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Nonlinear generalized fractional differential equations with generalized fractional integral conditions

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ABSTRACT

This research work is dedicated to an investigation of the existence and uniqueness of a class of nonlinear ψ-Caputo fractional differential equation on a finite interval $[0, T]$, equipped with nonlinear ψ-Riemann–Liouville fractional integral boundary conditions of different orders $0 < \alpha, \beta < 1$, we deal with a recently introduced ψ-Caputo fractional derivative of order $1 < q \leq 2$. The formulated problem will be transformed into an integral equation with the help of Green function. A full analysis of existence and uniqueness of solutions is proved using fixed point theorems: Leray–Schauder nonlinear alternative, Krasnoselskii and Schauder's fixed point theorems, Banach's and Boyd–Wong's contraction principles. We show that this class generalizes several other existing classes of fractional-order differential equations, and therefore the freedom of choice of the standard fractional operator. As an application, we provide an example to demonstrate the validity of our results.

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ψ-Fractional integral; ψ-Riemann–Liouville fractional derivative; ψ-Caputo fractional derivative

1. Introduction

The study of differentiation and integration to a fractional order has caught importance and popularity among researchers compared to classical differentiation and integration. Fractional operators used to illustrate better the reality of real-world phenomena with the hereditary property. For instance, various applications and comprehensive strategy of the fractional calculus are addressed in the works of Baleanu et al. [1,2], Abd-Elhameed et al. [3,4], Jarad et al. [5], Hafez et al. [6], and Youssri et al. [7]. A good review of different fractional operators can be found in [8]. It has been proved that differential equation with fractional-order process more accurately than integer-order differential equations do, and fractional arrangers provide excellent performance of the description of hereditary attributes than integer-order arrangers. Applications can be found in complex viscoelastic media, electrical spectroscopy, porous media, cosmology, environmental science, medicine (the modelling of infectious diseases), signal and image processing, materials, and many others. For detail, see analogous discussions to the topics in a comprehensive review by Sun et al. [9].

Fractional-order boundary value problems of nonlinear fractional differential equations have been extensively investigated by many authors. By applying various techniques of nonlinear analysis; many researchers have studied the existence of solutions of fractional-order differential equations supplemented by integral boundary conditions involve either classical, Riemann–Liouville, Hadamard, Erdélyi–Kober, or Katugampola type. For instance, in [10,11] Ahmad et al. applied the classical fixed point theory to nonlinear fractional differential equations with nonlocal generalized fractional integral boundary conditions. The author showed that the considered problems have a unique solution and unify some available results. In [12] Sun et al. investigated the existence of nonlinear fractional-order boundary value problems with nonlocal Erdélyi–Kober and generalized Riemann–Liouville type integral boundary conditions using Mawhin continuation theorem. We refer the reader to the survey by Agarwal et al. [13] which particularly had a chronological listing on major works in the investigation of the existence and uniqueness of differential equations and inclusions of fractional-orders with various boundary conditions.

In [14], Samko et al. presented fractional integrals and derivatives with unlike kernels. These fractional operators are now known as ψ-fractional operators, and it has been shown that these operators unify a wide class of fractional differentiations and integrations such as the aforementioned ones. As an application managing with the theory of ψ-fractional differentiation, we refer to [15], the reader will find other classifications of ψ-fractional differentiation with various applications in the work [16–18]

In [19], Almeida investigate the existence and uniqueness of solution for the following boundary value

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problem of fractional differential equations

$$^{c}\mathcal{D}^{\alpha}_{0+;\psi} = f(t, y(t)), \quad t \in [a, b],\ 2 < \alpha < 3, \tag{1}$$

$$y(a) = y_a, \quad y'(a) = y_{a_1}, \quad y(b) = K I^{\alpha}_{a+;\psi} y(\vartheta), \tag{2}$$

where $^{c}\mathcal{D}^{\alpha}_{0+;\psi}$ is the ψ-Caputo fractional derivative introduced in [20], $I^{\alpha}_{0+;\psi}$ is the ψ-Riemann–Liouville fractional integral presented in [14]. The existence results are obtained via the aid of some classical fixed point theorems.

We deal in this paper with a recently established ψ-Caputo fractional derivative within the framework of absolutely continuous functions proposed by Jarad and Abdeljawad [21]. We focus on the existence and uniqueness of solutions for a nonlinear fractional differential equation involving the ψ-Caputo fractional derivative, supplemented with ψ-Riemann–Liouville fractional integral boundary conditions of different orders

$$^{c}\mathcal{D}^{q}_{0^+;\psi} y(t) = f(t, y(t)), \quad t \in I = [0, T],\ 1 < q \leq 2, \tag{3}$$

$$y(0) - \delta_\psi y(0) = \frac{1}{\Gamma(\alpha)} \int_0^{\sigma} (\psi(\sigma) - \psi(s))^{\alpha-1} \psi'(s) g(s, y(s))\,\mathrm{d}s = I^{\alpha}_{0+;\psi} g(\sigma, y(\sigma)), \tag{4}$$

$$y(T) + \delta_\psi y(T) = \frac{1}{\Gamma(\beta)} \int_0^{\eta} (\psi(\eta) - \psi(s))^{\beta-1} \psi'(s) h(s, y(s))\mathrm{d}s = I^{\beta}_{0+;\psi} h(\eta, y(\eta)), \tag{5}$$

where $0 < \alpha, \beta \leq 1$, $\psi' > 0$ on $[0, T]$, $0 < \sigma, \eta < T$, f, g and h three real continuous functions defined on $[0, T] \times \mathbb{R}$, and $\delta_\psi = (1/\psi'(t))\mathrm{d}/\mathrm{d}t$

Our results here are new and generalize some known results in the literature for specific choices of the parameters involved. For instance, selecting $\psi(t) = t$, $\alpha = \beta = 1, \eta \to T^-$, and $\sigma \to T^-$ in the problem (3)–(5), the boundary conditions take the form

$$y(0) - y'(0) = \int_0^T g(s, y(s))\,\mathrm{d}s, \quad y(T) + y'(T) = \int_0^T h(s, y(s))\,\mathrm{d}s, \tag{6}$$

and the resulting problem meets the one studied in [22], reduces the one considered in [23] under a weakly sequentially continuity assumption imposed on f, g and h. Studies in [24] use the technique of measures of noncompactness and the Mönch's fixed point theorem. In case we choose $\delta_\psi y(0) = \delta_\psi y(T) = 0$ the problem (3)–(5) generalize the one considered in [13]. An integer-order version of (3)–(5) was considered in [25] with $q = 2, \psi(t) = t, \alpha = \beta = 1, \eta \to 1^-$, and $\sigma \to 1^-$, where the existence of solutions and extremal solutions are established. In summary, the present paper covers some interesting situations.

