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On continuity of the fractional derivative of the time-fractional semilinear pseudo-parabolic systems

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Abstract

In this work, we study an initial value problem for a system of nonlinear parabolic pseudo equations with Caputo fractional derivative. Here, we discuss the continuity which is related to a fractional order derivative. To overcome some of the difficulties of this problem, we need to evaluate the relevant quantities of the Mittag-Leffler function by constants independent of the derivative order. Moreover, we present an example to illustrate the theory.

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1 Introduction

In recent decades, fractional calculus has had many applications in various fields such as mechanic, biological, physical science, and applied science. This topic has increasingly asserted its role in the field of applied mathematics, especially the subjects are investigating the properties of the concept of derivative. Specifically, the situations with integer order in PDEs are not working well. Therefore, PDEs with fractional derivatives are a generalization equation with integer-order partial derivatives and a strong theoretical and practical interest. There have been many authors researching this field, for example, [1–15]. According to our search results, the extended results for a coupled nonlinear fractional pseudo-parabolic equation are still limited. This is the great impetus that motivated us to study the following model. In this paper, we extend the coupled system and consider the

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following coupled nonlinear fractional pseudo-parabolic equation:

$$\begin{cases} \partial_t^\alpha(u(x, t) - a\Delta u(x, t)) - \Delta u(x, t) = \mathcal{F}(u, v), & (x, t) \in \Omega \times (0, T), \\ \partial_t^\alpha(v(x, t) - a\Delta v(x, t)) - \Delta v(x, t) = \mathcal{G}(u, v), & (x, t) \in \Omega \times (0, T), \\ u(x, t) = v(x, t) = 0, & x \in \partial\Omega, t \in (0, T], \\ u_t(x, 0) = v_t(x, 0) = 0, & x \in \partial\Omega, \\ u(x, 0) = f(x), & x \in \Omega, \\ v(x, 0) = g(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where T, a are positive numbers and $\Omega \in \mathbb{R}^n, n \geq 1$, is an open bounded domain with a smooth boundary $\partial\Omega$. Note that $\mathbb{L}^2(\Omega), \mathbb{H}_0^1(\Omega), \mathbb{H}^2(\Omega)$ denote the usual Sobolev spaces. The symmetric uniform elliptic operator $\Delta : \mathbb{L}^2(\Omega) \rightarrow \mathbb{L}^2(\Omega)$ is defined by

$$\Delta u(x) = - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\Delta_{ij}(x) \frac{\partial}{\partial x_j} u(x) \right) + l(x)u(x, t), \quad x \in \overline{\Omega}.$$

With assumption $l(x) \in C(\overline{\Omega}, [0, \infty)), \Delta_{ij} \in C^1(\overline{\Omega}), \Delta_{ij} = \Delta_{ji}, 1 \leq i, j \leq n$, and there exists a positive constant $\tilde{\Delta} > 0$ for $x \in \overline{\Omega}, z = (z_1, z_2, \dots, z_n) \subset \mathbb{R}^n$, such that

$$\tilde{\Delta} \sum_{i=1}^n z_i^2 \leq \sum_{1 \leq i, j \leq n} z_i \Delta_{ij}(x) z_j,$$

see e.g. [16]. The constant $\alpha \in (1, 2)$ is the fractional order and ∂_t^α denotes the left-sided Caputo fractional derivative involving t , which is defined by

$$\partial_t^\alpha u(x, t) := \frac{1}{\Gamma(2 - \alpha)} \int_0^t (t - \eta)^{1-\alpha} \frac{\partial u^2}{\partial \eta}(x, \eta) d\eta,$$

where Γ is the gamma function.

If $\alpha = 2$, then ∂_t^2 is interpreted as a derivative of normal time. The second equation in the (1.1) called the fractional pseudo-parabolic equation has many practical applications, for example, the permeability of a homogeneous liquid through a cracked rock [17], one-way propagation of nonlinear dispersion long waves [18–20], and populations of [21] populations. Let Ω be an open and bounded domain \mathbb{R}^n with the boundary $\partial\Omega$. The functions $\mathcal{F}, \mathcal{G}, f, g$ satisfy some assumptions to be specified later.

In practice, many problems with space-time fraction equations are based on fractional parameters, that is, the order of fractions. However, these fractional parameters were not known during the modeling process. Therefore, the continuity of the solution on these parameters is very important for modeling purposes. To the best of our knowledge, there have been no results investigating continuity related to the fractional order of a system of pseudo-nonlinear parabolic equations.

This article is organized as follows. Part 2 provides some basic and preliminary definitions. In Part 3, we give the formula of a mild solution and some lemmas that may be related to the next section. In Part 4, we apply the results of Part 3 to establish the existence, uniqueness, and continuity of the solution to problem (1.1) in fractional order. Finally, we give an example to test the theory.

2 Preliminaries

2.1 Stability on the parameters of the Mittag-Leffler function

Consider the Mittag-Leffler function, which is defined by

$$E_{\alpha,\beta}(\xi) = \sum_{n=1}^{\infty} \frac{\xi^n}{\Gamma(n\alpha + \beta)} \quad \text{for } \xi \in \mathbb{C}, \alpha > 0, \text{ and } \beta \in \mathbb{R}.$$

We call to mind the following lemmas (see for example [3, 22, 23]), which will be useful for the main analysis of Sect. 3 and Sect. 4.

Lemma 2.1 *If $1 < \alpha < 2$, then, for all $\xi > 0$, where M is a positive constant depending only on α ,*

$$|E_{\alpha,1}(-\xi)| \leq M, \quad |E_{\alpha,\alpha}(-\xi)| \leq M.$$

Now, we have the following lemmas.

Lemma 2.2 *Let $\lambda > 0$ and $1 < \alpha < 2$. Then, for all $\xi > 0$, the following identities hold:*

$$\begin{aligned} \partial_{\xi} E_{\alpha,1}(-\lambda\xi^{\alpha}) &= -\lambda\xi^{\alpha-1} E_{\alpha,\alpha}(-\lambda\xi^{\alpha}), \\ \partial_{\xi} (\xi^{\alpha-1} E_{\alpha,\alpha}(-\lambda\xi^{\alpha})) &= -\xi^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda\xi^{\alpha}). \end{aligned} \tag{2.1}$$

Proof Apply Lemma 2.2 in [24]. □

Lemma 2.3 (see [25]) *Let $1 < \alpha < 2$. If T is large enough, then*

$$E_{\alpha,1}(-\lambda_j T^{\alpha}) \neq 0 \tag{2.2}$$

for all $j \in \mathbb{N}$, then there exist two constants m_{α} and M_{α} such that

$$\frac{m_{\alpha}}{1 + \lambda_j T^{\alpha}} \leq |E_{\alpha,1}(-\lambda_j T^{\alpha})| \leq \frac{M_{\alpha}}{1 + \lambda_j T^{\alpha}}. \tag{2.3}$$

From Lemma 2.3 in [26], we have the following lemmas.

Lemma 2.4 *Let $1 < \alpha_1 < \alpha_2 < 2$ and $\alpha \in (\alpha_1, \alpha_2)$. There exist two positive constants M_1, M_2 , and M_3 which just rely upon α_1, α_2 such that, for any $\xi \geq 0$, we get*

$$\begin{aligned} \frac{M_1(\alpha_1, \alpha_2)}{1 + \xi} \leq |E_{\alpha,1}(-\xi)| &\leq \frac{M_2(\alpha_1, \alpha_2)}{1 + \xi} \\ \text{if } \xi \text{ is large enough and } |E_{\alpha,\alpha}(-\xi)| &\leq \frac{M_3(\alpha_1, \alpha_2)}{1 + \xi}. \end{aligned} \tag{2.4}$$

Lemma 2.5 *Let $0 < \alpha_1 < \alpha < \alpha' < \alpha_2$ and $0 < \xi \leq T$. For any $\epsilon > 0$ independent of α , there always exists M_{ϵ} such that*

$$|\xi^{\alpha} - \xi^{\alpha'}| \leq \max(T^{\alpha_2+2\epsilon}, 1) M_{\epsilon} (\alpha' - \alpha)^{\epsilon} \xi^{\alpha-\epsilon}. \tag{2.5}$$

Proof See Lemma 3.2 in [24]. □

Using Lemmas 3.3 and 3.4, Section 3 in [24], we have the following lemmas.

Lemma 2.6 *Assume that $1 < \alpha_1 < \alpha < \alpha' < \alpha_2 < 2$ and $\epsilon > 0$. Then there exists a positive constant $A(\alpha_1, \alpha_2, \epsilon, \nu_0, T)$*

$$\begin{aligned} & |E_{\alpha,1}(-\lambda_j \xi^\alpha) - E_{\alpha',1}(-\lambda_j \xi^{\alpha'})| \\ & \leq A(\alpha_1, \alpha_2, \epsilon, \nu_0, T) \lambda_j^{\nu_0-1} \xi^{-\alpha_2(1-\nu_0)-\epsilon} [(\alpha' - \alpha)^\epsilon + (\alpha' - \alpha)] \end{aligned} \tag{2.6}$$

for any $0 \leq \nu_0 \leq 1$ and $0 < \xi \leq T$.

Proof See Lemma 3.3 for stability on parameters of the Mittag-Leffler function in [24]. □

Lemma 2.7 *Assume that $1 < \alpha_1 < \alpha < \alpha' < \alpha_2 < 2$. For any $0 \leq \nu_0 \leq 1$ and $\epsilon > 0$, there exists a positive constant $B(\alpha_1, \alpha_2, \epsilon, \nu_0, T)$*

$$\begin{aligned} & |\xi^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j t^\alpha) - \xi^{\alpha'-1} E_{\alpha',\alpha'}(-\lambda_j \xi^{\alpha'})| \\ & \leq B(\alpha_1, \alpha_2, \epsilon, \nu_0, T) \lambda_j^{\nu_0-1} \xi^{\alpha_1 \nu_0 - \epsilon - 1} [(\alpha' - \alpha)^\epsilon + (\alpha' - \alpha)]. \end{aligned} \tag{2.7}$$

Proof Use Lemma 3.4 from Section 3 in [24]. □

2.2 Some Sobolev spaces

In this section, we present some appropriate Sobolev space. Let the operator Δ be considered on $\mathbb{L}^2(\Omega)$ with domain $\mathbb{H}^2(\Omega) \cap \mathbb{H}_0^1(\Omega)$. Then the spectrum of Δ is a non-diminishing arrangement of positive real numbers $\{\lambda_j\}_{j \geq 1}$ which satisfy that $\lim_{j \rightarrow \infty} \lambda_j = \infty$. Let us denote by $\{\varphi_j\}_{j \geq 1}$ in $\mathbb{H}^2(\Omega) \cap \mathbb{H}_0^1(\Omega)$ the set of orthonormal eigenfunctions of Δ , which means that $\Delta \varphi_j = \lambda_j \varphi_j$. The sequence forms an orthonormal basis of $\mathbb{L}^2(\Omega)$, see e.g. [27]. For all $\gamma \geq 0$, the operator Δ^γ has the following representation:

$$\Delta^\gamma v := \sum_{j=1}^\infty \langle v, \varphi_j \rangle \lambda_j^\gamma \varphi_j, \quad v \in \mathbb{D}(\Delta^\gamma) = \left\{ v \in \mathbb{L}^2(\Omega) : \sum_{j=1}^\infty |\langle v, \varphi_j \rangle|^2 \lambda_j^{2\gamma} < \infty \right\}.$$

The domain $\mathbb{D}(\Delta^\gamma)$ is Banach spaces equipped with the norm

$$\|v\|_{\mathbb{D}(\Delta^\gamma)}^2 := \sum_{j=1}^\infty \lambda_j^{2\gamma} |\langle v, \varphi_j \rangle|^2. \tag{2.8}$$

If $\gamma = 1$, we have $\mathbb{D}(\Delta^1) = \mathbb{H}^2(\Omega)$.

For a given number $\gamma \geq 0$, the Hilbert space

$$\mathcal{H}^\gamma(\Omega) = \left\{ v \in \mathbb{L}^2(\Omega) : \sum_{j=1}^\infty |\langle v, \varphi_j \rangle|^2 \lambda_j^{2\gamma} < \infty \right\} \tag{2.9}$$

is endowed with the norm as follows:

$$\|v\|_{\mathcal{H}^\gamma}^2 := \sum_{j=1}^\infty \lambda_j^{2\gamma} |\langle v, \varphi_j \rangle|^2. \tag{2.10}$$

If $\gamma = 0$, then $\mathcal{H}^0(\Omega) = \mathbb{L}^2(\Omega)$. We identified a norm for $w(u, v) \in \mathcal{H}^\gamma(\Omega) = \mathcal{H}^\gamma(\Omega) \times \mathcal{H}^\gamma(\Omega)$ as follows:

$$\|w\|_{\mathcal{H}^\gamma(\Omega)} = \sqrt{\|u\|_{\mathcal{H}^\gamma(\Omega)}^2 + \|v\|_{\mathcal{H}^\gamma(\Omega)}^2}. \tag{2.11}$$

Let us denote by $\mathbb{C}((0, T]; \mathcal{H}^\gamma(\Omega))$ a space of all continuous functions with the map $(0, T] \rightarrow \mathcal{H}^\gamma(\Omega)$. For a given number $0 < \beta < 1$, we define by $\mathbb{C}^\beta((0, T]; \mathcal{H}^\gamma(\Omega))$ such that

$$\sup_{0 < t \leq T} t^\beta \|f(t)\|_{\mathcal{H}^\gamma(\Omega)} < \infty; \quad f \in \mathbb{C}((0, T]; \mathcal{H}^\gamma(\Omega)),$$

in which (see [26])

$$\|f\|_{\mathbb{C}^\beta((0, T]; \mathcal{H}^\gamma(\Omega))} := \sup_{0 < t \leq T} t^\beta \|f\|_{\mathcal{H}^\gamma(\Omega)}.$$

