Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2016, Article ID 4626940, 8 pages
http://dx.doi.org/10.1155/2016/4626940

Research Article

On Coupled Systems of Time-Fractional Differential Problems by Using a New Fractional Derivative

Ahmed Alsaedi,[1] Dumitru Baleanu,[2,3] Sina Etemad,[4] and Shahram Rezapour[4]

[1] *Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia*
[2] *Department of Mathematics, Cankaya University, Ogretmenler Caddesi 14, Balgat, 06530 Ankara, Turkey*
[3] *Institute of Space Sciences, Magurele, Bucharest, Romania*
[4] *Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran*

Correspondence should be addressed to Dumitru Baleanu; dumitru@cankaya.edu.tr

Received 19 August 2015; Revised 12 October 2015; Accepted 27 October 2015

Academic Editor: Ismat Beg

The existence of solutions for a coupled system of time-fractional differential equations including continuous functions and the Caputo-Fabrizio fractional derivative is examined. After that we investigated a coupled system of time-fractional differential inclusions including compact- and convex-valued L^1-Caratheodory multifunctions and the Caputo-Fabrizio fractional derivative.

1. Introduction

The fractional calculus is nowadays an excellent mathematical tool which opens the gates for finding hidden aspects of the dynamics of the complex processes which appear naturally in many branches of science and engineering [1–6]. The methods and techniques of this type of calculus are continuously generalized and improved especially during the last few decades. We recall that the existence and multiplicity of positive solutions corresponding to singular fractional boundary value problems were discussed in [7]. Also, the existence results for several nonlinear fractional differential equations were reported in [8]. Besides, the existence of positive solutions corresponding to a coupled system of multiterm singular fractional integrodifferential boundary value problems was shown in [9]. Inventing new derivatives and applying them to study the dynamics of complex systems are an important priority for researchers. As a result, very recently, a new fractional derivative without singular kernel has been provided [10, 11]. By using the main results presented in these two new works, we present the next definition.

Definition 1 (see [10]). The α order Caputo-Fabrizio time-fractional differential derivative of the function u is written as

$$\left({}^{\mathrm{CF}}D_t^{\alpha}u\right)(x,t) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)}\int_0^t \exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right]\frac{\partial u}{\partial t}ds, \quad (t \ge 0), \tag{1}$$

where $M(\alpha)$ represents a normalization function, $0 < \alpha < 1$, and $u \in H^1[(0,1)\times(0,1)]$.

Note that $({}^{\mathrm{CF}}D_t^{\alpha}u)(x,t) = 0$ whenever u is a constant function and the kernel has no singularity at $t = s$ [10, 11]. Also, Losada and Nieto defined the new time-fractional integral based on the new definition of Caputo-Fabrizio fractional derivative [11]. By using this idea, we provide the notion of Caputo-Fabrizio time-fractional integral.

Definition 2. The α order time-fractional integral of a function u has the form [11]

$$\left({}^{\mathrm{CF}}I_t^{\alpha}u\right)(x,t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}u(x,t) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t u(x,s)\,ds, \quad (t \geq 0), \tag{2}$$

where $M(\alpha)$ represents a normalization function and $0 < \alpha < 1$.

Losada and Nieto showed that $M(\alpha) = 2/(2-\alpha)$ for all $0 \leq \alpha \leq 1$ [11]. By substituting $M(\alpha)$ in (1), we obtain the definition of the time-fractional Caputo-Fabrizio derivative of order α for a function u as follows:

$$\left({}^{\mathrm{CF}}D_t^{\alpha}u\right)(x,t) = \frac{1}{1-\alpha}\int_0^t \exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right]\frac{\partial u}{\partial s}ds; \quad ((x,t) \in [0,1]\times[0,1]). \tag{3}$$

They proved that solution of $({}^{\mathrm{CF}}D_t^{\alpha}v)(x,t) = g(x,t)$ is given by

$$v(x,t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}g(x,t) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t g(x,s)\,ds + v(0,0), \tag{4}$$

where $0 < \alpha < 1$ and $(x,t) \in [0,1]\times[0,1]$ [11].

The next step is to consider (X,d) being a metric space. Let us denote by $\mathscr{P}(X)$ and 2^X the class of all subsets and the class of all nonempty subsets of X, respectively. Hence $\mathscr{P}_{\mathrm{cl}}(X), \mathscr{P}_{\mathrm{bd}}(X), \mathscr{P}_{\mathrm{cv}}(X), \mathscr{P}_{\mathrm{cp}}(X)$, and $\mathscr{P}_{\mathrm{cp,cv}}(X)$ are the class of all closed subsets, the class of all bounded subsets, the class of all convex subsets, the class of all compact subsets, and the class of all compact and convex subsets of X, respectively. We claim that $u \in X$ is a fixed point of the multifunction $F : X \to 2^X$ whenever $u \in Fu$ [12]. A multifunction $F : [0,1]\times[0,1] \to \mathscr{P}_{\mathrm{cl}}(\mathbb{R})$ is called measurable whenever the function $(x,t) \mapsto d(w, F(x,t)) = \inf\{\|w-v\| : v \in F(x,t)\}$ is measurable for all $w \in \mathbb{R}$ [12]. The Pompeiu-Hausdorff metric $H_d : 2^X \times 2^X \to [0,\infty)$ is defined by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\}, \tag{5}$$

such that $d(A,b) = \inf_{a\in A} d(a,b)$ [13]. $(\mathrm{CB}(X), H_d)$ is a metric space and $(\mathrm{CB}(X), H_d)$ depicts a generalized metric space. Here $\mathrm{CB}(X)$ denotes the set of closed and bounded subsets of X and $C(X)$ represents the set of closed subsets of X [12, 13]. We recall that F is said to be convex-valued (compact-valued) whenever Fu is convex (compact) set for each $u \in X$ [12]. We mention that a multifunction $F : X \to C(X)$ is a contraction whenever there exists a constant $\gamma \in (0,1)$ such that $H_d(Fu, Fv) \leq \gamma d(u,v)$ for all $u, v \in X$ [12]. In 1970, Covitz and Nadler proved that each closed-valued contractive multifunction on a complete metric space has a fixed point [14].

