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ON FRACTIONAL DIFFERENTIAL INCLUSION PROBLEMS INVOLVING FRACTIONAL ORDER DERIVATIVE WITH RESPECT TO ANOTHER FUNCTION

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Abstract

In this research work, we investigate the existence of solutions for a class of nonlinear boundary value problems for fractional-order differential inclusion with respect to another function. Endpoint theorem for φ-weak contractive maps is the main tool in determining our results. An example is presented in aim to illustrate the results.

Keywords: g-Riemann–Liouville Fractional Derivative; g-Caputo Fractional Derivative; Inclusion; Endpoint Theory.

1. INTRODUCTION

Over years, the investigation over fractional operators has been giving researchers numerous discoveries in many distinct technological fields.[1–4] Particularly, some remarkable applications in mechanics and physics have been done.[5,6] However, many authors have shown that some forms of such fractional differentiations and integrations may not be able to identify real-world problems suitably. For this purpose, a new fractional differential operator with unlike kernels has been recently developed in Ref. 7, the so-called Caputo fractional derivative of a function f with respect to another function ψ, or ψ-Caputo fractional derivative, in Ref. 8 the same fractional derivative was established on a set of absolute continuous functions. Since then several studies showed interest in the ψ-differentiation operators, we mention for example.[9–14]

The investigation of linear and nonlinear boundary value problems for differential equations and inclusions forms a substantial area of research mainly due to the appearance of such problems in a diversity of disciplines of applied sciences and engineering. In particular, problems with integral boundary conditions with a composition of different integral forms are an interesting and valuable class of problems as integral boundary conditions arise in the study of biomedical issues[15] and radiation conditions problems.[16]

Differential inclusions emerge in the mathematical modeling of certain problems in stochastic calculus and optimal controls.[17] Many authors contributed to developing theoretical aspects related to fractional differential inclusions like the study of existence, uniqueness, asymptotic behavior, etc. We mention for example.[18–22] More details on the theory and applications in this field, are found in the monograph by Ahmad *et al.*[23] and the survey by Agarwal *et al.*[24]

In this paper, we intend to investigate the following fractional differential inclusion with respect to another function g

$$ {}^{c}\mathcal{D}^{\eta}_{0^+;g}w(y) \in \Upsilon(y, w(y)), \quad y \in J := [0, l], \quad 1 < \eta \le 2, \tag{1.1} $$

subject to integral boundary value conditions with respect to another function g

$$ w(0) - \delta_g w(0) = \frac{a}{\Gamma(\theta)}\int_0^p (g(p) - g(\xi))^{\theta-1} g'(\xi) \times \kappa(\xi, w(\xi))d\xi = aI^{\theta}_{0^+;g}\kappa(p, w(p)), \tag{1.2} $$

$$ w(l) + \delta_g w(l) = \frac{b}{\Gamma(\mu)}\int_0^q (g(q) - g(\xi))^{\mu-1} g'(\xi)\omega \times (\xi, w(\xi))d\xi = bI^{\mu}_{0^+;g}\omega(q, w(q)), \tag{1.3} $$

where ${}^{c}\mathcal{D}^{\eta}_{g}$ is the Caputo fractional-order derivative with respect to another function g established by Jarad *et al.*,[8] $\Upsilon : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, $I^{z}_{0^+;\psi}$ is the ψ-Riemann–Liouville fractional integral of order $0 < z \le 1$, $z \in (\theta, \mu)$, on $[0, l]$, $0 < p, q < T$, κ and ω continuous functions defined on $[0, l] \times \mathbb{R}$, and $\delta_g = \frac{1}{g'(y)}\frac{d}{dy}$, while a and b are suitable chosen constants.

This paper proposes a new investigation in obtaining the existence criteria of solutions for the above inclusion problem by using endpoint theorem for φ-weak contractive multivalued mappings due to Moradi.[25]

The entire paper is designed as follows. In Sec. 2, Some preliminaries follow this section to allow the reader to grab an overview of the state-of-the-art. In Sec. 3, we establish the main result. We provide an example to illustrate the theory in Sec. 4.

2. PRELIMINARIES AND AUXILIARY RESULTS

The space of all continuous functions w from $[0, l]$ into $\mathbb{R}$ denoted by $E = C([0, l], \mathbb{R})$ endowed the supremum norm

$$\|w\| = \sup_{y\in[0,l]} |w(y)|.$$

$L^1([0, l], \mathbb{R})$ be the Banach space of measurable functions $w : [0, l] \to \mathbb{R}$ with the norm

$$\|w\|_1 = \int_0^l |w(\xi)|d\xi.$$

$AC([0, l], \mathbb{R})$ stands for the set of absolutely continuous functions from J into $\mathbb{R}$. We define $AC_g^n([0, l], \mathbb{R})$ by

$$AC_g^n([0,l],\mathbb{R}) = \left\{ w : [0,l] \to \mathbb{R}; (\delta_g^{n-1}w)(y) \in AC([0,l],\mathbb{R}), \delta_g = \frac{1}{g'(y)}\frac{d}{dy} \right\},$$

which is endowed with the norm given by

$$\|w\|_{C_g^n} = \sum_{k=0}^{n-1} \|\delta_g^k w(y)\|_\infty,$$

where $g \in C^n([0, l], \mathbb{R})$, with $g'(y) > 0$ on $[0, l]$, and

$$\delta_g^k = \underbrace{\delta_g\delta_g \ldots \delta_g}_{k \text{ times}}.$$

For a normed space $(E, \|.\|)$, let

$$\mathcal{P}_{\text{cp}}(E) = \{Q \in \mathcal{P}(E) : Q \text{ is compact}\},$$
$$\mathcal{P}_{\text{bd}}(E) = \{Q \in \mathcal{P}(E) : Q \text{ is bounded}\},$$
$$\mathcal{P}_{\text{cp,bd}}(E) = \{Q \in \mathcal{P}(E) : Q \text{ is closed and bounded}\}.$$

For more details on multivalued maps, see Ref. 26.

