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NON–CONFORMABLE INTEGRAL INEQUALITIES OF CHEBYSHEV–PÓLYA–SZEGÖ TYPE

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Abstract. Inequality studies involving new integrals and derivatives have been carried out recently. This article designed as follows, the results were obtained by using the non-conformable fractional integral operators to provide new inequalities of Polya-Szegö and Chebyshev type. Some special cases have been considered for our main findings.

1. Introduction and preliminaries

In the studies of theory of inequalities, many unique inequalities have been achieved with their aesthetic appearances and applications. These inequalities have been demonstrated in different forms including integral and derivatives of various mappings defined in normed spaces and different functional spaces. Moreover, new and useful inequalities have been produced, taking into account the additional features of mappings such as continuity, limitation and uniformity. We will start with Polya-Szegö inequality, one of the respected inequalities of inequality theory. This famous inequality can be given as follows (see [6]):

$$\frac{\int_{\theta}^{\vartheta} f^2(\rho)\,d\rho \int_{\theta}^{\vartheta} g^2(\rho)\,d\rho}{\left(\int_{\theta}^{\vartheta} f(\rho)\,g(\rho)\,d\rho\right)^2} \leqslant \frac{1}{4}\left(\sqrt{\frac{\kappa\tau}{\zeta\delta}}+\sqrt{\frac{\zeta\delta}{\kappa\tau}}\right)^2$$

In [7], Dragomir and Diamond have achieved a new and interesting inequality by combining Grüss inequality, another important inequality of the inequality theory, with the Polya-Szegö inequality. This inequality has been proved on the basis of the Polya-Szegö inequality as following:

THEOREM 1. *Let $f,g:[\theta,\vartheta]\to\mathbb{R}_+$ be two integrable mappings so that*

$$0<\zeta\leqslant f(\rho)\leqslant\kappa<\infty$$
$$0<\delta\leqslant g(\rho)\leqslant\tau<\infty$$

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for $\rho \in [\theta, \vartheta]$. *Then we have*

$$|T(f,g;\theta,\vartheta)| \leqslant \frac{1}{4}\frac{(\kappa-\zeta)(\tau-\delta)}{\sqrt{\zeta\delta\kappa\tau}}\left(\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta} f(\rho)\,d\rho\right)\left(\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta} g(\rho)\,d\rho\right) \tag{1.1}$$

The constant $\frac{1}{4}$ *is best possible in (1.1) in the sense it can not be replaced by a smaller constant.*

The functional in the above inequality is actually a key statement that will lead us to the Chebyshev inequality. First, let's define this functional as follows (see [4]):

$$T(f,g) = \frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta} f(\rho)g(\rho)\,d\rho - \left(\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta} f(\rho)\,d\rho\right)\left(\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta} g(\rho)\,d\rho\right) \tag{1.2}$$

where f ve g are two integrable mappings which are synchronous on $[\theta, \vartheta]$, i.e.

$$(f(\rho)-f(\eta))(g(\rho)-g(\eta)) \geqslant 0$$

for any $\rho, \eta \in [\theta, \vartheta]$, then the Chebyshev inequality states that $T(f,g) \geqslant 0$.

This functional (1.2) and the Chebyshev inequality produced through it has demonstrated its value through its use in applied sciences, numeric integration and probability. Apart from all these areas, many researchers have obtained new generalizations, enlargements and iterations for this inequality, which is widely used in inequality theory. For some recent counterparts, generalizations of Chebyshev inequality, the reader may refer to [4, 5, 10] and [20].

The crossing of inequalities with fractional analysis using fractional integral operators is a milestone. After this turning point, the theory of inequality has gained a new direction. In fractional analysis, the use of integral and derivative operators newly defined by the researchers in engineering, modeling, chaos theory, various branches of mathematics and mathematical biology, and the fact that mathematicians working in the field of inequality theory use these operators reinforced the great collaboration of these two areas. Now, we will continue by giving a integral definition called non-conformable integral that brings these two fields closer together.