The product space $\mathcal{C}^\beta(0, T, \mathcal{H}^\gamma(\Omega)) = \mathbb{C}^\beta((0, T], \mathcal{H}^\gamma(\Omega)) \times \mathbb{C}^\beta((0, T], \mathcal{H}^\gamma(\Omega))$ is also a Banach space endowed with the norm

$$\|w\|_{\mathcal{C}^\beta(0, T, \mathcal{H}^\gamma(\Omega))} = \sqrt{\|u\|_{\mathbb{C}^\beta((0, T], \mathcal{H}^\gamma(\Omega))}^2 + \|v\|_{\mathbb{C}^\beta((0, T], \mathcal{H}^\gamma(\Omega))}^2}$$

for $w = (u, v) \in \mathcal{C}^\beta(0, T, \mathcal{H}^\gamma(\Omega))$. For a given positive real number p , $\mathbb{L}_p^\infty(0, T, \mathcal{H}^\gamma(\Omega))$ is a Banach space with the norm

$$\|f\|_{\mathbb{L}_p^\infty(0, T, \mathcal{H}^\gamma(\Omega))} := \text{ess sup } e^{-pt} \|f\|_{\mathcal{H}^\gamma(\Omega)}. \tag{2.12}$$

The product space $\mathcal{L}_p^\infty(0, T, \mathcal{H}^\gamma(\Omega)) = \mathbb{L}_p^\infty(0, T, \mathcal{H}^\gamma(\Omega)) \times \mathbb{L}_p^\infty(0, T, \mathcal{H}^\gamma(\Omega))$ is a Banach space, and we also identified a norm for $w = (u, v) \in \mathcal{L}_p^\infty(0, T, \mathbb{H}^\gamma(\Omega))$ as follows:

$$\|w\|_{\mathcal{L}_p^\infty(0, T, \mathcal{H}^\gamma(\Omega))} = \sqrt{\|u\|_{\mathbb{L}_p^\infty(0, T, \mathcal{H}^\gamma(\Omega))}^2 + \|v\|_{\mathbb{L}_p^\infty(0, T, \mathcal{H}^\gamma(\Omega))}^2}$$

3 Relevant notations and a representation of solution

In perception of spectral decomposition

$$u(x, t) = \sum_{j=1}^\infty \langle u(\cdot, t), \varphi_j(\cdot) \rangle \varphi_j(x) \quad \text{and} \quad v(x, t) = \sum_{j=1}^\infty \langle v(\cdot, t), \varphi_j(\cdot) \rangle \varphi_j(x). \tag{3.1}$$

We can transform the first two equations of (1.1) into

$$\begin{cases} \langle \partial_t^\alpha (u(\cdot, t), \varphi_j) + a \langle \partial_t^\alpha \Delta u(\cdot, t), \varphi_j \rangle + \langle \Delta u(\cdot, t), \varphi_j \rangle = \langle \mathcal{F}(u(\cdot, t), v(\cdot, t)), \varphi_j \rangle, \\ \langle \partial_t^\alpha (v(\cdot, t), \varphi_j) + a \langle \partial_t^\alpha \Delta v(\cdot, t), \varphi_j \rangle + \langle \Delta v(\cdot, t), \varphi_j \rangle = \langle \mathcal{G}(u(\cdot, t), v(\cdot, t)), \varphi_j \rangle. \end{cases} \tag{3.2}$$

Using the formula $\Delta \varphi_j = \lambda_j \varphi_j$, we obtain

$$\begin{cases} (1 + a\lambda_j) \langle \partial_t^\alpha (u(\cdot, t), \varphi_j) + \lambda_j \langle u(\cdot, t), \varphi_j \rangle = \langle \mathcal{F}(u(\cdot, t), v(\cdot, t)), \varphi_j \rangle, \\ (1 + a\lambda_j) \langle \partial_t^\alpha (v(\cdot, t), \varphi_j) + \lambda_j \langle v(\cdot, t), \varphi_j \rangle = \langle \mathcal{G}(u(\cdot, t), v(\cdot, t)), \varphi_j \rangle. \end{cases} \tag{3.3}$$

The theory of fractional ordinary differential equations (see [3, 22, 23]) gives a unique function u_j, v_j as follows:

$$\begin{cases} u_j(t) = E_{\alpha,1}\left(\frac{-\lambda_j t^\alpha}{1+a\lambda_j}\right)f_j \\ \quad + \frac{1}{1+a\lambda_j} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(\frac{-\lambda_j(t-s)^\alpha}{1+a\lambda_j}\right) \mathcal{F}_j(s) \, ds, \\ v_j(t) = E_{\alpha,1}\left(\frac{-\lambda_j t^\alpha}{1+a\lambda_j}\right)g_j \\ \quad + \frac{1}{1+a\lambda_j} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(\frac{-\lambda_j(t-s)^\alpha}{1+a\lambda_j}\right) \mathcal{G}_j(s) \, ds. \end{cases} \tag{3.4}$$

Here, we denote $f_j := \langle f, \varphi_j \rangle$, $g_j := \langle g, \varphi_j \rangle$, $\mathcal{F}_j := \langle \mathcal{F}(u(\cdot, t), v(\cdot, t)), \varphi_j \rangle$ and $\mathcal{G}_j := \langle \mathcal{G}(u(\cdot, t), v(\cdot, t)), \varphi_j \rangle$. Hence solution (1.1) can be described as by Fourier series (3.1) and then given by

$$\begin{cases} u(x, t) = \sum_{j=1}^\infty E_{\alpha,1}\left(\frac{-\lambda_j t^\alpha}{1+a\lambda_j}\right)f_j \varphi_j \\ \quad + \sum_{j=1}^\infty \frac{1}{1+a\lambda_j} \left[\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(\frac{-\lambda_j(t-s)^\alpha}{1+a\lambda_j}\right) \mathcal{F}_j(s) \, ds \right] \varphi_j, \\ v(x, t) = \sum_{j=1}^\infty E_{\alpha,1}\left(\frac{-\lambda_j t^\alpha}{1+a\lambda_j}\right)g_j \varphi_j \\ \quad + \sum_{j=1}^\infty \frac{1}{1+a\lambda_j} \left[\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(\frac{-\lambda_j(t-s)^\alpha}{1+a\lambda_j}\right) \mathcal{G}_j(s) \, ds \right] \varphi_j. \end{cases} \tag{3.5}$$

It is obvious to see that the mild solution (1.1) is given by

$$\begin{cases} u_\alpha(\cdot, t) = \mathcal{P}_{\Delta,\alpha}(t)f + \int_0^t \mathcal{Q}_{\Delta,\alpha}(t-s)\mathcal{F}(u, v)(\cdot, s) \, ds, \\ v_\alpha(\cdot, t) = \mathcal{P}_{\Delta,\alpha}(t)g + \int_0^t \mathcal{Q}_{\Delta,\alpha}(t-s)\mathcal{G}(u, v)(\cdot, s) \, ds, \end{cases} \tag{3.6}$$

where

$$\begin{aligned} \mathcal{P}_{\Delta,\alpha}(t)h &:= \sum_{j=1}^\infty E_{\alpha,1}\left(\frac{-\lambda_j t^\alpha}{1+a\lambda_j}\right) \langle h, \varphi_j \rangle \varphi_j, \\ \mathcal{Q}_{\Delta,\alpha}(t-s)h &:= \sum_{j=1}^\infty \frac{1}{1+a\lambda_j} (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(\frac{-\lambda_j(t-s)^\alpha}{1+a\lambda_j}\right) \langle h, \varphi_j \rangle \varphi_j. \end{aligned}$$

Therefore, with $1 < \alpha < \alpha' < 2$, we also get

$$\begin{cases} u_{\alpha'}(\cdot, t) = \mathcal{P}_{\Delta,\alpha'}(t)f + \int_0^t \mathcal{Q}_{\Delta,\alpha'}(t-s)\mathcal{F}(u, v)(\cdot, s) \, ds, \\ v_{\alpha'}(\cdot, t) = \mathcal{P}_{\Delta,\alpha'}(t)g + \int_0^t \mathcal{Q}_{\Delta,\alpha'}(t-s)\mathcal{G}(u, v)(\cdot, s) \, ds. \end{cases} \tag{3.7}$$

Next, we give several lemmas related to Sect. 4 as follows.

Lemma 3.1 *Let $1 < \alpha_1 < \alpha < \alpha_2 < 2$, $\gamma \geq 0$, and $w \in \mathcal{H}^\gamma(\Omega)$. The following inequalities hold:*

$$\| \mathcal{P}_{\Delta,\alpha}(t)w \|_{\mathcal{H}^\gamma(\Omega)} \leq \overline{M}_2(\alpha_1, \alpha_2, a, \mu_0) t^{-\alpha_1 \mu_0} \|w\|_{\mathcal{H}^\gamma(\Omega)}, \tag{3.8}$$

$$\| \mathcal{Q}_\alpha(t-s)w \|_{\mathcal{H}^\gamma(\Omega)} \leq \overline{M}_3(\alpha_1, \alpha_2, a, \mu_0) (t-s)^{\alpha_1-1-\alpha_1 \mu_0} \|w\|_{\mathcal{H}^\gamma(\Omega)}, \tag{3.9}$$

where μ_0 is a positive number satisfying $0 < \mu_0 < 1$.

Proof Using Lemma 2.4, we get

$$\begin{aligned}
 \|\mathcal{P}_{\Delta,\alpha}(t)w\|_{\mathcal{H}^\gamma(\Omega)}^2 &= \sum_{j=1}^\infty \lambda_j^{2\gamma} E_{\alpha,1}^2\left(\frac{-\lambda_j t^\alpha}{1+a\lambda_j}\right) |\langle w, \varphi_j \rangle|^2 \\
 &\leq \sum_{j=1}^\infty \lambda_j^{2\gamma} \left(\frac{M_2(\alpha_1, \alpha_2)}{1+\frac{\lambda_j t^\alpha}{1+a\lambda_j}}\right)^2 |\langle w, \varphi_j \rangle|^2 \\
 &\leq \sum_{j=1}^\infty \lambda_j^{2\gamma} \frac{M_2^2(\alpha_1, \alpha_2)}{\left(1+\frac{\lambda_j t^\alpha}{1+a\lambda_j}\right)^{\mu_0}} |\langle w, \varphi_j \rangle|^2 \\
 &\leq \sum_{j=1}^\infty \lambda_j^{2\gamma} M_2^2(\alpha_1, \alpha_2) \left(\frac{\lambda_j t^\alpha}{1+a\lambda_j}\right)^{-2\mu_0} |\langle w, \varphi_j \rangle|^2 \\
 &\leq M_2^2(\alpha_1, \alpha_2) t^{-2\alpha\mu_0} (a+\lambda_1^{-1})^{2\mu_0} \sum_{j=1}^\infty \lambda_j^{2\gamma} |\langle w, \varphi_j \rangle|^2 \\
 &\leq M_2^2(\alpha_1, \alpha_2) t^{-2\alpha_1\mu_0} (\lambda_1^{-1}+a)^{2\mu_0} \|w\|_{\mathcal{H}^\gamma(\Omega)}^2.
 \end{aligned}$$

Therefore, with $\overline{M}_2(\alpha_1, \alpha_2, a, \mu_0) := M_2(\alpha_1, \alpha_2)(\lambda_1^{-1}+a)^{\mu_0}$, we have the following estimate:

$$\|\mathcal{P}_{\Delta,\alpha}(t)w\|_{\mathbb{H}^\gamma(\Omega)} \leq \overline{M}_2(\alpha_1, \alpha_2, a, \mu_0) t^{-\alpha_1\mu_0} \|w\|_{\mathcal{H}^\gamma(\Omega)}. \tag{3.10}$$

Likewise, utilizing Lemma 2.4, we can get the following estimation:

$$\begin{aligned}
 \|\mathcal{Q}_{\Delta,\alpha}(t-s)w\|_{\mathcal{H}^\gamma(\Omega)}^2 &= \sum_{j=1}^\infty \frac{\lambda_j^{2\gamma}}{(1+a\lambda_j)^2} (t-s)^{2(\alpha-1)} E_{\alpha,\alpha}^2\left(\frac{-\lambda_j(t-s)^\alpha}{1+a\lambda_j}\right) |\langle w, \varphi_j \rangle|^2 \\
 &\leq \sum_{j=1}^\infty \frac{\lambda_j^{2\gamma}}{(1+a\lambda_j)^2} (t-s)^{2(\alpha-1)} \left(\frac{M_3(\alpha_1, \alpha_2)}{1+\frac{\lambda_j(t-s)^\alpha}{1+a\lambda_j}}\right)^2 |\langle w, \varphi_j \rangle|^2 \\
 &\leq (t-s)^{2(\alpha-1-\alpha\mu_0)} M_3^2(\alpha_1, \alpha_2) \sum_{j=1}^\infty \lambda_j^{2\gamma} \left(\frac{1}{\lambda_j}+a\right)^{2(\mu_0-1)} \lambda_j^{-2} |\langle w, \varphi_j \rangle|^2 \\
 &\leq (t-s)^{2(\alpha-1-\alpha\mu_0)} M_3^2(\alpha_1, \alpha_2) (\lambda_1^{-1}+a)^{2(\mu_0-1)} \lambda_1^{-2} \sum_{j=1}^\infty \lambda_j^{2\gamma} |\langle w, \varphi_j \rangle|^2 \\
 &\leq (t-s)^{2(\alpha_1-1-\alpha_1\mu_0)} M_3^2(\alpha_1, \alpha_2) (\lambda_1^{-1}+a)^{2(\mu_0-1)} \lambda_1^{-2} \|w\|_{\mathcal{H}^\gamma(\Omega)}^2.
 \end{aligned} \tag{3.11}$$

Therefore, we deduce

$$\|\mathcal{Q}_\alpha(t-s)w\|_{\mathcal{H}^\gamma(\Omega)} \leq \overline{M}_3(\alpha_1, \alpha_2, a, \mu_0) (t-s)^{\alpha_1-1-\alpha_1\mu_0} \|w\|_{\mathcal{H}^\gamma(\Omega)}, \tag{3.12}$$

where $\overline{M}_3(\alpha_1, \alpha_2, a, \mu_0) := M_3(\alpha_1, \alpha_2)(\lambda_1^{-1}+a)^{\mu_0-1}\lambda_1^{-1}$.