Below we examine the existence of solutions for two coupled systems of nonlinear time-fractional differential equations and inclusions within Caputo-Fabrizio time-fractional derivative. First, we discuss the coupled system, namely,

$$\begin{gathered}\left({}^{\mathrm{CF}}D_t^{\alpha}u\right)(x,t) = f_1(x,t,u(x,t),v(x,t)),\\ \left({}^{\mathrm{CF}}D_t^{\beta}v\right)(x,t) = f_2(x,t,u(x,t),v(x,t))\end{gathered} \tag{6}$$

such that

$$\begin{gathered}u(0,0) = 0,\\ v(0,0) = 0,\end{gathered} \tag{7}$$

where $0 < \alpha < 1$, $0 < \beta < 1$, $(x,t) \in [0,1]\times[0,1]$, and the mappings $f_1, f_2 : [0,1]\times[0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous functions. In addition, we discuss the existence of solutions for the coupled system of nonlinear time-fractional differential inclusions

$$\begin{gathered}\left({}^{\mathrm{CF}}D_t^{\alpha}u\right)(x,t) \in F_1(x,t,u(x,t),v(x,t)),\\ \left({}^{\mathrm{CF}}D_t^{\beta}v\right)(x,t) \in F_2(x,t,u(x,t),v(x,t))\end{gathered} \tag{8}$$

such that

$$\begin{gathered}u(0,0) = 0,\\ v(0,0) = 0,\end{gathered} \tag{9}$$

where $F_1, F_2 : [0,1]\times[0,1]\times\mathbb{R}\times\mathbb{R} \to \mathscr{P}(\mathbb{R})$ are some multivalued maps.

We say that $F : [0,1]\times[0,1]\times\mathbb{R}\times\mathbb{R} \to 2^{\mathbb{R}}$ is a Caratheodory multifunction whenever $(x,t) \mapsto F(x,t,u_1,u_2)$ is measurable for all $u_i \in \mathbb{R}$ and $(u_1,u_2) \mapsto F(x,t,u_1,u_2)$ is upper semicontinuous (u.s.c) for almost all $(x,t) \in [0,1]\times[0,1]$ and $u_1, u_2 \in X$ [12]. A Caratheodory multifunction $F : [0,1]\times[0,1]\times\mathbb{R}\times\mathbb{R} \to 2^{\mathbb{R}}$ is said to be an L^1-Caratheodory whenever for each $\rho > 0$ there exists $\phi_\rho \in L^1([0,1]\times[0,1], \mathbb{R}^+)$ such that

$$\begin{aligned}&\|F(x,t,u_1,u_2)\|\\ &\quad = \sup_{(x,t)\in[0,1]\times[0,1]}\{|s| : s \in F(x,t,u_1,u_2)\} \leq \phi_\rho(x,t)\end{aligned} \tag{10}$$

for all $|u_i| \leq \rho$ and for almost all $(x,t) \in [0,1]\times[0,1]$ [12]. The set of selections of F_i at u_i is defined by

$$\begin{aligned}S_{F_i(u_i)} = \{&w_i \in L^1([0,1]\times[0,1], \mathbb{R}) : w_i(x,t)\\ &\in F(x,t,u_i(x,t),u_i'(x,t)) \text{ for almost all } (x,t)\\ &\in [0,1]\times[0,1]\},\end{aligned} \tag{11}$$

for all $u_i, u_i' \in C_{\mathbb{R}}([0,1]\times[0,1])$ for $i = 1, 2$. The sets $S_{F_i(u_i)}$ are nonempty for all $u_i \in C_K([0,1]\times[0,1])$ whenever $\dim K < \infty$ [12, 15]. The graph of the multifunction $F : X \to Y$ is defined by the set $\mathrm{Gr}(F) = \{(x, y) \in X \times Y : y \in F(x)\}$ (see [12, 13]). We say that the graph $\mathrm{Gr}(F)$ of $F : X \to \mathscr{P}_{cl}(Y)$ is a closed subset of $X \times Y$ whenever for all sequences $\{u_n\}_{n\in\mathbb{N}}$ in X and $\{y_n\}_{n\in\mathbb{N}}$ in Y with $u_n \to u_0$, $y_n \to y_0$, and $y_n \in F(u_n)$ for all n we have $y_0 \in F(u_0)$ [12]. Below we introduce the following results which will be required in our proofs.

Theorem 3 (see [12]). *Suppose that X is a Banach space, $T : X \to X$ is a completely continuous operator, and the set $K = \{u \in X : u = \lambda T u,\ \text{for some } \lambda \in [0, 1]\}$ is bounded. Then, T has a fixed point.*

Lemma 4 (see [12, Proposition 1.2]). *If $F : X \to \mathscr{P}_{cl}(Y)$ is upper semicontinuous, then $\mathrm{Gr}(F)$ is a closed subset of $X \times Y$. If F is completely continuous with a closed graph, then it is upper semicontinuous.*

Lemma 5 (see [12]). *Let X be a separable Banach space and $F : [0, 1] \times [0, 1] \times X \times X \to \mathscr{P}_{cp,cv}(X)$ an L^1-Caratheodory function. Then the operator $\Theta \cdot S_F : C_X([0, 1] \times [0, 1]) \to \mathscr{P}_{cp,cv}(C_X([0, 1]\times[0, 1]))$ defined by $u \mapsto (\Theta\cdot S_F)(u) = \Theta(S_{F,u})$ is a closed graph operator, where Θ is a linear continuous mapping from $L^1([0, 1] \times [0, 1], X)$ into $C_X([0, 1] \times [0, 1])$.*

