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Research Article

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On Numerical Solution Of The Time Fractional Advection-Diffusion Equation Involving Atangana-Baleanu-Caputo Derivative

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Abstract: A powerful algorithm is proposed to get the solutions of the time fractional Advection-Diffusion equation(TFADE): ${}^{ABC}\mathcal{D}^{\beta}_{0^+,t}u(x,t) = \zeta u_{xx}(x,t) - \kappa u_x(x,t) + F(x,t)$, $0 < \beta \leq 1$. The time-fractional derivative ${}^{ABC}\mathcal{D}^{\beta}_{0^+,t}u(x,t)$ is described in the Atangana-Baleanu Caputo concept. The basis of our approach is transforming the original equation into a new equation by imposing a transformation involving a fictitious coordinate. Then, a geometric scheme namely the group preserving scheme (GPS) is implemented to solve the new equation by taking an initial guess. Moreover, in order to present the power of the presented approach some examples are solved, successfully.

Keywords: Fictitious time integration method; Group preserving scheme; Time fractional Advection-Diffusion equation; Atangana-Baleanu Caputo derivative

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1 Introduction

Non-integer calculus is one of the most practical concepts. This issue has constructed since 1695. In fact, in the last few decade many researchers have been done important works in this field. The significant topic commenced recently to become very valuable in different areas such as science and engineering [1–6]. The development and gaining numerical and exact solutions of the partial and fractional equations, involving non-integer derivatives and integral, have obtained considerable importance. So, various approaches have been worked for such goal, see, [7–33]. In this study we attempt to solve the TFADE containing the Atangana-Baleanu Caputo derivative. Some methods are implemented to solve of such type of problems [34–43]. The TFADE arises in modeling the problems of biology and chemistry which contain diffusion process [44–46].

The structure of this work is based as follows. Preliminaries are supplied in section 2. Section 3 is dedicated to display the roles of the fictitious time integration method(FTIM) and group preserving scheme(GPS). Also, two examples are provided to show the capability of our scheme in section 4. Indeed, conclusion is provided in sections 5.

In this work we consider the following TFADE with the Atangana-Baleanu Caputo derivative of order β.

$$\begin{cases} {}^{ABC}\mathcal{D}^{\beta}_{0^+,t}u(x,t) = \zeta u_{xx}(x,t) - \kappa u_x(x,t) \\ \quad +F(x,t),\ (\mathbf{x},t) \in \Omega \subset \mathbb{R}^2, \\ u(x,0) = h_1(x),\ x \in \Omega_{\mathbf{x}}, \\ u(x,t_f) = h_2(x),\ x \in \Omega_{\mathbf{x}}, \\ u(a,t) = p_1(t),\ \ t \in \Omega_t, \\ u(b,t) = p_2(t),\ \ t \in \Omega_t, \end{cases} \tag{1}$$

where Ω_t and $\Omega_{\mathbf{x}}$ are boundaries of $\Omega := \{(x,t) : a \leq x \leq b,\ 0 \leq t \leq t_f\}$ in t and x, respectively. Also, ζ is a real parameter and κ is the average velocity.

2 Preliminaries

The the Atangana-Baleanu fractional(ABC) derivative in Caputo sense of order β and for $f \in H^1(0,1)$ is defined

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as :

$$ {}^{ABC}\mathfrak{D}_t^{\beta} f(t) = \frac{N(\beta)}{1-\beta}\int_0^t f^n(c)E_\beta\left(\frac{-\beta}{n-\beta}(t-c)^\beta\right)\mathrm{d}c, \quad (2) $$

$n-1<\beta\le n,$

where $E_\beta(z)$ is Mittag-Leffler function described as

$$ E_\beta(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta k+1)}. $$

and $N(\beta)$ is a standardization function defined as

$$ N(\beta)=1-\beta+\frac{\beta}{\Gamma(\beta)}. $$

With regard to the definition (2) for $0<\beta\le 1$ we have:

$$ {}^{ABC}\mathfrak{D}_t^{\beta} f(t) = \frac{N(\beta)}{1-\beta}\int_0^t f'(c)E_\beta\left(\frac{-\beta}{1-\beta}(t-c)^\beta\right)\mathrm{d}c. \quad (3) $$