Thus, we complete the proof of Lemma 3.1. □

Lemma 3.2 *Let $1 < \alpha_1 < \alpha < \alpha' < \alpha_2 < 2$, $\gamma \geq 0$ with $0 \leq \nu_0 \leq 1$ and $w \in \mathcal{H}^\gamma(\Omega)$. The following inequalities hold:*

$$\begin{aligned} & \| [\mathcal{P}_{\Delta,\alpha'}(t) - \mathcal{P}_{\Delta,\alpha}(t)]w \|_{\mathcal{H}^\gamma(\Omega)} \\ & \leq \bar{A}(\alpha_1, \alpha_2, \epsilon, \nu_0, a, T) [(\alpha' - \alpha)^\epsilon + (\alpha' - \alpha)] t^{-\alpha_2(1-\nu_0)-\epsilon} \|w\|_{\mathcal{H}^\gamma(\Omega)}. \end{aligned} \tag{3.13}$$

Besides, we have

$$\begin{aligned} & \| [\mathcal{Q}_{\Delta,\alpha'}(t-s) - \mathcal{Q}_{\Delta,\alpha}(t-s)]w \|_{\mathcal{H}^\gamma(\Omega)} \\ & \leq \bar{B}(\alpha_1, \alpha_2, \epsilon, \nu_0, a, T) [(\alpha' - \alpha)^\epsilon + (\alpha' - \alpha)] (t-s)^{\alpha_1\nu_0-\epsilon-1} \|w\|_{\mathcal{H}^\gamma(\Omega)}. \end{aligned}$$

Proof We get

$$\begin{aligned} & \| [\mathcal{P}_{\Delta,\alpha'}(t) - \mathcal{P}_{\Delta,\alpha}(t)]w \|_{\mathbb{H}^\gamma(\Omega)}^2 \\ & = \sum_{j=1}^\infty \lambda_j^{2\gamma} \left[E_{\alpha',1} \left(\frac{-\lambda_j t^{\alpha'}}{1 + a\lambda_j} \right) - E_{\alpha,1} \left(\frac{-\lambda_j t^\alpha}{1 + a\lambda_j} \right) \right]^2 |\langle w, \varphi_j \rangle|^2. \end{aligned}$$

By using Lemma 2.6, we obtain the following estimates:

$$\begin{aligned} & \| [\mathcal{P}_{\Delta,\alpha'}(t) - \mathcal{P}_{\Delta,\alpha}(t)]w \|_{\mathcal{H}^\alpha(\Omega)}^2 \\ & \leq \sum_{j=1}^\infty \lambda_j^{2\gamma} A^2(\alpha_1, \alpha_2, \epsilon, \nu_0, T) [(\alpha' - \alpha)^\epsilon + (\alpha' - \alpha)]^2 \\ & \quad \times \left(\frac{\lambda_j}{1 + a\lambda_j} \right)^{2(\nu_0-1)} t^{-2\alpha_2(1-\nu_0)-2\epsilon} |\langle w, \varphi_j \rangle|^2 \\ & \leq A^2(\alpha_1, \alpha_2, \epsilon, \nu_0, T) [(\alpha' - \alpha)^\epsilon + (\alpha' - \alpha)]^2 \\ & \quad \times t^{-2\alpha_2(1-\nu_0)-2\epsilon} (\lambda_1^{-1} + a)^{2(1-\nu_0)} \sum_{j=1}^\infty \lambda_j^{2\gamma} |\langle w, \varphi_j \rangle|^2 \\ & \leq A^2(\alpha_1, \alpha_2, \epsilon, \nu_0, T) [(\alpha' - \alpha)^\epsilon + (\alpha' - \alpha)]^2 \\ & \quad \times t^{-2\alpha_2(1-\nu_0)-2\epsilon} (\lambda_1^{-1} + a)^{2(1-\nu_0)} \|w\|_{\mathcal{H}^\gamma(\Omega)}^2. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} & \| [\mathcal{P}_{\Delta,\alpha'}(t) - \mathcal{P}_{\Delta,\alpha}(t)]w \|_{\mathcal{H}^\gamma(\Omega)} \\ & \leq \bar{A}(\alpha_1, \alpha_2, \epsilon, \nu_0, a, T) [(\alpha' - \alpha)^\epsilon + (\alpha' - \alpha)] t^{-\alpha_2(1-\nu_0)-\epsilon} \|w\|_{\mathcal{H}^\gamma(\Omega)}, \end{aligned}$$

where $\bar{A}(\alpha_1, \alpha_2, \epsilon, \nu_0, a, T) := A(\alpha_1, \alpha_2, \epsilon, \nu_0, T)(\lambda_1^{-1} + a)^{1-\nu_0}$.

Similarly, by applying Lemma 2.7, we also get

$$\begin{aligned} & \| [\mathcal{Q}_{\Delta,\alpha'}(t-s) - \mathcal{Q}_{\Delta,\alpha}(t-s)]w \|_{\mathcal{H}^\gamma(\Omega)}^2 \\ & = \sum_{j=1}^\infty \frac{\lambda_j^{2\gamma}}{(1 + a\lambda_j)^2} \left[(t-s)^{\alpha'-1} E_{\alpha',\alpha'} \left(\frac{-\lambda_j(t-s)^{\alpha'}}{1 + a\lambda_j} \right) \right]^2 \end{aligned}$$

$$\begin{aligned}
 & - (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(\frac{-\lambda_j(t-s)^\alpha}{1+a\lambda_j} \right) \Big]^2 |\langle w, \varphi_j \rangle|^2 \\
 \leq & \sum_{j=1}^\infty B^2(\alpha_1, \alpha_2, \epsilon, \nu_0, T) \left(\frac{\lambda_j}{1+a\lambda_j} \right)^{2(\nu_0-1)} (t-s)^{2\alpha_1\nu_0-2\epsilon-2} \\
 & \times [(\alpha' - \alpha)^\epsilon + (\alpha' - \alpha)]^2 \frac{\lambda_j^{2\gamma}}{(1+a\lambda_j)^2} |\langle w, \varphi_j \rangle|^2 \\
 \leq & B^2(\alpha_1, \alpha_2, \epsilon, \nu_0, T) [(\alpha' - \alpha_2)^\epsilon + (\alpha' - \alpha)]^2 \lambda_1^{-2} (t-s)^{2\alpha_1\nu_0-2\epsilon-2} \\
 & \times \sum_{j=1}^\infty \left(\frac{1}{\lambda_j} + a \right)^{-2\nu_0} \lambda_j^{2\gamma} |\langle w, \varphi_j \rangle|^2 \\
 \leq & B^2(\alpha_1, \alpha_2, \epsilon, \nu_0, T) [(\alpha' - \alpha)^\epsilon + (\alpha' - \alpha)]^2 \lambda_1^{-2} a^{-2\nu_0} (t-s)^{2\alpha_1\nu_0-2\epsilon-2} \|w\|_{\mathcal{H}^\gamma(\Omega)}^2.
 \end{aligned}$$

We denote $\bar{B}(\alpha_1, \alpha_2, \epsilon, \nu_0, a, T) := B(\alpha_1, \alpha_2, \epsilon, \nu_0, T) \lambda_1^{-1} a^{-\nu_0}$. Therefore, we obtain

$$\begin{aligned}
 & \| [\mathcal{Q}_{\Delta, \alpha'}(t-s) - \mathcal{Q}_{\Delta, \omega}(t-s)] w \|_{\mathcal{H}^\gamma(\Omega)} \\
 & \leq \bar{B}(\alpha_1, \alpha_2, \epsilon, \nu_0, a, T) [(\alpha' - \alpha)^\epsilon + (\alpha' - \alpha)] (t-s)^{\alpha_1\nu_0-\epsilon-1} \|w\|_{\mathcal{H}^\gamma(\Omega)}. \tag{3.14}
 \end{aligned}$$

We get all estimates of Lemma 3.2. This completes the proof. □

4 Stability of the fractional order of problem (1.1)

In this section, we are interested in studying the existence of a mild solution and the continuous dependence of the solution of problem (1.1) with input (the fractional-order α, α' and the initial condition f, g). We assume that \mathcal{F}, \mathcal{G} satisfy the following assumptions:

(S.1)

$$\| \mathcal{F}(u, v)(\cdot, t) \|_{\mathcal{H}^\gamma(\Omega)} \leq C_1 (1 + \|u(\cdot, t)\|_{\mathcal{H}^\gamma(\Omega)} + \|v(\cdot, t)\|_{\mathcal{H}^\gamma(\Omega)}), \tag{4.1}$$

$$\| \mathcal{G}(u, v)(\cdot, t) \|_{\mathcal{H}^\gamma(\Omega)} \leq C_2 (1 + \|u(\cdot, t)\|_{\mathcal{H}^\gamma(\Omega)} + \|v(\cdot, t)\|_{\mathcal{H}^\gamma(\Omega)}), \tag{4.2}$$

where $(u, v) \in \mathcal{H}^\gamma(\Omega) = \mathcal{H}^\gamma(\Omega) \times \mathcal{H}^\gamma(\Omega)$.

(S.2)

$$\| \mathcal{F}(u_1, v_1)(\cdot, t) - \mathcal{F}(u_2, v_2)(\cdot, t) \|_{\mathcal{H}^\gamma(\Omega)} \leq \mathcal{K}_1 (\|u_1 - u_2\|_{\mathcal{H}^\gamma(\Omega)} + \|v_1 - v_2\|_{\mathcal{H}^\gamma(\Omega)}), \tag{4.3}$$

$$\| \mathcal{G}(u_1, v_1)(\cdot, t) - \mathcal{G}(u_2, v_2)(\cdot, t) \|_{\mathcal{H}^\gamma(\Omega)} \leq \mathcal{K}_2 (\|u_1 - u_2\|_{\mathcal{H}^\gamma(\Omega)} + \|v_1 - v_2\|_{\mathcal{H}^\gamma(\Omega)}), \tag{4.4}$$

where $(u_1, v_1) \in \mathcal{H}^\gamma(\Omega), (u_2, v_2) \in \mathcal{H}^\gamma(\Omega)$.

Definition 4.1 $w = (u(\cdot, t), v(\cdot, t)) \in \mathcal{L}_p^\infty(0, T, \mathcal{H}^\gamma(\Omega)) = \mathbb{L}_p^\infty(0, T, \mathcal{H}^\gamma(\Omega)) \times \mathbb{L}_p^\infty(0, T, \mathcal{H}^\gamma(\Omega))$ is called a mild solution of problem (1.1) if it satisfies system (3.6).

Theorem 4.1 Let $w_0(f, g) \in \mathcal{H}^\gamma(\Omega)$. Assume that $1 < \alpha_1 < \alpha < \alpha' < \alpha_2 < 2$ and $0 < \nu_0 < 1$. The nonlinear integral equation (1.1) has a unique solution $w(u, v) \in \mathcal{L}_p^\infty(0, T; \mathcal{H}^\gamma(\Omega))$. Let $w_\alpha \in \mathcal{L}_p^\infty(0, T; \mathcal{H}^\gamma(\Omega))$ and $w_{\alpha'} \in \mathcal{L}_p^\infty(0, T; \mathcal{H}^\gamma(\Omega))$ be two solutions of (1.1) with

fractional order α and α' , respectively. If there exist numbers μ_0, ϵ satisfying $0 < \epsilon < \min(\frac{1}{2} - \alpha_2 + \alpha_2\nu_0, \alpha_1\nu_0 - \frac{1}{2})$ and $0 < \mu_0 < 1 - \frac{1}{2\alpha_1}$, then

$$\begin{aligned} & \| \mathbf{w} \|_{\mathcal{C}^{\alpha_1\mu_0}((0,T],\mathcal{H}^\nu(\Omega))} \\ & \leq \sqrt{\mathcal{M}_1(\alpha_1, \alpha_2, a, \mu_0, T, \|w_0\|_{\mathcal{H}^\nu(\Omega)})} \exp\left(\frac{\mathcal{M}_2(\alpha_1, \alpha_2, a, \mu_0, T)T}{2}\right), \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} & \| \mathbf{w}_{\alpha'} - \mathbf{w}_\alpha \|_{\mathcal{C}^{\alpha_2(1-\nu_0)+\epsilon}((0,T],\mathcal{H}^\nu(\Omega))} \\ & \leq \sqrt{\mathcal{M}_{\mu_0,\nu_0}^{\mathcal{C}_1,\mathcal{C}_2}(\alpha_1, \alpha_2, \epsilon, a, T, \|w_0\|_{\mathcal{H}^\nu(\Omega)})} [(\alpha' - \alpha)^\epsilon + (\alpha' - \alpha)] \\ & \quad \times \exp\left(\frac{\mathcal{M}_{\mu_0,\nu_0}^{\mathcal{K}_1,\mathcal{K}_2}(\alpha_1, \alpha_2, \epsilon, a, T)T}{2}\right). \end{aligned} \tag{4.6}$$

Proof of Theorem 4.1 We divide the proof into three parts.

