Hindawi
Journal of Mathematics
Volume 2020, Article ID 1632526, 8 pages
https://doi.org/10.1155/2020/1632526

Research Article

On p-Hybrid Wardowski Contractions

Erdal Karapınar,[1,2,3] **Hassen Aydi**,[4,5] **and Andreea Fulga**[6]

[1]ETSI Division of Applied Mathematics, Thu Dau Mot University, Thủ Dầu Một, Binh Duong Province, Vietnam
[2]Department of Medical Research, China Medical University, Taichung, Taiwan
[3]Department of Mathematics, Çankaya University, Ankara 06790, Turkey
[4]Nonlinear Analysis Research Group, Ton Duc Thang University, Ho Chi Minh City, Vietnam
[5]Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam
[6]Department of Mathematics and Computer Sciences, Transilvania University of Brasov, Brasov, Romania

Correspondence should be addressed to Hassen Aydi; hassen.aydi@tdtu.edu.vn

Received 11 May 2020; Accepted 25 July 2020; Published 24 August 2020

Academic Editor: Ljubisa Kocinac

The goal of this work is to introduce the concept of p-hybrid Wardowski contractions. We also prove related fixed-point results. Moreover, some illustrated examples are given.

1. Introduction

Let $\mathscr{G}$ represent the collection of functions $G\colon (0,\infty) \longrightarrow \mathbb{R}$ so that

(i) (G_1) G is strictly increasing
(ii) (G_2) for each sequence $\{\eta_n\}$ in $(0,\infty)$, $\lim_{n\longrightarrow\infty}\eta_n = 0$ iff $\lim_{n\longrightarrow\infty}G(\eta_n) = -\infty$
(iii) (G_3) there is $k \in (0,1)$ so that $\lim_{n\longrightarrow\infty}\eta^k G(\eta) = 0$

Definition 1 (see [1]). A mapping $\mathscr{T}\colon (\mathscr{M},d) \longrightarrow (\mathscr{M},d)$ is called a Wardowski contraction if there exist $\tau > 0$ and $G \in \mathscr{G}$ such that for all $\nu,\omega \in \mathscr{M}$,

$$d(\mathscr{T}\nu,\mathscr{T}\omega) > 0 \Longrightarrow \tau + G(d(\mathscr{T}\nu,\mathscr{T}\omega)) \leq G(d(\nu,\omega)). \tag{1}$$

Example 1 (see [1]). The functions $G\colon (0,\infty) \longrightarrow \mathbb{R}$ defined by

(1) $G(x) = \ln x$
(2) $G(x) = \ln x + x$
(3) $G(x) = -1/\sqrt{x}$
(4) $G(x) = \ln(x^2 + x)$

belong to $\mathscr{G}$.

Wardowski [1] introduced a new proper generalization of Banach contraction. For other related papers in the literature, see [2–10]. The main result of Wardowski is as follows.

Theorem 1 (see [1]). *Let $(\mathscr{M},d)$ be a complete metric space, and let $\mathscr{T}\colon \mathscr{M} \longrightarrow \mathscr{M}$ be an G-contraction. Then, Υ has a unique fixed point, say z, in $\mathscr{M}$ and for any point $\sigma \in \mathscr{M}$, the sequence $\{\Upsilon^j\sigma\}$ converges to z.*

Theorem 2 (see [11]). *Let $(\mathscr{M},d)$ be a complete metric space and $\mathscr{T}\colon \mathscr{M} \longrightarrow \mathscr{M}$ be a given mapping such that*

$$d(\mathscr{T}\nu,\mathscr{T}\omega) \leq \sigma_1 d(\nu,\omega) + \sigma_2 d(\nu,\mathscr{T}\nu) + \sigma_3 d(\omega,\mathscr{T}\omega) + \sigma_4\left[\frac{d(\nu,\mathscr{T}\omega) + d(\omega,\mathscr{T}\nu)}{2}\right], \tag{2}$$

for all $\nu,\omega \in \mathscr{M}$, where σ_i, $i = 1,2,3,4$, are nonnegative real numbers such that $\sum_{i=1}^4 \sigma_i < 1$. Then, $\mathscr{T}$ admits a unique fixed point in $\mathscr{M}$.