3 The fictitious time integration method(FTIM)

Now, we provide FTIM to convert the original time fractional Advection-Diffusion equation into a firsthand equation with one more dimension by introducing a fictitious damping coefficient μ. The structure of this method is as follows:

Using the definition (3) and $0<\beta\le 1$ for Eqs. (1) we have:

$$ \frac{N(\beta)}{\Gamma(1-\beta)}\int_0^t u_c(x,c)E_\beta\left(\frac{-\beta}{1-\beta}(t-c)^\beta\right)\mathrm{d}c - \zeta u_{xx}(x,t)+\kappa u_x(x,t)-F(x,t)=0. \quad (4) $$

We can increase the stablity of the method by proposing a fictitious damping coefficient μ in Eq. (4) as follows:

$$ \frac{\mu N(\beta)}{\Gamma(1-\beta)}\int_0^t u_c(x,c)E_\beta\left(\frac{-\beta}{1-\beta}(t-c)^\beta\right)\mathrm{d}c - \mu\zeta u_{xx}(x,t)+\mu\kappa u_x(x,t)-\mu F(x,t)=0. \quad (5) $$

Placing the following transformation in Eq. (5)

$$ \Xi(x,t,\eta)=(1+\eta)^\lambda u(x,t),\ 0<\lambda\le 1, \quad (6) $$

Results a new form of the original equation:

$$ \frac{\mu}{(1+\eta)^\lambda}\left[\frac{N(\beta)}{\Gamma(1-\beta)}\int_0^t \Xi_c(x,c,\eta)E_\beta\left(\frac{-\beta}{1-\beta}(t-c)^\beta\right)\mathrm{d}c - \zeta\Xi_{xx}(x,t,\eta)+\kappa\Xi_x(x,t,\eta)\right]-\mu F(x,t)=0. \quad (7) $$

Considering

$$ \frac{\partial\Xi}{\partial\eta}=\lambda(1+\eta)^{\lambda-1}u(x,t), \quad (8) $$

Eq. (7), can be written as:

$$ \frac{\partial\Xi}{\partial\eta}=\frac{\mu}{(1+\eta)^\lambda}\left[\frac{N(\beta)}{\Gamma(1-\beta)}\int_0^t \Xi_c(x,c,\eta)E_\beta\left(\frac{-\beta}{1-\beta}(t-c)^\beta\right)\mathrm{d}c - \zeta\Xi_{xx}(x,t,\eta)+\kappa\Xi_x(x,t,\eta)\right] - \mu F(x,t)+\lambda(1+\eta)^{\lambda-1}u. \quad (9) $$

Eq. (9) can be transformed to a new kind of functional PDE for Ξ, by setting $u=\frac{\Xi}{(1+\eta)^\lambda}$:

$$ \frac{\partial\Xi}{\partial\eta}=\frac{\mu}{(1+\eta)^\lambda}\left[\frac{N(\beta)}{\Gamma(1-\beta)}\int_0^t \Xi_c(x,c,\eta)E_\beta\left(\frac{-\beta}{1-\beta}(t-c)^\beta\right)\mathrm{d}c - \zeta\Xi_{xx}(x,t,\eta)+\kappa\Xi_x(x,t,\eta)\right] - \mu F(x,t)+\frac{\lambda\Xi(x,t,\eta)}{1+\eta}. \quad (10) $$

Using

$$ \frac{\partial}{\partial\eta}\left(\frac{\Xi}{(1+\eta)^\lambda}\right)=\frac{\Xi_\eta}{(1+\eta)^\lambda}-\frac{\lambda\Xi}{(1+\eta)^{\lambda+1}}, \quad (11) $$

and multiplying the factor $1/(1+\eta)^\lambda$ in Eq. (10), we obtain

$$ \frac{\partial}{\partial\eta}\left(\frac{\Xi}{(1+\eta)^\lambda}\right) = \frac{\mu}{(1+\eta)^\lambda}\left[\frac{N(\beta)}{\Gamma(1-\beta)}\int_0^t \Xi_c(x,c,\eta)E_\beta\left(\frac{-\beta}{1-\beta}(t-c)^\beta\right)\mathrm{d}c - \zeta\Xi_{xx}(x,t,\eta)+\kappa\Xi_x(x,t,\eta)\right]-\mu F(x,t). \quad (12) $$