The paper is organized as follows. In Section 2, we present the main concepts of the generalized fractional calculus and give some valuable preliminary results. In Section 3, we prove the existence and uniqueness of solution to the problem (3)–(5) by using the standard fixed-point theorem. In Section 4, we present an illustrative example. At last, we conclude our results.

2. Preliminaries and mathematical background

We review a few definitions, notations and results of ψ-fractional integrals and derivatives which will be used throughout this paper.

By $C([0, T], \mathbb{R})$ we denote the Banach space of all continuous functions from $[0, T]$ endowed with the norm defined by

$$\|u\|_\infty = \sup_{[0,T]} |u(x)|.$$

Let $AC([0, T], \mathbb{R})$ denotes the space of all absolutely continuous real valued function on $[0, T]$. Moreover, we define the space $AC^n_\psi([0, T], \mathbb{R})$ by

$$AC^n_\psi([0, T], \mathbb{R}) = \left\{ f : [0, T] \to \mathbb{R}; (\delta^{n-1}_\psi f)(t) \in AC([0, T], \mathbb{R}), \delta_\psi = \frac{1}{\psi'(t)} \frac{\mathrm{d}}{\mathrm{d}t} \right\},$$

which is endowed with the norm defined by

$$\|f\|_{C^n_\psi} = \sum_{k=0}^{n-1} \|\delta^k_\psi f(t)\|_\infty,$$

where $\psi \in C^n[0, T]$, $\psi'(t) > 0$ on $[0, T]$, and

$$\delta^k_\psi = \underbrace{\delta_\psi \delta_\psi \ldots \delta_\psi}_{\text{k times}}.$$

Definition 2.1 ([14]): Let $f : [0, T] \to \mathbb{R}$ be an integrable function. The ψ-Riemann–Liouville integral of order $\alpha > 0$ of f for $0 < t < T < +\infty$, is defined by

$$I^{\alpha}_{0+;\psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha-1} \psi'(s) f(s)\,\mathrm{d}s. \tag{7}$$

Definition 2.2 ([21]): For $0 < t < T < +\infty$, the ψ-Riemann–Liouville fractional derivative of order $\alpha > 0$ of a function $f \in AC^n_\psi([0, T])$ is defined by

$$\mathcal{D}^{\alpha}_{0+;\psi} f(t) = I^{n-\alpha}_{0+;\psi} (\delta^n_\psi f)(t) + \sum_{k=0}^{n-1} \frac{(\delta^k_\psi f)(0)}{\Gamma(k - \alpha + 1)} (\psi(t) - \psi(0))^{k-\alpha}$$

$$= \frac{1}{\Gamma(n-\alpha)} \int_0^t (\psi(t)-\psi(s))^{n-\alpha-1} \psi'(s)\delta_\psi^n f(s)\,ds + \sum_{k=0}^{n-1} \frac{(\delta_\psi^k f)(0)}{\Gamma(k-\alpha+1)} \times (\psi(t)-\psi(0))^{k-\alpha}, \quad (8)$$

if the integral exists, where $n=[\alpha]+1$, $\Gamma(.)$ is the Euler's gamma function defined by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$.

Definition 2.3 ([20,21]): For $f \in AC_\psi^n([0,T])$, the ψ-Caputo fractional derivative of order $\alpha > 0$, is defined by

$$^c\mathcal{D}^\alpha_{0+;\psi} f(t) = I^{n-\alpha}_{0+,\psi}(\delta_\psi^n f)(t) = \frac{1}{\Gamma(n-1)} \int_0^t (\psi(t)-\psi(s))^{n-\alpha-1} \times \psi'(s)(\delta_\psi^n f)(s)\,ds, \quad n=[\alpha]+1.$$

Thus, if $\alpha = n \in \mathbb{N}$ we have

$$^c\mathcal{D}^\alpha_{0+;\psi} f(t) = (\delta_\psi^n f)(t).$$

Lemma 2.4 ([21]): *Given a function $f \in AC_\psi^n[0,T]$, and $\alpha \in \mathbb{R}^+$, then*

$$I^\alpha_{0+;\psi}\,{}^c\mathcal{D}^\alpha_{0+;\psi} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(\delta_\psi^k f)(0)}{k!}(\psi(t)-\psi(0))^k, \quad (9)$$

in particular, for $0 < \alpha < 1$, we have

$$I^\alpha_{0+;\psi}\,{}^c\mathcal{D}^\alpha_{0+;\psi} f(t) = f(t) - f(0).$$

In the following auxiliary lemma, we solve the linear version of the problem (3)–(5).

Lemma 2.5: *Let $1 < q \le 2$ and $\varphi, \phi_1, \phi_2 : [0,T] \to \mathbb{R}$ be continuous. A function $y \in AC_\psi^2([0,T],\mathbb{R})$ is a solution of the fractional integral equation*

$$y(t) = L(t) + \int_0^T G_\psi(t,s)\varphi(s)\,ds, \quad (10)$$

where

$$L(t) = \frac{\psi(T)-\psi(t)+1}{\psi(T)-\psi(0)+2} I^\alpha_{0+;\psi}\phi_1(\sigma) + \frac{\psi(t)-\psi(0)+1}{\psi(T)-\psi(0)+2} I^\beta_{0+;\psi}\phi_2(\eta),$$

here, ψ is an increasing function on $[0,T]$ such that $\psi \in C^2[0,T]$, and $G_\psi(t,s)$ is a Green's function given by

$$G_\psi(t,s) = \psi'(s) \times \begin{cases} \dfrac{(\psi(t)-\psi(s))^{q-1}}{\Gamma(q)} - \dfrac{\psi(t)-\psi(0)+1}{(\psi(T)-\psi(0)+2)\Gamma(q)} (\psi(T)-\psi(s))^{q-1} - \frac{\psi(t)-\psi(0)+1}{(\psi(T)-\psi(0)+2)\Gamma(q-1)} (\psi(T)-\psi(s))^{q-2}, & 0 \le s \le t, \\ -\dfrac{\psi(t)-\psi(0)+1}{(\psi(T)-\psi(0)+2)\Gamma(q)}(\psi(T)-\psi(s))^{q-1} - \frac{\psi(t)-\psi(0)+1}{(\psi(T)-\psi(0)+2)\Gamma(q-1)}(\psi(T)-\psi(s))^{q-2}, & t \le s \le T, \end{cases} \quad (11)$$

if and only if y is a solution of the following ψ-fractional BVP

$$^c\mathcal{D}^q_{0+;\psi} y(t) = \varphi(t), \quad t \in I := [0,T], \quad (12)$$

$$y(0) - \delta_\psi y(0) = I^\alpha_{0+;\psi}\phi_1(\sigma), \quad (13)$$

$$y(T) + \delta_\psi y(T) = I^\beta_{0+;\psi}\phi_2(\eta). \quad (14)$$

Proof: Applying the ψ-Riemann–Liouville operator $I^\alpha_{0+;\psi}$ on both sides of Equation (12) and using Lemma 2.4, we obtain

$$y(t) = c_1 + c_2(\psi(t)-\psi(0)) + I^q_{0+;\psi}\varphi(t), \quad (15)$$

where c_1, c_2 are arbitrary constants. Taking the δ_ψ −derivative (15) we get

$$(\delta_\psi y)(t) = c_2 + I^{q-1}_{0+;\psi}\varphi(t). \quad (16)$$

From (13) and (14), we get

$$c_1 - c_2 = I^\alpha_{0+;\psi}\phi_1(\sigma), \quad (17)$$

and

$$c_1 + c_2(\psi(T)-\psi(0)+1) + I^q_{0+;\psi}\varphi(T) + I^{q-1}_{0+;\psi}\varphi(T) = I^\beta_{0+;\psi}\phi_2(\eta). \quad (18)$$