Now, basic definitions of fractional calculus with respect to another function are recalled.

Definition 2.1 (Ref. 27). The Riemann–Liouville fractional integral of order $r > 0$ of an integrable function f with respect to g is defined by

$$I_{0+;g}^r f(y) = \frac{1}{\Gamma(r)} \int_0^y (g(y) - g(\xi))^{r-1} g'(\xi) f(\xi) d\xi, \tag{2.1}$$

Definition 2.2 (Ref. 8). Let $f \in AC_g^n([0, l], \mathbb{R})$. The fractional derivative in Riemann–Liouville sense of order $r > 0$ with respect to g is defined as

$$\begin{aligned}\mathcal{D}_{0+;g}^r f(y) &= I_{0+;g}^{n-r}(\delta_g^n f)(y) + \sum_{k=0}^{n-1} \frac{(\delta_g^k f)(1)}{\Gamma(k-r+1)} \times (g(y)-g(0))^{k-r} \\ &= \frac{1}{\Gamma(n-r)} \int_0^y (g(y)-g(\xi))^{n-r-1} g'(\xi) \times \delta_g^n f(\xi) d\xi + \sum_{k=0}^{n-1} \frac{(\delta_g^k f)(0)}{\Gamma(k-r+1)} \times (g(y)-g(0))^{k-r},\end{aligned} \tag{2.2}$$

provided the integral exists. Γ is the Gamma function, and $n = [r] + 1$.

Definition 2.3. (Refs. 8 and 7) For a given $f \in AC_g^n([0, l], \mathbb{R})$. We define the Caputo fractional derivative of order $r > 0$ with respect to g by

$$\begin{aligned}{}^c\mathcal{D}_{0+;g}^r f(y) &= I_{0+;g}^{n-r}(\delta_g^n f)(y) \\ &= \frac{1}{\Gamma(n-1)} \int_0^y (g(y)-g(\xi))^{n-r-1} g'(\xi) \times (\delta_g^n f)(\xi) d\xi, \quad n = [r]+1.\end{aligned}$$

When $r = n \in \mathbb{N}$, we have

$${}^c\mathcal{D}_{0+;g}^r f(y) = (\delta_g^n f)(y).$$

Lemma 2.4 (Ref. 8). *For a given function $f \in AC_g^n([0, l], \mathbb{R})$, $r \in \mathbb{R}^+$, we have*

$$I_{0+;g}^r {}^c\mathcal{D}_{0+;g}^r f(y) = f(y) - \sum_{k=0}^{n-1} \frac{(\delta_g^k f)(0)}{k!} \times (g(y)-g(0))^k, \tag{2.3}$$

in particular, for $0 < r < 1$, we have

$$I_{0+;g}^r {}^c\mathcal{D}_{0+;g}^r f(y) = f(y) - f(0).$$

The next lemma is needed to investigate the existence of solutions to the problem (1.1)–(1.3).

Lemma 2.5 (Ref. 4). *Let ϑ, ρ_1 and ρ_2 be real continuous functions on $[0, l]$, and $1 \le \eta \le 2$. We say that $w \in AC_g^2([0, l], \mathbb{R})$ is a solution of the fractional integral equation*

$$w(y) = L(y) + \int_0^l G_g(y,\xi)\vartheta(\xi)d\xi, \tag{2.4}$$

where

$$L(y) = \frac{a(g(l)-g(y)+1)}{g(l)-g(0)+2} I^{\theta}_{0+;g}\rho_1(p) + \frac{b(g(y)-g(0)+1)}{g(l)-g(0)+2} I^{\mu}_{0+;g}\rho_2(q),$$

$g \in C^2([0,l],\mathbb{R})$ *with* $g' > 0$, *and* $G_g(y,\xi)$ *is the Green function with respect to g defined by*

$$G_g(y,\xi) = g'(\xi) \times \begin{cases} \frac{(g(y)-g(\xi))^{\eta-1}}{\Gamma(\eta)} - \frac{g(y)-g(0)+1}{(g(l)-g(0)+2)\Gamma(\eta)} \times (g(l)-g(\xi))^{\eta-1} \\ \quad - \frac{g(y)-g(0)+1}{(g(l)-g(0)+2)\Gamma(\eta-1)} \times (g(l)-g(\xi))^{\eta-2}, \quad 0 \le \xi \le y, \\ - \frac{g(y)-g(0)+1}{(g(l)-g(0)+2)\Gamma(\eta)} (g(l)-g(\xi))^{\eta-1} \\ \quad - \frac{g(y)-g(0)+1}{(g(l)-g(0)+2)\Gamma(\eta-1)} \times (g(l)-g(\xi))^{\eta-2}, \quad y \le \xi \le l, \end{cases} \tag{2.5}$$

if and only if w is a solution of the following fractional BVP:

$$^{c}\mathcal{D}^{\eta}_{0+;g} w(y) = \vartheta(y), \quad y \in J := [0,l], \tag{2.6}$$