DEFINITION 1. ([1]) Let $\alpha \in \mathbb{R}$ and $0 < \theta < \vartheta$. For each function $f \in L^1[\theta, \vartheta]$, we define

$${}_{N_3}J_u^{\alpha} f(\rho) = \int_u^{\rho} t^{-\alpha} f(t)\,dt$$

for every $\rho, u \in [\theta, \vartheta]$.

DEFINITION 2. ([1]) Let $\alpha \in \mathbb{R}$ and $\theta < \vartheta$. For each function $f \in L_{\alpha,0}[\theta, \vartheta]$ let us define the integrals

$${}_{N_3}J_{\theta+}^{\alpha} f(\rho) = \int_{\theta}^{\rho} (\rho - t)^{-\alpha} f(t)\,dt$$

$${}_{N_3}J_{\vartheta-}^{\alpha} f(\rho) = \int_{\rho}^{\vartheta} (t - \rho)^{-\alpha} f(t)\,dt$$

for every $\rho \in [\theta, \vartheta]$. Here, for $\alpha = 0$ $_{N_3}J^{\alpha}_{\theta+}f(\rho) = {}_{N_3}J^{\alpha}_{\vartheta-}f(\rho) = \int_{\theta}^{\vartheta} f(t)dt$.

For studies obtained using new operators in fractional analysis and inequalities for fractional integral operators in inequality theory, readers can examine studies in [2, 3, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 22, 23, 24, 25, 26, 27].

The main purpose of this study is to reveal some new integral inequalities by using non-conformable integral operators. These new inequalities of the Chebyshev-Polya-Szegö type have been obtained by using classical inequalities and basic analysis methods such as AM-GM inequality and Cauchy-Schwarz inequality for mappings that can be integrated. We have given various simple inequalities by selecting some special cases.

2. Main results

The following theorem contains some new inequalities of Pólya-Szegö type for positive mappings via non-conformable integral operators.

THEOREM 2. *f and g are two positive integrable mappings on $[0,\infty)$ and $\alpha \in \mathbb{R}$. Suppose that there exist four positive integrable mappings v_1, v_2, w_1 and w_2 such that:*

$$0 < v_1(\tau) \leqslant f(\tau) \leqslant v_2(\tau),\ 0 < w_1(\tau) \leqslant g(\tau) \leqslant w_2(\tau) \quad (\tau \in [0,\rho],\ \rho > 0) \tag{2.1}$$

Then one has:

$$\frac{N_3J^{\alpha}_{0^+}\left\{w_1w_2f^2\right\}(\rho)\, N_3J^{\alpha}_{0^+}\left\{v_1v_2g^2\right\}(\rho)}{\left(N_3J^{\alpha}_{0^+}\left\{(v_1w_1+v_2w_2)fg\right\}(\rho)\right)^2} \leqslant \frac{1}{4}. \tag{2.2}$$

Proof. By using the relations that are given in (2.1), for $\tau \in [0,\rho]$, $\rho > 0$, we can easily have

$$\left(\frac{v_2(\tau)}{w_1(\tau)} - \frac{f(\tau)}{g(\tau)}\right) \geqslant 0 \tag{2.3}$$

and

$$\left(\frac{f(\tau)}{g(\tau)} - \frac{v_1(\tau)}{w_2(\tau)}\right) \geqslant 0. \tag{2.4}$$

If we product the inequalities (2.3) and (2.4) side by side, we can write

$$\left(\frac{v_2(\tau)}{w_1(\tau)} - \frac{f(\tau)}{g(\tau)}\right)\left(\frac{f(\tau)}{g(\tau)} - \frac{v_1(\tau)}{w_2(\tau)}\right) \geqslant 0.$$

