\{\varphi_1,\varphi_1+\hbar,\varphi_1+2\hbar,\ldots\}$, we get

$$\begin{aligned}&\sum_{\kappa=\varphi_1/\hbar+1}^{\psi/\hbar}\frac{\vartheta_1(\psi-\breve{\rho}(\kappa\hbar))_{\hbar}^{\overline{\vartheta_1-1}}}{\mathcal{H}(\vartheta_1,\hbar)\Gamma(\vartheta_1)}\mathcal{U}^2(\kappa\hbar)\hbar\leq\\&\sum_{\kappa=\varphi_1/\hbar+1}^{\psi/\hbar}\frac{\vartheta_1(\psi-\breve{\rho}(\kappa\hbar))_{\hbar}^{\overline{\vartheta_1-1}}}{\mathcal{H}(\vartheta_1,\hbar)\Gamma(\vartheta_1)}\frac{\Lambda_2(\kappa\hbar)\hbar}{\Theta_1(\kappa\hbar)\hbar}\mathcal{U}(\kappa\hbar)\hbar\mathcal{V}(\kappa\hbar)\hbar.\end{aligned} \tag{3.23}$$

Adding (3.22) and (3.23), we have

$$\begin{aligned}&\frac{1-\vartheta_1}{\mathcal{H}(\vartheta_1,\hbar)(1-\vartheta_1+\vartheta_1\hbar)}\mathcal{U}^2(\lambda)+\sum_{\kappa=\varphi_1/\hbar+1}^{\psi/\hbar}\frac{\vartheta_1(\psi-\breve{\rho}(\kappa\hbar))_{\hbar}^{\overline{\vartheta_1-1}}}{\mathcal{H}(\vartheta_1,\hbar)\Gamma(\vartheta_1)}\mathcal{U}^2(\kappa\hbar)\hbar\\&\leq\frac{1-\vartheta_1}{\mathcal{H}(\vartheta_1,\hbar)(1-\vartheta_1+\vartheta_1\hbar)}\frac{\Lambda_2(\lambda)}{\Theta_1(\lambda)}\mathcal{U}(\lambda)\mathcal{V}(\lambda)+\sum_{\kappa=\varphi_1/\hbar+1}^{\psi/\hbar}\\&\frac{\vartheta_1(\psi-\breve{\rho}(\kappa\hbar))_{\hbar}^{\overline{\vartheta_1-1}}}{\mathcal{H}(\vartheta_1,\hbar)\Gamma(\vartheta_1)}\frac{\Lambda_2(\kappa\hbar)\hbar}{\Theta_1(\kappa\hbar)\hbar}\mathcal{U}(\kappa\hbar)\hbar\mathcal{V}(\kappa\hbar)\hbar.\end{aligned}$$

In view of Definition 2.9, we have

$${}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_{\hbar}^{-\vartheta_1}\big[\mathcal{U}^2(\psi)\big]\leq{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_{\hbar}^{-\vartheta_1}\Big[\frac{\Lambda_2\mathcal{UV}}{\Theta_1}(\psi)\Big]. \tag{3.24}$$

Analogously, we have

$${}^{\mathcal{AB}}_{\varsigma_1}\widehat{\nabla}_{\hbar}^{-\vartheta_2}\big[\mathcal{V}^2(\psi)\big]\leq{}^{\mathcal{AB}}_{\varsigma_1}\widehat{\nabla}_{\hbar}^{-\vartheta_2}\Big[\frac{\Theta_2\mathcal{UV}}{\Lambda_1}(\psi)\Big]. \tag{3.25}$$

Multiplying (3.24) and (3.25), we achieve the intended inequality in (3.20). □

Corollary 3.7. *For $0<\vartheta_1,\vartheta_2<1$ and let two positive functions $\mathcal{U}$ and $\mathcal{V}$ defined on $\mathbb{N}_{\varphi_1,\hbar}$ satisfying $\mathcal{A}_2$. Then, for all $\psi\in\{\varphi_1,\varphi_1+\hbar,\varphi_1+2\hbar,\ldots\}$, the following inequality holds:*

$$\frac{{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_{\hbar}^{-\vartheta_1}\big[\mathcal{U}^2(\psi)\big]\,{}^{\mathcal{AB}}_{\varsigma_1}\widehat{\nabla}_{\hbar}^{-\vartheta_2}\big[\mathcal{V}^2(\psi)\big]}{{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_{\hbar}^{-\vartheta_1}\big[\mathcal{UV}(\psi)\big]\,{}^{\mathcal{AB}}_{\varsigma_1}\widehat{\nabla}_{\hbar}^{-\vartheta_2}\big[\mathcal{UV}(\psi)\big]}\leq\frac{\mathcal{RT}}{r_1t_1}.$$

In what follow, some discrete Chebyshev type variants concerning the $\mathcal{AB}$-fractional sum defined in (2.9) are presented as follows.