Theorem 6 (see [12]). *Let E be a Banach space, C a closed convex subset of E, U an open subset of C, and $0 \in U$. Let us suppose that $F : \overline{U} \to \mathscr{P}_{cp,cv}(C)$ depicts an upper semicontinuous compact map, such that $\mathscr{P}_{cp,cv}(C)$ denotes the family of nonempty, compact convex subsets of C. Then either F admits a fixed point in $\overline{U}$ or there exist $u \in \partial U$ and $\lambda \in (0, 1)$ such that $u \in \lambda F(u)$.*

2. Main Results

First, we investigate the coupled system

$$\begin{aligned} \left({}^{\mathrm{CF}}D_t^{\alpha} u\right)(x,t) &= f_1(x,t,u(x,t),v(x,t)), \\ \left({}^{\mathrm{CF}}D_t^{\beta} v\right)(x,t) &= f_2(x,t,u(x,t),v(x,t)) \end{aligned} \tag{12}$$

equipped with the boundary value conditions $u(0, 0) = 0$ and $v(0, 0) = 0$, where $f_1, f_2 : [0, 1] \times [0, 1] \times X^2 \to X$ are continuous mappings, $\alpha, \beta \in (0, 1)$, $x, t \in [0, 1]$, and ${}^{\mathrm{CF}}D_t^{\alpha}$ and ${}^{\mathrm{CF}}D_t^{\beta}$ are the Caputo-Fabrizio time-fractional derivatives. Now, consider the Banach space $X = \{u : u \in C_{\mathbb{R}}([0, 1] \times [0, 1])\}$ endowed with the sup-norm $\|u\|_X = \sup_{(x,t)\in[0,1]\times[0,1]} |u(x, t)|$. Thus, the product space $(X \times X, \|\cdot\|_{X\times X})$ is also a Banach space via the product norm $\|(u, v)\|_{X\times X} = \|u\|_X + \|v\|_X$. First, we prove the next key lemma.

Lemma 7. *Suppose that $f \in L^1_X([0, 1]\times[0, 1])$ and $0 < \alpha < 1$. The function $u_0 \in C_X([0, 1] \times [0, 1])$ is a solution for the time-fractional integral equation*

$$\begin{aligned} u(x,t) = {} & \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\left(f(x,t) - f(0,0)\right) \\ & + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t f(x,s)\,ds \end{aligned} \tag{13}$$

if and only if u_0 is a unique solution of the time-fractional differential equation

$$\begin{aligned} \left({}^{\mathrm{CF}}D_t^{\alpha} u\right)(x,t) &= f(x,t), \quad (x,t) \in [0,1]\times[0,1], \\ u(0,0) &= 0. \end{aligned} \tag{14}$$

Proof. A solution of initial value problem (14) is denoted by u_0. As a result $({}^{\mathrm{CF}}D_t^{\alpha} u_0)(x, t) = f(x, t)$ and $u_0(0, 0) = 0$. By integrating both sides we get

$$\begin{aligned} & u_0(x,t) - u_0(0,0) \\ & \quad = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\left(f(x,t) - f(0,0)\right) \\ & \qquad + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t f(x,s)\,ds \end{aligned} \tag{15}$$

and so $u_0(x, t) = (2(1 - \alpha)/(2 - \alpha)M(\alpha))(f(x, t) - f(0, 0)) + (2\alpha/(2 - \alpha)M(\alpha)) \int_0^t f(x, s)ds$. This shows that u_0 represents the solution of time-fractional integral equation (13). If u_1 and u_2 are two distinct solutions for initial value problem (14), then ${}^{\mathrm{CF}}D_t^{\alpha} u_1(x, t) - {}^{\mathrm{CF}}D_t^{\alpha} u_2(x, t) = [{}^{\mathrm{CF}}D_t^{\alpha} u_1 - u_2](x, t) = 0$ and $(u_1 - u_2)(0, 0) = 0$. By the property of the Caputo-Fabrizio time-fractional derivative in [11], we get $u_1 = u_2$. Hence, u_0 is a unique solution of initial value problem (14). Now, suppose that u_0 is a solution of time-fractional integral equation (13). Then, we conclude that $u_0(x, t) = (2(1-\alpha)/(2-\alpha)M(\alpha))(f(x, t)- f(0, 0))+(2\alpha/(2-\alpha)M(\alpha)) \int_0^t f(x, s)ds$. By using (4), one can see that this function represents a solution for initial value problem (14). Note that $u_0(0, 0) = 0$. ☐

Now, we consider (1)-(2). For each $(x, t) \in [0, 1] \times [0, 1]$, define the operators $T_1, T_2 : X \to X$ by

$$\begin{aligned} & \left(T_1 v\right)(x,t) \\ & \quad = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} f_1(x,t,u(x,t),v(x,t)) \\ & \qquad - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} f_1(0,0,u(0,0),v(0,0)) \\ & \qquad + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t f_1(x,s,u(x,s),v(x,s))\,ds, \\ & \left(T_2 u\right)(x,t) \\ & \quad = \frac{2(1-\beta)}{(2-\beta)M(\beta)} f_2(x,t,u(x,t),v(x,t)) \\ & \qquad - \frac{2(1-\beta)}{(2-\beta)M(\beta)} f_2(0,0,u(0,0),v(0,0)) \\ & \qquad + \frac{2\beta}{(2-\beta)M(\beta)}\int_0^t f_2(x,s,u(x,s),v(x,s))\,ds \end{aligned} \tag{16}$$

and put

$$N_1 = \frac{4-2\alpha}{(2-\alpha)M(\alpha)},$$
$$N_2 = \frac{4-2\beta}{(2-\beta)M(\beta)}. \tag{17}$$