$$w(0) - \delta_g w(0) = a I^{\theta}_{0+;g}\rho_1(p), \tag{2.7}$$

$$w(l) + \delta_g w(l) = b I^{\mu}_{0+;g}\rho_2(q), \tag{2.8}$$

Proof. Applying the fractional operator $I^{\eta}_{0+;g}$ on both sides of Eq. (2.6) and making use of Lemma 2.4, we obtain

$$w(y) = k_1 + k_2(g(y)-g(0)) + I^{\eta}_{0+;g}\vartheta(y), \tag{2.9}$$

where k_1, k_2 are real constants. Taking the δ_g-derivative (2.9), we get

$$(\delta_g w)(y) = k_2 + I^{\eta-1}_{0+;g}\vartheta(y). \tag{2.10}$$

From (2.7) and (2.8), we get

$$k_1 - k_2 = a I^{\eta}_{0+;g}\rho_1(p), \tag{2.11}$$

and

$$k_1 + k_2(g(l)-g(0)+1) + I^{\eta}_{0+;g}\vartheta(l) + I^{\eta-1}_{0+;g}\vartheta(l) = b I^{\mu}_{0+;g}\rho_2(q). \tag{2.12}$$

Equations (2.9) and (2.12) give

$$k_2 = \frac{b}{g(l)-g(0)+2} I^{\mu}_{0+;g}\rho_2(q) - \frac{a}{g(l)-g(0)+2} I^{\theta}_{0+;g}\rho_1(p) - \frac{1}{g(l)-g(0)+2} I^{\eta}_{0+;g}\vartheta(l) - \frac{1}{g(l)-g(0)+2} I^{\eta-1}_{0+;g}\vartheta(l), \tag{2.13}$$

and

$$k_1 = \frac{a(g(l)-g(0)+1)}{g(l)-g(0)+2} I^{\theta}_{0+;g}\rho_1(p) + \frac{b}{g(l)-g(0)+2} I^{\mu}_{0+;g}\rho_2(q) - \frac{1}{g(l)-g(0)+2} \times \left[I^{\eta}_{0+;g}\vartheta(l) + I^{\eta-1}_{0+;g}\vartheta(l) \right]. \tag{2.14}$$

From (2.9), (2.13), (2.14) and using the fact that $\int_0^l = \int_0^y + \int_y^l$, we obtain

$$w(y) = L(y) + \int_0^l G_g(y,\xi)\vartheta(\xi)d\xi, \tag{2.15}$$

where

$$L(y) = \frac{a(g(l)-g(y)+1)}{g(l)-g(0)+2} I^{\theta}_{0+;g}\rho_1(p) + \frac{b(g(y)-g(0)+1)}{g(l)-g(0)+2} I^{\mu}_{0+;g}\rho_2(q), \tag{2.16}$$

and

$$G_g(y,\xi) = g'(\xi) \begin{cases} \frac{(g(y)-g(\xi))^{\eta-1}}{\Gamma(\eta)} \\ \quad - \frac{g(y)-g(0)+1}{(g(l)-g(0)+2)\Gamma(\eta)} \times (g(l)-g(\xi))^{\eta-1} \\ \quad - \frac{g(y)-g(0)+1}{(g(l)-g(0)+2)\Gamma(\eta-1)} \times (g(l)-g(\xi))^{\eta-2}, \quad 0 \le \xi \le y, \\ - \frac{g(y)-g(0)+1}{(g(l)-g(0)+2)\Gamma(\eta)} \times (g(l)-g(\xi))^{\eta-1} \\ \quad - \frac{g(y)-g(0)+1}{(g(l)-g(0)+2)\Gamma(\eta-1)} (g(l)-g(\xi))^{\eta-2}, \quad y \le \xi \le l. \end{cases}$$

Therefore, we have (2.4). Inversely, it is clear that if y verifies Eq. (2.4), then (2.6)–(2.8) hold. □

Remark 2.6. Green's function $G_g(y,\xi)$, is continuous, and thus is bounded. We let $\tilde{G}_g = \sup\{\int_0^l |G_g(y,\xi)|, y \in [0,l]\}$.

3. EXISTENCE RESULTS

In this work, we define the class Ψ by Ref. 25

$$\Psi := \{\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+, \text{ such that } \varphi^{-1}(0) = \{0\}, \varphi(y) < y \text{ for all } y \in [0,+\infty), \text{ and } \varphi(y_n) \to 0, \text{ whenever } y_n \to 0\}.$$

Let (E,d) be a metric space induced from the normed space $(E,\|.\|)$. Consider $H_d : \mathcal{P}(E) \times \mathcal{P}(E) \to \mathbb{R}^+ \cup \{\infty\}$ which defined as

$$H_d(Q,D) := \max\left\{\sup_{q\in Q} \rho(q,D), \sup_{d\in D} \rho(d,Q)\right\},$$

where $\rho(Q,d) = \inf_{q\in Q} \rho(q,d)$ and $\rho(q,D) = \inf_{d\in D} \rho(q,d)$. Then $(\mathcal{P}_{cl,bd}, H_d)$ is a metric space.[28]

Definition 3.1 (Ref. 25). Let (E,d) be a complete metric space. A multivalued operator $\Gamma : E \to \mathcal{P}_{\mathrm{cl,bd}}(E)$ is said to be a φ-weak contraction if there exists a function $\varphi \in \Psi$, such that

$$H_d(\Gamma\zeta, \Gamma\vartheta) \le d(\zeta,\vartheta) - \varphi(d(\zeta,\vartheta)),$$

for each $\zeta, \vartheta \in E$.

