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On new general versions of Hermite–Hadamard type integral inequalities via fractional integral operators with Mittag-Leffler kernel

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Abstract

The main motivation of this study is to bring together the field of inequalities with fractional integral operators, which are the focus of attention among fractional integral operators with their features and frequency of use. For this purpose, after introducing some basic concepts, a new variant of Hermite–Hadamard (HH-) inequality is obtained for *s*-convex functions in the second sense. Then, an integral equation, which is important for the main findings, is proved. With the help of this integral equation that includes fractional integral operators with Mittag-Leffler kernel, many HH-type integral inequalities are derived for the functions whose absolute values of the second derivatives are *s*-convex and *s*-concave. Some classical inequalities and hypothesis conditions, such as Hölder's inequality and Young's inequality, are taken into account in the proof of the findings.

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1 Introduction

Mathematics has basically started its adventure as a theoretical field with the efforts of researchers for centuries, and has continuously aimed to formulate events and phenomena in various fields such as physics, engineering, modeling, and mathematical biology into a form that can be calculated. Not content with this, it has always been looking for more effective and original solutions to problems. Fractional analysis is also one of the important tools that serve mathematics to find solutions to real world problems. In fact, recent studies have shown that fractional analysis serves this purpose more than classical analysis. The basic working principle of fractional analysis is to introduce new fractional derivatives and integral operators and to analyze the advantages of these operators with the help of real world problem solutions, modeling studies, and comparisons. New fractional derivatives and related integral operators are a quest to gain momentum to fractional analysis and to gain the most effective operators to the literature. This search is a dynamic process, and different features of kernel structures, time memory effect, and the desire to reach general forms are factors that differentiate fractional operators in this dynamic process. We will now take a look at some of the basic concepts of fractional analysis and build the basis for our work.

Definition 1 (see [1]) Let $\vartheta \in L[\varphi_1, \varphi_2]$. The Riemann–Liouville integrals $J^{\zeta}_{\varphi_1+}\vartheta$ and $J^{\zeta}_{\varphi_2-}\vartheta$ of order $\zeta > 0$ with $\varphi_1, \varphi_2 \geq 0$ are defined by

$$\left(J^{\zeta}_{\varphi_1+}\right)\vartheta(y) = \frac{1}{\Gamma(\zeta)} \int_{\varphi_1}^{y} (y-\sigma)^{\zeta-1}\vartheta(\sigma)\,d\sigma; \quad y > \varphi_1,$$

and

$$\left(J^{\zeta}_{\varphi_2-}\right)\vartheta(y) = \frac{1}{\Gamma(\zeta)} \int_{y}^{\varphi_2} (\sigma-y)^{\zeta-1}\vartheta(\sigma)\,d\sigma; \quad y < \varphi_2,$$

respectively, where $\Gamma(\cdot)$ is the gamma function and $(J^{0}_{\varphi_1^+})\vartheta(y) = (J^{0}_{\varphi_2-})\vartheta(y) = \vartheta(y)$.

The Riemann–Liouville fractional integral operator is a very useful operator and has been applied to many problems by researchers in both mathematical analysis and applied mathematics (see [2–4]). For many years, Caputo derivative and Riemann–Liouville integrals have been the best known operators in fractional analysis. Recently, the development of new fractional operators has accelerated and comparisons have been made by taking these operators as reference. We will now proceed with the definition of a new fractional integral operator that contains the kernel of the Riemann–Liouville integral operator.

Definition 2 (see [5]) The fractional integral related to the new fractional derivative with nonlocal kernel of a mapping $\vartheta \in H^1(\varphi_1, \varphi_2)$ is defined as follows:

$${}^{AB}_{\varphi_1}I^{\zeta}_{t}\left\{\vartheta(t)\right\} = \frac{1-\zeta}{B(\zeta)}\vartheta(t) + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_{\varphi_1}^{t} \vartheta(\sigma)(t-\sigma)^{\zeta-1}\,d\sigma,$$

where $\varphi_2 > \varphi_1$, $\zeta \in [0, 1]$.

In [6], the authors gave the right-hand side of integral operator as follows:

$$\left({}^{AB}I^{\zeta}_{\varphi_2}\right)\left\{\vartheta(t)\right\} = \frac{1-\zeta}{B(\zeta)}\vartheta(t) + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_{t}^{\varphi_2} \vartheta(\sigma)(\sigma-t)^{\zeta-1}\,d\sigma.$$

Here, $\Gamma(\zeta)$ is the gamma function. Due to $B(\zeta) > 0$ that is called the normalization function, this yields that the fractional Atangana–Baleanu integral of a positive function is positive. It should be noted that, when the order $\zeta \longrightarrow 1$, we recapture the standard integral. Also, the original function is recovered whenever the fractional order $\zeta \longrightarrow 0$.

This interesting integral operator owes its strong kernel to its associated fractional derivative operator. The Atangana–Baleanu fractional derivative operator is a nonsingular and nonlocal fractional integral operator with its kernel structure containing the Mittag-Leffler function. This rare operator is described in the Caputo sense and the Riemann–Liouville sense as follows.

Definition 3 (see [5]) Let $\vartheta \in H^1(\varphi_1,\varphi_2)$, $\varphi_2 > \varphi_1$, $\zeta \in [0,1]$. Then the definition of the new fractional derivative is given as follows:

$$ {}_{\varphi_1}^{ABC}D_t^{\zeta}\left[\vartheta(t)\right] = \frac{B(\zeta)}{1-\zeta}\int_{\varphi_1}^{t}\vartheta'(\sigma)E_{\zeta}\left[-\zeta\frac{(t-\sigma)^{\zeta}}{(1-\zeta)}\right]d\sigma. \tag{1.1} $$

Definition 4 (see [5]) Let $\vartheta \in H^1(\varphi_1,\varphi_2)$, $\varphi_2 > \varphi_1$, $\zeta \in [0,1]$. Then the definition of the new fractional derivative is given as follows:

$$ {}_{\varphi_1}^{ABR}D_t^{\zeta}\left[\zeta(t)\right] = \frac{B(\zeta)}{1-\zeta}\frac{d}{dt}\int_{\varphi_1}^{t}\vartheta(\sigma)E_{\zeta}\left[-\zeta\frac{(t-\sigma)^{\zeta}}{(1-\zeta)}\right]d\sigma. \tag{1.2} $$

To obtain more information related to structures and further properties of fractional operators, the interested readers can consider the following papers [3, 6–19].

After giving some basic information and concepts about fractional analysis, which is one of the basic foundations of the study, we will continue by reminding some basic concepts on convex functions and inequalities. Analytical and geometric inequalities are a topic that researchers focus on in mathematics both theoretically and practically. Especially in the last centuries, with the effect of convex analysis on theory, new inequalities and its applications have expanded the field. The contribution of different types of convex functions to the literature is supported by the inequalities proved based on them. The concept of convexity, which has a special position among functions with the aesthetics of its algebraic structure, its geometrical properties and the richness of its application areas, encounters the interest of researchers in many disciplines such as physics, engineering, economics, and approximation theory, as well as in mathematics. With the effect of this interest, many new types of convex functions have been introduced, and the concept of convexity has been carried to different spaces with multidimensional versions. The diverging and convergent aspects of each new convex function type have been identified, and enrichment has been added to the field of convex analysis.