Namely:

$$\begin{aligned} &(v_1(\tau)w_1(\tau) + v_2(\tau)w_2(\tau))\, f(\tau)g(\tau) \\ &\geqslant w_1(\tau)w_2(\tau)f^2(\tau) + v_1(\tau)v_2(\tau)g^2(\tau). \end{aligned} \tag{2.5}$$

To provide the non-conformable integral form, we multiply both sides of (2.5) by $(\rho - \tau)^{-\alpha}$ and then by integrating the resulting inequality with respect to τ over $(0,\rho)$, we can see that:

$$N_3J^{\alpha}_{0^+}\left\{(v_1w_1+v_2w_2)fg\right\}(\rho) \geqslant N_3J^{\alpha}_{0^+}\left\{w_1w_2f^2\right\}(\rho) + N_3J^{\alpha}_{0^+}\left\{v_1v_2g^2\right\}(\rho). \tag{2.6}$$

Lat us recall the AM-GM inequality, i.e. $(\theta+\vartheta \geqslant 2\sqrt{\theta\vartheta},\ \theta,\vartheta \in \mathbb{R}^+)$. By applying this classical inequality to (2.6), we obtain

$$N_3J^{\alpha}_{0^+}\{(v_1w_1+v_2w_2)fg\}(\rho) \geqslant 2\sqrt{N_3J^{\alpha}_{0^+}\{w_1w_2f^2\}(\rho)\times N_3J^{\alpha}_{0^+}\{v_1v_2g^2\}(\rho)}$$

By making use of some necessary operations, we deduce

$$N_3J^{\alpha}_{0^+}\left\{w_1w_2f^2\right\}(\rho)+N_3J^{\alpha}_{0^+}\left\{v_1v_2g^2\right\}(\rho) \leqslant \frac{1}{4}\left(N_3J^{\alpha}_{0^+}\{(v_1w_1+v_2w_2)fg\}(\rho)\right)^2.$$

This completes the proof of (2.1). □

Let us consider a special case for the result.

COROLLARY 1. *If we set $v_1=\zeta,\ v_2=\kappa,\ w_1=\delta$ and $w_2=\tau$ as special cases of these mappings, then we have*

$$\frac{\left(N_3J^{\alpha}_{0^+}f^2\right)(\rho)\left(N_3J^{\alpha}_{0^+}g^2\right)(\rho)}{\left(\left(N_3J^{\alpha}_{0^+}fg\right)(\rho)\right)^2} \leqslant \frac{1}{4}\left(\sqrt{\frac{\zeta\delta}{\kappa\tau}}+\sqrt{\frac{\kappa\tau}{\zeta\delta}}\right)^2.$$

THEOREM 3. *f and g are two positive integrable mappings on $[0,\infty)$ with α, $\theta\in\mathbb{R}-\{1\}$. Suppose that there exist four positive integrable mappings v_1,v_2,w_1 and w_2 satisfying condition (2.1). Then one has the following inequality for non-conformable integral operators:*

$$\begin{aligned} &N_3J^{\alpha}_{0^+}\{v_1v_2\}(\rho)N_3J^{\theta}_{0^+}\{w_1w_2\}(\rho)\times N_3J^{\alpha}_{0^+}\left\{f^2\right\}(\rho)N_3J^{\theta}_{0^+}\left\{g^2\right\}(\rho) \\ &\leqslant \frac{1}{4}\left(N_3J^{\alpha}_{0^+}\{v_1f\}(\rho)N_3J^{\theta}_{0^+}\{w_1g\}(\rho)+N_3J^{\alpha}_{0^+}\{v_2f\}(\rho)N_3J^{\theta}_{0^+}\{w_2g\}(\rho)\right)^2 \end{aligned} \tag{2.7}$$