Theorem 8. *Suppose that $f_1, f_2 : [0,1]\times[0,1]\times X\times X \to X$ are the continuous mappings in system (6)-(7) and there exist positive constants L_1 and L_2 fulfilling $|f_1(x,t,u_1,u_2)| \le L_1$ and $|f_2(x,t,u_1,u_2)| \le L_2$ for all $(x,t) \in [0,1]\times[0,1]$ and $u_1, u_2 \in X$. Then, system (6)-(7) possesses at least one solution.*

Proof. Let the operators $T_1, T_2 : X \to X$ defined by (16). We define the operator $T : X\times X \to X\times X$ by $T(u,v)(x,t) := ((T_1 v)(x,t), (T_2 u)(x,t))$ for all $(x,t) \in [0,1]\times[0,1]$. Note that T is continuous because the mappings f_1 and f_2 are continuous. We prove that the operator T maps bounded sets into the bounded subsets of $X\times X$. Let Ω be a bounded subset of $X\times X$, $(u,v)\in\Omega$, and $(x,t)\in[0,1]\times[0,1]$. Then, we have

$$\begin{aligned}
&|(T_1 v)(x,t)| \\
&= \left|\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} f_1(x,t,u(x,t),v(x,t)) - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} f_1(0,0,u(0,0),v(0,0)) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t f_1(x,s,u(x,s),v(x,s))\,ds\right| \\
&\le \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}|f_1(x,t,u(x,t),v(x,t))| + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}|f_1(0,0,0,0)| + \frac{2\alpha}{(2-\alpha)M(\alpha)} \cdot \int_0^t |f_1(x,s,u(x,s),v(x,s))|\,ds \\
&\le L_1\left\{\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)}t\right\} \le L_1\left\{\frac{4(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)}\right\} \le L_1\left\{\frac{4-2\alpha}{(2-\alpha)M(\alpha)}\right\} = L_1 N_1
\end{aligned} \tag{18}$$

and so $\|(T_1 v)(x,t)\|_X \le L_1 N_1$. Also, we have

$$\begin{aligned}
&|(T_2 u)(x,t)| \\
&= \left|\frac{2(1-\beta)}{(2-\beta)M(\beta)} f_2(x,t,u(x,t),v(x,t)) - \frac{2(1-\beta)}{(2-\beta)M(\beta)} f_2(0,0,u(0,0),v(0,0)) + \frac{2\beta}{(2-\beta)M(\beta)}\int_0^t f_2(x,s,u(x,s),v(x,s))\,ds\right| \\
&\le \frac{2(1-\beta)}{(2-\beta)M(\beta)}|f_2(x,t,u(x,t),v(x,t))| + \frac{2(1-\beta)}{(2-\beta)M(\beta)}|f_2(0,0,0,0)| + \frac{2\beta}{(2-\beta)M(\beta)} \cdot \int_0^t |f_2(x,s,u(x,s),v(x,s))|\,ds \\
&\le L_2\left\{\frac{2(1-\beta)}{(2-\beta)M(\beta)} + \frac{2(1-\beta)}{(2-\beta)M(\beta)} + \frac{2\beta}{(2-\beta)M(\beta)}t\right\} \le L_2\left\{\frac{4(1-\beta)}{(2-\beta)M(\beta)} + \frac{2\beta}{(2-\beta)M(\beta)}\right\} \le L_2\left\{\frac{4-2\beta}{(2-\beta)M(\beta)}\right\} \\
&= L_2 N_2
\end{aligned} \tag{19}$$

and so $\|(T_2 u)(x,t)\|_X \le L_2 N_2$. Thus, $\|T(u,v)(x,t)\|_{X\times X} \le L_1 N_1 + L_2 N_2$. This shows that the operator T maps bounded sets into the bounded sets of $X\times X$. Now, we show that the operator T is equicontinuous. Let $(x,t_1), (x,t_2) \in [0,1]\times[0,1]$ with $t_1 < t_2$. Then, we have

$$\begin{aligned}
&|(T_1 v)(x,t_2) - (T_1 v)(x,t_1)| \\
&= \left|\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} f_1(x,t_2,u(x,t_2),v(x,t_2)) - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} f_1(0,0,u(0,0),v(0,0)) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^{t_2} f_1(x,s,u(x,s),v(x,s))\,ds \right. \\
&\quad - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} f_1(x,t_1,u(x,t_1),v(x,t_1)) + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} f_1(0,0,u(0,0),v(0,0)) \left. - \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^{t_1} f_1(x,s,u(x,s),v(x,s))\,ds\right| \\
&\le \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}|f_1(x,t_2,u(x,t_2),v(x,t_2)) - f_1(x,t_1,u(x,t_1),v(x,t_1))| + \frac{2\alpha}{(2-\alpha)M(\alpha)} \cdot \int_{t_1}^{t_2} |f_1(x,s,u(x,s),v(x,s))|\,ds \\
&\le \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}|f_1(x,t_2,u(x,t_2),v(x,t_2)) - f_1(x,t_1,u(x,t_1),v(x,t_1))| + \frac{2\alpha L_1}{(2-\alpha)M(\alpha)}(t_2 - t_1).
\end{aligned} \tag{20}$$