Now let us refresh our memory by talking about the convex function, the s-convex function in the second sense, and the HH-inequality.

Definition 5 (see [20]) The function $\vartheta : [\varphi_1,\varphi_2] \subseteq \mathbb{R} \to \mathbb{R}$ is called a convex function if the inequality

$$ \vartheta\left(\sigma x + (1-\sigma)y\right) \le \sigma\vartheta(x) + (1-\sigma)\vartheta(y) \tag{1.3} $$

is satisfied for all $x,y \in [\varphi_1,\varphi_2]$ and $\sigma \in [0,1]$.

In [21], Orlicz has given the definition of s-convexity as follows.

Definition 6 A function $\vartheta : \mathbb{R}^+ \to \mathbb{R}$, where $\mathbb{R}^+ = [0,\infty)$, is called s-convex in the first sense if

$$ \vartheta(\kappa_1\varphi_1 + \kappa_2\varphi_2) \le \kappa_1^s\vartheta(\varphi_1) + \kappa_2^s\vartheta(\varphi_2) $$

for all $\varphi_1,\varphi_2 \in [0,\infty)$, $\kappa_1,\kappa_2 \ge 0$ with $\kappa_1^s + \kappa_2^s = 1$ and for some fixed $s \in (0,1]$.

Definition 7 A function $\vartheta : \mathbb{R}^+ \to \mathbb{R}$, where $\mathbb{R}^+ = [0,\infty)$, is said to be s-convex in the second sense if

$$\vartheta(\kappa_1\varphi_1 + \kappa_2\varphi_2) \leq \kappa_1^s\vartheta(\varphi_1) + \kappa_2^s\vartheta(\varphi_2)$$

for all $\varphi_1, \varphi_2 \in [0,\infty)$, $\kappa_1, \kappa_2 \geq 0$ with $\kappa_1 + \kappa_2 = 1$ and for some fixed $s \in (0,1]$.

Obviously, one can see that in case of $s = 1$, both definitions overlap with the standard concept of convexity.

The famous HH-inequality, which is built on convex functions with its different modifications, generalizations, and iterations, generates lower and upper limits for the mean value in the Cauchy sense and is given as follows.

Assume that $\vartheta : I \subset \mathbb{R} \to \mathbb{R}$ is a convex mapping on $I \subseteq \mathbb{R}$, where $\varphi_1, \varphi_2 \in I$, with $\varphi_1 < \varphi_2$. The HH-inequality for convex mappings can be presented as follows (see [20]):

$$\vartheta\left(\frac{\varphi_1 + \varphi_2}{2}\right) \leq \frac{1}{\varphi_2 - \varphi_1}\int_{\varphi_1}^{\varphi_2} \vartheta(\sigma)\,d\sigma \leq \frac{\vartheta(\varphi_1) + \vartheta(\varphi_2)}{2}. \tag{1.4}$$

In [22], a new variant of HH-inequality for s-convex mappings in the second sense has been performed by Dragomir and Fitzpatrick.

Theorem 1 *Assume that* $\vartheta : [0,\infty) \to [0,\infty)$ *is an* s*-convex function in the second sense, where* $s \in (0,1)$, *and let* $\varphi_1, \varphi_2 \in [0,\infty)$, $\varphi_1 < \varphi_2$. *If* $\vartheta \in L[\varphi_1, \varphi_2]$, *then one has the following*:

$$2^{s-1}\vartheta\left(\frac{\varphi_1 + \varphi_2}{2}\right) \leq \frac{1}{\varphi_2 - \varphi_1}\int_{\varphi_1}^{\varphi_2} \vartheta(\sigma)\,d\sigma \leq \frac{\vartheta(\varphi_1) + \vartheta(\varphi_2)}{s+1}. \tag{1.5}$$

Here, we must note that $k = \frac{1}{s+1}$ *is the best possible constant in* (1.5).

To provide more details related to different kinds of convex functions and generalizations, new variants and different forms of this important double inequality, we suggest to read the papers [20–42].

This study is organized as follows. First of all, the basic concepts to be used in the study were defined, and the scientific infrastructure required for the proof of the findings was created. In the main findings section, a new generalization of the HH-inequality, which includes Atangana–Baleanu integral operators for s-convex functions in the second sense, is obtained. Then, by giving an integral identity for differentiable s-convex functions in the second sense, new HH-type inequalities are proved for functions whose absolute value is s-convex in the second sense with the help of this identity. Also, a similar inequality is obtained for s-concave functions.

2 New results by Atangana–Baleanu fractional integral operators

We start this section by giving the following inequalities containing the versions of the HH-inequality for s-convex mappings in the second sense via new fractional integral operators defined by Atangana and Baleanu.

Throughout the study, we denote the terms $\Gamma(\zeta)$, $B(\zeta) > 0$, and β_x as gamma function, normalization function, and incomplete beta function, respectively.

Theorem 2 *Let $\vartheta : \mathbb{R}_+ \to \mathbb{R}_+$ be an s-convex function in the second sense, $s \in (0,1]$, and $\varphi_1, \varphi_2 \in \mathbb{R}_+$ with $\varphi_1 < \varphi_2$. If $\vartheta \in L[\varphi_1, \varphi_2]$, the inequalities for Atangana–Baleanu integral operators for all $\zeta \in (0,1]$ are obtained as follows*:

$$\begin{aligned} &2^s \frac{\vartheta(\frac{\varphi_1+\varphi_2}{2})}{B(\zeta)\Gamma(\zeta)} + \frac{1-\zeta}{(\varphi_2-\varphi_1)^\zeta}\left[\frac{\vartheta(\varphi_1)+\vartheta(\varphi_2)}{B(\zeta)}\right] \\ &\quad \le \frac{1}{(\varphi_2-\varphi_1)^\zeta}\left[{}^{AB}_{\varphi_1}I^{\zeta}_{\varphi_2}\{\vartheta(\varphi_2)\} + {}^{AB}I^{\zeta}_{\varphi_2}\{\vartheta(\varphi_1)\}\right] \\ &\quad \le \left[\frac{\vartheta(\varphi_1)+\vartheta(\varphi_2)}{B(\zeta)}\right]\left[\frac{\zeta}{\Gamma(\zeta)(\zeta+s)} + \frac{1-\zeta}{(\varphi_2-\varphi_1)^\zeta} + \frac{\zeta\beta(\zeta,s+1)}{\Gamma(\zeta)}\right]. \end{aligned} \tag{2.1}$$

Proof As ϑ is an s-convex function in the second sense, we can write

$$\vartheta\big(\sigma\varphi_1 + (1-\sigma)\varphi_2\big) \le \sigma^s\vartheta(\varphi_1) + (1-\sigma)^s\vartheta(\varphi_2)$$

for all $\sigma \in [0,1]$. Multiplying the above inequality with $\sigma^{\zeta-1}$ and then integrating the obtained inequality on $[0,1]$, we have

$$\begin{aligned} &\int_0^1 \sigma^{\zeta-1}\vartheta\big(\sigma\varphi_1 + (1-\sigma)\varphi_2\big)\,d\sigma \\ &\quad \le \left[\vartheta(\varphi_1)\int_0^1 \sigma^{\zeta+s-1}\,d\sigma + \vartheta(\varphi_2)\int_0^1 \sigma^{\zeta-1}(1-\sigma)^s\,d\sigma\right]. \end{aligned}$$