Proof. Similar to the proof of the previous theorem, if we consider the inequalities that are given in (2.1), we have

$$\left(\frac{v_2(\tau)}{w_1(\xi)}-\frac{f(\tau)}{g(\xi)}\right)\geqslant 0$$

and

$$\left(\frac{f(\tau)}{g(\xi)}-\frac{v_1(\tau)}{w_2(\xi)}\right)\geqslant 0.$$

This inequalities can be written as the following form:

$$\left(\frac{v_1(\tau)}{w_2(\xi)}+\frac{v_2(\tau)}{w_1(\xi)}\right)\frac{f(\tau)}{g(\xi)} \geqslant \frac{f^2(\tau)}{g^2(\xi)}+\frac{v_1(\tau)v_2(\tau)}{w_1(\xi)w_2(\xi)}. \tag{2.8}$$

If we product both sides of (2.8) by $w_1(\xi)w_2(\xi)g^2(\xi)$, we get

$$\begin{aligned} &v_1(\tau)f(\tau)w_1(\xi)g(\xi)+v_2(\tau)f(\tau)w_2(\xi)g(\xi) \\ &\geqslant w_1(\xi)w_2(\xi)f^2(\tau)+v_1(\tau)v_2(\tau)g^2(\xi). \end{aligned} \tag{2.9}$$

By a similar argument, if we multiply both sides of (2.9) by $(\rho-t)^{-\alpha}(\rho-\xi)^{-\theta}$ and then by integrating with respect to τ and ξ over $(0,\rho)^2$, we obtain

$$\begin{aligned}&N_3J_{0^+}^{\alpha}\left\{v_1f\right\}(\rho)N_3J_{0^+}^{\theta}\left\{w_1g\right\}(\rho)+N_3J_{0^+}^{\alpha}\left\{v_2f\right\}(\rho)N_3J_{0^+}^{\theta}\left\{w_2g\right\}(\rho)\\ &\geqslant N_3J_{0^+}^{\alpha}\left\{f^2\right\}(\rho)N_3J_{0^+}^{\theta}\left\{w_1w_2\right\}(\rho)+N_3J_{0^+}^{\alpha}\left\{v_1v_2\right\}(\rho)N_3J_{0^+}^{\theta}\left\{g^2\right\}(\rho).\end{aligned}$$

To arrange the resulting inequality, we can use the AM-GM inequality, hence we deduce

$$\begin{aligned}&N_3J_{0^+}^{\alpha}\left\{v_1f\right\}(\rho)N_3J_{0^+}^{\theta}\left\{w_1g\right\}(\rho)+N_3J_{0^+}^{\alpha}\left\{v_2f\right\}(\rho)N_3J_{0^+}^{\theta}\left\{w_2g\right\}(\rho)\\ &\geqslant 2\sqrt{N_3J_{0^+}^{\alpha}\left\{f^2\right\}(\rho)N_3J_{0^+}^{\theta}\left\{w_1w_2\right\}(\rho)\times N_3J_{0^+}^{\alpha}\left\{v_1v_2\right\}(\rho)N_3J_{0^+}^{\theta}\left\{g^2\right\}(\rho)}\end{aligned}$$

which is the inequality that is stated in (2.7). The proof is completed. □

COROLLARY 2. *If we choose $v_1=\zeta$, $v_2=\kappa$, $w_1=\delta$ and $w_2=\tau$, then we have*

$$\left(\frac{\rho^{1-\alpha}\rho^{1-\theta}}{(1-\alpha)(1-\theta)}\right)\times\frac{\left(N_3J_{0^+}^{\alpha}f^2\right)(\rho)\left(N_3J_{0^+}^{\theta}g^2\right)(\rho)}{\left(\left(N_3J_{0^+}^{\alpha}f\right)(\rho)\left(N_3J_{0^+}^{\theta}g\right)(\rho)\right)^2}\leqslant\frac{1}{4}\left(\sqrt{\frac{\zeta\delta}{\kappa\tau}}+\sqrt{\frac{\kappa\tau}{\zeta\delta}}\right)^2$$