This implies that $|(T_1v)(x,t_2) - (T_1v)(x,t_1)| \to 0$ whenever $(x,t_2) \to (x,t_1)$. By utilizing the Arzela-Ascoli theorem, T_1 is completely continuous. Similarly, we have

$$\begin{aligned}
&\left|(T_2u)(x,t_2) - (T_2u)(x,t_1)\right| \\
&= \left| \frac{2(1-\beta)}{(2-\beta)M(\beta)} f_2(x,t_2,u(x,t_2),v(x,t_2)) \right. \\
&- \frac{2(1-\beta)}{(2-\beta)M(\beta)} f_2(0,0,u(0,0),v(0,0)) \\
&+ \frac{2\beta}{(2-\beta)M(\beta)} \int_0^{t_2} f_2(x,s,u(x,s),v(x,s))\,ds \\
&- \frac{2(1-\beta)}{(2-\beta)M(\beta)} f_2(x,t_1,u(x,t_1),v(x,t_1)) \\
&+ \frac{2(1-\beta)}{(2-\beta)M(\beta)} f_2(0,0,u(0,0),v(0,0)) \\
&\left. - \frac{2\beta}{(2-\beta)M(\beta)} \int_0^{t_1} f_2(x,s,u(x,s),v(x,s))\,ds \right| \\
&\le \frac{2(1-\beta)}{(2-\beta)M(\beta)} \left| f_2(x,t_2,u(x,t_2),v(x,t_2)) \right. \\
&\left. - f_2(x,t_1,u(x,t_1),v(x,t_1)) \right| + \frac{2\beta}{(2-\beta)M(\beta)} \\
&\cdot \int_{t_1}^{t_2} \left| f_2(x,s,u(x,s),v(x,s)) \right| ds \\
&\le \frac{2(1-\beta)}{(2-\beta)M(\beta)} \left| f_2(x,t_2,u(x,t_2),v(x,t_2)) \right. \\
&\left. - f_2(x,t_1,u(x,t_1),v(x,t_1)) \right| + \frac{2\beta L_2}{(2-\beta)M(\beta)} (t_2 \\
&- t_1).
\end{aligned} \tag{21}$$

Again, by utilizing the Arzela-Ascoli theorem we observe that T_2 is completely continuous. Therefore, we get $\|T(u,v)(x,t_2) - T(u,v)(x,t_2)\|_{X\times X} \to 0$ whenever (x,t_2) tends to (x,t_1). Thus, T is completely continuous. In the next step we prove that

$$\Omega = \{(u,v) \in X \times X : (u,v) = \lambda T(u,v) \text{ for some } \lambda \in [0,1]\} \tag{22}$$

is bounded. Let (u,v) be an arbitrary element of Ω. Choose $\lambda \in [0,1]$ fulfilling $(u,v) = \lambda T(u,v)$. Hence, $v(x,t) = \lambda(T_1v)(x,t)$ and $u(x,t) = \lambda(T_2u)(x,t)$ for all $(x,t) \in [0,1] \times [0,1]$. Since

$$\frac{1}{\lambda}|v(x,t)| = \left|(T_1v)(x,t)\right| \le L_1N_1, \tag{23}$$

we get $|v(x,t)| \le \lambda L_1N_1$ and so $\|v(x,t)\|_X \le \lambda L_1N_1$. Similarly, we prove that $\|u(x,t)\|_X \le \lambda L_2N_2$. Thus, $\|(u,v)\|_{X\times X} \le \lambda[L_1N_1 + L_2N_2]$ and so Ω is a bounded set. Now, by using Theorem 3, we get that T has a fixed point which is a solution for the coupled system of the time-fractional differential equations. □

Next we study the existence of solution for the coupled system of time-fractional differential inclusions

$$\begin{aligned}
\left({}^{\mathrm{CF}}D_t^{\alpha}u\right)(x,t) &\in F_1(x,t,u(x,t),v(x,t)), \\
\left({}^{\mathrm{CF}}D_t^{\beta}v\right)(x,t) &\in F_2(x,t,u(x,t),v(x,t))
\end{aligned} \tag{24}$$

with the initial value conditions $u(0,0) = 0$ and $v(0,0) = 0$, where $F_1, F_2 : [0,1] \times [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ are some multivalued maps.

Definition 9. One says that $(u_1,u_2) \in C([0,1] \times [0,1], X) \times C([0,1] \times [0,1], X)$ is a solution for the system of the time-fractional differential inclusions whenever it satisfies the initial value conditions and there exists $(w_1,w_2) \in L^1([0,1] \times [0,1]) \times L^1([0,1] \times [0,1])$ such that $w_i(x,t) \in F_i(x,t,u(x,t),v(x,t))$ for almost all $(x,t) \in [0,1] \times [0,1]$ and $i = 1,2$ and also

$$\begin{aligned}
&u_i(x,t) \\
&= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} w_i(x,t,u(x,t),v(x,t)) \\
&- \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} w_i(0,0,u(0,0),v(0,0)) \\
&+ \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t w_i(x,s,u(x,s),v(x,s))\,ds,
\end{aligned} \tag{25}$$

for all $(x,t) \in [0,1] \times [0,1]$ and $i = 1,2$.