If we multiply both sides of the last inequality by $\frac{\zeta(\varphi_2-\varphi_1)^\zeta}{B(\zeta)\Gamma(\zeta)}$, and then if we add the term $\frac{1-\zeta}{B(\zeta)}\vartheta(\varphi_2)$, we get

$$\begin{aligned} &\frac{\zeta(\varphi_2-\varphi_1)^\zeta}{B(\zeta)\Gamma(\zeta)}\int_0^1 \sigma^{\zeta-1}\vartheta\big(\sigma\varphi_1 + (1-\sigma)\varphi_2\big)\,d\sigma + \frac{1-\zeta}{B(\zeta)}\vartheta(\varphi_2) \\ &\quad \le \frac{\zeta(\varphi_2-\varphi_1)^\zeta}{B(\zeta)\Gamma(\zeta)}\left[\vartheta(\varphi_1)\int_0^1 \sigma^{\zeta+s-1}\,d\sigma + \vartheta(\varphi_2)\int_0^1 \sigma^{\zeta-1}(1-\sigma)^s\,d\sigma\right] + \frac{1-\zeta}{B(\zeta)}\vartheta(\varphi_2). \end{aligned}$$

By making use of the change of variable $\sigma\varphi_1 + (1-\sigma)\varphi_2 = y$, we have

$$\begin{aligned} &{}^{AB}_{\varphi_1}I^{\zeta}_{\varphi_2}\{\vartheta(\varphi_2)\} \\ &\quad \le \vartheta(\varphi_2)\left[\frac{1-\zeta}{B(\zeta)} + \frac{\zeta(\varphi_2-\varphi_1)^\zeta\beta(\zeta,s+1)}{B(\zeta)\Gamma(\zeta)}\right] + \frac{\zeta(\varphi_2-\varphi_1)^\zeta}{B(\zeta)\Gamma(\zeta)(\zeta+s)}\vartheta(\varphi_1). \end{aligned} \tag{2.2}$$

And similarly we get

$$\begin{aligned} &\big({}^{AB}I^{\zeta}_{\varphi_2}\big)\{\vartheta(\varphi_1)\} \\ &\quad \le \vartheta(\varphi_1)\left[\frac{1-\zeta}{B(\zeta)} + \frac{\zeta(\varphi_2-\varphi_1)^\zeta\beta(\zeta,s+1)}{B(\zeta)\Gamma(\zeta)}\right] + \frac{\zeta(\varphi_2-\varphi_1)^\zeta}{B(\zeta)\Gamma(\zeta)(\zeta+s)}\vartheta(\varphi_2). \end{aligned} \tag{2.3}$$

If we consider the inequalities in (2.2) and (2.3), we conclude the second inequality in (2.1).

For obtaining the first inequality in (2.1), we use that, for all $u, v \in \mathbb{R}_+$, we have

$$\vartheta\left(\frac{u+v}{2}\right) \le \frac{\vartheta(u)+\vartheta(v)}{2^s}. \tag{2.4}$$

Now, let $u = \sigma\varphi_1 + (1-\sigma)\varphi_2$ and $v = (1-\sigma)\varphi_1 + \sigma\varphi_2$ with $\sigma \in [0,1]$. Then we get by (2.4) that

$$\vartheta\left(\frac{\varphi_1+\varphi_2}{2}\right) \le \frac{\vartheta(\sigma\varphi_1+(1-\sigma)\varphi_2)+\vartheta((1-\sigma)\varphi_1+\sigma\varphi_2)}{2^s}.$$

Multiplying the above inequality with $\sigma^{\zeta-1}$ and then integrating this inequality on $[0,1]$, we have

$$\begin{aligned}&2^s\frac{\vartheta(\frac{\varphi_1+\varphi_2}{2})}{\zeta}\\&\quad\le\int_0^1\sigma^{\zeta-1}\vartheta\bigl(\sigma\varphi_1+(1-\sigma)\varphi_2\bigr)\,d\sigma+\int_0^1\sigma^{\zeta-1}\vartheta\bigl((1-\sigma)\varphi_1+\sigma\varphi_2\bigr)\,d\sigma.\end{aligned}$$

If we multiply both sides of the last inequality $\frac{\zeta(\varphi_2-\varphi_1)^\zeta}{B(\zeta)\Gamma(\zeta)}$ and then if we add the term $\frac{1-\zeta}{B(\zeta)}[\vartheta(\varphi_1)+\vartheta(\varphi_2)]$ to two sides of the resulting inequality, we get

$$\begin{aligned}&2^s\frac{(\varphi_2-\varphi_1)^\zeta}{B(\zeta)\Gamma(\zeta)}\vartheta\left(\frac{\varphi_1+\varphi_2}{2}\right)+\frac{1-\zeta}{B(\zeta)}\bigl[\vartheta(\varphi_1)+\vartheta(\varphi_2)\bigr]\\&\quad\le\frac{\zeta(\varphi_2-\varphi_1)^\zeta}{B(\zeta)\Gamma(\zeta)}\int_0^1\sigma^{\zeta-1}\vartheta\bigl(\sigma\varphi_1+(1-\sigma)\varphi_2\bigr)\,d\sigma\\&\qquad+\frac{\zeta(\varphi_2-\varphi_1)^\zeta}{B(\zeta)\Gamma(\zeta)}\int_0^1\sigma^{\zeta-1}\vartheta\bigl((1-\sigma)\varphi_1+\sigma\varphi_2\bigr)\,d\sigma\\&\qquad+\frac{1-\zeta}{B(\zeta)}\bigl[\vartheta(\varphi_1)+\vartheta(\varphi_2)\bigr].\end{aligned}$$

The change of variables $\sigma\varphi_1+(1-\sigma)\varphi_2 = y$ and $\sigma\varphi_2+(1-\sigma)\varphi_1 = z$ gives us

$$\begin{aligned}&2^s\frac{(\varphi_2-\varphi_1)^\zeta}{B(\zeta)\Gamma(\zeta)}\vartheta\left(\frac{\varphi_1+\varphi_2}{2}\right)+\frac{1-\zeta}{B(\zeta)}\bigl[\vartheta(\varphi_1)+\vartheta(\varphi_2)\bigr]\\&\quad\le\bigl[{}_{\varphi_1}^{AB}I_{\varphi_2}^{\zeta}\bigl\{\vartheta(\varphi_2)\bigr\}+\bigl({}^{AB}I_{\varphi_2}^{\zeta}\bigr)\bigl\{\vartheta(\varphi_1)\bigr\}\bigr].\end{aligned}\tag{2.5}$$

If we multiply both sides of (2.5) by $\frac{1}{(\varphi_2-\varphi_1)^\zeta}$, we get the first inequality in (2.1). □

We continue this section by giving an equality containing second order derivatives for Atangana–Baleanu integral operators.