THEOREM 4. *f and g are two positive integrable mappings on $[0,\infty)$ with $\alpha\in\mathbb{R}$ and $\theta\in\mathbb{R}$. Suppose that there exist four positive integrable mappings v_1,v_2,w_1 and w_2 satisfying condition (2.1). Then the following inequality holds for non-conformable fractional integral operators:*

$$N_3J_{0^+}^{\alpha}\left\{f^2\right\}(\rho)N_3J_{0^+}^{\theta}\left\{g^2\right\}(\rho)\leqslant N_3J_{0^+}^{\alpha}\left\{\frac{v_2fg}{w_1}\right\}(\rho)N_3J_{0^+}^{\theta}\left\{\frac{w_2fg}{v_1}\right\}(\rho).\tag{2.10}$$

Proof. By a different variant of the inequality (2.1), we get

$$f^2(\tau)\leqslant\frac{v_2(\tau)}{w_1(\tau)}f(\tau)g(\tau).\tag{2.11}$$

By multiplying both sides of (2.11) by $(\rho-\tau)^{-\alpha}$ and by applying integration to the resulting inequality with respect to τ over $(0,\rho)$, we obtain

$$\int_0^{\rho}(\rho-\tau)^{-\alpha}f^2(t)d\tau\leqslant\int_0^{\rho}(\rho-\tau)^{-\alpha}\frac{v_2(t)}{w_1(t)}f(t)g(t)d\tau.$$

This inequality is obviously equal to

$$N_3J_{0^+}^{\alpha}\left\{f^2\right\}(\rho)\leqslant N_3J_{0^+}^{\alpha}\left\{\frac{v_2fg}{w_1}\right\}(\rho).\tag{2.12}$$

Similarly, we can use the following inequality

$$g^2(\xi)\leqslant\frac{w_2(\xi)}{v_1(\xi)}f(\xi)g(\xi).$$

By making use of the similar processes, we obtain

$$\int_0^\rho (\rho-\xi)^{-\theta} g^2(\xi)\,d\xi \leqslant \int_0^\rho (\rho-\xi)^{-\theta} \frac{w_2(\xi)}{v_1(\xi)} f(\xi) g(\xi)\,d\xi.$$

This result can be converted to non-conformable form as:

$$N_3J_{0^+}^{\theta}\left\{g^2\right\}(\rho) \leqslant N_3J_{0^+}^{\theta}\left\{\frac{w_2 f g}{v_1}\right\}(\rho). \tag{2.13}$$

By multiplying (2.12) and (2.13), we get the (2.10). The proof is completed. □

COROLLARY 3. *If $v_1=\zeta$, $v_2=\kappa$, $w_1=\delta$ and $w_2=\tau$, then we have*

$$\frac{\left(N_3J_{0^+}^{\alpha} f^2\right)(\rho)\left(N_3J_{0^+}^{\theta} g^2\right)(\rho)}{\left(\left(N_3J_{0^+}^{\alpha} fg\right)(\rho)\left(N_3J_{0^+}^{\theta} fg\right)(\rho)\right)^2} \leqslant \frac{\kappa\tau}{\zeta\delta}$$

THEOREM 5. *f and g are two positive integrable mappings on $[0,\infty)$ with α, $\theta \in \mathbb{R}-\{1\}$. Suppose that there exist four positive integrable mappings v_1, v_2, w_1 and w_2 satisfying condition (2.1). Then the following inequality holds for non-conformable integral operators:*