Theorem 3.8. *For $0<\vartheta<1$ and let two positive functions $\mathcal{U}$ and $\mathcal{V}$ defined on $\mathbb{N}_{\varphi_1,\hbar}$. Suppose that there exist four positive functions $\Lambda_1,\Lambda_2,\Theta_1$ and Θ_2 on $\mathbb{N}_{\varphi_1,\hbar}$. Then, for all $\psi\in\{\varphi_1,\varphi_1+\hbar,\varphi_1+2\hbar,\ldots\}$, the following inequality holds:*

$${}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_{\hbar}^{-\vartheta}\big[\mathcal{U}(\psi)\mathcal{V}(\psi)\big]\geq\frac{{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_{\hbar}^{-\vartheta}\big[\mathcal{U}(\psi)\big]\,{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_{\hbar}^{-\vartheta}\big[\mathcal{V}(\psi)\big]}{{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}_{\hbar}^{-\vartheta}\big[I(\psi)\big]}, \tag{3.26}$$

where I is the identity mapping.

Proof. It follows from the synchronism of the functions $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}_{\varphi_1,\hbar}$ that

$$\mathcal{U}(\lambda)\mathcal{V}(\lambda)+\mathcal{U}(\omega)\mathcal{V}(\omega)\geq\mathcal{U}(\lambda)\mathcal{V}(\omega)+\mathcal{U}(\omega)\mathcal{V}(\lambda). \tag{3.27}$$

Taking product both sides of (3.27) by $\frac{1-\vartheta}{\mathcal{H}(\vartheta,\hbar)(1-\vartheta+\vartheta\hbar)}$, we get

$$\begin{aligned}&\frac{1-\vartheta}{\mathcal{H}(\vartheta,\hbar)(1-\vartheta+\vartheta\hbar)}\mathcal{U}(\lambda)\mathcal{V}(\lambda)\\&\quad+\frac{1-\vartheta}{\mathcal{H}(\vartheta,\hbar)(1-\vartheta+\vartheta\hbar)}\mathcal{U}(\omega)\mathcal{V}(\omega)\\&\geq\frac{1-\vartheta}{\mathcal{H}(\vartheta,\hbar)(1-\vartheta+\vartheta\hbar)}\mathcal{U}(\lambda)\mathcal{V}(\omega)\\&\quad+\frac{1-\vartheta}{\mathcal{H}(\vartheta,\hbar)(1-\vartheta+\vartheta\hbar)}\mathcal{U}(\omega)\mathcal{V}(\lambda).\end{aligned} \tag{3.28}$$

Moreover, interchanging λ by ξ in (3.27) and conducting product both sides by $\frac{\vartheta(\psi-\breve{\rho}(\xi))_{\hbar}^{\overline{\vartheta-1}}}{\mathcal{H}(\vartheta,\hbar)\Gamma(\vartheta)}$. Also, summing both sides for $\xi\in\{\varphi_1,\varphi_1+\hbar,\varphi_1+2\hbar,\ldots\}$, we get

$$\sum_{\kappa=\varphi_1/\hbar+1}^{\psi/\hbar}\frac{\vartheta(\psi-\breve{\rho}(\kappa\hbar))_{\hbar}^{\overline{\vartheta-1}}}{\mathcal{H}(\vartheta,\hbar)\Gamma(\vartheta)}\mathcal{U}(\kappa\hbar)\hbar\mathcal{V}(\kappa\hbar)\hbar$$

$$
\begin{aligned}
&+\frac{\vartheta(\psi-\check{\rho}(\kappa\hbar))_{\hbar}^{\overline{\vartheta-1}}}{\mathcal{H}(\vartheta,\hbar)\Gamma(\vartheta)}\mathcal{U}(\omega)\mathcal{V}(\omega)\\
\geq&\frac{\vartheta(\psi-\check{\rho}(\kappa\hbar))_{\hbar}^{\overline{\vartheta-1}}}{\mathcal{H}(\vartheta,\hbar)\Gamma(\vartheta)}\mathcal{U}(\kappa\hbar)\hbar\mathcal{V}(\omega)\\
&+\frac{\vartheta(\psi-\check{\rho}(\kappa\hbar))_{\hbar}^{\overline{\vartheta-1}}}{\mathcal{H}(\vartheta,\hbar)\Gamma(\vartheta)}\mathcal{U}(\omega)\mathcal{V}(\kappa\hbar)\hbar. \qquad (3.29)
\end{aligned}
$$