Lemma 1 *Let $\varphi_1 < \varphi_2$, $\varphi_1,\varphi_2 \in I^\circ$ and $\vartheta : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on I°. If $\vartheta'' \in L[\varphi_1,\varphi_2]$, the identity for Atangana–Baleanu integral operators in equation (2.6) is valid for all $\zeta \in (0,1]$:*

$$\begin{aligned}&\frac{1}{\varphi_2-\varphi_1}\bigl[\bigl({}^{AB}I_{\frac{\varphi_1+\varphi_2}{2}}^{\zeta}\bigr)\bigl\{\vartheta(\varphi_1)\bigr\}+{}_{\frac{\varphi_1+\varphi_2}{2}}^{AB}I_{\varphi_2}^{\zeta}\bigl\{\vartheta(\varphi_2)\bigr\}\bigr]\\&\quad-\frac{1-\zeta}{(\varphi_2-\varphi_1)B(\zeta)}\bigl[\vartheta(\varphi_1)+\vartheta(\varphi_2)\bigr]\\&\quad-\frac{(\varphi_2-\varphi_1)^{\zeta-1}}{2^{\zeta-1}B(\zeta)\Gamma(\zeta)}\vartheta\left(\frac{\varphi_1+\varphi_2}{2}\right)\end{aligned}\tag{2.6}$$

$$= \frac{(\varphi_2-\varphi_1)^{\zeta+1}}{2(\zeta+1)B(\zeta)\Gamma(\zeta)} \times \int_0^1 m^{\zeta}(\sigma)\bigl[\vartheta''\bigl(\sigma\varphi_1+(1-\sigma)\varphi_2\bigr)+\vartheta''\bigl(\sigma\varphi_2+(1-\sigma)\varphi_1\bigr)\bigr]\,d\sigma,$$

where

$$m^{\zeta}(\sigma)=\begin{cases}\sigma^{\zeta+1}, & \sigma\in[0,\frac{1}{2}),\\ (1-\sigma)^{\zeta+1}, & \sigma\in[\frac{1}{2},1],\end{cases}$$

and also $\Gamma(\zeta)$ *is a gamma function and* $B(\zeta)>0$.

Proof By using the integration by parts, we can get

$$\begin{aligned}
&\frac{(\varphi_2-\varphi_1)^{\zeta+2}}{2(\zeta+1)B(\zeta)\Gamma(\zeta)}\int_0^1 m^{\zeta}(\sigma)\vartheta''\bigl(\sigma\varphi_1+(1-\sigma)\varphi_2\bigr)\,d\sigma\\
&=\frac{(\varphi_2-\varphi_1)^{\zeta+2}}{2(\zeta+1)B(\zeta)\Gamma(\zeta)}\biggl[\int_0^{\frac{1}{2}}\sigma^{\zeta+1}\vartheta''\bigl(\sigma\varphi_1+(1-\sigma)\varphi_2\bigr)\,d\sigma\\
&\quad+\int_{\frac{1}{2}}^1(1-\sigma)^{\zeta+1}\vartheta''\bigl(\sigma\varphi_1+(1-\sigma)\varphi_2\bigr)\,d\sigma\biggr]\\
&=\frac{(\varphi_2-\varphi_1)^{\zeta+2}}{2(\zeta+1)B(\zeta)\Gamma(\zeta)}\biggl\{\sigma^{\zeta+1}\frac{\vartheta'(\sigma\varphi_1+(1-\sigma)\varphi_2)}{(\varphi_1-\varphi_2)}\bigg|_0^{\frac{1}{2}}\\
&\quad-\int_0^{\frac{1}{2}}(\zeta+1)\sigma^{\zeta}\frac{\vartheta'(\sigma\varphi_1+(1-\sigma)\varphi_2)}{(\varphi_1-\varphi_2)}\,d\sigma\\
&\quad+(1-\sigma)^{\zeta+1}\frac{\vartheta'(\sigma\varphi_1+(1-\sigma)\varphi_2)}{(\varphi_1-\varphi_2)}\bigg|_{\frac{1}{2}}^1\\
&\quad+\int_{\frac{1}{2}}^1(\zeta+1)(1-\sigma)^{\zeta}\frac{\vartheta'(\sigma\varphi_1+(1-\sigma)\varphi_2)}{(\varphi_1-\varphi_2)}\,d\sigma\biggr\}\\
&=\frac{(\varphi_2-\varphi_1)^{\zeta+2}}{2(\zeta+1)B(\zeta)\Gamma(\zeta)}\biggl\{-\biggl(\frac{\zeta+1}{\varphi_1-\varphi_2}\biggr)\int_0^{\frac{1}{2}}\sigma^{\zeta}\vartheta'\bigl(\sigma\varphi_1+(1-\sigma)\varphi_2\bigr)\,d\sigma\\
&\quad+\biggl(\frac{\zeta+1}{\varphi_1-\varphi_2}\biggr)\int_{\frac{1}{2}}^1(1-\sigma)^{\zeta}\vartheta'\bigl(\sigma\varphi_1+(1-\sigma)\varphi_2\bigr)\,d\sigma\biggr\}.
\end{aligned}$$

If we use the integration by parts again, we can write

$$\begin{aligned}
&\frac{(\varphi_2-\varphi_1)^{\zeta+2}}{2(\zeta+1)B(\zeta)\Gamma(\zeta)}\biggl\{-\biggl(\frac{\zeta+1}{\varphi_1-\varphi_2}\biggr)\int_0^{\frac{1}{2}}\sigma^{\zeta}\vartheta'\bigl(\sigma\varphi_1+(1-\sigma)\varphi_2\bigr)\,d\sigma\\
&\quad+\biggl(\frac{\zeta+1}{\varphi_1-\varphi_2}\biggr)\int_{\frac{1}{2}}^1(1-\sigma)^{\zeta}\vartheta'\bigl(\sigma\varphi_1+(1-\sigma)\varphi_2\bigr)\,d\sigma\biggr\}\\
&=\frac{(\varphi_2-\varphi_1)^{\zeta+2}}{2(\zeta+1)B(\zeta)\Gamma(\zeta)}\biggl\{-\biggl(\frac{\zeta+1}{\varphi_1-\varphi_2}\biggr)\\
&\quad\times\biggl(\sigma^{\zeta}\frac{\vartheta(\sigma\varphi_1+(1-\sigma)\varphi_2)}{(\varphi_1-\varphi_2)}\bigg|_0^{\frac{1}{2}}-\int_0^{\frac{1}{2}}\zeta\sigma^{\zeta-1}\frac{\vartheta(\sigma\varphi_1+(1-\sigma)\varphi_2)}{(\varphi_1-\varphi_2)}\,d\sigma\biggr)
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{\zeta+1}{\varphi_1-\varphi_2}\right)\left((1-\sigma)^{\zeta}\frac{\vartheta(\sigma\varphi_1+(1-\sigma)\varphi_2)}{(\varphi_1-\varphi_2)}\bigg|_{\frac{1}{2}}^{1}\right.\\
&\left.+\int_{\frac{1}{2}}^{1}\zeta(1-\sigma)^{\zeta-1}\frac{\vartheta(\sigma\varphi_1+(1-\sigma)\varphi_2)}{(\varphi_1-\varphi_2)}\,d\sigma\right)\Bigg\}\\
&=\frac{(\varphi_2-\varphi_1)^{\zeta+2}}{2(\zeta+1)B(\zeta)\Gamma(\zeta)}\Bigg\{-\left(\frac{\zeta+1}{\varphi_1-\varphi_2}\right)\\
&\times\left(\frac{(\frac{1}{2})^{\zeta}\vartheta(\frac{\varphi_1+\varphi_2}{2})}{(\varphi_1-\varphi_2)}-\frac{\zeta}{(\varphi_1-\varphi_2)}\int_0^{\frac{1}{2}}\sigma^{\zeta-1}\vartheta\big(\sigma\varphi_1+(1-\sigma)\varphi_2\big)\,d\sigma\right)\\
&+\left(\frac{\zeta+1}{\varphi_1-\varphi_2}\right)\\
&\times\left(-\frac{(\frac{1}{2})^{\zeta}\vartheta(\frac{\varphi_1+\varphi_2}{2})}{(\varphi_1-\varphi_2)}+\frac{\zeta}{(\varphi_1-\varphi_2)}\int_{\frac{1}{2}}^{1}(1-\sigma)^{\zeta-1}\vartheta\big(\sigma\varphi_1+(1-\sigma)\varphi_2\big)\,d\sigma\right)\Bigg\}\\
&=\frac{(\varphi_2-\varphi_1)^{\zeta+2}}{2(\zeta+1)B(\zeta)\Gamma(\zeta)}\Bigg\{\frac{-2(\zeta+1)(\frac{1}{2})^{\zeta}\vartheta(\frac{\varphi_1+\varphi_2}{2})}{(\varphi_1-\varphi_2)^2}\\
&+\frac{\zeta(\zeta+1)}{(\varphi_1-\varphi_2)^2}\int_0^{\frac{1}{2}}\sigma^{\zeta-1}\vartheta\big(\sigma\varphi_1+(1-\sigma)\varphi_2\big)\,d\sigma\\
&+\frac{\zeta(\zeta+1)}{(\varphi_1-\varphi_2)^2}\int_{\frac{1}{2}}^{1}(1-\sigma)^{\zeta-1}\vartheta\big(\sigma\varphi_1+(1-\sigma)\varphi_2\big)\,d\sigma\Bigg\}.
\end{aligned}
$$