In the paper [12], the concept of interpolative Hardy–Rogers-type contractions was introduced.

Definition 2 (see [12]). On a metric space $(\mathscr{M}, d)$, a self-mapping $\mathscr{T}\colon \mathscr{M} \longrightarrow \mathscr{M}$ is an interpolative Hardy–Rogers-type contraction if there exist $\lambda \in [0,1)$ and $\sigma_1, \sigma_2, \sigma_3 \in (0,1)$ with $\sigma_1 + \sigma_2 + \sigma_3 < 1$, such that

$$d(\mathscr{T}\nu, \mathscr{T}\omega) \leq \lambda (d(\nu,\omega))^{\sigma_1} (d(\nu, \mathscr{T}\nu))^{\sigma_2} (d(\omega, \mathscr{T}\omega))^{\sigma_3} \cdot \left(\frac{d(\nu, \mathscr{T}\omega) + d(\omega, \mathscr{T}\nu)}{2}\right)^{1-\sigma_1-\sigma_2-\sigma_3}, \tag{3}$$

for all $\nu, \omega \in \mathscr{M}/\mathrm{F}_{\mathscr{T}}(\mathscr{M})$, where $_{\mathscr{T}}(\mathscr{M}) = \{\zeta \in \mathscr{M}\colon \mathscr{T}\zeta = \zeta\}$.

Theorem 3 (see [12]). *Let $(\mathscr{M}, d)$ be a complete metric space and $\mathscr{T}$ be an interpolative Hardy–Rogers-type contraction. Then, $\mathscr{T}$ has a fixed point in $\mathscr{M}$.*

The interpolation concept was used in other new papers related to fixed-point theory. For example, see [13–17]. In this paper, we consider new contractive type self-mappings, named as p-hybrid Wardowski contractions. Our fixed-point results will be supported by concrete examples.

2. Main Results

Let $(\mathscr{M}, d)$ be a metric space and $\mathscr{T}$ be a self-mapping on this space. For $p \geq 0$ and $\kappa_i \geq 0, i = 1,2,3,4$, such that $\sum_{i=1}^{4} \kappa_i = 1$, we define the following expression:

$$\mathscr{A}^p_{\mathscr{T}}(\nu,\omega) = \begin{cases} \left[\kappa_1 (d(\nu,\omega))^p + \kappa_2 (d(\nu,\mathscr{T}\nu))^p + \kappa_3 (d(\omega,\mathscr{T}\omega))^p + \kappa_4 \left(\dfrac{d(\omega,\mathscr{T}\nu) + d(\nu,\mathscr{T}\omega)}{2}\right)^p\right]^{1/p}, \\ \text{for } p > 0, \quad \nu,\omega \in \mathscr{M} \\ [d(\nu,\omega)]^{\kappa_1} [d(\nu,\mathscr{T}\nu)]^{\kappa_2} [d(\omega,\mathscr{T}\omega)]^{\kappa_3} \left[\dfrac{d(\nu,\mathscr{T}\omega) + d(\omega,\mathscr{T}\nu)}{2}\right]^{\kappa_4}, \\ \text{for } p = 0, \quad \nu,\omega \in \mathscr{M}/_{\mathscr{T}}(\mathscr{M}). \end{cases} \tag{4}$$

On the other hand, let $\mathscr{B}$ represent the set of functions $G\colon (0,\infty) \longrightarrow \mathbb{R}$ such that

(i) (G_a) G is strictly increasing

(ii) (G_b) there exists $\tau > 0$ such that $\tau + \lim_{t\longrightarrow t_0} \inf G(t) > \lim_{t\longrightarrow t_0} \sup G(t)$, for every $t_0 > 0$

Definition 3. A mapping $\mathscr{T}\colon (\mathscr{M}, d) \longrightarrow (\mathscr{M}, d)$ is called a p-hybrid Wardowski contraction, if there is $G \in \mathscr{B}$ such that

$$\begin{aligned} d(\mathscr{T}\nu, \mathscr{T}\omega) > 0 \text{ implies } \tau + G(d(\mathscr{T}\nu, \mathscr{T}\omega)) \\ \leq G\left(\mathscr{A}^p_{\mathscr{T}}(\nu,\omega)\right), \text{for every } p > 0. \end{aligned} \tag{5}$$

In particular, if inequality (5) holds for $p = 0$, we say the mapping $\mathscr{T}$ is a 0-hybrid Wardowski contraction.