Equation (15) and (18) give

$$c_2 = \frac{1}{\psi(T)-\psi(0)+2} I^\beta_{0+;\psi}\phi_2(\eta) - \frac{1}{\psi(T)-\psi(0)+2} I^\alpha_{0+;\psi}\phi_1(\sigma) - \frac{1}{\psi(T)-\psi(0)+2} I^q_{0+;\psi}\varphi(T) - \frac{1}{\psi(T)-\psi(0)+2} I^{q-1}_{0+;\psi}\varphi(T), \quad (19)$$

and

$$c_1 = \frac{\psi(T)-\psi(0)+1}{\psi(T)-\psi(0)+2} I^{\alpha}_{0+;\psi}\phi_1(\sigma) + \frac{1}{\psi(T)-\psi(0)+2} I^{\beta}_{0+;\psi}\phi_2(\eta) - \frac{1}{\psi(T)-\psi(0)+2}\left[I^{q}_{0+;\psi}\varphi(T) + I^{q-1}_{0+;\psi}\varphi(T)\right]. \tag{20}$$

From (15), (19), (20), and using the fact that $\int_0^T = \int_0^t + \int_t^T$ we get

$$y(t) = L(t) + \int_0^T G_\psi(t,s)\varphi(s)\,\mathrm{d}s, \tag{21}$$

where

$$L(t) = \frac{\psi(T)-\psi(t)+1}{\psi(T)-\psi(0)+2} I^{\alpha}_{0+;\psi}\phi_1(\sigma) + \frac{\psi(t)-\psi(0)+1}{\psi(T)-\psi(0)+2} I^{\beta}_{0+;\psi}\phi_2(\eta), \tag{22}$$

and $G_\psi(t,s)$ is the Green function such that defined by (11).

Therefore, we have (10). Inversely, it is obvious that if y satisfies Equation (10), then (12)–(14) hold. ■

3. Existence and uniqueness of solutions

In this section, we deal with the existence and uniqueness of solutions for the fractional-order boundary value problem (3)–(5) using certain fixed point theorems.

Remark 3.1: From the expression of $G_\psi(t,s)$, it is clear that $G_\psi(t,s)$ is continuous on $I \times I$, and hence is bounded. Thus, we let

$$G^*_\psi = \sup\left\{\int_0^T |G_\psi(t,s)|, t \in I\right\}.$$

In the following, for computational convenience, we set the following notations

$$\Omega_1 = \frac{(\psi(T)+1)(\psi(\sigma))^{\alpha}}{(\psi(T)-\psi(0)+2)\Gamma(\alpha+1)}, \tag{23}$$

$$\Omega_2 = \frac{((\psi(T)+1)(\psi(\eta))^{\beta}}{(\psi(T)-\psi(0)+2)\Gamma(\beta+1)}. \tag{24}$$

Based on Banach's fixed point theorem [26], we state the uniqueness of solutions of the problem (3)–(5).

Theorem 3.2: *Assume that the following hypotheses hold:*

(H1) *There exists a non-negative constant $k > 0$ such that*

$$|f(t,u) - f(t,v)| \le k|u-v|,$$
$$\text{for all } t \in I \text{ a.e } u,v \in \mathbb{R}.$$

(H2) *There exists a non-negative constant $k^* > 0$ such that*

$$|g(t,u) - g(t,v)| \le k^*|u-v|,$$
$$\text{for all } t \in I \text{ a.e } u,v \in \mathbb{R}.$$

(H3) *There exists a non-negative constant $k^{**} > 0$ such that*

$$|h(t,u) - h(t,v)| \le k^{**}|u-v|,$$
$$\text{for all } t \in I \text{ a.e } u,v \in \mathbb{R}.$$

Then the problem (3)–(5) has a unique solution on I, provided that

$$\left(\Omega_1 k^* + \Omega_2 k^{**} + TG^*_\psi k\right) < 1, \tag{25}$$

where Ω_1 and Ω_2 are given by (23) and (24) respectively.

Proof: We reformulate the problem (3)–(5) as a fixed point problem by considering the operator

$$N : C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R}),$$

defined by the formula

$$(Ny)(t) = L(t) + \int_0^T G_\psi(t,s) f(s,y(s))\,\mathrm{d}s, \tag{26}$$

where

$$L(t) = \frac{(\psi(T)-\psi(t)+1)}{(\psi(T)-\psi(0)+2)\Gamma(\alpha)} \int_0^{\sigma} (\psi(\sigma)-\psi(s))^{\alpha-1} \times \psi'(s) g(s,y(s))\,\mathrm{d}s + \frac{(\psi(t)-\psi(0)+1)}{(\psi(T)-\psi(0)+2)\Gamma(\beta)} \int_0^{\eta} (\psi(\eta)-\psi(s))^{\alpha-1} \times \psi'(s) h(s,y(s))\,\mathrm{d}s, \tag{27}$$

and the Green function $G_\psi(t,s)$ is given by (11). It is well known that the fixed points of the operator N are solutions of the problem (3)–(5). By using the Banach contraction mapping principle, we shall show that N has a fixed point.