Adding (3.28) and (3.29), we have

$$
\begin{aligned}
&\frac{1-\vartheta}{\mathcal{H}(\vartheta,\hbar)(1-\vartheta+\vartheta\hbar)}\mathcal{U}(\lambda)\mathcal{V}(\lambda)\\
&+\sum_{\kappa=\varphi_1/\hbar+1}^{\psi/\hbar}\frac{\vartheta(\psi-\check{\rho}(\kappa\hbar))_{\hbar}^{\overline{\vartheta-1}}}{\mathcal{H}(\vartheta,\hbar)\Gamma(\vartheta)}\mathcal{U}(\kappa\hbar)\hbar\mathcal{V}(\kappa\hbar)\hbar\\
&+\frac{1-\vartheta}{\mathcal{H}(\vartheta,\hbar)(1-\vartheta+\vartheta\hbar)}\mathcal{U}(\omega)\mathcal{V}(\omega)\\
&+\frac{\vartheta(\psi-\check{\rho}(\kappa\hbar))_{\hbar}^{\overline{\vartheta-1}}}{\mathcal{H}(\vartheta,\hbar)\Gamma(\vartheta)}\mathcal{U}(\omega)\mathcal{V}(\omega)\\
\geq&\frac{1-\vartheta}{\mathcal{H}(\vartheta,\hbar)(1-\vartheta+\vartheta\hbar)}\mathcal{U}(\lambda)\mathcal{V}(\omega)\\
&+\frac{\vartheta(\psi-\check{\rho}(\kappa\hbar))_{\hbar}^{\overline{\vartheta-1}}}{\mathcal{H}(\vartheta,\hbar)\Gamma(\vartheta)}\mathcal{U}(\kappa\hbar)\hbar\mathcal{V}(\omega)\\
&+\frac{1-\vartheta}{\mathcal{H}(\vartheta,\hbar)(1-\vartheta+\vartheta\hbar)}\mathcal{U}(\omega)\mathcal{V}(\lambda)\\
&+\frac{\vartheta(\psi-\check{\rho}(\kappa\hbar))_{\hbar}^{\overline{\vartheta-1}}}{\mathcal{H}(\vartheta,\hbar)\Gamma(\vartheta)}\mathcal{U}(\omega)\mathcal{V}(\kappa\hbar)\hbar. \qquad (3.30)
\end{aligned}
$$

In view of Definition 2.9, yields

$$
\begin{aligned}
&{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[\mathcal{U}(\psi)\mathcal{V}(\psi)\big]+{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[I(\psi)\big]\mathcal{U}(\omega)\mathcal{V}(\omega)\\
\geq&{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[\mathcal{U}(\psi)\big]\mathcal{V}(\omega)+{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[\mathcal{V}(\psi)\big]\mathcal{U}(\omega). \qquad (3.31)
\end{aligned}
$$

Again, taking product both sides of (3.31) by $\frac{1-\vartheta}{\mathcal{H}(\vartheta,\hbar)(1-\vartheta+\vartheta\hbar)}$, we get

$$
\begin{aligned}
&\frac{(1-\vartheta)\,{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[\mathcal{U}(\psi)\mathcal{V}(\psi)\big]}{\mathcal{H}(\vartheta,\hbar)(1-\vartheta+\vartheta\hbar)}+\frac{(1-\vartheta)\,{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[I(\psi)\big]}{\mathcal{H}(\vartheta,\hbar)(1-\vartheta+\vartheta\hbar)}\mathcal{U}(\omega)\mathcal{V}(\omega)\\
\geq&\frac{(1-\vartheta)\,{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[\mathcal{U}(\psi)\big]}{\mathcal{H}(\vartheta,\hbar)(1-\vartheta+\vartheta\hbar)}\mathcal{V}(\omega)+\frac{(1-\vartheta)\,{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[\mathcal{V}(\psi)\big]}{\mathcal{H}(\vartheta,\hbar)(1-\vartheta+\vartheta\hbar)}\mathcal{U}(\omega). \qquad (3.32)
\end{aligned}
$$

