Alexandria Engineering Journal (2020) 59, 2969–2974

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Alexandria Engineering Journal

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Numerical schemes for studying biomathematics model inherited with memory-time and delay-time

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Received 29 January 2020; revised 3 March 2020; accepted 11 March 2020
Available online 15 May 2020

KEYWORDS

Caputo-time-fractional derivative;
Delay time;
Fractional power series;
Homotopy perturbation method;
Mathematical biological model

MSC(2010)

35F25;
35C10

Abstract The effect of inherited memory-time and delay-time in the formulation of a mathematical population growth model is considered. Two different numerical schemes are introduced to study analytically the propagation of population growth. We provide a graphical analysis that shows the impact of both memory-time and delay-time acting on the behavior of population density. We concluded that both delay-time and time-fractional-derivative play the same role as delaying the propagation of the nonlinear population growth.

1. Introduction

A variety of nonlinear evolution equations that represent real-life applications are exposure for unexpected physical changes such as, delaying in its propagations, bifurcating its solution or stability issues. Linking such change with the theoretical considerations may lead to unknown. A wise thought on reasons behind such unexpected bifurcations is directed to the role of the rate of change of the field function with respect to time. Many studies emerged to review the rate of change as a fractional hospital with a topological relationship with the integer case. All scientific researches become interested in this area taking into consideration the use of the fractional derivative instead of the integer-derivative [1–9]. Extensive works have been accomplished to draw suitable mathematical methods for dealing with these fractal issues. Among the most important and widely used numerical methods are: decomposition schemes [10,11], variational iteration schema [12,13], fractional power series methods [14–17], finite difference scheme [18], Sumudu decomposition method [19] and other methods [20–22].

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Peer review under responsibility of Faculty of Engineering, Alexandria University.

https://doi.org/10.1016/j.aej.2020.03.038

One of the interesting mathematical models arise in science is the biology population growth model which is governed by the following form [23,24]

$$D_t P(x,t) - D_{xx}\left(P^2(x,t)\right) - f(P(x,t)) = 0, \;\; t > 0. \tag{1.1}$$

The function $P(x,t)$ stands for the population density and the functional $f(P)$ is the reaction source that describes the dynamics of the population growth. There are different types of the reaction source; $f(P) = cP$, is identified by Malthusian law. $f(P) = aP - bP^2$, is regarded as the Verhulst law. The most general form of such reaction source is defined as $f(P) = aP^n(1 - \frac{b}{a}P^m)$.

The main goal of this work is to study the above model under new considerations; the time-integer-derivative is to be replaced by fractional order and the reaction source is of Verhulst-type acting with delay type of form $\tau\, t: \;\; 0 < \tau < 1$. Therefore, the new biological model has the form:

$$D_t^\alpha P(x,t) - D_{xx}\left(P^2(x,t)\right) - aP(x,\tau t) + bP^2(x,\tau t) = 0, \;\; t > 0, \tag{1.2}$$

where $0 < \alpha \leqslant 1$ is the Caputo-fractional derivative [25–27].

The model given in (1.2) is to be presented for the first time. Whereas, previous studies concerned with only time-fractional derivatives. For example, in [28], the conformable fractional-derivative is used and exact solutions are found by means of Kudryashov method. Two-dimensional form of the time-fractional population growth of Malthusian law [29] is studied by means of fractional variational iteration method (FVIM). Finally, The finite difference method (FDM) and variational iteration method (VIM) have been successfully implemented [30] for solving (1.2) but with the absence of delay action.

The format of the current paper is as follows. In Section 2, we provide in details the t-spectrum scheme to solve the proposed model and give some graphical analysis. In Section 3, we present the homotopy perturbation method as an alternative scheme to handle (1.2). Finally, some concluding remarks and observations are given in the last section.

2. Time-coordinate spectrum function method

A power series of the form [31,32]

$$\sum_{j=0}^{\infty} b_j t^{j\alpha} = b_0 + b_1 t^\alpha + b_2 t^{2\alpha} + \ldots, \tag{2.1}$$

is called fractional Maclaurin series (FMS). Suppose that the function $v(t)$ has FMS of the form

$$v(t) = \sum_{j=0}^{\infty} b_j t^{j\alpha}, \quad 0 < t < R, \tag{2.2}$$

if $D^{j\alpha} v(t)$ are continuous on $I = (0, R)$, then $b_j = \frac{D^{j\alpha} v(0)}{\Gamma(j\alpha+1)}$ and R is called the radius of convergence.