Part 1. The existence and uniqueness of the solution of the nonlinear fractional pseudo-parabolic equation systems (1.1). For $\mathbf{w} \in \mathcal{L}_p^\infty(0, T; \mathcal{H}^\nu(\Omega))$, we consider the following function $\mathcal{H}\mathbf{w} := (\mathcal{H}_{\Delta,\alpha}u(\cdot, t), \mathcal{H}_{\Delta,\alpha}v(\cdot, t))$, where

$$\begin{cases} \mathcal{H}_{\Delta,\alpha}u(\cdot, t) = \mathcal{P}_{\Delta,\alpha}(t)f + \int_0^t \mathcal{Q}_{\Delta,\alpha}(t-s)\mathcal{F}(u, v)(\cdot, s) ds, \\ \mathcal{H}_{\Delta,\alpha}v(\cdot, t) = \mathcal{P}_{\Delta,\alpha}(t)g + \int_0^t \mathcal{Q}_{\Delta,\alpha}(t-s)\mathcal{G}(u, v)(\cdot, s) ds. \end{cases} \tag{4.7}$$

Let $\mathbf{w}_1(u_1(\cdot, t), v_1(\cdot, t)) \in \mathcal{L}_p^\infty(0, T; \mathcal{H}^\nu(\Omega))$, $\mathbf{w}_2(u_2(\cdot, t), v_2(\cdot, t)) \in \mathcal{L}_p^\infty(0, T; \mathcal{H}^\nu(\Omega))$, we have

$$\begin{aligned} \mathcal{H}_{\Delta,\alpha}u_1(\cdot, t) - \mathcal{H}_{\Delta,\alpha}u_2(\cdot, t) &= \int_0^t \mathcal{Q}_{\Delta,\alpha}(t-s)[\mathcal{F}(u_1, v_1)(\cdot, s) - \mathcal{F}(u_2, v_2)(\cdot, s)] ds, \\ \mathcal{H}_{\Delta,\alpha}v_1(\cdot, t) - \mathcal{H}_{\Delta,\alpha}v_2(\cdot, t) &= \int_0^t \mathcal{Q}_{\Delta,\alpha}(t-s)[\mathcal{G}(u_1, v_1)(\cdot, s) - \mathcal{G}(u_2, v_2)(\cdot, s)] ds. \end{aligned}$$

With $p > 0$, we get the following estimate:

$$\begin{aligned} & \| e^{-pt}(\mathcal{H}_{\Delta,\alpha}u_1(\cdot, t) - \mathcal{H}_{\Delta,\alpha}u_2(\cdot, t)) \|_{\mathcal{H}^\nu(\Omega)} \\ &= \left\| \int_0^t \mathcal{Q}_{\Delta,\alpha}(t-s)e^{-ps}[\mathcal{F}(u_1, v_1)(\cdot, s) - \mathcal{F}(u_2, v_2)(\cdot, s)] ds \right\|_{\mathcal{H}^\nu(\Omega)}. \end{aligned}$$

Applying Lemma 3.1, we get the following estimate:

$$\begin{aligned} & \| e^{-pt}(\mathcal{H}_{\Delta,\alpha}u_1(\cdot, t) - \mathcal{H}_{\Delta,\alpha}u_2(\cdot, t)) \|_{\mathcal{H}^\nu(\Omega)}^2 \\ & \leq \left[\int_0^t \overline{\mathcal{M}}_3(\alpha_1, \alpha_2, a, \mu_0)(t-s)^{\alpha_1-1-\alpha_1\mu_0} e^{-p(t-s)} \right. \\ & \quad \left. \times [e^{-ps} \|\mathcal{F}(u_1, v_1)(\cdot, s) - \mathcal{F}(u_2, v_2)(\cdot, s)\|_{\mathcal{H}^\alpha(\Omega)}] ds \right]^2 \end{aligned}$$

$$\begin{aligned} &\leq \overline{M}_3^2(\alpha_1, \alpha_2, a, \mu_0) \left[\int_0^t (t-s)^{\alpha_1-1-\alpha_1\mu_0} e^{-p(t-s)} \right. \\ &\quad \left. \times [e^{-ps} \|\mathcal{F}(u_1, v_1)(\cdot, s) - \mathcal{F}(u_2, v_2)(\cdot, s)\|_{\mathcal{H}^\gamma(\Omega)}] ds \right]^2. \end{aligned}$$

Using (4.3) and the inequality $(m + n)^2 \leq 2(m^2 + n^2)$, we obtain

$$\begin{aligned} &\|e^{-pt} (\mathcal{H}_{\Delta, \alpha} u_1(\cdot, t) - \mathcal{H}_{\Delta, \alpha} u_2(\cdot, t))\|_{\mathcal{H}^\gamma(\Omega)}^2 \\ &\leq \mathcal{K}_1^2 \overline{M}_3^2(\alpha_1, \alpha_2, a, \mu_0) \left[\int_0^t e^{-ps} (\|u_1 - u_2\|_{\mathcal{H}^\gamma(\Omega)} + \|v_1 - v_2\|_{\mathcal{H}^\gamma(\Omega)}) \right. \\ &\quad \left. \times (t-s)^{\alpha_1-1-\alpha_1\mu_0} e^{-p(t-s)} ds \right]^2. \end{aligned} \tag{4.8}$$

Using the inequality $(m + n)^2 \leq 2(m^2 + n^2)$, we deduce that

$$\begin{aligned} &\|\mathcal{H}_{\Delta, \alpha} u_1 - \mathcal{H}_{\Delta, \alpha} u_2\|_{\mathbb{L}_p^\infty(0, T, \mathcal{H}^\gamma(\Omega))}^2 \\ &\leq \mathcal{K}_1^2 \overline{M}_3^2(\alpha_1, \alpha_2, a, \mu_0) (\|u_1 - u_2\|_{\mathbb{L}_p^\infty(0, T, \mathcal{H}^\gamma(\Omega))} + \|v_1 - v_2\|_{\mathbb{L}_p^\infty(0, T, \mathcal{H}^\gamma(\Omega))})^2 \\ &\quad \times \left[\int_0^t (t-s)^{\alpha_1-1-\alpha_1\mu_0} e^{-p(t-s)} ds \right]^2 \\ &\leq 2\mathcal{K}_1^2 \overline{M}_3^2(\alpha_1, \alpha_2, a, \mu_0) (\|u_1 - u_2\|_{\mathbb{L}_p^\infty(0, T, \mathcal{H}^\gamma(\Omega))}^2 + \|v_1 - v_2\|_{\mathbb{L}_p^\infty(0, T, \mathcal{H}^\gamma(\Omega))}^2) \\ &\quad \times \left[\int_0^t (t-s)^{\alpha_1-1-\alpha_1\mu_0} e^{-p(t-s)} ds \right]^2 \\ &\leq 2\mathcal{K}_1^2 \overline{M}_3^2(\alpha_1, \alpha_2, a, \mu_0) \|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathcal{L}_p^\infty(0, T, \mathcal{H}^\gamma(\Omega))}^2 \\ &\quad \times \left[\int_0^t (t-s)^{\alpha_1-1-\alpha_1\mu_0} e^{-p(t-s)} ds \right]^2. \end{aligned}$$

Applying Hölder’s inequality with assumption $\mu_0 < 1 - \frac{1}{2\alpha_1}$, we obtain

$$\begin{aligned} &\|\mathcal{H}_{\Delta, \alpha} u_1 - \mathcal{H}_{\Delta, \alpha} u_2\|_{\mathbb{L}_p^\infty(0, T, \mathcal{H}^\gamma(\Omega))}^2 \\ &\leq 2\mathcal{K}_1^2 \overline{M}_3^2(\alpha_1, \alpha_2, a, \mu_0) \|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathcal{L}_p^\infty(0, T, \mathcal{H}^\gamma(\Omega))}^2 \\ &\quad \times \int_0^t (t-s)^{2\alpha_1-2-2\alpha_1\mu_0} ds \int_0^t e^{-2p(t-s)} ds \\ &\leq 2\mathcal{K}_1^2 \overline{M}_3^2(\alpha_1, \alpha_2, a, \mu_0) \|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathcal{L}_p^\infty(0, T, \mathcal{H}^\gamma(\Omega))}^2 \frac{t^{2\alpha_1-1-2\alpha_1\mu_0}}{2\alpha_1-1-2\alpha_1\mu_0} \frac{1-e^{-2pt}}{2p}. \end{aligned} \tag{4.9}$$

Hence

$$\begin{aligned} &\|\mathcal{H}_{\Delta, \alpha} u_1 - \mathcal{H}_{\Delta, \alpha} u_2\|_{\mathbb{L}_p^\infty(0, T, \mathcal{H}^\gamma(\Omega))}^2 \\ &\leq \frac{2\mathcal{K}_1^2 \overline{M}_3^2(\alpha_1, \alpha_2, a, \mu_0)}{2p(2\alpha_1-2\alpha_1\mu_0-1)} T^{2\alpha_1-2\alpha_1\mu_0-1} (1-e^{-2pT}) \|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathcal{L}_p^\infty(0, T, \mathcal{H}^\gamma(\Omega))}^2. \end{aligned} \tag{4.10}$$

We can obtain a similar estimate

$$\begin{aligned} & \| \mathcal{H}_{\Delta,\alpha} v_1 - \mathcal{H}_{\Delta,\alpha} v_2 \|_{L^p_\infty(0,T,\mathcal{H}^\gamma(\Omega))}^2 \\ & \leq \frac{2\mathcal{K}_2^2 \overline{M}_3^2(\alpha_1, \alpha_2, a, \mu_0)}{2p(2\alpha_1 - 2\alpha_1\mu_0 - 1)} T^{2\alpha_1 - 2\alpha_1\mu_0 - 1} (1 - e^{-2pT}) \| \mathbf{w}_1 - \mathbf{w}_2 \|_{L^p_\infty(0,T,\mathcal{H}^\gamma(\Omega))}^2. \end{aligned} \tag{4.11}$$

From (4.10)–(4.11), we find that

$$\begin{aligned} & \| \mathcal{H} \mathbf{w}_1 - \mathcal{H} \mathbf{w}_2 \|_{L^p_\infty(0,T,\mathcal{H}^\gamma(\Omega))} \\ & = \sqrt{ \| \mathcal{H} u_1 - \mathcal{H} u_2 \|_{L^p_\infty(0,T,\mathcal{H}^\gamma(\Omega))}^2 + \| \mathcal{H} v_1 - \mathcal{H} v_2 \|_{L^p_\infty(0,T,\mathcal{H}^\gamma(\Omega))}^2 } \\ & \leq \sqrt{ \frac{2(\mathcal{K}_1^2 + \mathcal{K}_2^2) \overline{M}_3^2(\alpha_1, \alpha_2, a, \mu_0)}{2p(2\alpha_1 - 2\alpha_1\mu_0 - 1)} T^{2\alpha_1 - \alpha_1\mu_0 - 1} (1 - e^{-2pT}) } } \\ & \quad \times \| \mathbf{w}_1 - \mathbf{w}_2 \|_{L^p_\infty(0,T,\mathcal{H}^\gamma(\Omega))}. \end{aligned} \tag{4.12}$$

If $\mathbf{w}_2 = 0$, then for any $\mathbf{w} \in L^p_\infty(0, T, \mathcal{H}^\gamma(\Omega))$

$$\begin{aligned} & \| \mathcal{H} \mathbf{w} \|_{L^p_\infty(0,T,\mathcal{H}^\gamma(\Omega))} \\ & \leq \| \mathcal{H} \mathbf{w} - \mathcal{H} \mathbf{w}_2 \|_{L^p_\infty(0,T,\mathcal{H}^\gamma(\Omega))} + \| \mathcal{H} \mathbf{w}_2 \|_{L^p_\infty(0,T,\mathcal{H}^\gamma(\Omega))} \\ & \leq \sqrt{ \frac{2(\mathcal{K}_1^2 + \mathcal{K}_2^2) \overline{M}_3^2(\alpha_1, \alpha_2, a, \mu_0)}{2p(2\alpha_1 - 2\alpha_1\mu_0 - 1)} T^{2\alpha_1 - \alpha_1\mu_0 - 1} (1 - e^{-2pT}) } } \| \mathbf{w} \|_{L^p_\infty(0,T,\mathcal{H}^\gamma(\Omega))} \\ & \quad + \| \mathcal{H} \mathbf{w}_2 \|_{L^p_\infty(0,T,\mathcal{H}^\gamma(\Omega))}. \end{aligned} \tag{4.13}$$

Note that $\mathcal{H} \mathbf{w}_2 := (\mathcal{H}_{\Delta,\alpha} u_2(\cdot, t), \mathcal{H}_{\Delta,\alpha} v_2(\cdot, t))$, where $u_2 = v_2 = 0$ and

$$\begin{cases} \mathcal{H}_{\Delta,\alpha} u_2(\cdot, t) = \mathcal{P}_{\Delta,\alpha}(t)f, \\ \mathcal{H}_{\Delta,\alpha} v_2(\cdot, t) = \mathcal{P}_{\Delta,\alpha}(t)g. \end{cases} \tag{4.14}$$

By applying Lemma 3.1, we can deduce that $\mathcal{H} \mathbf{w}_2 \in L^p_\infty(0, T, \mathcal{H}^\gamma(\Omega))$. Therefore, we conclude that if any $\mathbf{w} \in L^p_\infty(0, T, \mathcal{H}^\gamma(\Omega))$, then $\mathcal{H} \mathbf{w}$ is bounded.