Again, interchanging ω by ξ in (3.31) and conducting product both sides by $\frac{\vartheta(\psi-\check{\rho}(\xi))_{\hbar}^{\overline{\vartheta-1}}}{\mathcal{H}(\vartheta,\hbar)\Gamma(\vartheta)}$. Also, summing both sides for $\xi\in\{\varphi_1,\varphi_1+\hbar,\varphi_1+2\hbar,\ldots\}$, we get

$$
\begin{aligned}
&{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[\mathcal{U}(\psi)\mathcal{V}(\psi)\big]\sum_{\kappa=\varphi_1/\hbar+1}^{\psi/\hbar}\frac{\vartheta(\psi-\check{\rho}(\kappa\hbar))_{\hbar}^{\overline{\vartheta-1}}}{\mathcal{H}(\vartheta,\hbar)\Gamma(\vartheta)}I(\kappa\hbar)\hbar\\
&+{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[I(\psi)\big]\sum_{\kappa=\varphi_1/\hbar+1}^{\psi/\hbar}\frac{\vartheta(\psi-\check{\rho}(\kappa\hbar))_{\hbar}^{\overline{\vartheta-1}}}{\mathcal{H}(\vartheta,\hbar)\Gamma(\vartheta)}\mathcal{U}(\kappa\hbar)\hbar\mathcal{V}(\kappa\hbar)\hbar\\
\geq&{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[\mathcal{U}(\psi)\big]\sum_{\kappa=\varphi_1/\hbar+1}^{\psi/\hbar}\frac{\vartheta(\psi-\check{\rho}(\kappa\hbar))_{\hbar}^{\overline{\vartheta-1}}}{\mathcal{H}(\vartheta,\hbar)\Gamma(\vartheta)}\mathcal{V}(\kappa\hbar)\hbar\\
&+{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[\mathcal{V}(\psi)\big]\sum_{\kappa=\varphi_1/\hbar+1}^{\psi/\hbar}\frac{\vartheta(\psi-\check{\rho}(\kappa\hbar))_{\hbar}^{\overline{\vartheta-1}}}{\mathcal{H}(\vartheta,\hbar)\Gamma(\vartheta)}\mathcal{U}(\kappa\hbar)\hbar. \qquad (3.33)
\end{aligned}
$$

Adding (3.32) and (3.33), then this leads to the conclusion that

$$
\begin{aligned}
&{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[\mathcal{U}(\psi)\mathcal{V}(\psi)\big]{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[I(\psi)\big]+{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[I(\psi)\big]\\
&{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[\mathcal{U}(\psi)\mathcal{V}(\psi)\big]\geq{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[\mathcal{U}(\psi)\big]{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[\mathcal{V}(\psi)\big]\\
&+{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[\mathcal{V}(\psi)\big]{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}_{\hbar}\big[\mathcal{U}(\psi)\big].
\end{aligned}
$$

This completes the proof of Theorem 3.8. □

As a special case of Theorem 3.8 with the assumption of $\hbar=1$, we get the following result.

Corollary 3.9. *For $0<\vartheta<1$ and let two positive functions $\mathcal{U}$ and $\mathcal{V}$ defined on $\mathbb{N}_{\varphi_1}$. Suppose that there exist four positive functions $\Lambda_1,\Lambda_2,\Theta_1$ and Θ_2 on $\mathbb{N}_{\varphi_1}$. Then, for all $\psi\in\{\varphi_1,\varphi_1+1,\varphi_1+2,\ldots\}$, the following inequality holds:*

$$
{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}\big[\mathcal{U}(\psi)\mathcal{V}(\psi)\big]\geq\frac{{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}\big[\mathcal{U}(\psi)\big]{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}\big[\mathcal{V}(\psi)\big]}{{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta}\big[I(\psi)\big]}. \qquad (3.34)
$$

Theorem 3.10. *For $0<\vartheta_1,\vartheta_2<1$ and let two positive functions $\mathcal{U}$ and $\mathcal{V}$ defined on $\mathbb{N}_{\varphi_1,\hbar}$. Suppose that there exist four positive functions $\Lambda_1,\Lambda_2,\Theta_1$ and Θ_2 on $\mathbb{N}_{\varphi_1,\hbar}$. Then, for all $\psi\in\{\varphi_1,\varphi_1+\hbar,\varphi_1+2\hbar,\ldots\}$, the following inequality holds:*

$$
\begin{aligned}
&{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}_{\hbar}\big[\mathcal{U}(\psi)\mathcal{V}(\psi)\big]{}^{\mathcal{AB}}_{\varsigma_1}\widehat{\nabla}^{-\vartheta_2}_{\hbar}\big[I(\psi)\big]+{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}_{\hbar}\big[I(\psi)\big]\\
&{}^{\mathcal{AB}}_{\varsigma_1}\widehat{\nabla}^{-\vartheta_2}_{\hbar}\big[\mathcal{U}(\psi)\mathcal{V}(\psi)\big]\\
\geq&{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}_{\hbar}\big[\mathcal{U}(\psi)\big]{}^{\mathcal{AB}}_{\varsigma_1}\widehat{\nabla}^{-\vartheta_2}_{\hbar}\big[\mathcal{V}(\psi)\big]+{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}_{\hbar}\big[\mathcal{V}(\psi)\big]\\
&{}^{\mathcal{AB}}_{\varsigma_1}\widehat{\nabla}^{-\vartheta_2}_{\hbar}\big[\mathcal{U}(\psi)\big]. \qquad (3.35)
\end{aligned}
$$