An element $\vartheta \in E$ is said to be a fixed point of Γ, if $\vartheta \in \Gamma\vartheta$ and an endpoint or stationary point if $\Gamma\vartheta = \{\vartheta\}$. The set of all fixed points of Γ denotes by $\mathrm{Fix}(\Gamma)$, and $\mathrm{End}(\Gamma)$ stands for the set of all endpoints of Γ. We say that Γ has the approximation endpoint property if

$$\inf_{\zeta\in E} \sup_{\vartheta\in\Gamma\zeta} d(\zeta,\vartheta) = 0. \tag{3.1}$$

Lemma 3.2 (Ref. 25). *Let $\Gamma : E \to \mathcal{P}_{\mathrm{cl,bd}}$ be a multivalued φ-weak contractive. If Γ has the approximate endpoint property, then Γ has an unique endpoint. Moreover,* $\mathrm{End}(\Gamma) = \mathrm{Fix}(\Gamma)$.

Definition 3.3. A function $w \in AC_g^2([0,l],\mathbb{R})$ is called a solution of the inclusion problem (1.1) if there exists a function $\varrho \in L^1([0,l],\mathbb{R})$ with $\varrho(y) \in \Upsilon(y,w(y))$, such w satisfies conditions (1.2)–(1.3) and

$$^c\mathcal{D}^{\eta}_{0^+;g} w(y) = \varrho(y), \quad a.e\ y \in [0,l], 1 < \eta \le 2, \tag{3.2}$$

where $g \in C^2([0,l],\mathbb{R})$ with $g' > 0$ on $[0,l]$.

For each $w \in E$), we define the set of selections of F by

$$S_{\Upsilon,w} = \{\varrho \in L^1([0,l],\mathbb{R}), \varrho(y) \in \Upsilon(y,w(y)), \text{ a.e } y \in [0,l]\},$$

and the operator $K : E \to \mathcal{P}(E)$ associated with the problem (1.1)–(1.3) by

$$K(u) : \left\{ f \in E : \ f(y) = L_u(y) + \int_0^l G_g(y,\xi)\varrho(\xi)d\xi, \varrho \in S_{\Upsilon,u}\right\}, \tag{3.3}$$

where $G_g(y,\xi)$ is the Green function given by (2.5), and

$$L_u(y) = \frac{a(g(l)-g(y)+1)}{(g(l)-g(0)+2)\Gamma(\theta)} \times \int_0^p (g(p)-g(\xi))^{\theta-1} g'(\xi)\kappa(\xi,u(\xi))d\xi + \frac{b(g(y)-g(0)+1)}{(g(l)-g(0)+2)\Gamma(\mu)} \times \int_0^q (g(q)-g(\xi))^{\mu-1} g'(\xi)\omega(\xi,w(\xi))d\xi, \tag{3.4}$$

Throughout the paper, we set the following notations:

$$\Theta_1 = \frac{a(g(l)+1)(g(p))^{\theta}}{(g(l)-g(0)+2)\Gamma(\theta+1)}, \tag{3.5}$$

$$\Theta_2 = \frac{b(g(l)+1)(g(q))^{\mu}}{(g(l)-g(0)+2)\Gamma(\mu+1)}. \tag{3.6}$$

Theorem 3.4. *Let $\varphi \in \Psi$. Assume that the following hypotheses hold:*

(H1) $\Upsilon : [0,l] \times \mathbb{R} \to \mathcal{P}_{cp,bd}(\mathbb{R})$ *be a measurable multifunction map.*

(H2) *For $w, \bar{w} \in \mathbb{R}$, we have*

$$H_d(\Upsilon(y,w(y)), \Upsilon(y,\bar{w}(y))) \le \frac{1}{l\tilde{G}_\psi}(|w(y)-\bar{w}(y)| - \varphi(|w(y)-\bar{w}(y)|)).$$

(H3) *There exists $0 < \gamma \le 1$, such that*

$$|\kappa(y,w) - \kappa(y,\bar{w})| \le \gamma|w-\bar{w}|.$$

(H4) *There exists $0 < \varepsilon \le 1$, such that*

$$|\omega(y,w) - \omega(y,\bar{w})| \le \varepsilon|w-\bar{w}|.$$

If Υ verifies the approximation (3.1)*, then the inclusion problem* (1.1)–(1.3) *has a solution on* $[0,l]$*, provided that*

$$-1 < (\Theta_1\gamma + \Theta_2\varepsilon) \le 0, \tag{3.7}$$

where Θ_1 *and* Θ_2 *are given by* (3.5) *and* (3.6)*, respectively.*

Proof. We shall show that $K : E \to \mathcal{P}(E)$ given in (3.3) has an endpoint. The proof will be given in two steps as follows:

Step 1. K is a closed subset of $\mathcal{P}(E)$.