Part 2. From (3.6) and applying the inequality $(a + b)^2 \leq 2(a^2 + b^2)$, we have the following estimate:

$$\| u(\cdot, t) \|_{\mathcal{H}^\gamma(\Omega)}^2 \leq 2 \| \mathcal{P}_{\Delta,\alpha}(t)f \|_{\mathcal{H}^\gamma(\Omega)}^2 + 2 \left\| \int_0^t \mathcal{Q}_{\Delta,\alpha}(t-s) \mathcal{F}(u, v)(\cdot, s) ds \right\|_{\mathcal{H}^\gamma(\Omega)}^2.$$

Using Lemma 3.1, we get

$$\begin{aligned} \| u(\cdot, t) \|_{\mathcal{H}^\gamma(\Omega)}^2 & \leq 2 \overline{M}_2(\alpha_1, \alpha_2, a, \mu_0) t^{-2\alpha_1\mu_0} \| f \|_{\mathcal{H}^\gamma(\Omega)}^2 \\ & \quad + 2 \left(\int_0^t \overline{M}_3(\alpha_1, \alpha_2, a, \mu_0) (t-s)^{\alpha_1 - 1 - \alpha_1\mu_0} \| \mathcal{F}((u, v)(\cdot, s)) \|_{\mathcal{H}^\gamma(\Omega)} ds \right)^2. \end{aligned}$$

Multiplying both sides by $t^{2\alpha_1\mu_0}$ and using Hölder’s inequality with assumption (4.1), we can find that

$$\begin{aligned} & (t^{2\alpha_1\mu_0} \|u(\cdot, t)\|_{\mathcal{H}^{\nu}(\Omega)})^2 \\ & \leq 2\overline{M}_2(\alpha_1, \alpha_2, a, \mu_0) \|f\|_{\mathcal{H}^{\nu}(\Omega)}^2 + 2\overline{M}_3(\alpha_1, \alpha_2, a, \mu_0) t^{2\alpha_1\mu_0} \\ & \quad \times \left(\int_0^t (1 + \|u(\cdot, s)\|_{\mathcal{H}^{\nu}(\Omega)} + \|v(\cdot, s)\|_{\mathcal{H}^{\nu}(\Omega)}) s^{\alpha_1\mu_0} s^{-\alpha_1\mu_0} (t-s)^{\alpha_1-1-\alpha_1\mu_0} ds \right)^2 \\ & \leq 2\overline{M}_2(\alpha_1, \alpha_2, a, \mu_0) \|f\|_{\mathcal{H}^{\nu}(\Omega)}^2 + 2\overline{M}_3(\alpha_1, \alpha_2, a, \mu_0) t^{2\alpha_1\mu_0} \\ & \quad \times \int_0^t (1 + \|u(\cdot, s)\|_{\mathcal{H}^{\nu}(\Omega)} + \|v(\cdot, s)\|_{\mathcal{H}^{\nu}(\Omega)})^2 s^{2\alpha_1\mu_0} ds \int_0^t s^{-2\alpha_1\mu_0} (t-s)^{2\alpha_1-2-2\alpha_1\mu_0} ds. \end{aligned}$$

Using the beta function property $\int_0^t s^{\theta_1-1} (t-s)^{\vartheta_1-1} ds = t^{\theta_1+\vartheta_1-1} \mathbf{B}(\theta_1, \vartheta_1)$, $\theta_1 > 0$, $\vartheta_1 > 0$ with assumption $0 < \mu_0 < 1 - \frac{1}{2\alpha_1}$, then $2\alpha_1 - 1 - 2\alpha_1\mu_0 > 0$ and $-2\alpha_1\mu_0 + 1 > 0$, we obtain

$$\begin{aligned} (t^{2\alpha_1\mu_0} \|u(\cdot, t)\|_{\mathcal{H}^{\nu}(\Omega)})^2 & \leq 2\overline{M}_2(\alpha_1, \alpha_2, a, \mu_0) \|f\|_{\mathcal{H}^{\nu}(\Omega)}^2 \\ & \quad + 2\overline{M}_3(\alpha_1, \alpha_2, a, \mu_0) t^{2\alpha_1(1-\mu_0)} \mathbf{B}(\theta_1, \vartheta_1) \\ & \quad \times \int_0^t (1 + \|u(\cdot, s)\|_{\mathcal{H}^{\nu}(\Omega)} + \|v(\cdot, s)\|_{\mathcal{H}^{\nu}(\Omega)})^2 s^{2\alpha_1\mu_0} ds, \end{aligned}$$

where $\theta_1 := -2\alpha_1\mu_0 + 1$, $\vartheta_1 := 2\alpha_1 - 1 - 2\alpha_1\mu_0$. Using the inequality $(m + n + p)^2 \leq 3(m^2 + n^2 + p^2)$, we get

$$\begin{aligned} & (t^{2\alpha_1\mu_0} \|u(\cdot, t)\|_{\mathbb{H}^{\eta}(\Omega)})^2 \\ & \leq 2\overline{M}_2(\alpha_1, \alpha_2, a, \mu_0) \|f\|_{\mathcal{H}^{\nu}(\Omega)}^2 \\ & \quad + 6\overline{M}_3(\alpha_1, \alpha_2, a, \mu_0) t^{2\alpha_1(1-\mu_0)} \mathbf{B}(\theta_1, \vartheta_1) \\ & \quad \times \int_0^t (1 + \|u(\cdot, s)\|_{\mathcal{H}^{\nu}(\Omega)}^2 + \|v(\cdot, s)\|_{\mathcal{H}^{\nu}(\Omega)}^2) s^{2\alpha_1\mu_0} ds \\ & \leq 2\overline{M}_2(\alpha_1, \alpha_2, a, \mu_0) \|f\|_{\mathcal{H}^{\nu}(\Omega)}^2 + \frac{6\overline{M}_3(\alpha_1, \alpha_2, a, \mu_0) t^{2\alpha_1\mu_0+1} \mathbf{B}(\theta_1, \vartheta_1)}{2\alpha_1\mu_0 + 1} \\ & \quad + 6\overline{M}_3(\alpha_1, \alpha_2, a, \mu_0) t^{2\alpha_1(1-\mu_0)} \mathbf{B}(\theta_1, \vartheta_1) \\ & \quad \times \int_0^t (\|u(\cdot, s)\|_{\mathbb{H}^{\eta}(\Omega)}^2 + \|v(\cdot, s)\|_{\mathbb{H}^{\eta}(\Omega)}^2) s^{2\alpha_1\mu_0} ds. \tag{4.15} \end{aligned}$$

Similarly, we can also obtain

$$\begin{aligned} & (t^{2\alpha_1\mu_0} \|v(\cdot, t)\|_{\mathcal{H}^{\nu}(\Omega)})^2 \\ & \leq 2\overline{M}_2(\alpha_1, \alpha_2, a, \mu_0) \|g\|_{\mathcal{H}^{\nu}(\Omega)}^2 + \frac{6\overline{M}_3(\alpha_1, \alpha_2, a, \mu_0) t^{2\alpha_1\mu_0+1} \mathbf{B}(\theta_1, \vartheta_1)}{2\alpha_1\mu_0 + 1} \\ & \quad + 6\overline{M}_3(\alpha_1, \alpha_2, a, \mu_0) t^{2\alpha_1(1-\mu_0)} \mathbf{B}(\theta_1, \vartheta_1) \\ & \quad \times \int_0^t (\|u(\cdot, s)\|_{\mathcal{H}^{\nu}(\Omega)}^2 + \|v(\cdot, s)\|_{\mathcal{H}^{\nu}(\Omega)}^2) s^{2\alpha_1\mu_0} ds. \tag{4.16} \end{aligned}$$

From (4.15) and (4.16), we arrive at

$$\begin{aligned} & (\|u(\cdot, t)\|_{\mathcal{H}^{\gamma}(\Omega)}^2 + \|v(\cdot, t)\|_{\mathcal{H}^{\gamma}(\Omega)}^2) t^{2\alpha_1\mu_0} \\ & \leq \mathcal{M}_1(\alpha_1, \alpha_2, a, \mu_0, T, \|w_0\|_{\mathcal{H}^{\gamma}(\Omega)}) \\ & \quad + \mathcal{M}_2(\alpha_1, \alpha_2, a, \mu_0, T) \int_0^t (\|u(\cdot, s)\|_{\mathcal{H}^{\gamma}(\Omega)}^2 + \|v(\cdot, s)\|_{\mathcal{H}^{\gamma}(\Omega)}^2) s^{2\alpha_1\mu_0} ds, \end{aligned} \tag{4.17}$$

where

$$\begin{aligned} \mathcal{M}_1(\alpha_1, \alpha_2, a, \mu_0, T, \|w_0\|_{\mathcal{H}^{\gamma}(\Omega)}) & := 2\bar{\mathcal{M}}_2(\alpha_1, \alpha_2, a, \mu_0) \|w_0\|_{\mathcal{H}^{\gamma}(\Omega)}^2 \\ & \quad + \frac{12\bar{\mathcal{M}}_3(\alpha_1, \alpha_2, a, \mu_0) T^{2\alpha_1\mu_0+1} \mathbf{B}(\theta_1, \vartheta_1)}{2\alpha_1\mu_0 + 1}, \\ \mathcal{M}_2(\alpha_1, \alpha_2, a, \mu_0, T) & := 12\bar{\mathcal{M}}_3(\alpha_1, \alpha_2, a, \mu_0) T^{2\alpha_1(1-\mu_0)} \mathbf{B}(\theta_1, \vartheta_1). \end{aligned}$$

Applying $\|w_0\|_{\mathcal{H}^{\gamma}(\Omega)}^2 = \|f\|_{\mathcal{H}^{\gamma}(\Omega)}^2 + \|g\|_{\mathcal{H}^{\gamma}(\Omega)}^2$ and Gronwall’s inequality, we get

$$\begin{aligned} & \|u\|_{\mathbb{C}^{\alpha_1\mu_0}((0,T],\mathcal{H}^{\gamma}(\Omega))}^2 + \|v\|_{\mathbb{C}^{\alpha_1\mu_0}((0,T],\mathcal{H}^{\gamma}(\Omega))}^2 \\ & \leq \mathcal{M}_1(\alpha_1, \alpha_2, a, \mu_0, T, \|w_0\|_{\mathcal{H}^{\gamma}(\Omega)}) \exp(\mathcal{M}_2(\alpha_1, \alpha_2, a, \mu_0, T)T). \end{aligned}$$

Therefore, we get

$$\begin{aligned} & \|w\|_{\mathbb{C}^{\alpha_1\mu_0}((0,T],\mathcal{H}^{\gamma}(\Omega))} \\ & = \sqrt{\|u\|_{\mathbb{C}^{\alpha_1\mu_0}((0,T],\mathcal{H}^{\gamma}(\Omega))}^2 + \|v\|_{\mathbb{C}^{\alpha_1\mu_0}((0,T],\mathcal{H}^{\gamma}(\Omega))}^2} \\ & \leq \sqrt{\mathcal{M}_1(\alpha_1, \alpha_2, a, \mu_0, T, \|w_0\|_{\mathcal{H}^{\gamma}(\Omega)}) \exp\left(\frac{\mathcal{M}_2(\alpha_1, \alpha_2, a, \mu_0, T)T}{2}\right)}. \end{aligned} \tag{4.18}$$

Part 3. From equation (3.6), we then obtain

$$\begin{cases} u_{\alpha}(\cdot, t) = \mathcal{P}_{\Delta,\alpha}(t)f + \int_0^t \mathcal{Q}_{\Delta,\alpha}(t-s)\mathcal{F}(u_{\alpha}, v_{\alpha})(\cdot, s) ds, \\ v_{\alpha}(\cdot, t) = \mathcal{P}_{\Delta,\alpha}(t)g + \int_0^t \mathcal{Q}_{\Delta,\alpha}(t-s)\mathcal{G}(u_{\alpha}, v_{\alpha})(\cdot, s) ds, \end{cases} \tag{4.19}$$

and

$$\begin{cases} u_{\alpha'}(\cdot, t) = \mathcal{P}_{\Delta,\alpha'}(t)f + \int_0^t \mathcal{Q}_{\Delta,\alpha'}(t-s)\mathcal{F}(u_{\alpha'}, v_{\alpha'})(\cdot, s) ds, \\ v_{\alpha'}(\cdot, t) = \mathcal{P}_{\Delta,\alpha'}(t)g + \int_0^t \mathcal{Q}_{\Delta,\alpha'}(t-s)\mathcal{G}(u_{\alpha'}, v_{\alpha'})(\cdot, s) ds. \end{cases} \tag{4.20}$$

Using (4.19) and (4.20), we get

$$\begin{aligned} & u_{\alpha'}(\cdot, t) - u_{\alpha}(\cdot, t) \\ & = [\mathcal{P}_{\Delta,\alpha'}(t) - \mathcal{P}_{\Delta,\alpha}(t)]f + \int_0^t [\mathcal{Q}_{\Delta,\alpha'}(t-s) - \mathcal{Q}_{\Delta,\alpha}(t-s)]\mathcal{F}(u_{\alpha'}, v_{\alpha'})(\cdot, s) ds \\ & \quad + \int_0^t \mathcal{Q}_{\Delta,\alpha}(t-s)[\mathcal{F}(u_{\alpha'}, v_{\alpha'})(\cdot, s) - \mathcal{F}(u_{\alpha}, v_{\alpha})(\cdot, s)] ds, \end{aligned} \tag{4.21}$$

$$\begin{aligned}
 &v_{\alpha'}(\cdot, t) - v_{\alpha}(\cdot, t) \\
 &= [\mathcal{P}_{\Delta, \alpha'}(t) - \mathcal{P}_{\Delta, \alpha}(t)]g + \int_0^t [\mathcal{Q}_{\Delta, \alpha'}(t-s) - \mathcal{Q}_{\Delta, \alpha}(t-s)]\mathcal{G}(u_{\alpha'}, v_{\alpha'})(\cdot, s) \, ds \\
 &\quad + \int_0^t \mathcal{Q}_{\Delta, \alpha}(t-s)[\mathcal{G}(u_{\alpha'}, v_{\alpha'})(\cdot, s) - \mathcal{G}(u_{\alpha}, v_{\alpha})(\cdot, s)] \, ds. \tag{4.22}
 \end{aligned}$$

Applying the inequality $(m + n + p)^2 \leq 3(m^2 + n^2 + p^2)$, we get the following estimate:

$$\begin{aligned}
 &\|u_{\alpha'}(\cdot, t) - u_{\alpha}(\cdot, t)\|_{\mathcal{H}^{\nu}(\Omega)}^2 \\
 &\leq 3\|[\mathcal{P}_{\Delta, \alpha'}(t) - \mathcal{P}_{\Delta, \alpha}(t)]f\|_{\mathcal{H}^{\nu}(\Omega)}^2 \\
 &\quad + 3\left\|\int_0^t [\mathcal{Q}_{\Delta, \alpha'}(t-s) - \mathcal{Q}_{\Delta, \alpha}(t-s)]\mathcal{F}(u_{\alpha'}, v_{\alpha'})(\cdot, s) \, ds\right\|_{\mathcal{H}^{\nu}(\Omega)}^2 \\
 &\quad + 3\left\|\int_0^t \mathcal{Q}_{\Delta, \alpha}(t-s)[\mathcal{F}(u_{\alpha'}, v_{\alpha'})(\cdot, s) - \mathcal{F}(u_{\alpha}, v_{\alpha})(\cdot, s)] \, ds\right\|_{\mathcal{H}^{\nu}(\Omega)}^2. \tag{4.23}
 \end{aligned}$$