Using again the transformation $u=\frac{\Xi}{(1+\eta)^\lambda}$, we get:

$$ u_\eta=\frac{\mu}{(1+\eta)^\lambda}\left[\frac{N(\beta)}{\Gamma(1-\beta)}\int_0^t u_c(x,c,\eta)E_\beta\left(\frac{-\beta}{1-\beta}(t-c)^\beta\right)\mathrm{d}c - \zeta u_{xx}(x,t,\eta)+\kappa u_x(x,t,\eta)\right]-\mu F(x,t). \quad (13) $$

Suppose $u_i^j(\eta) := u(x_i,t_j,\eta)$ as the values of u at a point (x_i,t_j), Eq.(12) converts to the following form:

$$ \frac{d}{d\eta}u_i^j(\eta) = \frac{\mu}{(1+\eta)^\lambda}\left[\frac{N(\beta)}{\Gamma(1-\beta)}\int_0^{t_j} u_c(x_i,c,\eta)E_\beta\left(\frac{-\beta}{1-\beta}(t_j-c)^\beta\right)\mathrm{d}c - \zeta\frac{u_{i+1}^j(\eta)-2u_i^j(\eta)+u_{i-1}^j(\eta)}{\Delta x^2}+\kappa\frac{u_{i+1}^j(\eta)-u_i^j(\eta)}{\Delta x}\right] - \mu F(x_i,t_j). \quad (14) $$

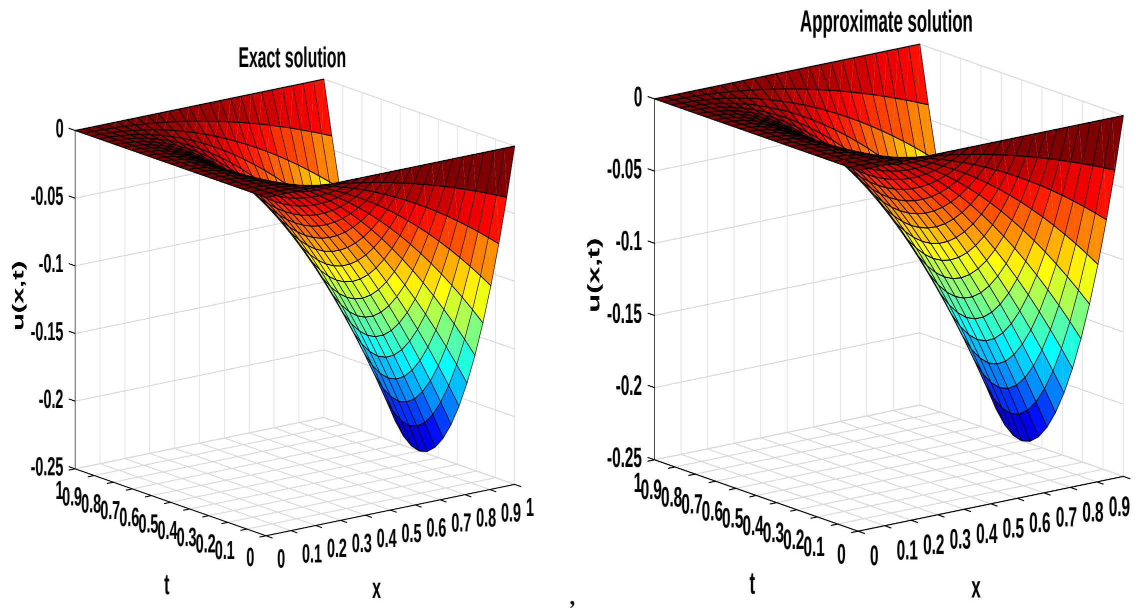

Figure 1: Plots of the exact solution and the numerical solution under $\beta = 0.5$ for example 1.