Proof. Interchanging ϑ by ϑ_1 in (3.31), gives

$$
\begin{aligned}
&{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}_{\hbar}\big[\mathcal{U}(\psi)\mathcal{V}(\psi)\big]+{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}_{\hbar}\big[I(\psi)\big]\mathcal{U}(\omega)\mathcal{V}(\omega)\geq{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}_{\hbar}\\
&\big[\mathcal{U}(\psi)\big]\mathcal{V}(\omega)+{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}_{\hbar}\big[\mathcal{V}(\psi)\big]\mathcal{U}(\omega). \qquad (3.36)
\end{aligned}
$$

Taking product both sides of (3.36) by $\frac{1-\vartheta_2}{\mathcal{H}(\vartheta_2,\hbar)(1-\vartheta_2+\vartheta_2\hbar)}$, we get

$$
\begin{aligned}
&\frac{(1-\vartheta_2)\,{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}_{\hbar}\big[\mathcal{U}(\psi)\mathcal{V}(\psi)\big]}{\mathcal{H}(\vartheta_2,\hbar)(1-\vartheta_2+\vartheta_2\hbar)}\\
&+\frac{(1-\vartheta_2)\,{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}_{\hbar}\big[I(\psi)\big]}{\mathcal{H}(\vartheta_2,\hbar)(1-\vartheta_2+\vartheta_2\hbar)}\mathcal{U}(\omega)\mathcal{V}(\omega)\\
\geq&\frac{(1-\vartheta_2)\,{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}_{\hbar}\big[\mathcal{U}(\psi)\big]}{\mathcal{H}(\vartheta_2,\hbar)(1-\vartheta_2+\vartheta_2\hbar)}\mathcal{V}(\omega)\\
&+\frac{(1-\vartheta_2)\,{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}_{\hbar}\big[\mathcal{V}(\psi)\big]}{\mathcal{H}(\vartheta_2,\hbar)(1-\vartheta_2+\vartheta_2\hbar)}\mathcal{U}(\omega). \qquad (3.37)
\end{aligned}
$$

Again, interchanging ω by ξ_2 in (3.36) and conducting product both sides by $\frac{\vartheta_2(\psi-\check{\rho}(\xi))_{\hbar}^{\overline{\vartheta_2-1}}}{\mathcal{H}(\vartheta_2,\hbar)\Gamma(\vartheta_2)}$. Also, summing both sides for $\xi_2\in\{\varsigma_1,\varsigma_1+\hbar,\varsigma_1+2\hbar,\ldots\}$, we get

$$
\begin{aligned}
&{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}_{\hbar}\big[\mathcal{U}(\psi)\mathcal{V}(\psi)\big]\sum_{\kappa=\varsigma_1/\hbar+1}^{\psi/\hbar}\frac{\vartheta_2(\psi-\check{\rho}(\kappa\hbar))_{\hbar}^{\overline{\vartheta_2-1}}}{\mathcal{H}(\vartheta_2,\hbar)\Gamma(\vartheta_2)}I(\kappa\hbar)\hbar\\
&+{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}_{\hbar}\big[I(\psi)\big]\sum_{\kappa=\varsigma_1/\hbar+1}^{\psi/\hbar}\frac{\vartheta_2(\psi-\check{\rho}(\kappa\hbar))_{\hbar}^{\overline{\vartheta_2-1}}}{\mathcal{H}(\vartheta_2,\hbar)\Gamma(\vartheta_2)}\mathcal{U}(\kappa\hbar)\hbar\mathcal{V}(\kappa\hbar)\hbar\\
\geq&{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}_{\hbar}\big[\mathcal{U}(\psi)\big]\sum_{\kappa=\varsigma_1/\hbar+1}^{\psi/\hbar}\frac{\vartheta_2(\psi-\check{\rho}(\kappa\hbar))_{\hbar}^{\overline{\vartheta_2-1}}}{\mathcal{H}(\vartheta_2,\hbar)\Gamma(\vartheta_2)}\mathcal{V}(\kappa\hbar)\hbar\\
&+{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}_{\hbar}\big[\mathcal{V}(\psi)\big]\sum_{\kappa=\varsigma_1/\hbar+1}^{\psi/\hbar}\frac{\vartheta_2(\psi-\check{\rho}(\kappa\hbar))_{\hbar}^{\overline{\vartheta_2-1}}}{\mathcal{H}(\vartheta_2,\hbar)\Gamma(\vartheta_2)}\mathcal{U}(\kappa\hbar)\hbar. \qquad (3.38)
\end{aligned}
$$