Let $w_n \in E$ such that $w_n \to w$, and $(f_n)_{n\ge1} \in K(w_n)$ be a sequence such that $f_n \to \tilde{f}$. Then there exists a $\varrho_n \in S_{\Upsilon,w_n}$ such that, for each $y \in [0,l]$, we get

$$f_n(y) = L_{w_n}(y) + \int_0^l G_g(y,\xi)\varrho_n(\xi)d\xi.$$

We have to show that there exists $\varrho \in S_{\Upsilon,w}$ such that for each $y \in [0,l]$

$$\tilde{f}(y) = L_w(y) + \int_0^l G_g(y,\xi)\varrho(\xi)d\xi.$$

Since Υ has compact values, the sequence $(\varrho_n)_{n\ge1}$ has a sub-sequence, denoted by ϱ_{n_m} which converges strongly to some $\varrho \in L^1([0,l],\mathbb{R})$. Indeed, for every $\varpi \in \Upsilon(y,u(y))$, we have

$$|\varrho_{n_m}(y) - \varrho(y)| \le |\varrho_{n_m}(y) - \varpi| + |\varpi - \varrho(y)|,$$

it follows that

$$\begin{aligned}|\varrho_{n_m}(y) - \varrho(y)| &\le H_d(\Upsilon(y,w_n),\Upsilon(y,w))\\ &\le \frac{1}{l\tilde{G}_g}(\|w_n - w\| - \varphi(\|w_n - w\|)),\end{aligned}$$

we have $\|w_n - w\| \to 0$, and hence $\varrho \in S_{\Upsilon,w}$. Consequently, for each $y \in [0,l]$

$$\begin{aligned}&|f_{n_m}(y) - \tilde{f}(y)|\\ &\quad\le \frac{a(g(l)+1)}{(g(l)-g(0)+2)\Gamma(\theta)}\int_0^p\\ &\qquad\times |(g(p)-g(\xi))^{\theta-1}g'(\xi)||\kappa(\xi,w_{n_m}(\xi))\\ &\qquad-\kappa(\xi,w(\xi))|d\xi + \frac{b(g(l)+1)}{(g(l)-g(0)+2)\Gamma(\mu)}\\ &\qquad\times\int_0^q |(g(q)-g(\xi))^{\mu-1}g'(\xi)||\omega(\xi,w_{n_m}(\xi))\\ &\qquad-\omega(\xi,w(\xi))|d\xi + \int_0^l |G_g(y,\xi)|\\ &\qquad\times|\varrho_{n_m}(\xi) - \varrho(\xi)|d\xi\\ &\quad\le \frac{a(g(l)+1)(g(p))^\theta}{(g(l)-g(0)+2)\Gamma(\theta+1)}\gamma\|w_{n_m} - w\|\\ &\qquad+\frac{b(g(l)+1)(g(q))^\mu}{(g(l)-g(0)+2)\Gamma(\mu+1)}\varepsilon\|w_{n_m} - w\|\\ &\qquad+l\tilde{G}_g\frac{1}{l\tilde{G}_g}(\|w_{n_m} - w\| - \varphi(\|w_{n_m} - w\|))\\ &\quad\le(\Theta_1\gamma+\Theta_2\varepsilon)\|w_{n_m} - w\| + \|w_{n_m} - w\|\\ &\qquad-\varphi(\|w_{n_m} - w\|).\end{aligned}$$

Since $\|w_{n_m} - w\| \to 0$ then $\varphi(\|w_{n_m} - w\|) \to 0$, κ, and ω are continuous, so $\|f_{n_m} - \tilde{f}\| \to 0$ whenever $m \to 0$. Therefore $\tilde{f} \in K(w)$ and K is closed multifunction.

Step 2. K is a φ-weak contractive multifunction, i.e. for $w, \bar{w} \in E$, we show

$$H_d(K(w),K(\bar{w})) \le \|w - \bar{w}\| - \varphi(\|w - \bar{w}\|).$$

Let $w, \bar{w} \in E$ and $f_1 \in K(w)$. Then, there exists $\varrho_1(y) \in S_{\Upsilon,w}$ such that, for each $y \in [0,l]$

$$f_1(y) = L_w(y) + \int_0^l G_g(y,\xi)\varrho_1(\xi)d\xi.$$

From (H2), it follows that

$$\begin{aligned}H_d(\Upsilon(y,w),\Upsilon(y,\bar{w})) \le \frac{1}{l\tilde{G}_\psi}(&|w(y) - \bar{w}(y)|\\ &-\varphi(|w(y) - \bar{w}(y)|)).\end{aligned}$$

Thus, there exists $z \in \Upsilon(y,\bar{w}(y))$ provided that

$$\begin{aligned}|\varrho_1(y) - z| \le \frac{1}{l\tilde{G}_g}(&|w(y) - \bar{w}(y)| - \varphi\\ &\times(|w(y) - \bar{w}(y)|)), \quad y \in [0,l].\end{aligned}$$

Define $U : [0,l] \to \mathcal{P}(\mathbb{R})$ given by

$$U(y) = \left\{z \in \mathbb{R} : |\varrho_1(y) - z| \le \frac{1}{l\tilde{G}_g}(|w(y) - \bar{w}(y)| - \varphi(|w(y) - \bar{w}(y)|))\right\}.$$

Since $U(y) \cap \Upsilon(y,\bar{w})$ is measurable, then we can find a measurable selection $\varrho_2(y)$ for $U(y)\cap\Upsilon(y,\bar{w})$. Thus $\varrho_2(y) \in \Upsilon(y,\bar{w}(y))$, and for each $y \in [0,l]$, we have

$$|\varrho_1(y)-\varrho_2(y)| \leq \frac{1}{l\tilde{G}_g}(|w(y)-\bar{w}(y)| - \varphi(|w(y)-\bar{w}(y)|)).$$