Discretization of the above equation needs to calculate the approximation of the following integral:

$$\int_0^{t_j} u_c(x_i, c, \eta) E_\beta\left(\frac{-\beta}{1-\beta}(t_j - c)^\beta\right) \mathrm{d}c$$
$$\approx \frac{N(\beta)}{\Gamma(1-\beta)} \sum_{k=1}^{j} \frac{u_i^{j+1} - u_i^j}{\Delta t}$$
$$\times \int_{(k-1)\Delta t}^{k\Delta t} E_\beta\left(\frac{-\beta}{1-\beta}(t_k - c)^\beta\right) \mathrm{d}c.$$

where

$$\int_{(k-1)\Delta t}^{k\Delta t} E_\beta\left(\frac{-\beta}{1-\beta}(t_k - c)^\beta\right) \mathrm{d}c$$
$$\approx (t_j - t_{k+1}) E_\beta\left(\frac{-\beta}{1-\beta}(t_j - t_{k+1})^\beta\right)$$
$$- (t_j - t_k) E_\beta\left(\frac{-\beta}{1-\beta}(t_j - t_k)^\beta\right)$$

where $\Delta t = \frac{T}{n}$, $x_i = a + i\Delta x$ and $t_j = j\Delta t$.

Considering $\mathbf{u} = (u_1^1, u_1^2, \ldots, u_m^n)^T$, Eq. (13) can be written as:

$$\mathbf{u}' = \mathbf{Z}(\mathbf{u}, \eta), \ \mathbf{u} \in \mathbb{R}^{m \times n}, \eta \in \mathbb{R}, \quad M = m \times n, \qquad (15)$$

where $\mathbf{u}$ is M-dimensional vector and $\mathbf{Z} \in \mathbb{R}^M$ is a vector function of $\mathbf{u}$ and η. Now, we are ready to use the group preserving scheme(GPS) introduced in [47] to solve Eq. (14):

$$\mathbf{u}_{s+1} = \mathbf{u}_s + \qquad (16)$$
$$\frac{\left[\cosh\left(\frac{\Delta\eta \|\mathbf{Z}_s\|}{\|\mathbf{u}_s\|}\right) - 1\right] \mathbf{Z}_s.\mathbf{u}_s + \sinh\left(\frac{\Delta\eta \|\mathbf{Z}_s\|}{\|\mathbf{u}_s\|}\right) \|\mathbf{u}_s\| \|\mathbf{Z}_s\|}{\|\mathbf{Z}_s\|^2} \mathbf{Z}_s.$$

by taking the initial value of $u_i^j(0)$ from fictitious time $\eta = 0$ to a chosen fictitious time η_f. Also, stopping criterion for this numerical integration is:

$$\sqrt{\sum_{i,j=1}^{m,n} [u_i^j(s+1) - u_i^j(s)]^2} \leq \varepsilon, \qquad (17)$$

where ε is a selected convergence criterion.

4 Numerical examples

Now, we some two examples to demonstrate the power of FTIM for solving the TFADE.

Example 1: Take the following problem [47] by order $\beta = 0.5$, $\zeta = 1$ and $\kappa = 1$

$${}^{ABC}\mathfrak{D}_{0^+,t}^{\beta} u(x,t) = \zeta u_{xx}(x,t) - \kappa u_x(x,t) + F(x,t),$$

where

$$F(x,t) = 2\left(\frac{N(\beta)}{1-\beta}\right) x(x-1)t^2 E_{\beta,3}\left[\frac{-\beta}{1-\beta} t^\beta\right] - 2t^2 + (2x-1)t^2,$$

We apply our method to solve this example for parameters $\mu = 111$ and $\lambda = 0.1$. Also, we use the number of grids