Theorem 10. *Let $F_1, F_2 : [0,1] \times [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathscr{P}_{cp,cv}(\mathbb{R})$ be L^1-Caratheodory multifunctions. Suppose that there exist a nondecreasing bounded continuous map $\psi : [0,\infty) \to (0,\infty)$ and a continuous function $p : [0,1] \times [0,1] \to (0,\infty)$ such that $\|F_i(x,t,u_i(x,t),u_i'(x,t))\| \le p(x,t)\psi(\|u_i\|)$ for all $(x,t) \in [0,1] \times [0,1]$, $u_i, u_i' \in X$ for $i = 1,2$. Then, coupled system of time-fractional differential inclusions (8)-(9) has at least one solution.*

Proof. Define the operator $N : X\times X \to 2^{X\times X}$ by $N(u_1,u_2) = \binom{N_1(u_1,u_2)}{N_2(u_1,u_2)}$, where

$$\begin{aligned}
N_1(u_1,u_2) = \{&h_1 \in X \times X : \text{ there exists } v_1 \\
&\in S_{F_1,u_1} \text{ such that } h_1(x,t) = v_1(x,t)\ \forall (x,t) \\
&\in [0,1] \times [0,1]\}, \\
N_2(u_1,u_2) = \{&h_2 \in X \times X : \text{ there exists } v_2 \\
&\in S_{F_2,u_2} \text{ such that } h_2(x,t) = v_2(x,t)\ \forall (x,t) \\
&\in [0,1] \times [0,1]\},
\end{aligned}$$

$$h_1(x,t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} v_1(x,t) - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \cdot v_1(0,0) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t v_1(x,s)\,ds,$$
$$h_2(x,t) = \frac{2(1-\beta)}{(2-\beta)M(\beta)} v_2(x,t) - \frac{2(1-\beta)}{(2-\beta)M(\beta)} \cdot v_2(0,0) + \frac{2\beta}{(2-\beta)M(\beta)} \int_0^t v_2(x,s)\,ds. \quad (26)$$

By Lemma 7, it is clear that each fixed point of the operator N is a solution for system of time-fractional differential inclusions (8). First, we prove that the multifunction N is convex-valued. Let $(u_1,u_2) \in X \times X$, $(h_1,h_2), (h_1',h_2') \in N(u_1,u_2)$. Choose $v_i, v_i' \in S_{F_i(u_1,u_2)}$ such that

$$h_i(x,t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} v_i(x,t) - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} v_i(0,0) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t v_i(x,s)\,ds,$$
$$h_i'(x,t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} v_i'(x,t) - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} v_i'(0,0) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t v_i'(x,s)\,ds \quad (27)$$

for almost all $(x,t) \in [0,1]\times[0,1]$ and $i = 1,2$. Let $0 \le \lambda \le 1$ be given. Then, we have

$$\left[\lambda h_i + (1-\lambda)h_i'\right](x,t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \left[\lambda v_i(x,t) + (1-\lambda)v_i'(x,t)\right] - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \left[\lambda v_i(0,0) + (1-\lambda)v_i'(0,0)\right] + \frac{2\alpha}{(2-\alpha)M(\alpha)} \cdot \int_0^t \left[\lambda v_i(x,s) + (1-\lambda)v_i'(x,s)\right] ds \quad (28)$$

for $i = 1,2$. Since the operator F_i has convex values, $S_{F_i(u_i)}$ is a convex set and $[\lambda h_i + (1-\lambda)h_i'] \in N_i(u_1,u_2)$ for $i = 1,2$. This implies that the operator N has convex values. Now, we prove that N maps bounded sets of X into bounded sets. Let $r > 0$, $B_r = \{(u_1,u_2) \in X \times X : \|(u_1,u_2)\| \le r\}$ be a bounded subset of $X \times X$, $(h_1,h_2) \in N(u_1,u_2)$, and $(u_1,u_2) \in B_r$. Then, there exists $(v_1,v_2) \in S_{F_1(u_1)} \times S_{F_2(u_2)}$ such that

$$h_1(x,t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} v_1(x,t) - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} v_1(0,0) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t v_1(x,s)\,ds \quad (29)$$

and $h_2(x,t) = (2(1-\beta)/(2-\beta)M(\beta))v_2(x,t) - (2(1-\beta)/(2-\beta)M(\beta))v_2(0,0) + (2\beta/(2-\beta)M(\beta)) \int_0^t v_2(x,s)ds$ for almost all $(x,t) \in [0,1]\times[0,1]$. If $\|p\|_\infty = \sup_{(x,t)\in[0,1]\times[0,1]} |p(x,t)|$, then we obtain

$$\begin{aligned} \left|(h_1)(x,t)\right| &= \left| \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} v_1(x,t) - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} v_1(0,0) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t v_1(x,s)\,ds \right| \\ &\le \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \left|v_1(x,t)\right| + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \left|v_1(0,0)\right| + \frac{2\alpha}{(2-\alpha)M(\alpha)} \cdot \int_0^t \left|v_1(x,s)\right| ds \le p(x,t)\,\psi(\|u_1\|) \\ &\cdot \left\{ \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)} t \right\} \le \|p\|_\infty \cdot \psi(\|u_1\|) \left\{ \frac{4(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)} \right\} \\ &\le \|p\|_\infty \psi(\|u_1\|) \left\{ \frac{4-2\alpha}{(2-\alpha)M(\alpha)} \right\} = \|p\|_\infty \cdot \psi(\|u_1\|) N_1, \end{aligned} \quad (30)$$

where the constant N_1 is defined by (17). This implies that $\|h_1\| \le \|p\|_\infty \psi(\|u_1\|) N_1$. Similarly, we get $\|h_2\| \le \|p\|_\infty \psi(\|u_2\|) N_2$, where the constant N_2 is defined by (17). Thus, $\|(h_1,h_2)\| \le \|p\|_\infty \psi(\|(u_1,u_2)\|)(N_1 + N_2)$. Now, we prove that N maps bounded sets into equicontinuous subsets of $X \times X$. Let $(u_1, u_2) \in B_r$ and $(x, t_1), (x, t_2) \in [0,1] \times [0,1]$ with $t_1 < t_2$. Then, we have

$$\begin{aligned}
&\left|(h_1)(x,t_2) - (h_1)(x,t_1)\right| = \left|\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} v_1(x,t_2)\right. \\
&\quad - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} v_1(0,0) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \\
&\quad \cdot \int_0^{t_2} v_1(x,s)\,ds - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} v_1(x,t_1) \\
&\quad + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} v_1(0,0) \\
&\quad \left. - \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^{t_1} v_1(x,s)\,ds\right| \\
&\quad \leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \left|v_1(x,t_2) - v_1(x,t_1)\right| \\
&\quad + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_{t_1}^{t_2} \left|v_1(x,s)\right| ds \\
&\quad \leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \left|v_1(x,t_2) - v_1(x,t_1)\right| \\
&\quad + \frac{2\alpha \|p\|_\infty \psi(\|u_1\|)}{(2-\alpha)M(\alpha)} (t_2 - t_1).
\end{aligned} \tag{31}$$