Consider that the solution of (1.2) can be represented as a product of two single variable functions, $P(x,t) = u(x)v(t)$, where $u(x)$ is analytic function and $v(t)$ has FMS representation. Then,

$$P(x,t) = \left(\sum_{i=0}^{\infty} a_i x^i\right)\left(\sum_{j=0}^{\infty} b_j t^{j\alpha}\right), = \sum_{k=0}^{\infty} U_k(x) t^{k\alpha}. \tag{2.3}$$

where $U_k(x) = \frac{D_t^{j\alpha} v(x,0)}{\Gamma(j\alpha+1)}$. We refer to $U_k(x)$ as the transformed function or the t-coordinate spectrum function. Also, the inverse of the spectrum $U_k(x)$ is given by (2.3). Apply the t-fractional derivative on (2.3) gives

$$D_t^\alpha P(x,t) = \sum_{k=0}^{\infty}\left(\frac{\Gamma((k+1)\alpha+1)}{\Gamma(k\alpha+1)} U_{k+1}(x)\right) t^{k\alpha}. \tag{2.4}$$

Also,

$$P(x,\tau t) = \sum_{k=0}^{\infty}\left(\tau^{k\alpha} U_k(x)\right) t^{k\alpha}. \tag{2.5}$$

Apply the cauchy product of two infinite series and by (2.5) we get the following expansion

$$P^2(x,\tau t) = \sum_{k=0}^{\infty}\left(\tau^{k\alpha}\sum_{n=0}^{k} U_{k-n}(x) U_n(x)\right) t^{k\alpha}. \tag{2.6}$$

Table 1, provides the necessary transformed spectrum functions that will be used to solve the problem under investigation. Applying the rules given in Table 1, the following spectrum transformed form of Eq. (1.2) is obtained

$$\frac{\Gamma((k+1)\alpha+1)}{\Gamma(k\alpha+1)} U_{k+1}(x) - \sum_{n=0}^{k}\left(U_n(x)U_{k-n}(x)\right)'' - a\tau^{k\alpha} U_k(x) + b\tau^{k\alpha}\sum_{n=0}^{k} U_n(x) U_{k-n}(x) = 0. \tag{2.7}$$

From (2.7), we deduce the recurrence relation regarding $U_k(x)$ as

$$U_{k+1}(x) = \frac{\Gamma(k\alpha+1)}{\Gamma((k+1)\alpha+1)}\left(\sum_{n=0}^{k}\left(U_n(x)U_{k-n}(x)\right)'' + a\tau^{k\alpha} U_k(x) - b\tau^{k\alpha}\sum_{n=0}^{k} U_n(x)U_{k-n}(x)\right). \tag{2.8}$$

For clarity, we list the first few terms of the spectrum sequence of the growth model

$$\begin{aligned}
U_1 &= \tfrac{1}{\Gamma(\alpha+1)}\left((U_0^2)'' + aU_0 - bU_0^2\right),\\
U_2 &= \tfrac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\left(2(U_0U_1)'' + a\tau^\alpha U_1 - 2b\tau^\alpha U_0U_1\right),\\
U_3 &= \tfrac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\left(2(U_0U_2)'' + (U_1^2)'' + a\tau^{2\alpha} U_2 - b\tau^{2\alpha}(2U_0U_2 + U_1^2)\right),\\
U_4 &= \tfrac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}\left(2(U_0U_3)'' + 2(U_1U_2)'' + a\tau^{3\alpha} U_3 - 2b\tau^{3\alpha}(U_0U_3 + U_1U_2)\right).
\end{aligned} \tag{2.9}$$

Now, we are ready to discuss some graphical analysis regarding the accuracy of t-spectrum function method, and the impact of both time-fractional and time-delay acting on the

Table 1 Transformed t-coordinate spectrum functions.