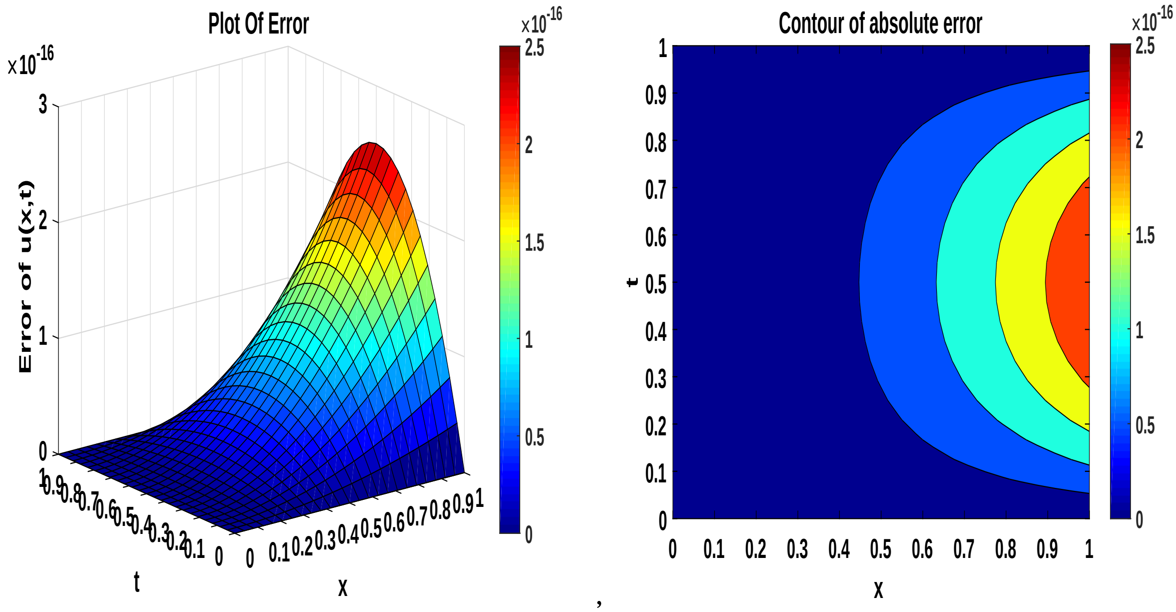

Figure 2: Plots of the absolute errors and contour plot under $\beta = 0.5$ for example 1.

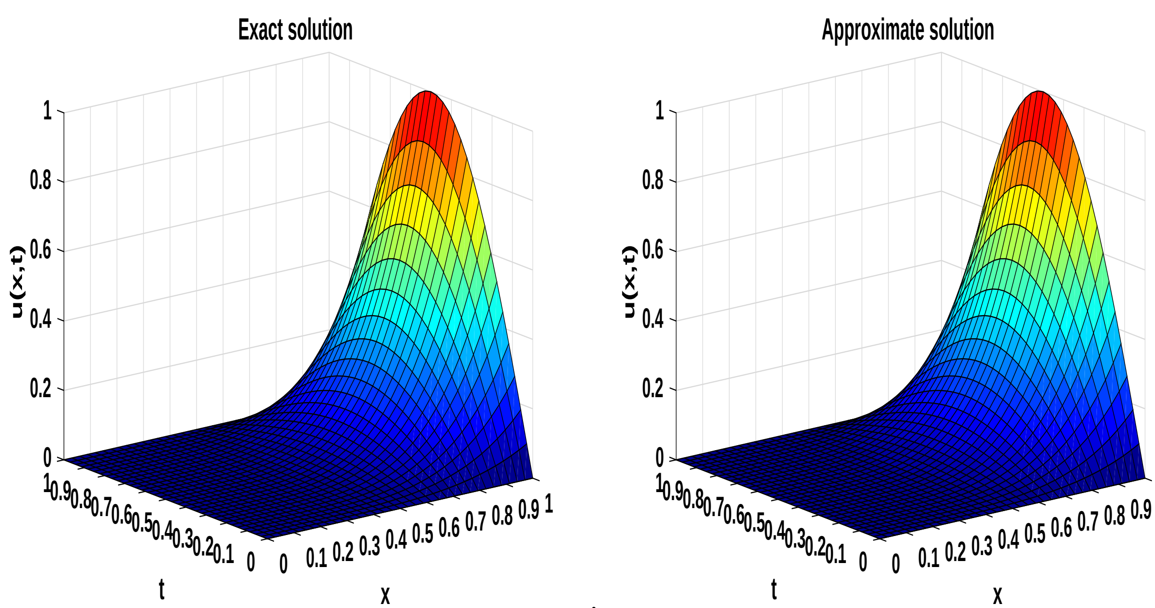

Figure 3: Plots of the exact-solutions by $\beta = 0.4$ for example 2.

$m = 25$ and $n = 25$ in each coordinates of space and time, respectively The initial guess and step size for η are considered as $u_i^j(0) = 10^{-5}$ and $\Delta\eta = 10^{-6}$. Indeed, supposed domainfor this problem is $\Omega = [0, 1] \times [0, 1]$. Figure 1 is assigned to depict the exact solution $u(x, t) = x(x-1)t^2$ and the approximate solutions derived by the cuurent scheme. One can see the capability of the presented method for solving this problem in Figure 2. This figure depicts that the error gained by our algorithm is about 2.5×10^{-16}. This error is much nicer than the error of the described method in [43] which is about 1×10^{-7}.