By using a similar method, we obtain

$$\begin{aligned}
&\left|(h_2)(x,t_2) - (h_2)(x,t_1)\right| \\
&\quad \leq \frac{2(1-\beta)}{(2-\beta)M(\beta)} \left|v_2(x,t_2) - v_2(x,t_1)\right| \\
&\quad + \frac{2\beta \|p\|_\infty \psi(\|u_2\|)}{(2-\beta)M(\beta)} (t_2 - t_1).
\end{aligned} \tag{32}$$

Hence, $|h_i(x,t_2) - h_i(x,t_1)| \to 0$ as $(x,t_2) \to (x,t_1)$. By using the Arzela-Ascoli theorem we get that N is completely continuous. Here, we prove that N is upper semicontinuous. By using Lemma 4, N is upper semicontinuous whenever it has a closed graph. Since N is completely continuous, we must show that N has a closed graph.

Let $\{(u_1^n, u_2^n)\}$ be a sequence in $X \times X$ with $(u_1^n, u_2^n) \to (u_1^0, u_2^0)$ and $(h_1^n, h_2^n) \in N(u_1^n, u_2^n)$ with $(h_1^n, h_2^n) \to (h_1^0, h_2^0)$. We show that $(h_1^0, h_2^0) \in N(u_1^0, u_2^0)$. For each $(h_1^n, h_2^n) \in N(u_1^n, u_2^n)$, we can choose $(v_1^n, v_2^n) \in S_{F_1(u_1^n)} \times S_{F_2(u_2^n)}$ such that

$$\begin{aligned}
h_1^n(x,t) &= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} v_1^n(x,t) \\
&\quad - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} v_1^n(0,0) \\
&\quad + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t v_1^n(x,s)\,ds
\end{aligned} \tag{33}$$

and $h_2^n(x,t) = (2(1-\beta)/(2-\beta)M(\beta))v_2^n(x,t) - (2(1-\beta)/(2-\beta)M(\beta))v_2^n(0,0) + (2\beta/(2-\beta)M(\beta))\int_0^t v_2^n(x,s)ds$ for all $(x,t) \in [0,1] \times [0,1]$. It is sufficient to show that there exists $(v_1^0, v_2^0) \in S_{F_1(u_1^0)} \times S_{F_2(u_2^0)}$ such that

$$\begin{aligned}
h_1^0(x,t) &= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} v_1^0(x,t) \\
&\quad - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} v_1^0(0,0) \\
&\quad + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t v_1^0(x,s)\,ds,
\end{aligned} \tag{34}$$

and $h_2^0(x,t) = (2(1-\beta)/(2-\beta)M(\beta))v_2^0(x,t) - (2(1-\beta)/(2-\beta)M(\beta))v_2^0(0,0) + (2\beta/(2-\beta)M(\beta))\int_0^t v_2^0(x,s)ds$ for all $(x,t) \in [0,1] \times [0,1]$. Now, consider the linear operators $\Theta_1, \Theta_2 : L^1([0,1] \times [0,1], X) \to C([0,1] \times [0,1], X)$ defined by

$$\begin{aligned}
\Theta_1(v)(x,t) &= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} v(x,t) \\
&\quad - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} v(0,0) \\
&\quad + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t v(x,s)\,ds
\end{aligned} \tag{35}$$

and $\Theta_2(v)(x,t) = (2(1-\beta)/(2-\beta)M(\beta))v(x,t) - (2(1-\beta)/(2-\beta)M(\beta))v(0,0) + (2\beta/(2-\beta)M(\beta))\int_0^t v(x,s)ds$. Note that

$$\begin{aligned}
&\left\|h_1^n(x,t) - h_1^0(x,t)\right\| \\
&\quad = \left\|\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\left[v_1^n(x,t) - v_1^0(x,t)\right]\right. \\
&\quad - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\left[v_1^n(0,0) - v_1^0(0,0)\right] \\
&\quad \left. + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t \left[v_1^n(x,s) - v_1^0(x,s)\right] ds\right\| \\
&\quad \longrightarrow 0, \\
&\left\|h_2^n(x,t) - h_2^0(x,t)\right\| \\
&\quad = \left\|\frac{2(1-\beta)}{(2-\beta)M(\beta)}\left[v_2^n(x,t) - v_2^0(x,t)\right]\right. \\
&\quad - \frac{2(1-\beta)}{(2-\beta)M(\beta)}\left[v_2^n(0,0) - v_2^0(0,0)\right] \\
&\quad \left. + \frac{2\beta}{(2-\beta)M(\beta)} \int_0^t \left[v_2^n(x,s) - v_2^0(x,s)\right] ds\right\| \\
&\quad \longrightarrow 0.
\end{aligned} \tag{36}$$

By using Lemma 5, we get that $\Theta_i \cdot S_{F_i}$ is a closed graph operator for $i = 1, 2$. Also, we get $h_i^n(x,t) \in \Theta_i(S_{F_i(u_i^n)})$ for all n. Since $u_i^n \to u_i^0$, we get

$$\begin{aligned} h_1^0(x,t) = {} & \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} v_1^0(x,t) \\ & - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} v_1^0(0,0) \\ & + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t v_1^0(x,s)\,ds \end{aligned} \tag{37}$$

and $h_2^0(x,t) = (2(1-\beta)/(2-\beta)M(\beta))v_2^0(x,t) - (2(1-\beta)/(2-\beta)M(\beta))v_2^0(0,0) + (2\beta/(2-\beta)M(\beta)) \int_0^t v_2^0(x,s)ds$ for some $v_i^0 \in S_{F_i(u_i^0)}$ $(i = 1, 2)$. Thus, N has a closed graph.