By using the changing of variable, we get the term for Atangana–Baleanu integral operators

$$
\begin{aligned}
&\frac{(\varphi_2-\varphi_1)^{\zeta+2}}{2(\zeta+1)B(\zeta)\Gamma(\zeta)}\Bigg\{\frac{-2(\zeta+1)(\frac{1}{2})^{\zeta}\vartheta(\frac{\varphi_1+\varphi_2}{2})}{(\varphi_1-\varphi_2)^2}\\
&+\frac{\zeta(\zeta+1)}{(\varphi_1-\varphi_2)^2}\int_{\varphi_2}^{\frac{\varphi_1+\varphi_2}{2}}\left(\frac{y-\varphi_2}{\varphi_1-\varphi_2}\right)^{\zeta-1}\vartheta(y)\frac{dy}{\varphi_1-\varphi_2}\\
&+\frac{\zeta(\zeta+1)}{(\varphi_1-\varphi_2)^2}\int_{\frac{\varphi_1+\varphi_2}{2}}^{\varphi_1}\left(\frac{\varphi_1-y}{\varphi_1-\varphi_2}\right)^{\zeta-1}\vartheta(y)\frac{dy}{\varphi_1-\varphi_2}\Bigg\}\\
&=-\frac{(\varphi_2-\varphi_1)^{\zeta}}{B(\zeta)\Gamma(\zeta)}\left(\frac{1}{2}\right)^{\zeta}\vartheta\left(\frac{\varphi_1+\varphi_2}{2}\right)+\frac{1}{2}\big({}^{AB}_{\frac{\varphi_1+\varphi_2}{2}}I^{\zeta}_{\varphi_2}\big\{\vartheta(\varphi_2)\big\}\big)\\
&-\frac{1-\zeta}{2B(\zeta)}\big\{\vartheta(\varphi_2)\big\}+\frac{1}{2}\big({}^{AB}I^{\zeta}_{\frac{\varphi_1+\varphi_2}{2}}\big)\big\{\vartheta(\varphi_1)\big\}-\frac{1-\zeta}{2B(\zeta)}\big\{\vartheta(\varphi_1)\big\}.
\end{aligned}
\tag{2.7}
$$

As a similar calculation of (2.7), we get

$$
\begin{aligned}
&\frac{(\varphi_2-\varphi_1)^{\zeta+2}}{2(\zeta+1)B(\zeta)\Gamma(\zeta)}\int_0^1 m^{\zeta}(\sigma)\vartheta''\big(\sigma\varphi_2+(1-\sigma)\varphi_1\big)\,d\sigma\\
&=-\frac{(\varphi_2-\varphi_1)^{\zeta}}{B(\zeta)\Gamma(\zeta)}\left(\frac{1}{2}\right)^{\zeta}\vartheta\left(\frac{\varphi_1+\varphi_2}{2}\right)+\frac{1}{2}\big({}^{AB}_{\frac{\varphi_1+\varphi_2}{2}}I^{\zeta}_{\varphi_2}\big\{\vartheta(\varphi_2)\big\}\big)\\
&-\frac{1-\zeta}{2B(\zeta)}\big\{\vartheta(\varphi_2)\big\}+\frac{1}{2}\big({}^{AB}I^{\zeta}_{\frac{\varphi_1+\varphi_2}{2}}\big)\big\{\vartheta(\varphi_1)\big\}-\frac{1-\zeta}{2B(\zeta)}\big\{\vartheta(\varphi_1)\big\}.
\end{aligned}
\tag{2.8}
$$

If we add (2.7) and (2.8), and after this step if we multiply the resulting equality by $\frac{1}{(\varphi_2-\varphi_1)}$, we complete the proof of the inequality in (2.6). □

Now, we are going to produce generalizations of the HH-type inequalities for Atangana–Baleanu fractional integral operators by using the new integral equation and s-convexity identity. Throughout the study, we denote the following terms with F:

$$F = \frac{1}{\varphi_2-\varphi_1}\Big[{}^{AB}I^{\zeta}_{\frac{\varphi_1+\varphi_2}{2}}\{\vartheta(\varphi_1)\} + {}^{AB}_{\frac{\varphi_1+\varphi_2}{2}}I^{\zeta}_{\varphi_2}\{\vartheta(\varphi_2)\}\Big] - \frac{1-\zeta}{(\varphi_2-\varphi_1)B(\zeta)}\big[\vartheta(\varphi_1)+\vartheta(\varphi_2)\big] - \frac{(\varphi_2-\varphi_1)^{\zeta-1}}{2^{\zeta-1}B(\zeta)\Gamma(\zeta)}\vartheta\left(\frac{\varphi_1+\varphi_2}{2}\right).$$

Theorem 3 *Let $\varphi_1 < \varphi_2$, $\varphi_1,\varphi_2 \in I^\circ$ and $\vartheta : I \subset [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function on I° and $\vartheta'' \in L[\varphi_1,\varphi_2]$. If $|\vartheta''|$ is an s-convex function in the second sense on $[\varphi_1,\varphi_2]$ for some fixed $s \in (0,1]$, we obtain the following inequality for Atangana–Baleanu integral operators*:

$$|F| \le \frac{(\varphi_2-\varphi_1)^{\zeta+1}}{(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\frac{(\frac{1}{2})^{\zeta+s+2}}{\zeta+s+2} + \beta_{\frac{1}{2}}(\zeta+2,s+1)\right)\big(\big|\vartheta''(\varphi_1)\big| + \big|\vartheta''(\varphi_2)\big|\big), \tag{2.9}$$

where $\zeta \in (0,1]$.