Let $x, y \in C([0,T],\mathbb{R})$. Then, for each $t \in [0,T]$, one can obtain

$$|(Nx)(t) - (Ny)(t)| \le \frac{(\psi(T)+1)}{(\psi(T)-\psi(0)+2)\Gamma(\alpha)} \int_0^{\sigma} |(\psi(\sigma)-\psi(s))^{\alpha-1} \psi'(s)||g(s,x(s)) - g(s,y(s))|\,\mathrm{d}s + \frac{(\psi(T)+1)}{(\psi(T)-\psi(0)+2)\Gamma(\beta)} \int_0^{\eta} |(\psi(\eta)-\psi(s))^{\beta-1} \psi'(s)||h(s,x(s)) - h(s,y(s))|\,\mathrm{d}s + \int_0^T |G_\psi(t,s)||f(s,x(s)) - f(s,y(s))|\,\mathrm{d}s$$

$$\leq \frac{(\psi(T)+1)(\psi(\sigma))^{\alpha}}{(\psi(T)-\psi(0)+2)\Gamma(\alpha+1)} k^{*}\|x-y\|_{\infty}$$
$$+ \frac{(\psi(T)+1)(\psi(\eta))^{\beta}}{(\psi(T)-\psi(0)+2)\Gamma(\beta+1)} k^{**}\|x-y\|_{\infty}$$
$$+ TG_{\psi}^{*} k\|x-y\|_{\infty}$$
$$\leq (\Omega_1 k^{*} + \Omega_2 k^{**} + TG_{\psi}^{*} k)\|x-y\|_{\infty}.$$

Taking the supremum, we obtain

$$\|N(x)-N(y)\|_{\infty} \leq (\Omega_1 k^{*} + \Omega_2 k^{**} + TG_{\psi}^{*} k)\|x-y\|_{\infty}.$$

Hence, by (25), N is a contraction. By the Banach fixed point theorem N has a fixed point which is a solution of the problem (3)–(5). The proof is now complete. ■

Now, our next uniqueness result for the problem (3)–(5) relies on Boyd–Wong Contraction Principle [27].

Theorem 3.3: *Let X be a complete metric space and suppose $T : X \to X$ satisfies*

$$d(Tx, Ty) \leq \Psi(d(x,y)), \quad \text{for each } x, y \in X,$$

where $\Psi : [0, +\infty) \to [0, +\infty)$ is upper semi-continuous function from the right (i.e $r_j \downarrow r \geq 0 \Rightarrow \limsup_{n\to+\infty} \Psi(r_j) \leq \Psi(r)$) and satisfies $0 \leq \Psi < t$ for $t > 0$. Then T has a unique fixed point $\bar{x}$ and $T_n(x)$ converges to $\bar{x}$ for each $x \in X$.

Theorem 3.4: *Assume that the following hypothesis hold*

(H4) *there exist χ_1, χ_2 and $\chi_3 : \mathbb{R}^+ \to \mathbb{R}^+$ upper semi-continuous from the right and non-decreasing functions such that*

$$|f(t,u) - f(t,v)| \leq \chi_1(|u-v|), \tag{28}$$
$$|g(t,u) - g(t,v)| \leq \chi_2(|u-v|), \tag{29}$$
$$|h(t,u) - h(t,v)| \leq \chi_3(|u-v|), \tag{30}$$

for all $t \in [0,T]$ a.e u and $v \in \mathbb{R}$, and

$$\Psi(t) := \left(\Omega_1 \chi_2(t) + \Omega_2 \chi_3(t) + TG_{\psi}^{*} \chi_1(t)\right) < t, \quad \text{for all } t > 0, \tag{31}$$

where Ω_1 and Ω_2 are given by (23) and (24) respectively. Then the problem (3)–(5) has a unique solution.

Proof: We define the operator N as in (26). For any $x, y \in C([0,T], \mathbb{R})$, and $t \in [0,T]$, by using (H4), we get

$$|(Nx)(t) - (Ny)(t)|$$
$$\leq \frac{(\psi(T)+1)}{(\psi(T)-\psi(0)+2)\Gamma(\alpha)} \int_0^{\sigma} |(\psi(\sigma)-\psi(s))^{\alpha-1}$$
$$\psi'(s)||g(s,x(s)) - g(s,y(s))| \, ds$$
$$+ \frac{(\psi(T)+1)}{(\psi(T)-\psi(0)+2)\Gamma(\beta)} \int_0^{\eta} |(\psi(\eta)-\psi(s))^{\beta-1}$$
$$\psi'(s)||h(s,x(s)) - h(s,y(s))| \, ds$$
$$+ \int_0^{T} |G_{\psi}(t,s)||f(s,x(s)) - f(s,y(s))| \, ds$$
$$\leq \frac{(\psi(T)+1)(\psi(\sigma))^{\alpha}}{(\psi(T)-\psi(0)+2)\Gamma(\alpha+1)} \chi_2(\|x-y\|_{\infty})$$
$$+ \frac{(\psi(T)+1)(\psi(\eta))^{\beta}}{(\psi(T)-\psi(0)+2)\Gamma(\beta+1)} \chi_3(\|x-y\|_{\infty})$$
$$+ TG_{\psi}^{*} \chi_1(\|x-y\|_{\infty})$$
$$\leq \Omega_1 \chi_2(\|x-y\|_{\infty}) + \Omega_2 \chi_3(\|x-y\|_{\infty})$$
$$+ TG_{\psi}^{*} \chi_1(\|x-y\|_{\infty})$$
$$\leq \Psi(\|x-y\|_{\infty}).$$

By taking the supremum, we obtain,

$$\|(Nx)(t) - (Ny)(t)\|_{\infty} \leq \Psi(\|x-y\|_{\infty}).$$

Since $\Psi(t) < t$ for all $t > 0$, then Boyd–Wong's contraction principle can be applied and N has a unique fixed point which is the unique solution of the problem (3)–(5). ■

Remark 3.5: Theorem 3.4 is a generalization of Theorem 3.2. Indeed for

$$\chi_1(t) = kt, \quad \chi_2(t) = k^{*}t, \quad \text{and} \quad \chi_3(t) = k^{**}t,$$

condition (H4) becomes (H1–H3), and (31) is satisfied if and only if (25) holds.

Our next existence result for the problem (3)–(5) is based on Leray–Schauder nonlinear alternative [28].

Theorem 3.6: *Assume that the following hypotheses hold*

(H5) *$f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function.*

(H6) *There exists $\phi_f \in L^1(I, \mathbb{R}^+)$, and a continuous and non-decreasing function $\chi_1 : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$|f(t,u)| \leq \phi_f(t)\chi_1(|u|), \quad \text{for all } t \in I \text{ a.e } u \in \mathbb{R}.$$

(H7) *There exists $\phi_g \in L^1(I, \mathbb{R}^+)$, and a continuous and non-decreasing function $\chi_2 : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$|g(t,u)| \leq \phi_g \chi_2(|u|), \quad \text{for all } t \in I \text{ a.e } u \in \mathbb{R}.$$

(H8) *There exists $\phi_h \in L^1(I,\mathbb{R}^+)$, and a continuous and non-decreasing function $\chi_3 : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$|h(t,u)| \le \phi_h \chi_3(|u|), \quad \text{for all } t \in I \text{ a.e } u \in \mathbb{R}.$$

(H9) *There exists a number $C > 0$ such that*

$$\frac{C}{a\chi_2(C) + b\chi_3(C) + cG^*_\psi \chi_1(C)} > 1, \tag{32}$$

where

$$a = \frac{\psi(T)+1}{(\psi(T)-\psi(0)+2)} I^\alpha_{0+;\psi}\phi_g(t),$$
$$b = \frac{\psi(T)+1}{(\psi(T)-\psi(0)+2)} I^\beta_{0+;\psi}\phi_h(t), \tag{33}$$
$$c = \int_0^T \phi_f(s)\,ds.$$

Then the problem (3)–(5) has at least one solution on I.