Using Lemmas 3.1 and 3.2, we obtain

$$\begin{aligned}
 &\|u_{\alpha'}(\cdot, t) - u_{\alpha}(\cdot, t)\|_{\mathcal{H}^{\nu}(\Omega)}^2 \\
 &\leq 3(\overline{A}(\alpha_1, \alpha_2, \epsilon, \nu_0, a, T)[(\alpha' - \alpha)^{\epsilon} + (\alpha' - \alpha)]t^{-\alpha_2(1-\nu_0)-\epsilon}\|f\|_{\mathcal{H}^{\nu}(\Omega)}^2 \\
 &\quad + 3\overline{B}^2(\alpha_1, \alpha_2, \epsilon, \nu_0, a, T)[(\alpha' - \alpha)^{\epsilon} + (\alpha' - \alpha)] \\
 &\quad \times \left(\int_0^t (t-s)^{\alpha_1\nu_0-\epsilon-1}\|\mathcal{F}(u_{\alpha'}, v_{\alpha'})(\cdot, s)\|_{\mathcal{H}^{\nu}(\Omega)} \, ds\right)^2 \\
 &\quad + 3\overline{M}_3^2(\alpha_1, \alpha_2, a, \mu_0) \\
 &\quad \times \left(\int_0^t (t-s)^{\alpha_1-1-\alpha_1\mu_0}\|\mathcal{F}(u_{\alpha'}, v_{\alpha'})(\cdot, s) - \mathcal{F}(u_{\alpha}, v_{\alpha})(\cdot, s)\|_{\mathcal{H}^{\nu}(\Omega)} \, ds\right)^2. \tag{4.24}
 \end{aligned}$$

Using assumptions (4.1) and (4.3), we obtain

$$\begin{aligned}
 &\|u_{\alpha'}(\cdot, t) - u_{\alpha}(\cdot, t)\|_{\mathbb{H}^{\nu}(\Omega)}^2 \\
 &\leq 3(\overline{A}(\alpha_1, \alpha_2, \epsilon, \nu_0, a, T)[(\alpha' - \alpha)^{\epsilon} + (\alpha' - \alpha)]t^{-\alpha_2(1-\nu_0)-\epsilon}\|f\|_{\mathcal{H}^{\nu}(\Omega)}^2 \\
 &\quad + 3\mathcal{C}_1^2\overline{B}^2(\alpha_1, \alpha_2, \epsilon, \nu_0, a, T)[(\alpha' - \alpha)^{\epsilon} + (\alpha' - \alpha)]^2I_1 \\
 &\quad + 3\mathcal{K}_1^2\overline{M}_3^2(\alpha_1, \alpha_2, a, \mu_0)I_2, \tag{4.25}
 \end{aligned}$$

where

$$I_1 := \left(\int_0^t (t-s)^{\alpha_1\nu_0-\epsilon-1}(1 + \|u_{\alpha'}(\cdot, t)\|_{\mathcal{H}^{\nu}(\Omega)} + \|v_{\alpha'}(\cdot, t)\|_{\mathcal{H}^{\nu}(\Omega)}) \, ds\right)^2, \tag{4.26}$$

$$I_2 := \left(\int_0^t (t-s)^{\alpha_1-1-\alpha_1\mu_0}(\|u_{\alpha'} - u_{\alpha}\|_{\mathcal{H}^{\nu}(\Omega)} + \|v_{\alpha'} - v_{\alpha}\|_{\mathcal{H}^{\nu}(\Omega)}) \, ds\right)^2. \tag{4.27}$$

Multiplying both sides by $t^{2\alpha_2(1-\nu_0)+2\epsilon}$, we get

$$\begin{aligned}
 & t^{2\alpha_2(1-\nu_0)+2\epsilon} \|u_{\alpha'}(\cdot, t) - u_{\alpha}(\cdot, t)\|_{\mathbb{H}^{\eta}(\Omega)}^2 \\
 & \leq 3(\overline{A}(\alpha_1, \alpha_2, \epsilon, \nu_0, a, T)[(\alpha' - \alpha)^{\epsilon} + (\alpha' - \alpha)] \|f\|_{\mathcal{H}^{\gamma}(\Omega)})^2 \\
 & \quad + 3\mathcal{K}_1^2 \overline{M}_3^2(\alpha_1, \alpha_2, a, \mu_0) t^{2\alpha_2(1-\mu_0)+2\epsilon} I_2 \\
 & \quad + 3\mathcal{C}_1^2 \overline{B}^2(\alpha_1, \alpha_2, \epsilon, \nu_0, a, T)[(\alpha' - \alpha)^{\epsilon} + (\alpha' - \alpha)]^2 t^{2\alpha_2(1-\nu_0)+2\epsilon} I_1.
 \end{aligned} \tag{4.28}$$

Now we estimate I_1 , from (4.26) we get

$$\begin{aligned}
 I_1 & = \left(\int_0^t (t-s)^{\alpha_1\nu_0-2\epsilon-1} s^{-\alpha_1\mu_0} \right. \\
 & \quad \left. \times s^{\alpha_1\mu_0} (1 + \|u_{\alpha'}(\cdot, t)\|_{\mathcal{H}^{\gamma}(\Omega)} + \|v_{\alpha'}(\cdot, t)\|_{\mathcal{H}^{\gamma}(\Omega)}) ds \right)^2.
 \end{aligned}$$

We assume that $0 < \epsilon < \alpha_1\nu_0 - \frac{1}{2}$ and $\mu_0 < \frac{1}{2\alpha_1}$, then $2\alpha_1\nu_0 - 2\epsilon - 1 > 0$, $1 - 2\alpha_1\mu_0 > 0$. Using Hölder’s inequality and the properties of beta function, we obtain that

$$\begin{aligned}
 I_1 & \leq \int_0^t (t-s)^{2\alpha_1\nu_0-2\epsilon-2} s^{-2\alpha_1\mu_0} ds \\
 & \quad \times \int_0^t s^{2\alpha_1\mu_0} (1 + \|u_{\alpha'}(\cdot, t)\|_{\mathcal{H}^{\gamma}(\Omega)} + \|v_{\alpha'}(\cdot, t)\|_{\mathcal{H}^{\gamma}(\Omega)})^2 ds \\
 & \leq t^{2\alpha_1\nu_0-2\alpha_1\mu_0-2\epsilon} \mathbf{B}(\theta_2, \vartheta_2) \\
 & \quad \times \int_0^t s^{2\alpha_1\mu_0} (1 + \|u_{\alpha'}(\cdot, t)\|_{\mathcal{H}^{\gamma}(\Omega)} + \|v_{\alpha'}(\cdot, t)\|_{\mathcal{H}^{\gamma}(\Omega)})^2 ds,
 \end{aligned} \tag{4.29}$$

where $\theta_2 := 2\alpha_1\nu_0 - 2\epsilon - 1$, $\vartheta_2 := 1 - 2\alpha_1\mu_0$.

Applying the inequality $(m + n + p)^2 \leq 3(m^2 + n^2 + p^2)$, we can deduce

$$\begin{aligned}
 I_1 & \leq 3t^{2\alpha_1\nu_0-2\alpha_1\mu_0-2\epsilon} \mathbf{B}(\theta_2, \vartheta_2) \int_0^t s^{2\alpha_1\mu_0} (1 + \|u_{\alpha'}\|_{\mathcal{H}^{\gamma}(\Omega)}^2 + \|v_{\alpha'}\|_{\mathcal{H}^{\gamma}(\Omega)}^2) ds \\
 & \leq 3t^{2\alpha_1\nu_0-2\alpha_1\mu_0-2\epsilon} \mathbf{B}(\theta_2, \vartheta_2) \\
 & \quad \times \int_0^t (s^{2\alpha_1\mu_0} + \|u_{\alpha'}\|_{\mathbb{C}^{\alpha_1\mu_0}((0,T],\mathcal{H}^{\gamma}(\Omega))}^2 + \|v_{\alpha'}\|_{\mathbb{C}^{\alpha_1\mu_0}((0,T],\mathcal{H}^{\gamma}(\Omega))}^2) ds.
 \end{aligned} \tag{4.30}$$

Therefore, from (4.18), we have the following estimate:

$$\begin{aligned}
 I_1 & \leq \frac{3t^{2\alpha_1\nu_0+1-2\epsilon} \mathbf{B}(\theta_2, \vartheta_2)}{2\alpha_1\mu_0 + 1} \\
 & \quad + 3t^{2\alpha_1\nu_0-2\alpha_1\mu_0-2\epsilon+1} \mathbf{B}(\theta_2, \vartheta_2) \|w_{\alpha'}\|_{\mathbb{C}^{\alpha_1\mu_0}((0,T],\mathcal{H}^{\gamma}(\Omega))}^2 \\
 & \leq \frac{3T^{2\alpha_1\nu_0+1-2\epsilon} \mathbf{B}(\theta_2, \vartheta_2)}{2\alpha_1\mu_0 + 1} + 3T^{2\alpha_1\nu_0-2\alpha_1\mu_0-2\epsilon+1} \mathbf{B}(\theta_2, \vartheta_2) \\
 & \quad \times \mathcal{M}_1(\alpha_1, \alpha_2, a, \mu_0, T, \|w_0\|_{\mathcal{H}^{\gamma}(\Omega)}) \exp(\mathcal{M}_2(\alpha_1, \alpha_2, a, \mu_0, T)T).
 \end{aligned}$$

To facilitate the calculation, we set

$$\begin{aligned} & \mathcal{M}(\alpha_1, \alpha_2, a, \mu_0, \nu_0, T, \|w_0\|_{\mathcal{H}^\gamma(\Omega)}) \\ & := \frac{3T^{2\alpha_1\nu_0+1-2\epsilon} \mathbf{B}(\theta_2, \vartheta_2)}{2\alpha_1\mu_0 + 1} \\ & \quad + 3T^{2\alpha_1\nu_0-2\alpha_1\mu_0-2\epsilon+1} \mathbf{B}(\theta_2, \vartheta_2) \\ & \quad \times \mathcal{M}_1(\alpha_1, \alpha_2, a, \mu_0, T, \|w_0\|_{\mathcal{H}^\gamma(\Omega)}) \exp(\mathcal{M}_2(\alpha_1, \alpha_2, a, \mu_0, T)T). \end{aligned}$$

Hence

$$I_1 \leq \mathcal{M}(\alpha_1, \alpha_2, a, \mu_0, \nu_0, T, \|w_0\|_{\mathcal{H}^\gamma(\Omega)}). \tag{4.31}$$

Next, estimate I_2 . From (4.27), applying Hölder’s inequality, we deduce

$$\begin{aligned} I_2 & = \left(\int_0^t (t-s)^{\alpha_1-1-\alpha_1\mu_0} s^{-\alpha_2(1-\nu_0)-\epsilon} \right. \\ & \quad \left. \times s^{2\alpha_2(1-\nu_0)+\epsilon} (\|u_{\alpha'} - u_\alpha\|_{\mathcal{H}^\gamma(\Omega)} + \|v_{\alpha'} - v_\alpha\|_{\mathcal{H}^\gamma(\Omega)}) \, ds \right)^2 \\ & \leq \int_0^t (t-s)^{2\alpha_1-2-2\alpha_1\mu_0} s^{-2\alpha_2(1-\nu_0)-2\epsilon} \, ds \\ & \quad \times \int_0^t s^{2\alpha_2(1-\nu_0)+2\epsilon} (\|u_{\alpha'}(\cdot, s) - u_\alpha(\cdot, s)\|_{\mathcal{H}^\gamma(\Omega)} + \|v_{\alpha'}(\cdot, s) - v_\alpha(\cdot, s)\|_{\mathcal{H}^\gamma(\Omega)})^2 \, ds \\ & \leq \int_0^t (t-s)^{2\alpha_1-2-2\alpha_1\mu_0} s^{-2\alpha_2(1-\nu_0)-2\epsilon} \, ds \\ & \quad \times 2 \int_0^t s^{2\alpha_2(1-\nu_0)+2\epsilon} (\|u_{\alpha'}(\cdot, s) - u_\alpha(\cdot, s)\|_{\mathcal{H}^\gamma(\Omega)}^2 + \|v_{\alpha'}(\cdot, s) - v_\alpha(\cdot, s)\|_{\mathcal{H}^\gamma(\Omega)}^2) \, ds. \end{aligned}$$

Using the beta function property with assumption $\mu_0 < 1 - \frac{1}{2\alpha_1}$ and $0 < \epsilon < \min(\frac{1}{2} - \alpha_2 + \alpha_2\nu_0, \alpha_1\nu_0 - \frac{1}{2})$, we obtain

$$\begin{aligned} I_2 & \leq 2t^{2\alpha_1-2\alpha_1\mu_0-2\alpha_2+2\alpha_2\nu_0-2\epsilon} \mathbf{B}(\theta_3, \vartheta_3) \\ & \quad \times \int_0^t s^{2\alpha_2(1-\nu_0)+2\epsilon} (\|u_{\alpha'}(\cdot, s) - u_\alpha(\cdot, s)\|_{\mathcal{H}^\gamma(\Omega)}^2 + \|v_{\alpha'}(\cdot, s) - v_\alpha(\cdot, s)\|_{\mathcal{H}^\gamma(\Omega)}^2) \, ds, \end{aligned}$$

where

$$\theta_3 := -2\alpha_1 - 1 - 2\alpha_1\mu_0, \quad \vartheta_3 := 1 - 2\alpha_1(1 - \nu_0) - 2\epsilon.$$