Proof By using the equality in (2.6) and the s-convexity of $|\vartheta''|$, we have

$$\begin{aligned}|F| &\le \frac{(\varphi_2-\varphi_1)^{\zeta+1}}{2(\zeta+1)B(\zeta)\Gamma(\zeta)} \\ &\quad\times \int_0^1 \big|m^{\zeta}(\sigma)\big|\big[\big|\vartheta''(\sigma\varphi_1+(1-\sigma)\varphi_2)\big| + \big|\vartheta''(\sigma\varphi_2+(1-\sigma)\varphi_1)\big|\big]\,d\sigma \\ &\le \frac{(\varphi_2-\varphi_1)^{\zeta+1}}{2(\zeta+1)B(\zeta)\Gamma(\zeta)}\left\{\int_0^{\frac{1}{2}} \sigma^{\zeta+1}\big[\sigma^s\big|\vartheta''(\varphi_1)\big| + (1-\sigma)^s\big|\vartheta''(\varphi_2)\big|\big]\,d\sigma \right. \\ &\quad + \int_{\frac{1}{2}}^1 (1-\sigma)^{\zeta+1}\big[\sigma^s\big|\vartheta''(\varphi_1)\big| + (1-\sigma)^s\big|\vartheta''(\varphi_2)\big|\big]\,d\sigma \\ &\quad + \int_0^{\frac{1}{2}} \sigma^{\zeta+1}\big[\sigma^s\big|\vartheta''(\varphi_2)\big| + (1-\sigma)^s\big|\vartheta''(\varphi_1)\big|\big]\,d\sigma \\ &\quad \left. + \int_{\frac{1}{2}}^1 (1-\sigma)^{\zeta+1}\big[\sigma^s\big|\vartheta''(\varphi_2)\big| + (1-\sigma)^s\big|\vartheta''(\varphi_1)\big|\big]\,d\sigma\right\}.\end{aligned}$$

Afterwards, by getting the necessary calculations, we complete the proof of the inequality in (2.9). □

Corollary 1 *In Theorem 3, if we choose $s = 1$, we have the following inequality*:

$$|F| \le \frac{(\varphi_2-\varphi_1)^{\zeta+1}}{(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\frac{(\frac{1}{2})^{\zeta+3}}{\zeta+3} + \beta_{\frac{1}{2}}(\zeta+2,2)\right)\big(\big|\vartheta''(\varphi_1)\big| + \big|\vartheta''(\varphi_2)\big|\big).$$

Corollary 2 *In Theorem 3, if $|\vartheta''| \le M$ on I°, $M > 0$, we have the following inequality*:

$$|F| \le \frac{2M(\varphi_2-\varphi_1)^{\zeta+1}}{(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\frac{(\frac{1}{2})^{\zeta+s+2}}{\zeta+s+2} + \beta_{\frac{1}{2}}(\zeta+2,s+1)\right).$$

Theorem 4 *Let $\varphi_1 < \varphi_2$, $\varphi_1, \varphi_2 \in I^\circ$ and $\vartheta : I \subset [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable mapping on I° and $\vartheta'' \in L[\varphi_1, \varphi_2]$. If $|\vartheta''|^q$ is an s-convex function in the second sense on $[\varphi_1, \varphi_2]$ for some fixed $s \in (0,1]$, we obtain the following inequality for Atangana–Baleanu integral operators*:

$$|F| \leq \frac{(\varphi_2 - \varphi_1)^{\zeta+1}}{(\zeta+1)B(\zeta)\Gamma(\zeta)} \left(\frac{(\frac{1}{2})^{\zeta p + p}}{\zeta p + p + 1} \right)^{\frac{1}{p}} \frac{1}{(s+1)^{\frac{1}{q}}} \left(\left|\vartheta''(\varphi_1)\right| + \left|\vartheta''(\varphi_2)\right| \right), \tag{2.10}$$

where $\zeta \in (0,1]$, $q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof By using the equality in (2.6) and Hölder's inequality, we get

$$\begin{aligned} |F| &\leq \frac{(\varphi_2 - \varphi_1)^{\zeta+1}}{2(\zeta+1)B(\zeta)\Gamma(\zeta)} \\ &\quad \times \int_0^1 \left|m^\zeta(\sigma)\right| \left[\left|\vartheta''(\sigma\varphi_1 + (1-\sigma)\varphi_2)\right| + \left|\vartheta''(\sigma\varphi_2 + (1-\sigma)\varphi_1)\right| \right] d\sigma \\ &\leq \frac{(\varphi_2 - \varphi_1)^{\zeta+1}}{2(\zeta+1)B(\zeta)\Gamma(\zeta)} \left(\int_0^1 \left|m^\zeta(\sigma)\right|^p d\sigma \right)^{\frac{1}{p}} \\ &\quad \times \left[\left(\int_0^1 \left|\vartheta''(\sigma\varphi_1 + (1-\sigma)\varphi_2)\right|^q d\sigma \right)^{\frac{1}{q}} + \left(\int_0^1 \left|\vartheta''(\sigma\varphi_2 + (1-\sigma)\varphi_1)\right|^q d\sigma \right)^{\frac{1}{q}} \right]. \end{aligned}$$

To reach the result, we use the s-convexity in the second sense on $[\varphi_1, \varphi_2]$, and then we use the fact that

$$\sum_{k=1}^n (u_k + v_k)^m \leq \sum_{k=1}^n u_k^m + \sum_{k=1}^n v_k^m$$

for $0 \leq m < 1$, $u_1, u_2, \ldots, u_n \geq 0$, $v_1, v_2, \ldots, v_n \geq 0$. So, we obtained the inequality in (2.10). The proof is completed. □

Corollary 3 *In Theorem 4, if we choose $s = 1$, we have the following inequality*:

$$|F| \leq \frac{(\varphi_2 - \varphi_1)^{\zeta+1}}{(\zeta+1)B(\zeta)\Gamma(\zeta)} \left(\frac{(\frac{1}{2})^{\zeta p + p}}{\zeta p + p + 1} \right)^{\frac{1}{p}} \frac{1}{2^{\frac{1}{q}}} \left(\left|\vartheta''(\varphi_1)\right| + \left|\vartheta''(\varphi_2)\right| \right).$$

Corollary 4 *In Theorem 4, if $|\vartheta''| \leq M$ on I°, $M > 0$, we have the following inequality*:

$$|F| \leq \frac{2M(\varphi_2 - \varphi_1)^{\zeta+1}}{(\zeta+1)B(\zeta)\Gamma(\zeta)} \left(\frac{(\frac{1}{2})^{\zeta p + p}}{\zeta p + p + 1} \right)^{\frac{1}{p}} \frac{1}{(s+1)^{\frac{1}{q}}}.$$