Theorem 4. *A p-hybrid Wardowski contraction self-mapping on a complete metric space admits exactly one fixed point in $\mathscr{M}$.*

Proof. Taking an arbitrary point $\nu_0 \in \mathscr{M}$, we consider the sequence $\{\nu_n\}$ defined by the relation $\nu_n = \mathscr{T}\nu_{n-1}$, $n \geq 1$. According to this construction, it is easy to see that if there is n_0 so that $\nu_{n_0} = \nu_{n_0+1} = \mathscr{T}\nu_{n_0}$, ν_{n_0} turns into a fixed point of T. We shall presume that for all $n \in \mathbb{N}_0$,

$$\nu_{n+1} \neq \nu_n \Longleftrightarrow d(\nu_{n+1}, \nu_n) = d(\mathscr{T}\nu_n, \mathscr{T}\nu_{n-1}) > 0. \tag{6}$$

On account of (4), for $\nu = \nu_n$ and $\omega = \nu_{n-1}$, we have that

$$\begin{aligned} \mathscr{A}^p_{\mathscr{T}}(\nu_n, \nu_{n-1}) &= \left[\kappa_1 (d(\nu_n,\nu_{n-1}))^p + \kappa_2 (d(\nu_n,\mathscr{T}\nu_n))^p + \kappa_3 (d(\nu_{n-1},\mathscr{T}\nu_{n-1}))^p + \kappa_4 \left(\frac{d(\nu_n,\mathscr{T}\nu_{n-1}) + d(\nu_{n-1},\mathscr{T}\nu_n)}{2}\right)^p\right]^{1/p} \\ &= \left[\kappa_1 (d(\nu_n,\nu_{n-1}))^p + \kappa_2 (d(\nu_n,\nu_{n+1}))^p + \kappa_3 (d(\nu_{n-1},\nu_n))^p + \kappa_4 \left(\frac{d(\nu_n,\nu_n) + d(\nu_{n-1},\nu_{n+1})}{2}\right)^p\right]^{1/p}, \\ &\leq \left[\kappa_1 (d(\nu_n,\nu_{n-1}))^p + \kappa_2 (d(\nu_n,\nu_{n+1}))^p + \kappa_3 (d(\nu_{n-1},\nu_n))^p + \kappa_4 \left(\frac{d(\nu_{n-1},\nu_n) + d(\nu_n,\nu_{n+1})}{2}\right)^p\right]. \end{aligned} \tag{7}$$

Denoting by $\chi_n = d(\nu_{n-1}, \nu_n)$, we have

$$\mathcal{A}_{\mathcal{T}}^{p}(\nu_n, \nu_{n-1}) = \left[(\kappa_1 + \kappa_3)\chi_n^p + \kappa_2\chi_{n+1}^p + \kappa_4\left(\frac{\chi_n+\chi_{n+1}}{2}\right)^p\right]^{1/p}, \quad (8)$$

and from (5), it follows that

$$\tau + G(d(\mathcal{T}\nu_{n-1}, \mathcal{T}\nu_n)) \leq G\left(\mathcal{A}_{\mathcal{T}}^{p}(\nu_{n-1}, \nu_n)\right) \leq G\left(\left[\kappa_1 (d(\nu_n, \nu_{n-1}))^p + \kappa_2 (d(\nu_n, \nu_{n+1}))^p + \kappa_3 (d(\nu_{n-1}, \nu_n))^p + \kappa_4\left(\frac{d(\nu_{n-1}, \nu_n) + d(\nu_n, \nu_{n+1})}{2}\right)^{1/p}\right]\right), \quad (9)$$

which gives us

$$G(\chi_{n+1}) = G\,d(\nu_n, \nu_{n+1}) = G(d(\mathcal{T}\nu_{n-1}, \mathcal{T}\nu_n)) \leq G\left(\left[(\kappa_1 + \kappa_3)\chi_n^p + \kappa_2\chi_{n+1}^p + \kappa_4\left(\frac{\chi_n + \chi_{n+1}}{2}\right)^p\right]^{1/p}\right) - \tau. \quad (10)$$