And so, we get

$$\begin{aligned} I_2 & \leq 2t^{2\alpha_1-2\alpha_1\mu_0-2\alpha_2+2\alpha_2\nu_0-2\epsilon} \mathbf{B}(\theta_3, \vartheta_3) \\ & \quad \times \int_0^t s^{2\alpha_2(1-\nu_0)+2\epsilon} (\|u_{\alpha'}(\cdot, s) - u_\alpha(\cdot, s)\|_{\mathcal{H}^\gamma(\Omega)}^2 + \|v_{\alpha'}(\cdot, s) - v_\alpha(\cdot, s)\|_{\mathcal{H}^\gamma(\Omega)}^2) \, ds. \tag{4.32} \end{aligned}$$

From (4.28)–(4.32), we obtain

$$\begin{aligned}
 & t^{2\alpha_2(1-\nu_0)+2\epsilon} \|u_{\alpha'}(\cdot, t) - u_\alpha(\cdot, t)\|_{\mathcal{H}^\nu(\Omega)}^2 \\
 & \leq 3(\bar{A}(\alpha_1, \alpha_2, \epsilon, \nu_0, a, T)[(\alpha' - \alpha)^\epsilon + (\alpha' - \alpha)]) \|f\|_{\mathcal{H}^\nu(\Omega)}^2 \\
 & \quad + 3C_1^2 \bar{B}^2(\alpha_1, \alpha_2, \epsilon, \nu_0, a, T) \mathcal{M}(\alpha_1, \alpha_2, a, \mu_0, T)[(\alpha' - \alpha)^\epsilon + (\alpha' - \alpha)]^2 t^{2\alpha_2(1-\nu_0)+2\epsilon} \\
 & \quad + 6K_1^2 \bar{M}_3^2(\alpha_1, \alpha_2, a, \mu_0) t^{2\alpha_2(1-\mu_0)+2\epsilon} 2t^{2\alpha_1-2\alpha_1\mu_0-2\alpha_2+2\alpha_2\nu_0-2\epsilon} \mathbf{B}(\theta_3, \vartheta_3) \\
 & \quad \times \int_0^t s^{2\alpha_2(1-\nu_0)+2\epsilon} (\|u_{\alpha'}(\cdot, s) - u_\alpha(\cdot, s)\|_{\mathcal{H}^\nu(\Omega)}^2 + \|v_{\alpha'}(\cdot, s) - v_\alpha(\cdot, s)\|_{\mathcal{H}^\nu(\Omega)}^2) ds. \tag{4.33}
 \end{aligned}$$

Hence, we get the following estimate:

$$\begin{aligned}
 & t^{2\alpha_2(1-\nu_0)+2\epsilon} \|u_{\alpha'}(\cdot, t) - u_\alpha(\cdot, t)\|_{\mathcal{H}^\nu(\Omega)}^2 \\
 & \leq 3(\bar{A}(\alpha_1, \alpha_2, \epsilon, \nu_0, a, T)[(\alpha' - \alpha)^\epsilon + (\alpha' - \alpha)]) \|f\|_{\mathcal{H}^\nu(\Omega)}^2 \\
 & \quad + 3C_1^2 \bar{B}^2(\alpha_1, \alpha_2, \epsilon, \nu_0, a, T) \mathcal{M}(\alpha_1, \alpha_2, a, \mu_0, T)[(\alpha' - \alpha)^\epsilon + (\alpha' - \alpha)]^2 T^{2\alpha_2(1-\nu_0)+2\epsilon} \\
 & \quad + 6K_1^2 \bar{M}_3^2(\alpha_1, \alpha_2, a, \mu_0) T^{2\alpha_1(1-\mu_0)+2\alpha_2(\nu_0-\mu_0)} \mathbf{B}(\theta_3, \vartheta_3) \\
 & \quad \times \int_0^t s^{2\alpha_2(1-\nu_0)+2\epsilon} (\|u_{\alpha'}(\cdot, s) - u_\alpha(\cdot, s)\|_{\mathcal{H}^\nu(\Omega)}^2 + \|v_{\alpha'}(\cdot, s) - v_\alpha(\cdot, s)\|_{\mathcal{H}^\nu(\Omega)}^2) ds. \tag{4.34}
 \end{aligned}$$

In the same way as above, we obtain

$$\begin{aligned}
 & t^{2\alpha_2(1-\nu_0)+2\epsilon} \|v_{\alpha'}(\cdot, t) - v_\alpha(\cdot, t)\|_{\mathcal{H}^\nu(\Omega)}^2 \\
 & \leq 3(\bar{A}(\alpha_1, \alpha_2, \epsilon, \nu_0, a, T)[(\alpha' - \alpha)^\epsilon + (\alpha' - \alpha)]) \|g\|_{\mathcal{H}^\nu(\Omega)}^2 \\
 & \quad + 3C_2^2 \bar{B}^2(\alpha_1, \alpha_2, \epsilon, \nu_0, a, T) \mathcal{M}(\alpha_1, \alpha_2, a, \mu_0, T)[(\alpha' - \alpha)^\epsilon + (\alpha' - \alpha)]^2 T^{2\alpha_2(1-\nu_0)+2\epsilon} \\
 & \quad + 6K_2^2 \bar{M}_3^2(\alpha_1, \alpha_2, a, \mu_0) T^{2\alpha_1(1-\mu_0)+2\alpha_2(\nu_0-\mu_0)} \mathbf{B}(\theta_3, \vartheta_3) \\
 & \quad \times \int_0^t s^{2\alpha_2(1-\nu_0)+2\epsilon} (\|u_{\alpha'}(\cdot, s) - u_\alpha(\cdot, s)\|_{\mathcal{H}^\nu(\Omega)}^2 + \|v_{\alpha'}(\cdot, s) - v_\alpha(\cdot, s)\|_{\mathcal{H}^\nu(\Omega)}^2) ds. \tag{4.35}
 \end{aligned}$$

For simplicity to some math formulas, one should put

$$\begin{aligned}
 & \mathcal{M}_{\mu_0, \nu_0}^{C_1, C_2}(\alpha_1, \alpha_2, \epsilon, a, T, \|w_0\|_{\mathcal{H}^\nu(\Omega)}) \\
 & := 3\bar{A}(\alpha_1, \alpha_2, \epsilon, \nu_0, a, T) \|w_0\|_{\mathcal{H}^\nu(\Omega)}^2 \\
 & \quad + 3(C_1^2 + C_2^2) \bar{B}^2(\alpha_1, \alpha_2, \epsilon, \nu_0, a, T) \mathcal{M}(\alpha_1, \alpha_2, a, \mu_0, T) T^{2\alpha_2(1-\nu_0)+2\epsilon}
 \end{aligned}$$

and

$$\begin{aligned}
 & \mathcal{M}_{\mu_0, \nu_0}^{K_1, K_2}(\alpha_1, \alpha_2, \epsilon, a, T) \\
 & := 3(K_1^2 + K_2^2) \bar{M}_3^2(\alpha_1, \alpha_2, a, \mu_0) T^{2\alpha_1(1-\mu_0)+2\alpha_2(\nu_0-\mu_0)} \mathbf{B}(\theta_3, \vartheta_3).
 \end{aligned}$$

Combining estimates (4.33) and (4.35), we have

$$\begin{aligned}
 & t^{2\alpha_2(1-\nu_0)+2\epsilon} \left(\|u_{\alpha'}(\cdot, t) - u_{\alpha}(\cdot, t)\|_{\mathcal{H}^{\nu}(\Omega)}^2 + \|v_{\alpha'}(\cdot, t) - v_{\alpha}(\cdot, t)\|_{\mathcal{H}^{\nu}(\Omega)}^2 \right) \\
 & \leq 3(\overline{A}(\alpha_1, \alpha_2, \epsilon, \nu_0, a, T)[(\alpha' - \alpha)^{\epsilon} + (\alpha' - \alpha)])^2 (\|f\|_{\mathcal{H}^{\nu}(\Omega)}^2 + \|g\|_{\mathcal{H}^{\nu}(\Omega)}^2) \\
 & \quad + 3(C_1^2 + C_2^2)\overline{B}^2(\alpha_1, \alpha_2, \epsilon, \nu_0, a, T) \\
 & \quad \times \mathcal{M}(\alpha_1, \alpha_2, a, \mu_0, T)T^{2\alpha_2(1-\nu_0)+2\epsilon} [(\alpha' - \alpha)^{\epsilon} + (\alpha' - \alpha)]^2 \\
 & \quad + 3(\mathcal{K}_1^2 + \mathcal{K}_2^2)\overline{M}_3^2(\alpha_1, \alpha_2, a, \mu_0)T^{2\alpha_1(1-\mu_0)+2\alpha_2(\nu_0-\mu_0)}\mathbf{B}(\theta_3, \vartheta_3) \\
 & \quad \times \int_0^t s^{2\alpha_2(1-\nu_0)+2\epsilon} \left(\|u_{\alpha'}(\cdot, s) - u_{\alpha}(\cdot, s)\|_{\mathcal{H}^{\nu}(\Omega)}^2 + \|v_{\alpha'}(\cdot, s) - v_{\alpha}(\cdot, s)\|_{\mathcal{H}^{\nu}(\Omega)}^2 \right) ds. \tag{4.36}
 \end{aligned}$$

Hence, we get the following estimate:

$$\begin{aligned}
 & t^{2\alpha_2(1-\nu_0)+2\epsilon} \left(\|u_{\alpha'}(\cdot, t) - u_{\alpha}(\cdot, t)\|_{\mathcal{H}^{\nu}(\Omega)}^2 + \|v_{\alpha'}(\cdot, t) - v_{\alpha}(\cdot, t)\|_{\mathcal{H}^{\nu}(\Omega)}^2 \right) \\
 & \leq \mathcal{M}_{\mu_0, \nu_0}^{C_1, C_2}(\alpha_1, \alpha_2, \epsilon, a, T, \|w_0\|_{\mathcal{H}^{\nu}(\Omega)}) [(\alpha' - \alpha)^{\epsilon} + (\alpha' - \alpha)]^2 \\
 & \quad + \mathcal{M}_{\mu_0, \nu_0}^{\mathcal{K}_1, \mathcal{K}_2}(\alpha_1, \alpha_2, \epsilon, a, T) \\
 & \quad \times \int_0^t s^{2\alpha_2(1-\nu_0)+2\epsilon} \left(\|u_{\alpha'}(\cdot, s) - u_{\alpha}(\cdot, s)\|_{\mathcal{H}^{\nu}(\Omega)}^2 + \|v_{\alpha'}(\cdot, s) - v_{\alpha}(\cdot, s)\|_{\mathcal{H}^{\nu}(\Omega)}^2 \right) ds. \tag{4.37}
 \end{aligned}$$

Applying Gronwall’s inequality, we have the following estimate:

$$\begin{aligned}
 & t^{2\alpha_2(1-\nu_0)+2\epsilon} \left(\|u_{\alpha'}(\cdot, t) - u_{\alpha}(\cdot, t)\|_{\mathcal{H}^{\nu}(\Omega)}^2 + \|v_{\alpha'}(\cdot, t) - v_{\alpha}(\cdot, t)\|_{\mathcal{H}^{\nu}(\Omega)}^2 \right) \\
 & \leq \mathcal{M}_{\mu_0, \nu_0}^{C_1, C_2}(\alpha_1, \alpha_2, \epsilon, a, T, \|w_0\|_{\mathcal{H}^{\nu}(\Omega)}) [(\alpha' - \alpha)^{\epsilon} + (\alpha' - \alpha)]^2 \\
 & \quad \times \exp(\mathcal{M}_{\mu_0, \nu_0}^{\mathcal{K}_1, \mathcal{K}_2}(\alpha_1, \alpha_2, \epsilon, a, T)t). \tag{4.38}
 \end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}
 & \|u_{\alpha'} - u_{\alpha}\|_{\mathbb{C}^{\alpha_2(1-\nu_0)+\epsilon}((0, T], \mathcal{H}^{\nu}(\Omega))}^2 + \|v_{\alpha'} - v_{\alpha}\|_{\mathbb{C}^{\alpha_2(1-\nu_0)+\epsilon}((0, T], \mathcal{H}^{\nu}(\Omega))}^2 \\
 & \leq \mathcal{M}_{\mu_0, \nu_0}^{C_1, C_2}(\alpha_1, \alpha_2, \epsilon, a, T, \|w_0\|_{\mathcal{H}^{\nu}(\Omega)}) [(\alpha' - \alpha)^{\epsilon} + (\alpha' - \alpha)]^2 \\
 & \quad \times \exp(\mathcal{M}_{\mu_0, \nu_0}^{\mathcal{K}_1, \mathcal{K}_2}(\alpha_1, \alpha_2, \epsilon, a, T)T). \tag{4.39}
 \end{aligned}$$

Finally, we obtain

$$\begin{aligned}
 & \|w_{\alpha'} - w_{\alpha}\|_{\mathcal{C}^{\alpha_2(1-\nu_0)+\epsilon}((0, T], \mathcal{H}^{\nu}(\Omega))} \\
 & \leq \sqrt{\mathcal{M}_{\mu_0, \nu_0}^{C_1, C_2}(\alpha_1, \alpha_2, \epsilon, a, T, \|w_0\|_{\mathcal{H}^{\nu}(\Omega)}) [(\alpha' - \alpha)^{\epsilon} + (\alpha' - \alpha)]} \\
 & \quad \times \exp\left(\frac{\mathcal{M}_{\mu_0, \nu_0}^{\mathcal{K}_1, \mathcal{K}_2}(\alpha_1, \alpha_2, \epsilon, a, T)T}{2}\right). \tag{4.40}
 \end{aligned}$$