Proof: Consider the operator N defined by (26), by applying Leray–Schauder nonlinear alternative we will prove that N has a fixed point. The proof will be given in several steps

Step 1: N is continuous. Let y_n be a sequence such that $y_n \to y$ in $C([0,T],\mathbb{R})$. Then for each $t \in [0,T]$

$$\begin{aligned}
&|(Ny_n)(t) - (Ny)(t)| \\
&\le \frac{(\psi(T)+1)}{(\psi(T)-\psi(0)+2)\Gamma(\alpha)} \int_0^\sigma |(\psi(\sigma)-\psi(s))^{\alpha-1} \\
&\psi'(s)|\|g(s,y_n(s)) - g(s,y(s))\|_\infty\,ds \\
&\quad + \frac{(\psi(T)+1)}{(\psi(T)-\psi(0)+2)\Gamma(\beta)} \int_0^\eta |(\psi(\eta)-\psi(s))^{\beta-1} \\
&\psi'(s)|\|h(s,y_n(s)) - h(s,y(s))\|_\infty\,ds \\
&\quad + \int_0^T |G_\psi(t,s)|\|f(s,y_n(s)) - f(s,y(s))\|_\infty\,ds \\
&\le \Omega_1\|g(.,y_n(.)) - g(.,y(.))\|_\infty \\
&+ \Omega_2\|h(.,y_n(.)) - h(.,y(.))\|_\infty \\
&+ TG^*_\psi\|f(.,y_n(.)) - f(.,y(.))\|_\infty.
\end{aligned}$$

Since f, g and h are continuous functions, we have

$$\|(Ny_n)(t) - (Ny)(t)\| \to 0, \quad \text{as } n \to +\infty.$$

Step 2: F maps bounded sets into bounded sets in $C([0,T],\mathbb{R})$. For any $\nu > 0$ there exists a positive constant l such that for each $y \in B_\nu := \{y \in C([0,T],\mathbb{R}), \|y\|_\infty \le \nu\}$, we have $\|N(y)\| \le l$. By using (H6)-(H8) for each $t \in [0,T]$, one can obtain

$$\begin{aligned}
&|(Ny)(t)| \\
&\le \frac{(\psi(T)+1)}{(\psi(T)-\psi(0)+2)\Gamma(\alpha)} \\
&\quad \times \int_0^\sigma (\psi(\sigma)-\psi(s))^{\alpha-1}\psi'(s)|g(s,y(s))|\,ds \\
&\quad + \frac{(\psi(T)+1)}{(\psi(T)-\psi(0)+2)\Gamma(\beta)} \\
&\quad \times \int_0^\eta (\psi(\eta)-\psi(s))^{\beta-1}\psi'(s)|h(s,y(s))|\,ds \\
&+ \int_0^T G_\psi(t,s)|f(s,y(s))|\,ds \\
&\le \frac{(\psi(T)+1)}{(\psi(T)-\psi(0)+2)\Gamma(\alpha)}\chi_2(\|y\|_\infty) \\
&\quad \times \int_0^\sigma (\psi(\sigma)-\psi(s))^{\alpha-1}\psi'(s)\phi_g(s)\,ds \\
&\quad + \frac{(\psi(T)+1)}{(\psi(T)-\psi(0)+2)\Gamma(\beta)}\chi_3(\|y\|_\infty) \\
&\quad + \int_0^\eta (\psi(\eta)-\psi(s))^{\beta-1}\psi'(s)\phi_h(s)\,ds \\
&+ \chi_1(\|y\|_\infty)G^*_\psi \int_0^T \phi_f(s)\,ds.
\end{aligned}$$

Thus, for each $t \in [0,T]$

$$\begin{aligned}
\|(Ny)(t)\|_\infty \le{}& \chi_2(\nu)\frac{(\psi(T)+1)}{(\psi(T)-\psi(0)+2)} I^\alpha_{0+;\psi}\phi_g(\sigma) \\
&+ \chi_3(\nu)\frac{(\psi(T)+1)}{(\psi(T)-\psi(0)+2)} I^\beta_{0+;\psi}\phi_h(\eta) \\
&+ \chi_1(\nu)G^*_\psi \int_0^T \phi_f(s)\,ds := l.
\end{aligned}$$

Step 3: N maps bounded sets into equicontinuous sets of $C([0,T],\mathbb{R})$. Let $t_1, t_2 \in (0,T]$. If $t_1 < t_2$ then $\psi(t_1) < \psi(t_2)$, and let $y \in B_\nu$ where B_ν is a bounded set of $C([0,T],\mathbb{R})$ as given in step 2. Then we obtain

$$\begin{aligned}
&|(Ny)(t_2) - (Ny)(t_1)| \\
&\le \frac{\psi(t_2)-\psi(t_1)}{(\psi(T)-\psi(0)+2)\Gamma(\alpha)} \\
&\times \int_0^\sigma (\psi(\sigma)-\psi(s))^{\alpha-1}\psi'(s)|g(s,y(s))|\,ds \\
&\quad + \frac{\psi(t_2)-\psi(t_1)}{(\psi(T)-\psi(0)+2)\Gamma(\beta)} \\
&\times \int_0^\eta (\psi(\eta)-\psi(s))^{\beta-1}\psi'(s)|h(s,y(s))|\,ds \\
&\quad + \int_0^T |G_\psi(t_2,s) - G_\psi(t_1,s)||f(s,y(s))|\,ds \\
&\le \frac{\psi(t_2)-\psi(t_1)}{(\psi(T)-\psi(0)+2)\Gamma(\alpha)}\chi_2(\nu) \\
&\times \int_0^\sigma (\psi(\sigma)-\psi(s))^{\alpha-1}\psi'(s)\phi_g(s)\,ds \\
&\quad + \frac{\psi(t_2)-\psi(t_1)}{(\psi(T)-\psi(0)+2)\Gamma(\beta)}\chi_3(\nu) \\
&\times \int_0^\eta (\psi(\eta)-\psi(s))^{\beta-1}\psi'(s)\phi_h(s)\,ds \\
&\quad + \chi_1(\nu)\|G_\psi(t_2,s) - G_\psi(t_1,s)\|_\infty \int_0^T \phi_f(s)\,ds,
\end{aligned}$$

which tend to 0, since $\psi(t_2)-\psi(t_1)\to 0$ when $t_2-t_1\to 0$ independently of y. In view of the Ascoli-Arzelà theorem and the consequence of Steps 1 to 3, we conclude that N is completely continuous.