Theorem 5 *Under the assumptions of Theorem 4, we get the inequality in (2.11)*:

$$\begin{aligned} |F| &\leq \frac{(\varphi_2 - \varphi_1)^{\zeta+1}}{(\zeta+1)B(\zeta)\Gamma(\zeta)} \left(\frac{1}{2^{\zeta+1}(\zeta+2)} \right)^{\frac{1}{p}} \\ &\quad \times \left(\frac{(\frac{1}{2})^{\zeta+s+2}}{\zeta+s+2} + \beta_{\frac{1}{2}}(\zeta+2, s+1) \right)^{\frac{1}{q}} \left(\left|\vartheta''(\varphi_1)\right|^q + \left|\vartheta''(\varphi_2)\right|^q \right)^{\frac{1}{q}}, \end{aligned} \tag{2.11}$$

where $\zeta \in (0,1]$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof When we use Hölder's inequality from a different point of view, we can write

$$
\begin{aligned}
|F| \leq & \frac{(\varphi_2-\varphi_1)^{\zeta+1}}{2(\zeta+1)B(\zeta)\Gamma(\zeta)} \\
& \times \int_0^1 \left|m^{\zeta}(\sigma)\right| \left[\left|\vartheta''\left(\sigma\varphi_1+(1-\sigma)\varphi_2\right)\right| + \left|\vartheta''\left(\sigma\varphi_2+(1-\sigma)\varphi_1\right)\right|\right] d\sigma \\
\leq & \frac{(\varphi_2-\varphi_1)^{\zeta+1}}{2(\zeta+1)B(\zeta)\Gamma(\zeta)} \left(\int_0^1 \left|m^{\zeta}(\sigma)\right| d\sigma\right)^{\frac{1}{p}} \\
& \times \left[\left(\int_0^1 \left|m^{\zeta}(\sigma)\right| \left|\vartheta''\left(\sigma\varphi_1+(1-\sigma)\varphi_2\right)\right|^q d\sigma\right)^{\frac{1}{q}}\right. \\
& \left. + \left(\int_0^1 \left|m^{\zeta}(\sigma)\right| \left|\vartheta''\left(\sigma\varphi_2+(1-\sigma)\varphi_1\right)\right|^q d\sigma\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

If we apply the s-convexity of $|\vartheta''|^q$ and calculate the above integrals, we get the desired. □

Corollary 5 *In Theorem 5, if we choose $s = 1$, we have the following inequality*:

$$
\begin{aligned}
|F| \leq & \frac{(\varphi_2-\varphi_1)^{\zeta+1}}{(\zeta+1)B(\zeta)\Gamma(\zeta)} \left(\frac{1}{2^{\zeta+1}(\zeta+2)}\right)^{\frac{1}{p}} \\
& \times \left(\frac{(\frac{1}{2})^{\zeta+3}}{\zeta+3} + \beta_{\frac{1}{2}}(\zeta+2,2)\right)^{\frac{1}{q}} \left(\left|\vartheta''(\varphi_1)\right|^q + \left|\vartheta''(\varphi_2)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}
$$

Corollary 6 *In Theorem 5, if $|\vartheta''| \leq M$ on I°, $M > 0$, we have the following inequality*:

$$
\begin{aligned}
|F| \leq & \frac{2^{\frac{1}{q}} M(\varphi_2-\varphi_1)^{\zeta+1}}{(\zeta+1)B(\zeta)\Gamma(\zeta)} \left(\frac{1}{2^{\zeta+1}(\zeta+2)}\right)^{\frac{1}{p}} \\
& \times \left(\frac{(\frac{1}{2})^{\zeta+s+2}}{\zeta+s+2} + \beta_{\frac{1}{2}}(\zeta+2,s+1)\right)^{\frac{1}{q}}.
\end{aligned}
$$

Theorem 6 *Let $\varphi_1 < \varphi_2$, $\varphi_1, \varphi_2 \in I^\circ$ and $\vartheta : I \subset [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function on I° and $\vartheta'' \in L[\varphi_1, \varphi_2]$. If $|\vartheta''|^q$ is an s-convex function in the second sense on $[\varphi_1, \varphi_2]$ for some fixed $s \in (0, 1]$, we obtain the following inequality in (2.12) for Atangana–Baleanu integral operators*:

$$
\begin{aligned}
|F| \leq & \frac{(\varphi_2-\varphi_1)^{\zeta+1}}{(\zeta+1)B(\zeta)\Gamma(\zeta)} \left(\frac{(\frac{1}{2})^{(\zeta+1)(\frac{q-p}{q-1})}(q-1)}{(\zeta+1)(q-p)+q-1}\right)^{1-\frac{1}{q}} \left(\left|\vartheta''(\varphi_1)\right|^q + \left|\vartheta''(\varphi_2)\right|^q\right)^{\frac{1}{q}} \\
& \times \left(\frac{(\frac{1}{2})^{(\zeta+1)p+s+1}}{(\zeta+1)p+s+1} + \beta_{\frac{1}{2}}\left((\zeta+1)p+1, s+1\right)\right)^{\frac{1}{q}},
\end{aligned}
\tag{2.12}
$$

where $\zeta \in (0, 1]$, $q \geq p > 1$.

Proof By using Hölder's inequality in a different way, we can write

$$
|F| \leq \frac{(\varphi_2-\varphi_1)^{\zeta+1}}{2(\zeta+1)B(\zeta)\Gamma(\zeta)} \left\{\left(\int_0^1 \left|m^{\zeta}(\sigma)\right|^{\frac{q-p}{q-1}} d\sigma\right)^{1-\frac{1}{q}}\right.
$$

$$\times\left(\int_0^1 \left|m^\zeta(\sigma)\right|^p \left|\vartheta''\bigl(\sigma\varphi_1+(1-\sigma)\varphi_2\bigr)\right|^q d\sigma\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 \left|m^\zeta(\sigma)\right|^{\frac{q-p}{q-1}} d\sigma\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 \left|m^\zeta(\sigma)\right|^p \left|\vartheta''\bigl(\sigma\varphi_2+(1-\sigma)\varphi_1\bigr)\right|^q d\sigma\right)^{\frac{1}{q}}\Bigg\}.$$

If we use the s-convexity of $|\vartheta''|^q$ above, we have

$$|F| \le \frac{(\varphi_2-\varphi_1)^{\zeta+1}}{2(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\int_0^1 \left|m^\zeta(\sigma)\right|^{\frac{q-p}{q-1}} d\sigma\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 \left|m^\zeta(\sigma)\right|^p \bigl[\sigma^s\left|\vartheta''(\varphi_1)\right|^q+(1-\sigma)^s\left|\vartheta''(\varphi_2)\right|^q\bigr] d\sigma\right)^{\frac{1}{q}}$$
$$+\frac{(\varphi_2-\varphi_1)^{\zeta+1}}{2(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\int_0^1 \left|m^\zeta(\sigma)\right|^{\frac{q-p}{q-1}} d\sigma\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 \left|m^\zeta(\sigma)\right|^p \bigl[\sigma^s\left|\vartheta''(\varphi_2)\right|^q+(1-\sigma)^s\left|\vartheta''(\varphi_1)\right|^q\bigr] d\sigma\right)^{\frac{1}{q}}.$$

By making the necessary integral calculations, the proof is completed. □

Corollary 7 *In Theorem 6, if we choose $s=1$, we have the following inequality*:

$$|F| \le \frac{(\varphi_2-\varphi_1)^{\zeta+1}}{(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\frac{(\frac{1}{2})^{(\zeta+1)(\frac{q-p}{q-1})}(q-1)}{(\zeta+1)(q-p)+q-1}\right)^{1-\frac{1}{q}}$$
$$\times\left(\left|\vartheta''(\varphi_1)\right|^q+\left|\vartheta''(\varphi_2)\right|^q\right)^{\frac{1}{q}}\left(\frac{(\frac{1}{2})^{(\zeta+1)p+2}}{(\zeta+1)p+2}+\beta_{\frac{1}{2}}\bigl((\zeta+1)p+1,2\bigr)\right)^{\frac{1}{q}}.$$