This completes the proof. □

5 Numerical results

In this section, we show an example which shows the effectiveness of our method. Let the operator $-\Delta$ on the domain $\Omega = (0, \pi)$ with the Dirichlet boundary condition and $t \in [0, 1]$, $a = 1$, we have the eigenvalues of $-\Delta$ given by $\lambda_j = j^2$ ($j \in \mathbb{Z}^+$) and $\varphi_j(x) = \sqrt{\frac{2}{\pi}} \sin(jx)$, respectively. We consider the problem to find $(u(x, t), v(x, t))$ as follows:

$$\begin{cases} \partial_t^\alpha(u(x, t) + \partial_{xx}^2 u(x, t)) + \partial_{xx}^2 u(x, t) = \mathcal{F}(u, v, x, t), & (x, t) \in (0, \pi) \times (0, 1), \\ \partial_t^\alpha(v(x, t) + \partial_{xx}^2 v(x, t)) + \partial_{xx}^2 v(x, t) = \mathcal{G}(u, v, x, t), & (x, t) \in (0, \pi) \times (0, 1), \end{cases} \tag{5.1}$$

where $(x, t) \in (0, \pi) \times (0, 1)$, the source functions are given by

$$\begin{cases} \mathcal{F}(u, v, x, t) = v^2 - 4u - \frac{6t^{2-\alpha} \sin(2x)}{\Gamma(3-\alpha)} + (\frac{t^6}{2} - t^3 + \frac{1}{2})(\cos(8x) - 1), \\ \mathcal{G}(u, v, x, t) = u^2 - 16v - \frac{90t^{3-\alpha} \sin(4x)}{\Gamma(4-\alpha)} - (\frac{t^4}{2} - t^2 + \frac{1}{2})(1 - \cos(4x)), \end{cases} \tag{5.2}$$

and the Dirichlet boundary condition as follows:

$$\begin{cases} u(0, t) = u(\pi, t) = 0, & t \in (0, 1), \\ v(0, t) = v(\pi, t) = 0, & t \in (0, 1). \end{cases} \tag{5.3}$$

Assume that the values of u, v at the initial time $t = 0$ are given by

$$\begin{cases} u(x, 0) = f(x) := -\sin(2x), & u_t(x, 0) = 0, & x \in (0, \pi), \\ v(x, 0) = g(x) := \sin(4x), & v_t(x, 0) = 0, & x \in (0, \pi). \end{cases} \tag{5.4}$$

Then the exact solution of problem (5.1)–(5.4) is given by

$$\begin{cases} u(x, t) = (t^2 - 1) \sin(2x), \\ v(x, t) = (1 - t^3) \sin(3x). \end{cases}$$

At the discretization level, a uniform grid of mesh-points (x_m, t_n) is used to discretize the space and time intervals

$$x_m = \frac{(m - 1)\pi}{h_x}, \quad m = \overline{1, h_x + 1}, \quad \text{and} \quad t_n = \frac{n - 1}{h_t}, \quad n = \overline{1, h_t + 1}.$$

Next, by using Simpson’s rule of numerical integration, we have the following approximate integration of $z \in L^2(0, \pi)$:

$$\int_0^\pi z(s) ds \approx \frac{\Delta s}{3} z(s_1) + \frac{2\Delta s}{3} \sum_{l=1}^{(N+1)/2-1} z(s_{2l}) + \frac{4\Delta s}{3} \sum_{l=1}^{(N+1)/2} z(s_{2l-1}) + \frac{\Delta s}{3} z(s_{N+1}).$$

In code Matlab, we have the solution of problem (5.1) which can be written in a matrix form as follows:

$$U = \begin{bmatrix} u(x_1, t_1) & u(x_2, t_1) & u(x_3, t_1) & \cdots & u(x_{m+1}, t_1) \\ u(x_1, t_2) & u(x_2, t_2) & u(x_3, t_2) & \cdots & u(x_{m+1}, t_2) \\ u(x_1, t_3) & u(x_2, t_3) & u(x_3, t_3) & \cdots & u(x_{m+1}, t_3) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u(x_1, t_{n+1}) & u(x_2, t_{n+1}) & u(x_3, t_{n+1}) & \cdots & u(x_{m+1}, t_{n+1}) \end{bmatrix}_{(n+1) \times (m+1)},$$

and

$$V = \begin{bmatrix} v(x_1, t_1) & v(x_2, t_1) & v(x_3, t_1) & \cdots & v(x_{m+1}, t_1) \\ v(x_1, t_2) & v(x_2, t_2) & v(x_3, t_2) & \cdots & v(x_{m+1}, t_2) \\ v(x_1, t_3) & v(x_2, t_3) & v(x_3, t_3) & \cdots & v(x_{m+1}, t_3) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ v(x_1, t_{n+1}) & v(x_2, t_{n+1}) & v(x_3, t_{n+1}) & \cdots & v(x_{m+1}, t_{n+1}) \end{bmatrix}_{(n+1) \times (m+1)},$$

where the numerical solutions are presented in the following (J is a truncation parameter of series):

$$\begin{aligned} u(x_m, t_n) &= \frac{2}{\pi} \sum_{j=1}^J E_{\alpha,1} \left(\frac{-j^2 t_n^\alpha}{1+j^2} \right) \int_0^\pi f(\zeta) \sin(j\zeta) d\zeta \sin(jx_m) \\ &+ \frac{2}{\pi} \sum_{j=1}^J \frac{1}{1+j^2} \int_0^{t_n} (t_n-s)^{\alpha-1} E_{\alpha,\alpha} \left(\frac{-j^2(t_n-s)^\alpha}{1+j^2} \right) \int_0^\pi \mathcal{F}(\zeta) \sin(j\zeta) d\zeta ds \sin(jx_m) \end{aligned}$$

and

$$\begin{aligned} v(x_m, t_n) &= \frac{2}{\pi} \sum_{j=1}^J E_{\alpha,1} \left(\frac{-j^2 t_n^\alpha}{1+j^2} \right) \int_0^\pi g(\zeta) \sin(j\zeta) d\zeta \sin(jx_m) \\ &+ \frac{2}{\pi} \sum_{j=1}^J \frac{1}{1+j^2} \int_0^{t_n} (t_n-s)^{\alpha-1} E_{\alpha,\alpha} \left(\frac{-j^2(t_n-s)^\alpha}{1+j^2} \right) \int_0^\pi \mathcal{G}(\zeta) \sin(j\zeta) d\zeta ds \sin(jx_m). \end{aligned}$$

By fixing t , we consider the following estimations with fractional derivative orders α and α^* to compare the regularity of the solution, which are given by

$$ESTU_{\alpha^*}^\alpha(t) = \left(\sum_{m=1}^{h_x+1} |u_\alpha(x_m, t) - u_{\alpha^*}(x_m, t)|^2 / (h_x + 1) \right)^{1/2}, \tag{5.5}$$

$$ESTV_{\alpha^*}^\alpha(t) = \left(\sum_{m=1}^{h_x+1} |v_\alpha(x_m, t) - v_{\alpha^*}(x_m, t)|^2 / (h_x + 1) \right)^{1/2}, \tag{5.6}$$

where $(u_{\alpha^*}, v_{\alpha^*})$ and (u_α, v_α) are the solutions in the case α^* and α , respectively.

Table 1 The error estimates at $t = 0.2$ for $\alpha = 1.1, \alpha^* \in \{1.11, 1.12, 1.13, 1.14, 1.15\}$

$t = 0.2$	$\alpha^* = 1.11$	$\alpha^* = 1.12$	$\alpha^* = 1.13$	$\alpha^* = 1.14$	$\alpha^* = 1.15$
$ESTU_{\alpha}^{\alpha^*}(0.2)$	0.05274	0.00283	0.00027	2.67385e-05	3.23890e-06
$ESTV_{\alpha}^{\alpha^*}(0.2)$	0.05903	0.00178	0.00019	2.25538e-05	1.91779e-06

Table 2 The error estimates at $t = 0.4$ for $\alpha = 1.5, \alpha^* \in \{1.51, 1.52, 1.53, 1.54, 1.55\}$

$t = 0.4$	$\alpha^* = 1.51$	$\alpha^* = 1.52$	$\alpha^* = 1.53$	$\alpha^* = 1.54$	$\alpha^* = 1.55$
$ESTU_{\alpha}^{\alpha^*}(0.4)$	0.09677	0.00563	0.00073	7.57786e-05	6.87516e-06
$ESTV_{\alpha}^{\alpha^*}(0.4)$	0.16414	0.06821	0.06635	0.066176	0.0642495

Table 3 The error estimates at $t = 0.8$ for $\alpha = 1.8, \alpha^* \in \{1.81, 1.82, 1.83, 1.84, 1.85\}$

$t = 0.8$	$\alpha^* = 1.81$	$\alpha^* = 1.82$	$\alpha^* = 1.83$	$\alpha^* = 1.84$	$\alpha^* = 1.85$
$ESTU_{\alpha}^{\alpha^*}(0.8)$	0.01754	0.00266	0.00020	2.05406e-05	1.75455e-06
$ESTV_{\alpha}^{\alpha^*}(0.8)$	0.01971	0.00251	0.00029	2.41501e-05	3.50684e-06

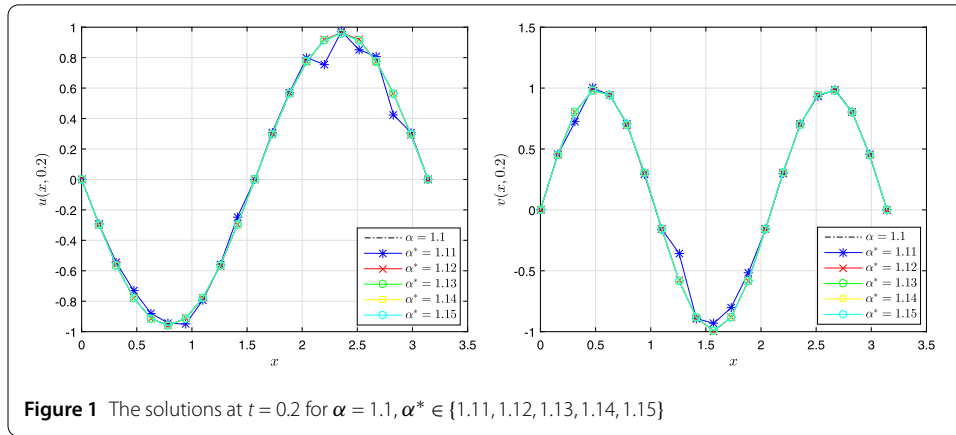


Figure 1 The solutions at $t = 0.2$ for $\alpha = 1.1, \alpha^* \in \{1.11, 1.12, 1.13, 1.14, 1.15\}$

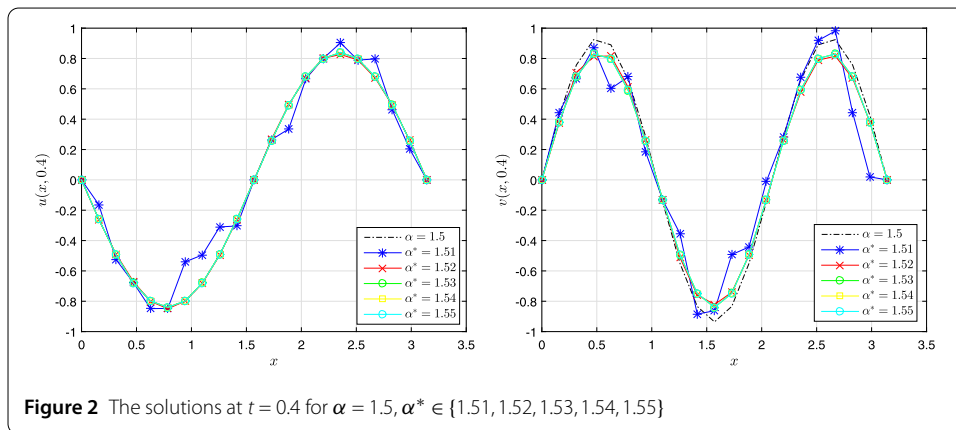
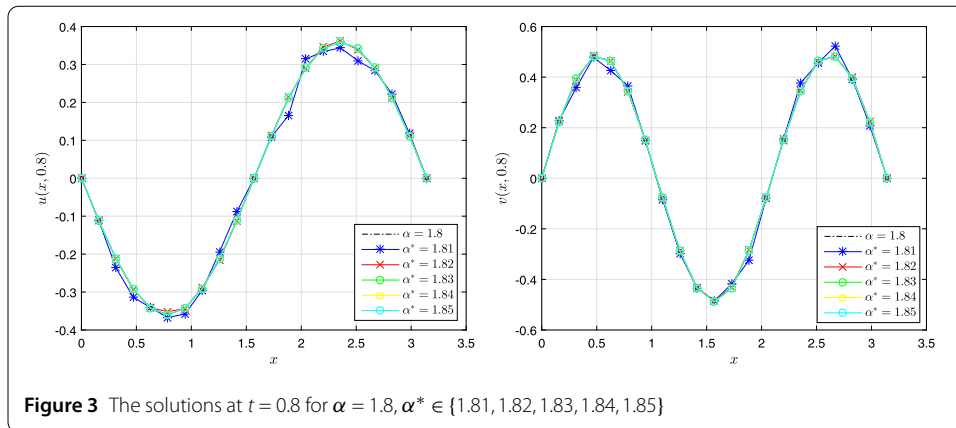


Figure 2 The solutions at $t = 0.4$ for $\alpha = 1.5, \alpha^* \in \{1.51, 1.52, 1.53, 1.54, 1.55\}$

Tables 1–3 present the error estimates between the solutions for α and α^* . It clearly shows that the solution for α^* converges to the solution for α as the deflection of the fractional order tends to zero. For a more intuitive look, we can see the graphs of the solutions in Figs. 1, 2, and 3.



6 Conclusion

In this paper, we considered the initial value problem for a system of nonlinear parabolic pseudo equations with Caputo fractional derivative. Here, we discuss the continuity which is related to a fractional order derivative. To illustrate the theoretical results, we gave an example in some cases. In the future work, we will expand the problem to the case of other derivative definitions such as Riemann–Liouville and conformable. Especially, the operators affecting the spatial variable of two equations are different, this is an open and difficult problem.

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