Step 4: There exists an open sets $U\subset C([0,T],\mathbb{R})$ with $y\neq\theta N(y)$ for $\theta\in(0,1)$ and $y\in\partial U$. Let y be a solution of $y-\theta N(y)=0$ for $\theta\in(0,1)$. Then for $t\in[0,T]$ we get

$$\begin{aligned}|y(t)| &= |\theta(Ny)(t)|\\ &\leq \frac{(\psi(T)+1)}{(\psi(T)-\psi(0)+2)\Gamma(\alpha)}\\ &\quad\times\int_0^{\sigma}(\psi(\sigma)-\psi(s))^{\alpha-1}\psi'(s)|g(s,y(s))|\,ds\\ &\quad+\frac{(\psi(T)+1)}{(\psi(T)-\psi(0)+2)\Gamma(\beta)}\\ &\quad\times\int_0^{\eta}(\psi(\eta)-\psi(s))^{\beta-1}\psi'(s)|h(s,y(s))|\,ds\\ &\quad\times+\int_0^{T}G_{\psi}(s,t)|f(s,y(s))|\,ds\\ &\leq \frac{(\psi(T)+1)}{(\psi(T)-\psi(0)+2)\Gamma(\alpha)}\chi_2(\|y\|_\infty)\\ &\quad\times\int_0^{\sigma}(\psi(\sigma)-\psi(s))^{\alpha-1}\psi'(s)\phi_g(s)\,ds\\ &\quad+\frac{(\psi(T)+1)}{(\psi(T)-\psi(0)+2)\Gamma(\beta)}\chi_3(\|y\|_\infty)\\ &\quad\times\int_0^{\eta}(\psi(\eta)-\psi(s))^{\beta-1}\psi'(s)\phi_h(s)\\ &\quad ds+\chi_1(\|y\|_\infty)G^*_{\psi}\int_0^{T}\phi_f(s)\,ds,\end{aligned}$$

taking the supremum for $t\in[0,T]$, yields

$$\frac{\|y\|_\infty}{a\chi_2(\|y\|_\infty)+b\chi_3(\|y\|_\infty)+cG^*_{\psi}\chi_1(\|y\|)}\leq 1.$$

■

By the condition (H9), there exists a positive constant C such that $\|y\|\neq C$. Next we define $U=\{y\in C([0,T],\mathbb{R}),\|y\|<C\}$ and note that the operator $N:\partial U\to C([0,T],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $y\in\partial U$ such that $y-\theta Ny=0$ for some $\theta\in(0,1)$. Therefore by the nonlinear alternative of Leray–Schauder type, we deduce that N has a fixed point $y\in\bar{U}$ which is a solution of the problem (3)–(5).

Our next existence result for the problem (3)–(5) based on Leray–Schauder's Degree theory [29].

Theorem 3.7: *Along with condition* (H4), *assume that the following hypotheses hold*

(H10) *there exist constants $\kappa\geq 0$ and $M>0$ such that*

$$|f(t,u)|\leq\kappa|u|+M,\quad \text{for } t\in I,\ \text{a.e } u\in\mathbb{R},$$

(H11) *there exist constants $\vartheta\geq 0$ and M^* such that*

$$|g(t,u)|\leq\vartheta|u|+M^*,\quad \text{for } t\in I,\ \text{a.e } u\in\mathbb{R},$$

(H12) *there exist constants $\nu\geq 0$ and M^{**} such that*

$$|h(t,u)|\leq\nu|u|+M^{**},\quad \text{for } t\in I,\ \text{a.e } u\in\mathbb{R},$$

where κ, ϑ and ν satisfying the following condition

$$\vartheta\Omega_1+\nu\Omega_2+\kappa TG^*_{\psi}<1,$$

Ω_1 and Ω_2 are given by (23) and (24), respectively. Then the problem (3)–(5) has at least one solution on $[0,T]$.

Proof: Based on the fixed point problem, we have

$$y=Ny, \tag{34}$$

if and only if y is solution of the problem (3)–(5) where $N:C([0,T],\mathbb{R})\to C([0,T],\mathbb{R})$ defined by 26. we simply need to establish the existence of at least one solution $y\in C([0,T])$ for 34. In order to complete the proof we set a ball $B_R\subset C([0,T])$ with a constant radius $R>0$ as follows

$$B_R=\{y\in C[0,T]:\max_{t\in[0,T]}|y(t)|<R\}.$$

We need to demonstrate that, $N:\bar{B}_R\to C[0,T]$ satisfies

$$y\neq\lambda Ny,\quad \text{for all } u\in\partial B_\nu, \text{ and all } \lambda\in[0,1]. \tag{35}$$

Let us define

$$H(\lambda,y)=\lambda Ny,$$

where $\lambda\in[0,1]$ and $y\in C([0,T])$. From Theorem 3.6, we know that the operator N is continuous, and completely continuous. Then, by the Arzèla–Ascoli theorem, a continuous map h_λ defined by $h_\lambda(y)=y-H(\lambda,y)=y-\lambda N(y)$ is also completely continuous. The following Leray–Schauder degrees are well defined unless (35) holds, then by the homotopy in-variance of topological degree, it is concluded that

$$\begin{aligned}deg(h_\lambda,B_R,0)&=deg(I-\lambda N,B_R,0)=deg(h_1,B_R,0)\\ &=deg(h_0,B_R,0)=deg(I,B_R,0)=1\neq 0,\end{aligned}$$

for $0\in B_R$, and I indicates the unit operator. By the non-zero property of Leray–Schauder degree, we have $h_1(y)=y-Ny=0$ for at least $x\in B_R$. In order to prove 35, let us assume that $y=\lambda Ny$ for some $\lambda\in[0,1]$ and for all $t\in[0,T]$, thus

$$\begin{aligned}|y(t)|&=|\lambda(Ny)(t)|\\ &\leq\frac{(\psi(T)+1)}{(\psi(T)-\psi(0)+2)\Gamma(\alpha)}\\ &\quad\times\int_0^{\sigma}|(\psi(\sigma)-\psi(s))^{\alpha-1}\psi'(s)||g(s,y(s))|\\ &\quad\times ds+\frac{(\psi(T)+1)}{(\psi(T)-\psi(0)+2)\Gamma(\beta)}\\ &\quad\times\int_0^{\eta}|(\psi(\eta)-\psi(s))^{\beta-1}\psi'(s)||h(s,y(s))|\end{aligned}$$

$$\times\,\mathrm{d}s+\int_0^T|G_\psi(s,t)||f(s,y(s))|\,\mathrm{d}s$$
$$\leq\frac{(\psi(T)+1)(\psi(\sigma))^\alpha}{(\psi(T)-\psi(0)+2)\Gamma(\alpha+1)}(\vartheta|y|+M^*)$$
$$+\frac{(\psi(T)+1)(\psi(\eta))^\beta}{(\psi(T)-\psi(0)+2)\Gamma(\beta+1)}(\nu|y|+M^{**})$$
$$+(\kappa|y|+M)TG_\psi^*$$
$$=\Omega_1(\vartheta|y|+M^*)+\Omega_2(\nu|y|+M^{**})$$
$$+(\kappa|y|+M)TG_\psi^*,$$

on which, taking the supremum for $t\in[0,T]$, and solving for $\|y\|_\infty$ yields

$$\|y\|_\infty\leq\frac{M^*\Omega_1+M^{**}\Omega_2+MTG_\psi^*}{(1-\vartheta\Omega_1--\nu\Omega_2-\kappa TG_\psi^*)}.$$

If

$$R=\frac{M^*\Omega_1+M^{**}\Omega_2+MTG_\psi^*}{(1-\vartheta\Omega_1-\nu\Omega_2-\kappa TG_\psi^*)}+1,$$

the inequality 35 holds. This completes the proof. ∎

The last existence result depends on the Krasnoselskii–Schaefer type fixed point theorem [30].