We define $f_2(y)$ for each $y \in [0,l]$, as follows:

$$f_2(y) = L_{\bar{w}}(y) + \int_0^l G_g(y,\xi)\varrho_2(\xi)d\xi.$$

Then for $y \in [0,l]$

$$\begin{aligned}
&|f_1(y)-f_2(y)| \\
&\quad \leq \frac{a(g(l)+1)}{(g(l)-g(0)+2)\Gamma(\theta)} \\
&\qquad \times \int_0^p |(g(p)-g(\xi))^{\theta-1}g'(\xi)|| \\
&\qquad \times \kappa(\xi,w(\xi)) - \kappa(\xi,\bar{w}(\xi))|d\xi \\
&\qquad + \frac{b(g(l)+1)}{(g(l)-g(0)+2)\Gamma(\mu)} \\
&\qquad \times \int_0^q |(g(q)-g(\xi))^{\mu-1}g'(\xi)|| \\
&\qquad \times \omega(\xi,w(\xi)) - \omega(\xi,\bar{w}(\xi))|d\xi \\
&\qquad + \int_0^l |G_g(y,\xi)||\varrho_1(\xi)-\varrho_2(\xi)|d\xi \\
&\quad \leq \frac{a(g(l)+1)(g(p))^{\theta}}{(g(l)-g(0)+2)\Gamma(\theta+1)}\gamma|w-\bar{w}| \\
&\qquad + \frac{b(g(l)+1)(g(q))^{\mu}}{(g(l)-g(0)+2)\Gamma(\mu+1)} \\
&\qquad \times \varepsilon|w-\bar{w}| + l\tilde{G}_g \\
&\quad \leq (\Theta_1\gamma + \Theta_2\varepsilon)|w(y)-\bar{w}(y)| + l\tilde{G}_g\frac{1}{l\tilde{G}_g} \\
&\qquad \times (|w(y)-\bar{w}(y)| - \varphi(|w(y)-\bar{w}(y)|)). \\
&\quad \leq \|w(y)-\bar{w}(y)\| - \varphi(\|w(y)-\bar{w}(y)\|).
\end{aligned}$$

Therefore,

$$\|f_1 - f_2\| \leq \|w-\bar{w}\| - \varphi(\|w-\bar{w}\|).$$

It follows that $H_d(K(w),K(\bar{w})) \leq \|w-\bar{w}\| - \varphi(\|w-\bar{w}\|)$, for all $w,\bar{w} \in E$. By hypothesis, since the operator Υ has an approximate endpoint property then by Lemma 3.2 K has an endpoint $w^* \in E$, i.e. $Kw^* = \{w^*\}$, which is also a fixed point, which is a solution to the inclusion problem (1.1)–(1.3). The proof is now complete. □

Special cases. This work covers two new existence results follow as special cases. By taking $g(y) = y$, $\theta = \mu = 1$, in (1.1)–(1.3), we obtain the results for the classical Caputo fractional inclusions with classical nonlocal integral boundary conditions of the form:

$$w(0) - w'(0) = a\int_0^p \kappa(\xi,w(\xi))d\xi,$$

$$w(l) + w'(l) = b\int_0^q \omega(\xi,w(\xi))d\xi,$$

by setting $q \to l^-$, $p \to l^-$, and $a = b = 1$, the resulting inclusion problem considered in Ref. 1, where the existence criteria for the solution were discussed for convex and nonconvex multivalued maps by applying the standard fixed-point theorems for multivalued maps. While the results for fractional inclusions equipped with generalized Caputo fractional derivative[29] and nonlocal Katugampola fractional integrals[30] as boundary conditions of the form

$$w(0) - \delta_\rho w(0) = a^\rho I^{\theta}_{0+}\kappa(p,w(p)),$$

$$w(l) + \delta_\rho w(l) = b^\rho I^{\mu}_{0+}\omega(q,w(q)),$$

follow by setting $g(y) = y^\rho/\rho$. While the corresponding boundary value problem to (1.1)–(1.3) with $a = b = 1$ for fractional differential equations has been studied recently in Ref. 4

4. EXAMPLE

As an application of the main results, we consider the following fractional differential inclusion:

$$^{c}\mathcal{D}^{\eta}_{0+;g}w(y) \in \Upsilon(y,w(y)), \quad y \in [0,e], \tag{4.1}$$

$$w(0) - 2w'(0) = -\frac{1}{2}I^{\frac{1}{4}}_{0+,g}\kappa\left(\frac{2}{7}, w\left(\frac{2}{7}\right)\right), \tag{4.2}$$

$$w(e) + 2w'(e) = -\frac{2}{7}I^{\frac{1}{7}}_{0+,g}\omega\left(\frac{1}{2}, w\left(\frac{1}{2}\right)\right), \tag{4.3}$$

where $1 < \eta \leq 2$, $p = \frac{2}{7}$, $q = \frac{1}{2}$, $a = -\frac{1}{2}$, $b = -\frac{3}{7}$, and $l = e$. Here, $\Upsilon : [0,e] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$\Upsilon(y,w(y)) = \left[0, \frac{|w|}{2(1+|w|)}\right]. \tag{4.4}$$

Set

$$\kappa(y,w) = \sin(y)\tan^{-1}(|w|), \quad (y,w) \in [0,e] \times \mathbb{R},$$

$$\omega(y,w) = \frac{e^{-y}|w|}{(|w|+1)(6+e^y)}, \quad (y,w) \in [0,e] \times \mathbb{R}.$$