$P(x,t)$	$U_k(x)$
$P(x,\tau t)$	$\tau^{k\alpha} U_k(x)$
$P^2(x,t)$	$\sum_{n=0}^{k} U_{(k-n)}(x) U_n(x))$
$P^2(x,\tau t)$	$\tau^{k\alpha}\sum_{n=0}^{k} U_{(k-n)}(x) U_n(x)$
$D_t^\alpha P(x,t)$	$\frac{\Gamma((k+1)\alpha+1)}{\Gamma(k\alpha+1)} U_{k+1}(x)$
$D_{xx} P^2(x,t)$	$\sum_{n=0}^{k}\left(U_{(k-n)}(x) U_n(x)\right)''$

population density. The analysis will be performed on a numerical example of (1.2); the case of $a = b = 1$; subject to the initial population density $P(x,0) = f(x) = 1 + e^{\frac{1}{2}x}$. We should point here that the presence of the delay factor τ in the calculations of the spectrum functions $U_i(x)$ will increase the size of the resulting terms. Moreover, it is hard to find or even guess a closed form solution. Therefore, we decide to consider $\phi = \phi(x,t) = \sum_{n=0}^{6} U_n(x)t^{n\alpha}$, to represent the approximation solution for the population problem. Fig. 1, presents, respectively, the exact solution $P(x,t) = 1 + e^{\frac{x-t}{2}}$ (See [28]), the approximate function ϕ, and the obtained absolute error $|P(x,t) - \phi(x,t)|$ under the case of $\alpha = \tau = 1$. These obtained 3D plots show that the assigned approximation expansion is in good agreement with the exact one. Fig. 2, shows the impact and the interaction of both fractional order α and the delay factor τ on the population density. Four drawn relations can be observed:

- The order of α : $0 < \alpha < 1$ is proportional with the population density.
- An increase of the delay factor leads to an increase in the population density. Moreover, for any value of τ : $0 < \tau < 1$, it delays the expected value of the population growth.
- In the presence of the fractional-derivative "$\alpha < 1$", the delay factor affects the delay of population growth much faster if compared with the integer-case "$\alpha = 1$".
- Both delay-time and time-fractional-derivative play the same role as delaying the propagation of nonlinear evolutionary models.

In conclusion, we speculate that the fractional derivative order or/and the time-delay have the potential to affect real-life observation of the dynamics of interacting populations.

2.1. Convergence of spectrum function method

In this part, we validate numerically the convergence of the transformed t-spectrum function method applied to the previous example. Therefore, to study the convergence of the obtained approximate solution $\phi(x,t)$ to its exact solution, we use the approach proposed by Hosseini and Nasabzadeh [33]. We determine the values of γ_i where $\gamma_i = \max\{\frac{U_{i+1}(x)}{U_i(x)}\} : x \in (-\infty, \infty)$ and therefore, the solution converges when $|\gamma_i| < 1$. For the proposed example, we obtained the following bounds:

- $-\frac{1}{2} < \gamma_0 < 0$.
- $\gamma_1 = -\frac{1}{4}$.
- $\gamma_2 = -\frac{1}{6}$.
- $\gamma_3 = -\frac{1}{8}$.
- $\gamma_4 = -\frac{1}{10}$.
- $\gamma_5 = -\frac{1}{12}$.
- $\gamma_k = -\frac{1}{2(k+1)}$: $k \geqslant 1$.

The above findings are regarded as a numerical evidence that the solution $\phi(x,t)$ obtained by using transformed spectrum method converges to $P(x,t)$.