Adding (3.37) and (3.38), then this leads to the conclusion that

$$
\begin{aligned}
&{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}_{\hbar}\big[\mathcal{U}(\psi)\mathcal{V}(\psi)\big]{}^{\mathcal{AB}}_{\varsigma_1}\widehat{\nabla}^{-\vartheta_2}_{\hbar}\big[I(\psi)\big]+{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}_{\hbar}\big[I(\psi)\big]\\
&{}^{\mathcal{AB}}_{\varsigma_1}\widehat{\nabla}^{-\vartheta_2}_{\hbar}\big[\mathcal{U}(\psi)\mathcal{V}(\psi)\big]\\
\geq&{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}_{\hbar}\big[\mathcal{U}(\psi)\big]{}^{\mathcal{AB}}_{\varsigma_1}\widehat{\nabla}^{-\vartheta_2}_{\hbar}\big[\mathcal{V}(\psi)\big]+{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}_{\hbar}\big[\mathcal{V}(\psi)\big]\\
&{}^{\mathcal{AB}}_{\varsigma_1}\widehat{\nabla}^{-\vartheta_2}_{\hbar}\big[\mathcal{U}(\psi)\big]. \qquad (3.39)
\end{aligned}
$$

□

As a special case of Theorem 3.10 with the assumption of $\hbar = 1$, we get the following result.

Corollary 3.11. *For* $0 < \vartheta_1, \vartheta_2 < 1$ *and let two positive functions* $\mathcal{U}$ *and* $\mathcal{V}$ *defined on* $\mathbb{N}_{\varphi_1,\hbar}$. *Suppose that there exist four positive functions* $\Lambda_1, \Lambda_2, \Theta_1$ *and* Θ_2 *on* $\mathbb{N}_{\varphi_1,\hbar}$. *Then, for all* $\psi \in \{\varphi_1, \varphi_1 + \hbar, \varphi_1 + 2\hbar, \ldots\}$, *the following inequality holds:*

$$\begin{aligned} &{}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}\big[\mathcal{U}(\psi)\mathcal{V}(\psi)\big]\,{}^{\mathcal{AB}}_{\varsigma_1}\widehat{\nabla}^{-\vartheta_2}\big[I(\psi)\big] + {}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}\big[I(\psi)\big] \\ &{}^{\mathcal{AB}}_{\varsigma_1}\widehat{\nabla}_{\hbar}^{-\vartheta_2}\big[\mathcal{U}(\psi)\mathcal{V}(\psi)\big] \\ &\geq {}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}\big[\mathcal{U}(\psi)\big]\,{}^{\mathcal{AB}}_{\varsigma_1}\widehat{\nabla}^{-\vartheta_2}\big[\mathcal{V}(\psi)\big] + {}^{\mathcal{AB}}_{\varphi_1}\widehat{\nabla}^{-\vartheta_1}\big[\mathcal{V}(\psi)\big] \\ &{}^{\mathcal{AB}}_{\varsigma_1}\widehat{\nabla}^{-\vartheta_2}\big[\mathcal{U}(\psi)\big]. \end{aligned}$$

Remark 2. If we take $\vartheta_1 = \vartheta_2$, then Theorem 3.10 reduces to Theorem 3.8.

4. Application

In this note, we illustrate how to build the four bounding functions and how to employ them to determine Chebyshev form discrete fractional variants of two unknown functions.