Taking $g(y) = 2y$, which is differentiable and increasing function on $[0,e]$ with $g'(y) = 2$ on $[0,e]$. From the expression (2.5), the function G_g with respect to g is given by

$$G_g(y,\xi) = \begin{cases} \frac{2^\eta (y-\xi)^{\eta-1}}{\Gamma(\eta)} - \frac{2^{\eta-1}(2y+1)}{(l+1)\Gamma(\eta)}(l-\xi)^{\eta-1} \\ \quad - \frac{2^{\eta-2}(2y+1)}{(l+1)\Gamma(\eta-1)}(l-\xi)^{\eta-2}, \quad 0 \le \xi \le y, \\ - \frac{2^{\eta-1}(2y+1)}{(l+1)\Gamma(\eta)}(l-\xi)^{\eta-1} \\ \quad - \frac{2^{\eta-1}(2y+1)}{(l+1)\Gamma(\eta-1)}(l-\xi)^{\eta-2}, \quad y \le \xi \le e. \end{cases} \tag{4.5}$$

From 4.5, we get

$$\begin{aligned} \int_0^e G_g(y,\xi)d\xi &= \int_0^y G_g(y,\xi)d\xi + \int_y^e G_g(y,\xi)d\xi \\ &= -\frac{2^\eta y}{\Gamma(\eta+1)} - \frac{2^{\eta-1}(2y+1)}{(e+1)\Gamma(\eta+1)}(e-y)^\eta \\ &\quad + \frac{2^{\eta-1}e^\eta(2y+1)}{(e+1)\Gamma(\eta+1)} - \frac{2^{\eta-1}(2y+1)}{(e+1)\Gamma(\eta)}(e-y)^{q-1} \\ &\quad + \frac{2^{q-1}e^{\eta-1}(2y+1)}{(e+1)\Gamma(\eta)} + \frac{2^{\eta-1}(2y+1)}{(e+1)\Gamma(\eta+1)}(e-y)^\eta \\ &\quad + \frac{2^{\eta-1}(2y+1)}{(e+1)\Gamma(\eta)}(e-y)^{\eta-1}, \end{aligned}$$

which leads to

$$\tilde{G}_g < \frac{2^\eta e^{\eta+1} + 2^{\eta-1}e^\eta}{\Gamma(\eta+1)} + \frac{(2e)^\eta + (2e)^{(\eta-1)}}{\Gamma(\eta)}.$$

Now, choosing $\varphi(u) = \frac{u}{2}$. Clearly, the function $\varphi \in \Psi$

$$\begin{aligned} H_d(\Upsilon(y,w),\Upsilon(y,\bar{w})) &\le \frac{1}{2}\left|\frac{w-\bar{w}}{(1+w)(1+\bar{w})}\right| \\ &\le \frac{1}{2}|w-\bar{w}| \\ &< \left(\frac{2^\eta e^{\eta+1} + 2^{\eta-1}e^\eta}{\Gamma(\eta+1)} + \frac{(2e)^\eta + (2e)^{(\eta-1)}}{\Gamma(\eta)}\right)^{-1} \\ &\quad \times (\|w-\bar{w}\| - \varphi(\|w-\bar{w}\|)). \end{aligned}$$

Hence, the condition (H2) holds for $w, \bar{w} \in \mathbb{R}$. We have, on the other hand,

$$\begin{aligned} |\kappa(y,w) - \kappa(y,\bar{w})| &\le |\sin(y)(w-\bar{w})| \\ &\le \sin(e)|w-\bar{w}| \\ &\le \gamma|w-\bar{w}|, \end{aligned}$$

and

$$\begin{aligned} |\omega(y,w) - \omega(y,\bar{w})| &= \frac{e^{-y}}{(6+e^y)}\left|\frac{w}{w+1} - \frac{\bar{w}}{\bar{w}+1}\right| \\ &= \frac{e^{-y}}{(6+e^y)}\frac{|w-\bar{w}|}{(w+1)(1+\bar{w})} \\ &\le \frac{e^{-e}}{(6+e^e)}|w-\bar{w}| \\ &\le \varepsilon|w-\bar{w}|. \end{aligned}$$

Therefore, conditions (H3) and (H4) hold.

$$\begin{aligned} \Theta_1\gamma + \Theta_2\varepsilon &= -\frac{(1+2e)\sin(e)}{\sqrt{2}\sqrt[4]{7}(2+2e)\Gamma(5/4)} \\ &\quad - \frac{2e^e(1+2e)}{7(2+2e)(6+e^e)\Gamma(8/7)} \\ &\approx -0,3593 < 0, \end{aligned}$$

thus condition (3.7) hold.

We define an operator $K : C([0,e],\mathbb{R}) \to \mathcal{P}(C([0,e],\mathbb{R}))$

$$\begin{aligned} K(u) = \{\chi \in C([0,e],\mathbb{R}) : \chi(y) = u(y), \\ \text{for all } y \in [0,e], \varrho \in S_{\Upsilon,u}\}, \end{aligned}$$

where

$$u(y) = L_u(y) + \int_0^l G_g(y,\xi)\varrho(\xi)d\xi.$$

Note that 0 is a unique endpoint of K, i.e. $K(0) = \{0\}$, which implies that $\sup_{u\in K(0)}\|u\| = 0$, hence $\inf_{u\in E}\sup_{\chi\in K(u)}\|u - \chi\| = 0$. Accordingly, the multifunction K verifies the approximate endpoint property. Since all the hypotheses of theorem (3.4) are fulfilled, then the inclusion problem (4.1)–(4.3) with Υ given by (4.4) has at least one solution on $[0,e]$.