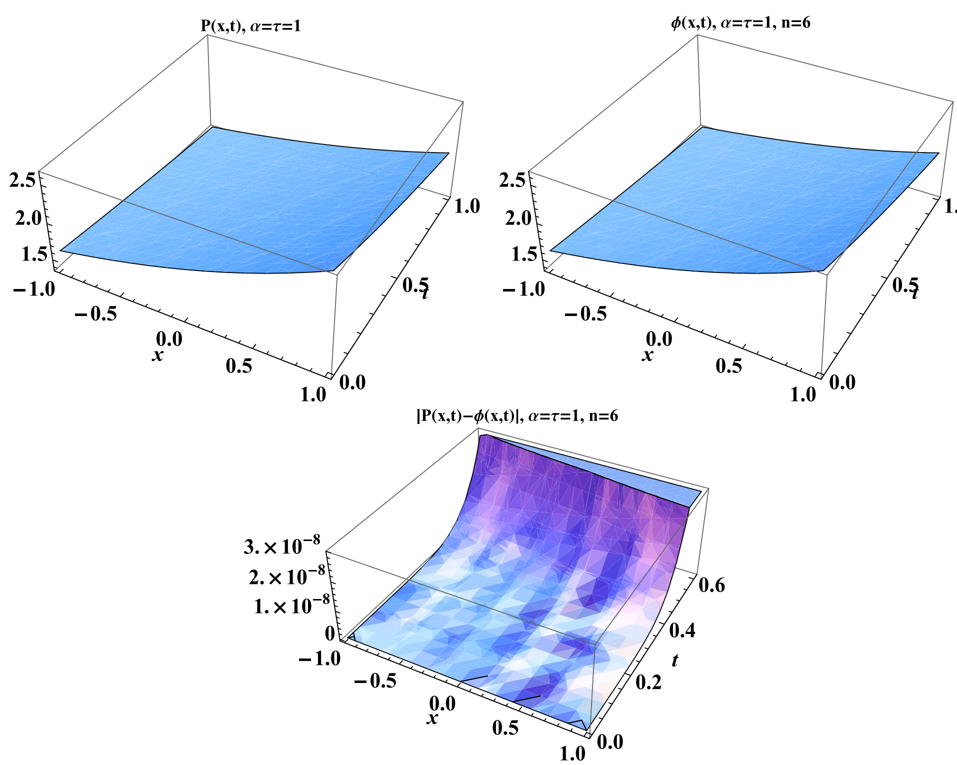

Fig. 1 3D plots of, respectively, $P(x,t), \phi(x,t)$ and $|P(x,t) - \phi(x,t)|$. The case for $\alpha = \tau = 1$ and $n = 6$.

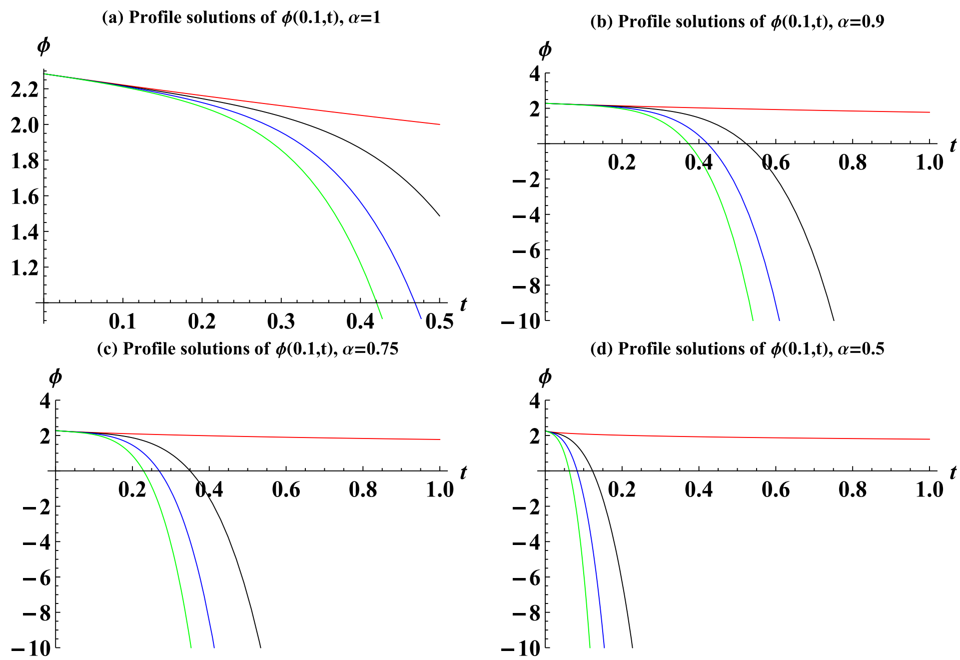

Fig. 2 The interaction of both fractional order α and the delay factor τ on the population density. Where, Red $\equiv (\tau = 1)$, Black $\equiv (\tau = 0.75)$, Blue $\equiv (\tau = 0.5)$, Green $\equiv (\tau = 0.25)$.