Now, we prove that there is an open set $U \subseteq X$ with $(u_1, u_2) \notin N(u_1, u_2)$ for all $\lambda \in (0,1)$ and $(u_1, u_2) \in \partial U$. Let $\lambda \in (0,1)$ and $(u_1, u_2) \in \lambda N(u_1, u_2)$. Then, there exists $v_i \in L^1([0,1] \times [0,1], \mathbb{R})$ with $v_i \in S_{F_i(u_i)}$ $(i = 1, 2)$ such that

$$\begin{aligned} u_1(x,t) = {} & \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} v_1(x,t) \\ & - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} v_1(0,0) \\ & + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t v_1(x,s)\,ds \end{aligned} \tag{38}$$

and $u_2(x,t) = (2(1-\beta)/(2-\beta)M(\beta))v_2(x,t) - (2(1-\beta)/(2-\beta)M(\beta))v_2(0,0) + (2\beta/(2-\beta)M(\beta)) \int_0^t v_2(x,s)ds$ for all $(x,t) \in [0,1] \times [0,1]$. By using the above computed values, we obtain $\|u_i\| \le \|p\|_\infty \psi(\|u_i\|) \sum_{i=1}^n N_i$ for $i = 1, 2$. This follows that $\|u_i\| / \|p\|_\infty \psi(\|u_i\|) \sum_{i=1}^n N_i \le 1$ for $i = 1, 2$. Choose $M_i > 0$ with $\|u_i\| \ne M_i$ in such a way that $M_i / \|p\|_\infty \psi(\|u_i\|) \sum_{i=1}^n N_i > 1$ for $i = 1, 2$. Put $U = \{(u_1, u_2) \in X \times X : \|(u_1, u_2)\| < \min\{M_1, M_2\}\}$. We note that the operator $N : \overline{U} \to \mathcal{P}(X)$ is upper semicontinuous and completely continuous. Also, we showed that there is no $(u_1, u_2) \in \partial U$ such that $(u_1, u_2) \in \lambda N(u_1, u_2)$ for some $\lambda \in (0,1)$. Hence, with the help of Theorem 6, we get that N has a fixed point $(u_1, u_2) \in \overline{U}$ which is a solution for time-fractional differential inclusion (8)-(9). □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

Research of the last two authors was supported by Azarbaijan Shahid Madani University.

References

[1] R. P. Agarwal, M. Benchohra, and S. Hamani, "A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions," *Acta Applicandae Mathematicae*, vol. 109, no. 3, pp. 973–1033, 2010.

[2] D. Baleanu, H. Mohammadi, and S. Rezapour, "On a nonlinear fractional differential equation on partially ordered metric spaces," *Advances in Difference Equations*, vol. 2013, article 83, 2013.

[3] D. Baleanu, S. Rezapour, S. Etemad, and A. Alsaedi, "On a time-fractional integrodifferential equation via three-point boundary value conditions," *Mathematical Problems in Engineering*, vol. 2015, Article ID 785738, 12 pages, 2015.

[4] K. Diethelm, *The Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics, Springer, Berlin, Germany, 2010.

[5] R. Gorenflo and F. Mainardi, "Fractional calculus: integral and differential equations of fractional order," in *Fractals and Fractional Calculus in Continuum Mechanics*, A. Carpinteri and F. Mainardi, Eds., Springer, New York, NY, USA, 1997.

[6] G. Samko, A. Kilbas, and O. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, 1993.

[7] Z. Bai and W. Sun, "Existence and multiplicity of positive solutions for singular fractional boundary value problems," *Computers & Mathematics with Applications*, vol. 63, no. 9, pp. 1369–1381, 2012.

[8] D. Baleanu, R. P. Agarwal, H. Mohammadi, and S. Rezapour, "Some existence results for a nonlinear fractional differential equation on partially ordered Banach spaces," *Boundary Value Problems*, 2013:112, 8 pages, 2013.

[9] D. Baleanu, S. Zahra Nazemi, and S. Rezapour, "The existence of positive solutions for a new coupled system of multiterm singular fractional integrodifferential boundary value problems," *Abstract and Applied Analysis*, vol. 2013, Article ID 368659, 15 pages, 2013.

[10] M. Caputo and M. Fabrizio, "A new definition of fractional derivative without singular kernel," *Progress in Fractional Differentiation and Applications*, vol. 1, no. 2, pp. 73–85, 2015.

[11] J. Losada and J. J. Nieto, "Properties of a new fractional derivative without singular kernel," *Progress in Fractional Differentiation and Applications*, vol. 1, no. 2, pp. 87–92, 2015.

[12] D. R. Smart, *Fixed Point Theorems*, Cambridge University Press, Cambridge, UK, 1980.

[13] S. M. Aleomraninejad, S. Rezapour, and N. Shahzad, "On fixed point generalizations of Suzuki's method," *Applied Mathematics Letters*, vol. 24, no. 7, pp. 1037–1040, 2011.

[14] H. Covitz and S. B. Nadler, "Multi-valued contraction mappings in generalized metric spaces," *Israel Journal of Mathematics*, vol. 8, pp. 5–11, 1970.

[15] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Eqautions*, John Wiley & Sons, 1993.