$$\begin{aligned}
&\left|\left(\frac{\rho^{1-\alpha}}{1-\alpha}\right)\left(N_3J_{0^+}^{\theta} fg\right)(\rho) + \left(\frac{\rho^{1-\theta}}{1-\theta}\right)\left(N_3J_{0^+}^{\alpha} fg\right)(\rho)\right. \\
&\left. - \left(N_3J_{0^+}^{\alpha} f\right)(\rho)\left(N_3J_{0^+}^{\theta} g\right)(\rho) - \left(N_3J_{0^+}^{\theta} f\right)(\rho)\left(N_3J_{0^+}^{\alpha} g\right)(\rho)\right| \\
&\leqslant \left|A_1(f,v_1,v_2)(\rho) + A_2(f,v_1,v_2)(\rho)\right|^{1/2}\left|A_1(g,w_1,w_2)(\rho) + A_2(g,w_1,w_2)(\rho)\right|^{1/2}
\end{aligned} \tag{2.14}$$

where

$$A_1(u,v,w)(\rho) = \left(\frac{\rho^{1-\theta}}{1-\theta}\right) \times \frac{\left(N_3J_{0^+}^{\alpha}\{(v+w)u\}(\rho)\right)^2}{4N_3J_{0^+}^{\alpha}\{vw\}(\rho)} - \left(N_3J_{0^+}^{\alpha} u\right)(\rho)\left(N_3J_{0^+}^{\theta} u\right)(\rho)$$

and

$$A_2(u,v,w)(\rho) = \left(\frac{\rho^{1-\alpha}}{1-\alpha}\right) \times \frac{\left(N_3J_{0^+}^{\theta}\{(v+w)u\}(\rho)\right)^2}{4N_3J_{0^+}^{\theta}\{vw\}(\rho)} - \left(N_3J_{0^+}^{\alpha} u\right)(\rho)\left(N_3J_{0^+}^{\theta} u\right)(\rho).$$

Proof. Let us define the functional $H(\tau,\xi)$ for $\tau,\xi \in (0,\rho)$ with $\rho>0$ by using f and g that are two positive integrable mappings on $[0,\infty)$ as following:

$$H(\tau,\xi) = (f(\tau)-f(\xi))(g(\tau)-g(\xi)).$$

This functional can be represented as:

$$H(\tau,\xi) = f(\tau)g(\tau) + f(\xi)g(\xi) - f(\tau)g(\xi) - f(\xi)g(\tau). \tag{2.15}$$

Multiplying both sides by $(\rho-t)^{-\alpha}(\rho-t)^{-\theta}$ and double integrating the resulting inequality with respect to τ and ξ over $(0,\rho)^2$, we have

$$\begin{aligned}
&\int_0^{\rho}\int_0^{\rho}(\rho-\tau)^{-\alpha}(\rho-\xi)^{-\theta}H(\tau,\xi)\,d\tau d\xi\\
=&\left(\frac{\rho^{1-\theta}}{1-\theta}\right)\left(N_3J_{0+}^{\alpha}fg\right)(\rho)+\left(\frac{\rho^{1-\alpha}}{1-\alpha}\right)\left(N_3J_{0+}^{\theta}fg\right)(\rho)\\
&-\left(N_3J_{0+}^{\alpha}f\right)(\rho)\left(N_3J_{0+}^{\theta}g\right)(\rho)-\left(N_3J_{0+}^{\theta}f\right)(\rho)\left(N_3J_{0+}^{\alpha}g\right)(\rho).
\end{aligned}$$

Therefore, the Cauchy-Schwarz inequality will play a key role to simplify the above inequality as follows:

$$\begin{aligned}
&\left|\int_0^{\rho}\int_0^{\rho}(\rho-\tau)^{-\alpha}(\rho-\xi)^{-\theta}H(\tau,\xi)\,d\tau d\xi\right|\\
\leqslant&\left[\int_0^{\rho}\int_0^{\rho}(\rho-\tau)^{-\alpha}(\rho-\xi)^{-\theta}f^2(\tau)d\tau d\xi\right]\\
&+\int_0^{\rho}\int_0^{\rho}(\rho-\tau)^{-\alpha}(\rho-\xi)^{-\theta}f^2(\xi)\,d\tau d\xi\\
&-2\int_0^{\rho}\int_0^{\rho}(\rho-\tau)^{-\alpha}(\rho-\xi)^{-\theta}f(\tau)f(\xi)\,d\tau d\xi\Bigg]^{1/2}\\
&\times\Bigg[\int_0^{\rho}\int_0^{\rho}(\rho-\tau)^{-\alpha}(\rho-\xi)^{-\theta}g^2(\tau)\,d\tau d\xi\\
&+\int_0^{\rho}\int_0^{\rho}(\rho-\tau)^{-\alpha}(\rho-\xi)^{-theta}g^2(\xi)\,d\tau d\xi\\
&-2\int_0^{\rho}\int_0^{\rho}(\rho-\tau)^{-\alpha}(\rho-\xi)^{-\theta}g(\tau)g(\xi)\,d\tau d\xi\Bigg]^{1/2}.
\end{aligned}$$

By making use of the necessary modifications with the definition of non-conformable integrals, we get

$$\begin{aligned}
&\left|\int_0^{\rho}\int_0^{\rho}(\rho-\tau)^{-\alpha}(\rho-\xi)^{-\theta}H(\tau,\xi)\,d\tau d\xi\right|\\
\leqslant&\Bigg[\left(\frac{\rho^{1-\theta}}{1-\theta}\right)\left(N_3J_{0+}^{\alpha}f^2\right)(\rho)+\left(\frac{\rho^{1-\alpha}}{1-\alpha}\right)\left(N_3J_{0+}^{\theta}f^2\right)(\rho)\\
&-2\left(N_3J_{0+}^{\alpha}f\right)(\rho)\left(N_3J_{0+}^{\theta}f\right)(\rho)\Bigg]^{1/2}\\
&\times\Bigg[\left(\frac{\rho^{1-\theta}}{1-\theta}\right)\left(N_3J_{0+}^{\alpha}g^2\right)(\rho)+\left(\frac{\rho^{1-\alpha}}{1-\alpha}\right)\left(N_3J_{0+}^{\theta}g^2\right)(\rho)\\
&-2\left(N_3J_{0+}^{\alpha}g\right)(\rho)\left(N_3J_{0+}^{\theta}g\right)(\rho)\Bigg]^{1/2}
\end{aligned}$$

Applying Theorem 2 with $w_1(\tau) = w_2(\tau) = g(\tau) = 1$, we deduce

$$\left(\frac{\rho^{1-\theta}}{1-\theta}\right) N_3 J_{0^+}^{\alpha}\left\{f^2\right\}(\rho) \leqslant \left(\frac{\rho^{1-\theta}}{1-\theta}\right) \frac{\left(N_3 J_{0^+}^{\alpha}\left\{(v_1+v_2) f\right\}(\rho)\right)^2}{4 N_3 J_{0^+}^{\alpha}\left\{v_1 v_2\right\}(\rho)}.$$

It is obvious to see that:

$$\begin{aligned} &\left(\frac{\rho^{1-\theta}}{1-\theta}\right) N_3 J_{0^+}^{\alpha}\left\{f^2\right\}(\rho) - \left(N_3 J_{0^+}^{\alpha} f\right)(\rho)\left(N_3 J_{0^+}^{\theta} f\right)(\rho) \\ \leqslant &\left(\frac{\rho^{1-\theta}}{1-\theta}\right) \frac{\left(N_3 J_{0^+}^{\alpha}\left\{(v_1+v_2) f\right\}(\rho)\right)^2}{4 N_3 J_{0^+}^{\alpha}\left\{v_1 v_2\right\}(\rho)} - \left(N_3 J_{0^+}^{\alpha} f\right)(\rho)\left(N_3 J_{0^+}^{\theta} f\right)(\rho) \\ = &A_1(f, v_1, v_2) \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} &\left(\frac{\rho^{1-\alpha}}{1-\alpha}\right) N_3 J_{0^+}^{\theta}\left\{f^2\right\}(\rho) - \left(N_3 J_{0^+}^{\alpha} f\right)(\rho)\left(N_3 J_{0^+}^{\theta} f\right)(\rho) \\ \leqslant &\left(\frac{\rho^{1-\alpha}}{1-\alpha}\right) \frac{\left(N_3 J_{0^+}^{\theta}\left\{(v_1+v_2) f\right\}(\rho)\right)^2}{4 N_3 J_{0^+}^{\theta}\left\{v_1 v_2\right\}(\rho)} - \left(N_3 J_{0^+}^{\alpha} f\right)(\rho)\left(N_3 J_{0^+}^{\theta} f\right)(\rho) \\ = &A_2(f, v_1, v_2). \end{aligned} \tag{2.17}$$