Theorem 3.8: *Let A and B be two mappings of a Banach space X, such that*

(i) *A is a contraction, and*
(ii) *B is completely continuous.*

Then, either

(a) *the operator equation $y=A(y)+B(y)$ has a solution, or*
(b) *the set $\mathbb{E}=\{u\in X:\ u=\lambda A(u/\lambda)+\lambda B(u)\}$ is unbounded for $0<\lambda<1$.*

Theorem 3.9: *Assume that* (H2), (H3) *and* (H6) *hold. Furthermore, if*

$$\left(\Omega_1k^*+\Omega_2k^{**}\right)<1, \tag{36}$$

and

$$\frac{\|y\|_\infty(1-\Omega_1k^*-\Omega_2k^{**})}{g^*\alpha^{-1}(\psi(\sigma))^\alpha+h^*\beta^{-1}(\psi(\eta))^\beta+cG_\psi^*\psi(\|y\|_\infty)}>1, \tag{37}$$

holds, where Ω_1, and Ω_2 defined by (23), and (24) respectively, $g^*=\sup_{t\in[0,T]}|g(t,0)|$, $h^*=\sup_{t\in[0,T]}|h(t,0)|$, the problem (3)–(5) has at least one solution on I.

Proof: We define the operators $A,B:C([0,T],\mathbb{R})\to C([0,T],\mathbb{R})$ by

$$(Ay)(t)=\frac{\psi(T)-\psi(t)+1}{(\psi(T)-\psi(0)+2)\Gamma(\alpha)}I^\alpha_{0^+;\psi}g(\sigma,y(\sigma))$$
$$+\frac{\psi(t)-\psi(0)+1}{(\psi(T)-\psi(0)+2)\Gamma(\beta)}I^\beta_{0^+;\psi}h(\eta,y(\eta)),$$

$$(By)(t)=\int_0^TG_\psi(t,s)f(s,y(s))\,\mathrm{d}s,$$

where G_ψ is a Green's function given by (11). The operator A is a contraction map from the Banach space $C([0,T],\mathbb{R})$ into itself. Indeed, by using conditions (H1) and (H2), for $x,y\in C([0,T],\mathbb{R})$, we get

$$|Ay(t)-Ax(t)|$$
$$\leq\frac{(\psi(T)+1)}{(\psi(T)-\psi(0)+2)\Gamma(\alpha)}\int_0^\sigma|(\psi(\sigma)$$
$$-\psi(s))^{\alpha-1}\psi'(s)||g(s,y(s))-g(s,x(s))|\,\mathrm{d}s$$
$$+\frac{(\psi(T)+1)}{(\psi(T)-\psi(0)+2)\Gamma(\beta)}\int_0^\eta|(\psi(\eta)$$
$$-\psi(s))^{\beta-1}\psi'(s)||h(s,y(s))-h(s,x(s))|\,\mathrm{d}s$$
$$\leq\frac{(\psi(T)+1)(\psi(\sigma))^\alpha}{(\psi(T)-\psi(0)+2)\Gamma(\alpha+1)}k^*\|y$$
$$-x\|_\infty|+\frac{(\psi(T)+1)(\psi(\eta))^\beta}{(\psi(T)-\psi(0)+2)\Gamma(\beta+1)}k^{**}\|y-x\|_\infty$$
$$\leq(\Omega_1k^*+\Omega_2k^{**})\|y-x\|_\infty,$$

which, on taking supremum over $t\in[0,T]$, yields

$$\|Ay(t)-Ax(t)\|_\infty\leq(\Omega_1k^*+\Omega_2k^{**})\|y-x\|_\infty.$$

Therefore A is a contraction as (36) holds. Clearly B is completely continuous by using condition (H6). Thus, we just need to prove that $\mathbb{E}:=\{y\in C([0,T])\ y=\lambda A(y/\lambda)+\lambda B(y),\lambda\in(0,T)\}$ is bounded. Let $y\in\mathbb{E}$; then for each $t\in[0,T]$, we are led to the homotopy equation

$$y(t)=\lambda A(\frac{y}{\lambda})(t)+\lambda B(y)(t).$$

From (H2), (H3) and (H6) we have

$$|y(t)|\leq\frac{\lambda(\psi(T)+1)}{(\psi(T)-\psi(0)+2)\Gamma(\alpha)}\int_0^\sigma|(\psi(\sigma)$$
$$-\psi(s))^{\alpha-1}\psi'(s)|$$
$$\times\left(\left|g(s,\frac{y(s)}{\lambda})-g(t,0)\right|+|g(t,0)|\right)\mathrm{d}s$$
$$+\frac{\lambda(\psi(T)+1)}{(\psi(T)-\psi(0)+2)\Gamma(\beta)}\int_0^\eta|(\psi(\eta)$$
$$-\psi(s))^{\beta-1}\psi'(s)|$$
$$\times\left(\left|h(s,\frac{y(s)}{\lambda})-h(t,0)\right|+|h(t,0)|\right)\mathrm{d}s$$
$$+\lambda\int_0^T|G_\psi(s,t)||f(s,y(s))|\,\mathrm{d}s$$
$$\leq\frac{(\psi(T)+1)(\psi(\sigma))^\alpha}{(\psi(T)-\psi(0)+2)\Gamma(\alpha+1)}k^*\|y\|_\infty$$
$$+g^*\sup_{0<\sigma<T}\left|\frac{1}{\alpha}(\psi(\sigma)-\psi(s))^\alpha\Big|_0^\sigma\right|$$
$$+\frac{(\psi(T)+1)(\psi(\eta))^\beta}{(\psi(T)-\psi(0)+2)\Gamma(\beta+1)}k^{**}\|y\|_\infty$$

$$+ h^* \sup_{0<\eta<T} \left| \frac{1}{\beta}(\psi(\eta) - \psi(s))^\beta \Big|_0^\eta \right|$$
$$+ \chi_1(\|y\|_\infty) G_\psi^* \int_0^T \phi_f(s)\, ds$$
$$\leq \Omega_1 k^* \|y\|_\infty + \Omega_2 k^{**} \|y\|_\infty + g^* \frac{(\psi(\sigma))^\alpha}{\alpha}$$
$$+ h^* \frac{(\psi(\eta))^\beta}{\beta} + c G_\psi^* \chi_1(\|y\|_\infty),$$

on which, taking the supremum for $t \in [0, T]$, yields

$$\frac{\|y\|_\infty (1 - \Omega_1 k^* - \Omega_2 k^{**})}{g^* \alpha^{-1} (\psi(\sigma))^\alpha + h^* \beta^{-1} (\psi(\eta))^\beta + c G_\psi^* \chi_1(\|y\|_\infty)} \leq 1. \tag{38}$$