Example 2: Suppose the below equation [43] by $\zeta = 1$ and $\kappa = 1$

$$ {}^{ABC}\mathcal{D}^{\beta}_{0^+,t} u(x, t) = \zeta u_{xx}(x, t) - \kappa u_x(x, t) + F(x, t), $$

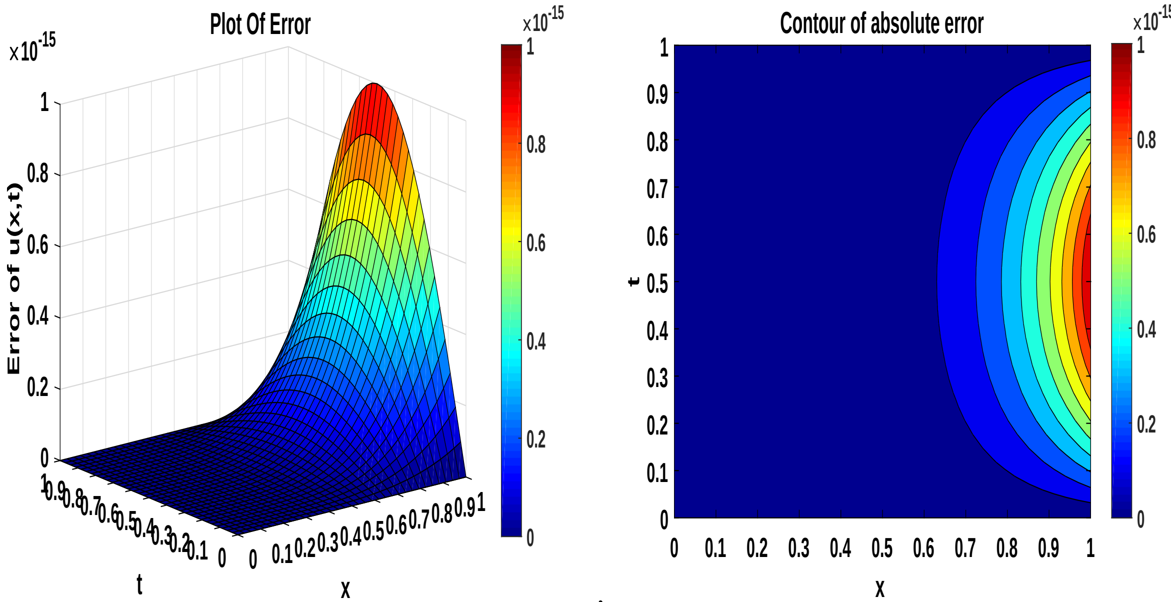

Figure 4: Plots of the absolute errors and contour plot under $\beta = 0.4$ for example 2.

where

$$F(x,t) = 120\left(\frac{N(\beta)}{1-\beta}\right)t^5 sin(\pi x)E_{\beta,6}\left[\frac{-\beta}{1-\beta}t^\beta\right] + \pi t^5(\pi sin(\pi x) + cos(\pi x)),$$

With the time fractional order $\beta = 0.4$. By selecting the important parameters $\mu = 18$ and $\lambda = 1$ we are able to manage the stableness and convergency rate of the scheme, respectively. To implement the GPS we choose the initial guess $u_i^j(0) = 0.0001$. The approximate solutions and the exact solution $u(x,t) = t^5 sin(\pi x)$ for $m = n = 35$ and $\Delta\eta = 10^{-3}$ are shown in Figure 3. Figure 4 is dedicated to reveal the gained low error by our method under the mentionad parameters. This figure illustrates that the error gained by the presented scheme is about 1×10^{-15}. This error is much reliable than the error of the utilized scheme in [43] which is about 1×10^{-6}.

5 Conclusion

In this study the fractional Advection-Diffusion equation is transformed into a new type of functional partial differential equations in a new space with one additional dimension by introducing a fictitious coordinate which has an important role in the presented method. After that, a semi-discretization is implemented on the new equation. Then the group preserving scheme as a numerical approach was applied to integrate a system of the first order of ordinary differential equations(ODEs) by selecting an initial guess. Some numerical examples were solved, which show that the current scheme is applicable and powerful for solving the TFADE involving Atangana-Baleanu-Caputo Derivative.

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