Let us define a unit step function $\mathfrak{u}$:

$$\mathfrak{u}(\xi) = \begin{cases} 1, & \xi > 0, \\ 0, & \xi \leq 0. \end{cases} \tag{4.1}$$

Also, the Heaviside unit step function $\mathfrak{u}_c(\xi)$ is stated as

$$\mathfrak{u}_c(\xi) = \mathfrak{u}(\xi - c)\begin{cases} 1, & \xi \geq c, \\ 0, & \xi < c. \end{cases} \tag{4.2}$$

Consider a piece-wise continuous function Λ_1 on $\mathbb{N}_{0,\hbar}$ presented as

$$\begin{aligned} \Lambda_1(\xi) &= \epsilon_1\big(\mathfrak{u}_0(\xi) - \mathfrak{u}_{\xi_1}(\xi)\big) + \epsilon_2\big(\mathfrak{u}_{\xi_1}(\xi) - \mathfrak{u}_{\xi_2}(\xi)\big) \\ &\quad + \epsilon_3\big(\mathfrak{u}_{\xi_2}(\xi) - \mathfrak{u}_{\xi_3}(\xi)\big) + \ldots + \epsilon_{w+1}\mathfrak{u}_{\xi_{w+1}}(\xi) \\ &= \sum_{\jmath=0}^{w}(\epsilon_{\jmath+1} - \epsilon_{\jmath})\mathfrak{u}_{\xi_\jmath}(\xi), \end{aligned} \tag{4.3}$$

where $\epsilon_0 \equiv 0$ and $0 \in \{\xi_0, \xi_1, \ldots, \xi_{w+1}\} = \mathbb{N}_{0,\hbar}$.

Similarly, the functions Υ_2, Υ_3 and υ_4 are presented as

$$\Lambda_2(\xi) = \sum_{\jmath=0}^{w}(\tilde{\epsilon}_{\jmath+1} - \tilde{\epsilon}_{\jmath})\mathfrak{u}_{\xi_\jmath}(\xi), \tag{4.4}$$

$$\Theta_3(\xi) = \sum_{\jmath=0}^{w}(\check{\epsilon}_{\jmath+1} - \check{\epsilon}_{\jmath})\mathfrak{u}_{\xi_\jmath}(\xi), \tag{4.5}$$

$$\Theta_4(\xi) = \sum_{\jmath=0}^{w}(\grave{\epsilon}_{\jmath+1} - \grave{\epsilon}_{\jmath})\mathfrak{u}_{\xi_\jmath}(\xi), \tag{4.6}$$

where constants $\tilde{\epsilon}_0 = \check{\epsilon}_0 = \grave{\epsilon} \equiv= 0$. If the condition $(\mathcal{A}_1)$ is satisfied by an integrable function $\mathcal{U}$ on $\mathbb{N}_{0,\hbar}$, then we have $\epsilon_{\jmath+1} \leq \mathcal{U}(\xi) \leq \tilde{\epsilon}_{\jmath+1}$ for every $\xi \in \{\xi_0, \xi_1, \ldots, \xi_{w+1}\}$, $\jmath = 0, 1, 2, \ldots, w$.

Proposition 4.1. *For* $0 < \vartheta < 1$ *and let there be two positive functions* $\mathcal{U}$ *and* $\mathcal{V}$ *defined on* $\mathbb{N}_{0,\hbar}$. *Suppose that the mappings defined in (4.3), (4.4), (4.5) and (4.6), respectively, satisfy* $(\mathcal{A}_1)$, *then*

$$\frac{\Big[\sum_{\jmath=0}^{w}\check{\epsilon}_{\jmath+1}\grave{\epsilon}_{\jmath+1}\,{}^{\mathcal{AB}}_{\xi_\jmath\xi_{\jmath+1}}\widehat{\nabla}_{\hbar}^{-\vartheta}\mathcal{U}^2\Big](\psi)\Big[\sum_{\jmath=0}^{w}\epsilon_{\jmath+1}\tilde{\epsilon}_{\jmath+1}\,{}^{\mathcal{AB}}_{\xi_\jmath\xi_{\jmath+1}}\widehat{\nabla}_{\hbar}^{-\vartheta}\mathcal{V}^2\Big](\psi)}{\Big[\sum_{\jmath=0}^{w}\big(\epsilon_{\jmath+1}\check{\epsilon}_{\jmath+1} + \grave{\epsilon}_{\jmath+1}\tilde{\epsilon}_{\jmath+1}\big)\,{}^{\mathcal{AB}}_{\xi_\jmath\xi_{\jmath+1}}\widehat{\nabla}_{\hbar}^{-\vartheta}\mathcal{U}\mathcal{V}\Big]^2(\psi)} \leq \frac{1}{4}. \tag{4.7}$$