3. Homotopy perturbation schema

The Homotopy perturbation method (HPM) will be used as an alternative technique to solve (1.2). This numerical scheme suggest to write the problem in two homotopy forms [34,35]; either

$$D_t P(x,t) = q\big(D_t P(x,t) + D_t^\alpha P(x,t) - D_{xx}\big(P^2(x,t)\big) - aP(x,\tau t) + bP^2(x,\tau t)\big), \quad (3.1)$$

or

$$D_t^\alpha P(x,t) = q\big(D_{xx}\big(P^2(x,t)\big) + aP(x,\tau t) - bP^2(x,\tau t)\big), \quad (3.2)$$

with a perturbation form of the function $P(x,t)$ as

$$P(x,t) = \sum_{i=0}^{\infty} q^i P_i(x,t). \quad (3.3)$$

To solve (1.2) subject to $P(x,0) = f(x)$, we substitute (3.3) in either (3.1) or (3.2) and in the initial condition. Then, in a sequence order, we solve sub-equations resulting from collecting same coefficients of q^i. By considering (3.1) we use the rule $D_t^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}$ whereas in the case of (3.2) we use the rule $J_t^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} t^{\beta+\alpha}$. In this context, we prefer the form given in (3.2). Now, we proceed as the following steps:

Step I: q^0: We solve

$$D_t^\alpha P_0(x,t) = 0, \quad P_0(x,0) = f(x).$$

Which leads to the first component

$$P_0(x,t) = f(x). \quad (3.4)$$

Step II: q^1: We solve

$$D_t^\alpha P_1(x,t) = D_{xx}\big(P_0^2(x,t)\big) + aP_0(x,\tau t) - bP_0^2(x,\tau t),$$
$$P_1(x,0) = 0.$$

Which leads to the second component

$$P_1(x,t) = h(x)\frac{t^\alpha}{\Gamma(\alpha+1)}, \quad (3.5)$$

where $h(x) = f''(x) + af(x) - bf^2(x)$.

Step III: q^2: We solve

$$D_t^\alpha P_2(x,t) = D_{xx}(2P_0(x,t)P_1(x,t)) + aP_1(x,\tau t) - 2bP_0(x,\tau t)P_1(x,\tau t), \quad P_2(x,0) = 0.$$

Which leads to the third component

$$P_2(x,t) = \{2(f(x)h(x))'' + (a - 2bf(x))h(x)\tau^\alpha\} \times \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}. \quad (3.6)$$

Step IV: q^3: We solve

$$\begin{aligned} D_t^\alpha P_3(x,t) = {} & D_{xx}\big(2P_0(x,t)P_2(x,t) + P_1^2(x,t)\big) \\ & + aP_2(x,\tau t) - b\big(2P_0(x,\tau t)P_2(x,\tau t) + P_1^2(x,\tau t)\big), \\ P_3(x,0) = {} & 0. \end{aligned}$$

Which leads to the fourth component

$$\begin{aligned} P_3(x,t) = {} & \Big\{\big(2f(x)\big(2(f(x)h(x))'' + (a - 2bf(x))h(x)\tau^\alpha\big)\big)'' \\ & + \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}(h^2(x))''\Big\}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ & + \big\{a\big(2(f(x)h(x))'' + (a - 2bf(x))h(x)\tau^\alpha\big)\tau^{2\alpha}\big\}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ & - b\big\{2f(x)\big(2(f(x)h(x))'' + (a - 2bf(x))h(x)\tau^\alpha\big) \\ & + h^2(x)\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\big\}\tau^{2\alpha}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}. \end{aligned} \quad (3.7)$$

We observe the difficulty in determining the components $P_i(x,t)$ as we proceed further for $i \geqslant 3$. To this end, we con-