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REFERENCES

1. S. Hamani, M. Benchohra and John R. Graef, Existence results for boundary value problems with nonlinear fractional inclusions and integral conditions, *Electron. J. Diff. Equ.* **2010**(20) (2010) 1–16
2. R. P. Agarwal, M. Benchohra and S. Hamani, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, *Acta Appl. Math.* **109**(3) (2010) 973–1033.
3. M. Benchohra, J. R. Graef and S. Hamani, Existence results for boundary value problems with nonlinear fractional differential equations, *Appl. Anal.* **87** (2008) 851–863.
4. S. Belmor, C. Ravichandran and F. Jarad, Nonlinear generalized fractional differential equations with generalized fractional integral conditions, *J. Taibah Univ. Sci.* **14**(1) (2020) 114–123, doi:10.1080/16583655.2019.1709265.
5. T. M. Atanacković, S. Pilipović, B. Stanković and D. Zorica, *F*ractional Calculus with Applications in Mechanics (Wiley-ISTE, 2014).
6. H. Rudolf, *A*pplications of Fractional Calculus in Physics (World Scientific, 2000).
7. R. Almeida, A Caputo fractional derivative of a function with respect to another function, *Commun. Nonlinear Sci. Numer. Simul.* **44** (2017) 460–481.
8. F. Jarad and T. Abdeljawad, Generalized fractional derivatives and Laplace transform, *Discrete Contin. Dyn. Syst.* **13**(3) (2020) 709–722.
9. T. Abdeljawad, F. Madjidi, F. Jarad and N. Sene, On dynamic systems in the frame of singular function dependent kernel fractional derivatives, *Mathematics* **7**(10) (2019) 946.
10. R. Almeida, Fractional differential equations with mixed boundary conditions, *B. Malays. Math. Sci. Soc.* **42**(4) (2019) 1687–1697.
11. R. Ameen, F. Jarad and T. Abdeljawad, Ulam stability for delay fractional differential equations with a generalized Caputo derivative, *Filomat* **32**(15) (2018) 5265–5274.
12. F. Jarad, S. Harikrishnan, K. Shah and K. Kanagarajan, Existence and stability results to a class of fractional random implicit differential equations involving a generalized Hilfer fractional derivative, *Discrete Contin. Dyn. Syst.* **13**(3) (2020) 723–739, doi:10.3934/dcdss.2020040.
13. B. Samet and H. Aydi, Lyapunov-type inequalities for an anti-periodic fractional boundary value problem involving ψ-Caputo fractional derivative, *J. Inequal. Appl.* **2018**(286) (2018) 9 pp., https://doi.org/10.1186/s13660-018-1850-4.
14. J. Sousa, C. Vanterler da, K. D. Kucche and E. C. De Oliveira, Stability of ψ-Hilfer impulsive fractional differential equations, *Appl. Math. Lett.* **88** (2019) 73–80.
15. F. Nicoud and T. Schönfeld, Integral boundary conditions for unsteady biomedical CFD applications, *Int. J. Numer. Methods Fluids* **40** (2002) 457–465.
16. S. H. Schot, Eighty years of Sommerfeld's radiation condition, *Historia Math.* **19**(4) (1992) 385–401.
17. M. Benaïm, J. Hofbauer and S. Sorin, Stochastic approximations and differential inclusions, *SIAM J. Control Optim.* **44**(1) (2005) 328–348.
18. S. Abbas, M. Benchohra and M. A. Darwish, Fractional differential inclusions of Hilfer type under weak topologies in Banach spaces, *Asian-European J. Math.* **13**(1) (2018) 2050015.
19. S. Abbas, M. Benchohra, S. Hamani and J. Henderson, Upper and lower solutions method for Caputo–Hadamard fractional differential inclusions, *Math. Morav.* **23**(1) (2019) 107–118.
20. B.Ahmad, A. Alsaedi, S. K. Ntouyas and H. H. Al-Sulami, On neutral functional differential inclusions involving hadamard fractional derivatives, *Mathematics* **7**(11) (2019) 1084.
21. B. Ahmad, S. K. Ntouyas and J. Tariboon, A study of mixed Hadamard and Riemann-Liouville fractional integro-differential inclusions via endpoint theory, *Appl. Math. Lett.* **52** (2016) 9–14.
22. B. Ahmad, S. K. Ntouyas and H. H. Alsulami, Existence results for n-th order multipoint integral boundary-value problems of differential inclusions, *Electron. J. Differ. Equ.* **2013**(203) (2013) 1–13.
23. B. Ahmad, A. Alsaedi, S. K. Ntouyas and J. Tariboon, *Hadamard-Type Fractional Differential Equations*, *Inclusions and Inequalities* (Springer, Cham, 2017).
24. R. P. Agarwal, M. Benchohra and S. Hamani, Boundary value problems for fractional differential equations, *Adv. Stud. Contemp. Math.* **16**(2) (2008) 181–196.
25. S. Moradi and F. Khojasteh, Endpoints of φ-weak and generalized φ-weak contractive mappings, *Filomat* **26** (2012) 725–732.
26. S. Hu, N. Papageorgiou, *Handbook of Multivalued Analysis*, Vol. 1 (Theory, Kluwer, Dordrecht, 1997).
27. S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives*, Theory and Applications (Gordon and Breach, Yverdon, 1993).
28. M. Kisielewicz, *Differential Inclusions and Optimal Control* (Kluwer, Dordrecht, The Netherlands, 1991).
29. F. Jarad, T. Abdeljawad and D. Baleanu, On the generalized fractional derivatives and their Caputo modification, *J. Nonlinear Sci. Appl.* **10**(5) (2017) 2607–2619.
30. U. N. Katugampola, New approach to a genaralized fractional integral, *Appl. Math. Comput.* **218**(3) (2011) 860–865.