Proof. In view of Definition 2.9, we have

$${}^{\mathcal{AB}}_{0}\widehat{\nabla}_{\hbar}^{-\vartheta}\big[\Theta_1\Theta_2\mathcal{U}^2\big](\psi) = \Big[\sum_{\jmath=0}^{w}\check{\epsilon}_{\jmath+1}\grave{\epsilon}_{\jmath+1}\,{}^{\mathcal{AB}}_{\xi_\jmath\xi_{\jmath+1}}\widehat{\nabla}_{\hbar}^{-\vartheta}\mathcal{U}^2\Big](\psi),$$

$${}^{\mathcal{AB}}_{0}\widehat{\nabla}_{\hbar}^{-\vartheta}\big[\Lambda_1\Lambda_2\mathcal{V}^2\big](\psi) = \Big[\sum_{\jmath=0}^{w}\epsilon_{\jmath+1}\tilde{\epsilon}_{\jmath+1}\,{}^{\mathcal{AB}}_{\xi_\jmath\xi_{\jmath+1}}\widehat{\nabla}_{\hbar}^{-\vartheta}\mathcal{V}^2\Big](\psi)$$

and

$$\begin{aligned} &\Big[{}^{\mathcal{AB}}_{0}\widehat{\nabla}_{\hbar}^{-\vartheta}\big[(\Theta_1\Lambda_1 + \Theta_2\Lambda_2)\mathcal{U}\mathcal{V}\big](\psi)\Big]^2 \\ &= \Big[\sum_{\jmath=0}^{w}\big(\epsilon_{\jmath+1}\check{\epsilon}_{\jmath+1} + \grave{\epsilon}_{\jmath+1}\tilde{\epsilon}_{\jmath+1}\big)\,{}^{\mathcal{AB}}_{\xi_\jmath\xi_{\jmath+1}}\widehat{\nabla}_{\hbar}^{-\vartheta}\mathcal{U}\mathcal{V}\Big]^2(\psi). \end{aligned}$$

Employing Lemma 3.1, we get the inequality (4.7). □

Proposition 4.2. *For* $0 < \vartheta < 1$ *and let there be two positive functions* $\mathcal{U}$ *and* $\mathcal{V}$ *defined on* $\mathbb{N}_{0,\hbar}$. *Suppose that the mappings defined in (4.3), (4.4), (4.5) and (4.6), respectively, satisfy* $(\mathcal{A}_1)$, *then*

$$\begin{aligned} &{}^{\mathcal{AB}}_{0}\widehat{\nabla}_{\hbar}^{-\vartheta_1}\big[\mathcal{U}^2(\psi)\big]\,{}^{\mathcal{AB}}_{0}\widehat{\nabla}_{\hbar}^{-\vartheta_2}\big[\mathcal{V}^2(\psi)\big] \leq {}^{\mathcal{AB}}_{\xi_\jmath\xi_{\jmath+1}}\widehat{\nabla}_{\hbar}^{-\vartheta_1} \\ &\left\{\frac{\sum_{\jmath=0}^{w}\tilde{\epsilon}_{\jmath+1}\mathcal{U}\mathcal{V}}{\sum_{\jmath=0}^{w}\grave{\epsilon}_{\jmath+1}}(\psi)\right\}{}^{\mathcal{AB}}_{\xi_\jmath\xi_{\jmath+1}}\widehat{\nabla}_{\hbar}^{-\vartheta_2}\left\{\frac{\sum_{\jmath=0}^{w}\check{\epsilon}_{\jmath+1}\mathcal{U}\mathcal{V}}{\sum_{\jmath=0}^{w}\epsilon_{\jmath+1}}(\psi)\right\}. \end{aligned} \tag{4.8}$$

Proof. Taking into account (4.3)-(4.6) and employing Theorem 3.6, we get the immediate consequence (4.8). □

5. Conclusion

The $\mathcal{DFC}$ expansions will help researchers implement increasingly effective analytical models developed by discrete fractional physical phenomena in the context of fractional derivatives with various discrete kernels. In this note, we proposed novel diverse kinds of inequalities such as the Pólya-Szegö and Chebyshev type variants that are accomplished with $\mathcal{AB}$-fractional sums involving discrete $\hbar$-Mittag-Leffler kernel, whose compensations concluded all novelties in literature. The previously described implications may also be applied to the discrete $\mathcal{AB}$-fractional case. From an application viewpoint, we have presented several generalizations with the aid of the Heaviside function, which appears relatively harmonious with the results. Consequently, one can forthrightly create the sense that present implications can be accomplished for $\hbar = 1$. In order to ascertain the intensity of the presented repercussions, we leverage them to explore a collection of inequalities such as the Agarwal-Ryoo-Kim type inequality, the Agarwal-Thandapani type inequality, one-dimensional Ou-Yang inequality, the Nanko inequality and several other variants incorporating $\mathcal{AB}$-fractional sums involving discrete $\hbar$-Mittag-Leffler kernel.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Declaration of Competing Interest

The authors declare that they have no competing interests.

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