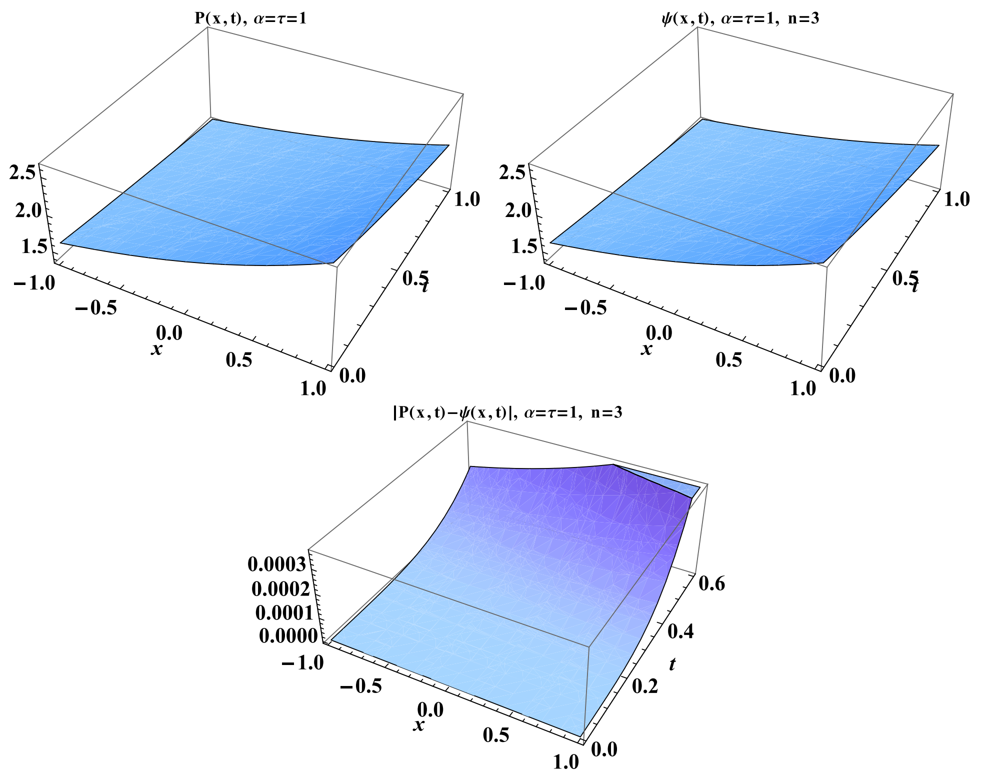

Fig. 3 3D plots of, respectively, $P(x,t), \psi(x,t)$ and $|P(x,t) - \psi(x,t)|$. The case for $\alpha = \tau = 1$ and $n = 3$.

sider $\psi(x,t) = \sum_{i=0}^{3} P_i(x,t)$ to represent the homotopy approximate solution to (1.2). Fig. 3, shows that HPM is a good alternative scheme even considering a few components of its series' representation.

4. Conclusions

In this manuscript, we suggested a new version of the mathematical biology growth model. The new model consider the presence of two actions on the time-coordinate. These actions are categorized as fractional-time derivative to represent the rate of change in the population density, and a delay-time of pantograph type acting on the reaction source term which is responsible for the dynamics of the population growth.

We offered two numerical schemes to find an approximation solution of the population growth model. The first method is derived from the fractional powers series whose coefficients are addressed as the time-coordinate spectrum functions. This approach leads to a recursive relation among its spectrum functions which give the advantages of determining "as many as needed" the spectrum functions with delicate computational process and in turns leads to a high accuracy approximation. In contrast, the alternative homotopy perturbation technique which decomposes the solution as infinite analytical series, its components to be determined by solving successive initial value fractional problems. These homotopy steps become difficult task as the calculation processes go farther. In most cases, only a few terms are considered for the approximate solution with limited accuracy size.

The obtained findings for the t-spectrum function method are used to study the impact of both time-fractional derivative and the delay-time. We concluded that the presence of fractional derivative helps to understand the fact that the delay occurs in the population density "due to a delay-factor embedded in the problem" could be less than the expected values. We may say that integer-derivative can be regarded as theoretical-perspective, whereas the fractional-derivative is a realistic-perspective.

As future work, we will considered the n-dimensional extension of the fractional power series [36–40] to study multivariate-fractional models arise in sciences and engineering.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The first and the third authors would like to thank Ajman University, Ajman, UAE for providing the financial support.

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