Corollary 8 *In Theorem 6, if $|\vartheta''|\le M$ on I°, $M>0$, we have the following inequality*:

$$|F| \le \frac{2^{\frac{1}{q}}M(\varphi_2-\varphi_1)^{\zeta+1}}{(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\frac{(\frac{1}{2})^{(\zeta+1)(\frac{q-p}{q-1})}(q-1)}{(\zeta+1)(q-p)+q-1}\right)^{1-\frac{1}{q}}$$
$$\times[\left(\frac{(\frac{1}{2})^{(\zeta+1)p+s+1}}{(\zeta+1)p+s+1}+\beta_{\frac{1}{2}}\bigl((\zeta+1)p+1,s+1\bigr)\right)^{\frac{1}{q}}.$$

Theorem 7 *Let $\varphi_1<\varphi_2$, $\varphi_1,\varphi_2\in I^\circ$ and $\vartheta: I\subset[0,\infty)\longrightarrow\mathbb{R}$ be a differentiable function on I° and $\vartheta''\in L[\varphi_1,\varphi_2]$. If $|\vartheta''|^q$ is an s-convex function in the second sense on $[\varphi_1,\varphi_2]$ for some fixed $s\in(0,1]$, we obtain the following inequality in (2.13) for Atangana–Baleanu integral operators*:

$$|F| \le \frac{(\varphi_2-\varphi_1)^{\zeta+1}}{(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\frac{(\frac{1}{2})^{(\zeta+1)p}}{((\zeta+1)p+1)p}+\frac{|\vartheta''(\varphi_1)|^q+|\vartheta''(\varphi_2)|^q}{(s+1)q}\right), \tag{2.13}$$

where $\zeta\in(0,1]$ and $q>1$.

Proof By using Lemma 1, we have

$$
\begin{aligned}
|F| \le{} & \frac{(\varphi_2-\varphi_1)^{\zeta+1}}{2(\zeta+1)B(\zeta)\Gamma(\zeta)} \\
& \times \int_0^1 \left|m^{\zeta}(\sigma)\right|\left[\left|\vartheta''\left(\sigma\varphi_1+(1-\sigma)\varphi_2\right)\right|+\left|\vartheta''\left(\sigma\varphi_2+(1-\sigma)\varphi_1\right)\right|\right]d\sigma.
\end{aligned}
$$

By using Young's inequality as $xy \le \frac{1}{p}x^p + \frac{1}{q}y^q$, we get

$$
\begin{aligned}
|F| \le{} & \frac{(\varphi_2-\varphi_1)^{\zeta+1}}{2(\zeta+1)B(\zeta)\Gamma(\zeta)} \\
& \times \left\{\frac{1}{p}\int_0^1 \left|m^{\zeta}(\sigma)\right|^p d\sigma + \frac{1}{q}\int_0^1 \left|\vartheta''\left(\sigma\varphi_1+(1-\sigma)\varphi_2\right)\right|^q d\sigma \right. \\
& \left. + \frac{1}{p}\int_0^1 \left|m^{\zeta}(\sigma)\right|^p d\sigma + \frac{1}{q}\int_0^1 \left|\vartheta''\left(\sigma\varphi_2+(1-\sigma)\varphi_1\right)\right|^q d\sigma \right\}.
\end{aligned}
$$

By using the s-convexity of $|\vartheta''|^q$ and by simple calculations, we provide the result. □

Corollary 9 *In Theorem 7, if we choose $s = 1$, we have the following inequality*:

$$
|F| \le \frac{(\varphi_2-\varphi_1)^{\zeta+1}}{(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\frac{(\frac{1}{2})^{(\zeta+1)p}}{((\zeta+1)p+1)p} + \frac{|\vartheta''(\varphi_1)|^q+|\vartheta''(\varphi_2)|^q}{2q}\right).
$$

Corollary 10 *In Theorem 7, if $|\vartheta''| \le M$ on $I^\circ, M > 0$, we have the following inequality*:

$$
|F| \le \frac{(\varphi_2-\varphi_1)^{\zeta+1}}{(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\frac{(\frac{1}{2})^{(\zeta+1)p}}{((\zeta+1)p+1)p} + \frac{2M^q}{(s+1)q}\right).
$$

Theorem 8 *Let $\varphi_1 < \varphi_2$, $\varphi_1, \varphi_2 \in I^\circ$ and $\vartheta : I \subset [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function on I° and $\vartheta'' \in L[\varphi_1, \varphi_2]$. If $|\vartheta''|^q$ is an s-concave function in the second sense on $[\varphi_1, \varphi_2]$ for some fixed $s \in (0, 1]$, we obtain the following inequality in (2.14) for Atangana–Baleanu integral operators*:

$$
|F| \le \frac{(\varphi_2-\varphi_1)^{\zeta+1}}{(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\frac{(\frac{1}{2})^{\zeta p+p}}{\zeta p+p+1}\right)^{\frac{1}{p}} 2^{\frac{s-1}{q}}\left|\vartheta''\left(\frac{\varphi_1+\varphi_2}{2}\right)\right|, \tag{2.14}
$$

where $\zeta \in (0, 1]$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof If we apply Hölder's inequality, we have

$$
\begin{aligned}
|F| \le{} & \frac{(\varphi_2-\varphi_1)^{\zeta+1}}{2(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\int_0^1 \left|m^{\zeta}(\sigma)\right|^p d\sigma\right)^{\frac{1}{p}} \\
& \times \left[\left(\int_0^1 \left|\vartheta''\left(\sigma\varphi_1+(1-\sigma)\varphi_2\right)\right|^q d\sigma\right)^{\frac{1}{q}} + \left(\int_0^1 \left|\vartheta''\left(\sigma\varphi_2+(1-\sigma)\varphi_1\right)\right|^q d\sigma\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Since $|\vartheta''|^q$ is s-concave on $[\varphi_1, \varphi_2]$, we can write the following results by taking into account the variant of the HH-inequality for s-concave functions:

$$\int_0^1 \left|\vartheta''\big(\sigma\varphi_1 + (1-\sigma)\varphi_2\big)\right|^q d\sigma \leq 2^{s-1}\left|\varphi''\left(\frac{\varphi_1 + \varphi_2}{2}\right)\right|^q,$$
$$\int_0^1 \left|\varphi''\big(\sigma\varphi_2 + (1-\sigma)\varphi_1\big)\right|^q d\sigma \leq 2^{s-1}\left|\varphi''\left(\frac{\varphi_1 + \varphi_2}{2}\right)\right|^q.$$

By using these results in the above inequality, we complete the proof. □

3 Conclusion

We see that the main idea for most of the studies in the field of inequalities is to generalize, to reveal new boundaries, and to create findings that will allow different applications. In this direction, sometimes the features of the function, sometimes new methods, and sometimes new operators are used, and these choices add original value to the studies. In this context, in the paper, which includes reflections of fractional analysis to inequality theory, the main motivation point is to obtain new integral inequalities for s-convex and s-concave functions that involve Atangana–Baleanu fractional integral operators. First, a general form of the HH-inequality for Atangana–Baleanu fractional integral operators has been obtained. Then, using a newly established integral identity, various HH-type inequalities have been derived. The special cases of these inequalities, which are presented in general forms, have been taken into consideration.

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