Similarly, applying Theorem 2 with $v_1(\tau) = v_2(\tau) = f(\tau) = 1$, we have

$$\begin{aligned} &\left(\frac{\rho^{1-\theta}}{1-\theta}\right) N_3 J_{0^+}^{\alpha}\left\{g^2\right\}(\rho) - \left(N_3 J_{0^+}^{\alpha} g\right)(\rho)\left(N_3 J_{0^+}^{\theta} g\right)(\rho) \\ \leqslant &A_1(g, w_1, w_2) \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} &\left(\frac{\rho^{1-\alpha}}{1-\alpha}\right) N_3 J_{0^+}^{\theta}\left\{g^2\right\}(\rho) - \left(N_3 J_{0^+}^{\alpha} g\right)(\rho)\left(N_3 J_{0^+}^{\theta} g\right)(\rho) \\ \leqslant &A_2(g, w_1, w_2). \end{aligned} \tag{2.19}$$

Using (2.16)–(2.19), we conclude the result. □

THEOREM 6. *f and g are two positive integrable mappings on $[0,\infty)$ with α, $\theta \in \mathbb{R} - \{1\}$. Suppose that there exist four positive integrable mappings v_1, v_2, w_1 and w_2 satisfying condition (2.1). Then the following inequality holds for non-conformable integral operators:*

$$\begin{aligned} &\left|\left(\frac{\rho^{1-\alpha}}{1-\alpha}\right) N_3 J_{0^+}^{\alpha}\{fg\}(\rho) - \left(N_3 J_{0^+}^{\alpha} f\right)(\rho)\left(N_3 J_{0^+}^{\alpha} g\right)(\rho)\right| \\ \leqslant &\left|A(f, v_1, v_2)(\rho) A(g, w_1, w_2)(\rho)\right|^{1/2} \end{aligned} \tag{2.20}$$

where

$$A(u, v, w)(\rho) = \left(\frac{\rho^{1-\alpha}}{1-\alpha}\right) \times \frac{\left(N_3 J_{0^+}^{\alpha}\{(v+w) u\}(\rho)\right)^2}{4 N_3 J_{0^+}^{\alpha}\{vw\}(\rho)} - \left(\left(N_3 J_{0^+}^{\alpha} u\right)(\rho)\right)^2.$$

Proof. If we choose $\alpha = \theta$ in (2.14), we get (2.20). □

COROLLARY 4. *If we set $v_1 = \zeta$, $v_2 = \kappa$, $w_1 = \delta$ and $w_2 = \tau$, then we have*

$$\left| \left(\frac{\rho^{1-\alpha}}{1-\alpha} \right) N_3 J^{\alpha}_{0+} \{fg\}(\rho) - \left(N_3 J^{\alpha}_{0+} f \right)(\rho) \left(N_3 J^{\alpha}_{0+} g \right)(\rho) \right|$$
$$\leqslant \frac{(\kappa - \zeta)(\tau - \delta)}{4\sqrt{\zeta\delta\kappa\tau}} \times \left(N_3 J^{\alpha}_{0+} f \right)(\rho) \left(N_3 J^{\alpha}_{0+} g \right)(\rho).$$

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