By means of (37), it follows that there exist $R > 0$ such that $\|y\| > R$ for each $y \in \mathbb{E}$. Therefore $\|y\| \leq R$ for each $y \in \mathbb{E}$, and the Krasnoselskii–Schaefer type fixed point theorem indicates that the set $\mathbb{E}$ is bounded. ■

4. Example

In this section, we give an example to illustrate the usefulness of the main results. Let us consider the following fractional boundary value problem

$$ {}^c\mathcal{D}^q_{0^+;\psi} y(t) = \frac{\sin(\frac{\pi}{2} t)}{5\pi} \frac{\log(y)}{(1 + \log(y))},$$
$$(t, y) \in [0, 1] \times [e, +\infty), \tag{39}$$
$$y(0) - \frac{1}{3t^2} y'(0) = I^{1/3}_{0+,\psi} g\left(\frac{1}{7}, y\left(\frac{1}{7}\right)\right), \tag{40}$$
$$y(T) + \frac{1}{3t^2} y'(T) = I^{2/7}_{0+,\psi} h\left(\frac{1}{2}, y\left(\frac{1}{2}\right)\right), \tag{41}$$

where $1 < q \leq 2$, $\sigma = \frac{1}{5}$, $\eta = \frac{1}{2}$, and $T = 1$. Set

$$f(t, x) = \frac{\sin(\frac{\pi}{2} t) \log(x)}{5\pi(1 + \log(x))}, \quad (t, x) \in [0, 1] \times [e, \infty),$$
$$g(t, x) = \frac{x}{(1 + 5t)^2}, \quad (t, x) \in [0, 1] \times [e, \infty),$$
$$h(t, x) = \frac{x}{(1 + 3t)^2}, \quad (t, x) \in [0, 1] \times [e, \infty).$$

Let us take $\psi(t) = t^3 + 1$. Clearly ψ is an increasing function on $[0, 1]$ and $\psi'(t) = 3t^2$ is a continuous function on $[0, 1]$.

Let $u, v \in [e, +\infty)$, and $t \in [0, 1]$. Then, we have

$$|f(t, u) - f(t, v)| \leq \frac{\sin(\frac{\pi}{2} t)}{5\pi} \left| \frac{\log(u)}{(1 + \log(u))} - \frac{\log(v)}{(1 + \log(v))} \right|$$
$$= \frac{\sin(\frac{\pi}{2} t)}{5\pi} \frac{|\log(u) - \log(v)|}{(1 + \log(u))(1 + \log(v))}$$
$$\leq \frac{\sin(\frac{\pi}{2} t)}{5\pi} |\log(u) - \log(v)|$$
$$\leq \frac{\sin(\frac{\pi}{2} t)}{5\pi} |u - v| \leq \frac{1}{5\pi} |u - v|.$$

Thus, condition (H1) holds with $k = 1/5\pi$. It is easy to show that conditions (H2) –(H3) are satisfied with $k^* = \frac{1}{36}$, $k^{**} = \frac{1}{16}$.

From (11) Green's function G_ψ is given by

$$G_\psi(t, s)$$
$$= 3t^2 \times \begin{cases} \frac{(t^3 - s^3)^{q-1}}{\Gamma(q)} - \frac{(t^3 + 1)}{(T^3 + 2)\Gamma(q)}(1 - s^3)^{q-1} \\ -\frac{(t^3+1)}{(T^3+2)\Gamma(q-1)}(1 - s^3)^{q-2}, \quad 0 \leq s \leq t, \\ -\frac{(t^3 + 1)}{(T^3 + 2)\Gamma(q)}(1 - s^3)^{q-1} \\ -\frac{(t^3+1)}{(T^3+2)\Gamma(q-1)}(1 - s^3)^{q-2}, \quad t \leq s \leq 1. \end{cases} \tag{42}$$

From 42 we have

$$\int_0^1 G_\psi(t, s)\, ds = \int_0^t G_\psi(t, s)\, ds + \int_t^1 G_\psi(t, s)\, ds$$
$$= -\frac{3t^5}{\Gamma(q + 1)} - \frac{3t^2(t^3 + 1)}{3\Gamma(q + 1)}(1 - t^3)^q$$
$$+ \frac{3t^2(t^3 + 1)}{3\Gamma(q + 1)}$$
$$- \frac{3t^2(t^3 + 1)}{3\Gamma(q)}(1 - t^3)^{q-1}$$
$$+ \frac{3t^2(t^3 + 1)}{3\Gamma(q)} + \frac{3t^2(t^3 + 1)}{3\Gamma(q + 1)}(1 - t^3)^q$$
$$+ \frac{3t^2(t^3 + 1)}{3\Gamma(q)}(1 - t^3)^{q-1}.$$

A simple calculation yields

$$G_\psi^* < \frac{3}{\Gamma(q + 1)} + \frac{3}{\Gamma(q)}.$$

Now

$$\Omega_1 k^* + \Omega_2 k^{**} + T G_\psi^* k$$
$$= \frac{(\psi(\frac{1}{5}))^\alpha}{36\Gamma(\alpha + 1)} + \frac{(\psi(\frac{1}{2}))^\beta}{16\Gamma(\beta + 1)}$$
$$+ \frac{3}{5\pi\Gamma(q + 1)} + \frac{3}{5\pi\Gamma(q)} < 1,$$

thus condition (25) is satisfied with $T = 1$, $q \in (1, 2]$, and for each α and $\beta \in (0, 1)$. By Theorem 3.2, the problem (39)–(41) has a unique solution on $[0, 1]$.

5. Conclusion

The fractional calculus has been attracting many scientists because of the good results obtained when the traditional derivatives are replaced with fractional derivatives. Hence, studying the qualitative properties such as the existence and uniqueness of solutions to differential equations in the framework of fractional derivatives has gained importance. In this article, we discussed the existence and uniqueness of a certain class of boundary value problem in the frame of ψ-Caputo fractional derivatives and with boundary conditions expressed in terms of ψ-Riemann–Liouville fractional integrals. This class of boundary value problem is new and is the generalization of some systems discussed in the literature. In fact, the ψ-fractional operators contain within themselves some conventional fractional operators such as the Riemann–Liouville and Hadamard fractional operators and many others.

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