If $\max\{\chi_n, \chi_{n+1}\} = \chi_{n+1}$, then the above inequality becomes

$$G(\chi_{n+1}) \leq G\left(\left[(\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4)\chi_{n+1}^p\right]^{1/p}\right) - \tau < G(\chi_{n+1}), \quad (11)$$

which is a contradiction. Consequently, $\max\{\chi_n, \chi_{n+1}\} = \chi_n$ and then there exists $\chi \geq 0$ such that

$$\lim_{n\longrightarrow\infty} \chi_n = \chi. \quad (12)$$

Supposing that $\chi > 0$, we have $\lim_{n\longrightarrow\infty}\mathcal{A}_{\mathcal{T}}^{p}(\nu_{n-1}, \nu_n) = \chi$ and by (G_b), we obtain

$$\tau + G(\chi + 0) \leq G(\chi + 0), \quad (13)$$

which is a contradiction. Therefore,

$$\lim_{n\longrightarrow\infty} d(\nu_{n-1}, \nu_n) = 0. \quad (14)$$

In order to prove that $\{\nu_n\}$ is a Cauchy sequence in $(\mathcal{M}, d)$, we suppose that there exist $\epsilon > 0$ and the sequences $\{n_*(k)\}, \{m_*(k)\}$ of positive integers, with $n_*(k) > m_*(k) > k$ such that

$$d\left(\nu_{n_*(k)}, \nu_{m_*(k)}\right) \geq \varepsilon, \quad d\left(\nu_{n_*(k)-1}, \nu_{m_*(k)}\right) < \varepsilon, \quad (15)$$

for any $k \in \mathbb{N}$.

Thus, we have

$$\varepsilon \leq d\left(\nu_{n_*(k)}, \nu_{m_*(k)}\right) \leq d\left(\nu_{n_*(k)}, \nu_{n_*(k)-1}\right) + d\left(\nu_{n_*(k)-1}, \nu_{m_*(k)}\right) < d\left(\nu_{n_*(k)}, \nu_{n_*(k)-1}\right) + \varepsilon. \quad (16)$$

When $k \longrightarrow \infty$, using (14) and (15), it follows

$$\lim_{k\longrightarrow\infty} d\left(\nu_{n_*(k)}, \nu_{m_*(k)}\right) = \varepsilon. \quad (17)$$

By using the triangle inequality, we have

$$0 \leq \left|d\left(\nu_{n_*(k)+1}, \nu_{m_*(k)+1}\right) - d\left(\nu_{n_*(k)}, \nu_{m_*(k)}\right)\right|, \leq d\left(\nu_{n_*(k)+1}, \nu_{n_*(k)}\right) + d\left(\nu_{m_*(k)}, \nu_{m_*(k)+1}\right), \quad (18)$$

$$\lim_{k\longrightarrow\infty}\left|d\left(\nu_{n_*(k)+1}, \nu_{m^*(k)+1}\right) - d\left(\nu_{n_*(k)}, \nu_{m_*(k)}\right)\right| \leq \lim_{k\longrightarrow\infty}\left[d\left(\nu_{n_*(k)+1}, \nu_{n_*(k)}\right) + d\left(\nu_{m_*(k)}, \nu_{m_*(k)+1}\right)\right] = 0. \quad (19)$$

So,

$$\lim_{k\longrightarrow\infty} d\left(\nu_{n_*(k)+1}, \nu_{m_*(k)+1}\right) = \lim_{k\longrightarrow\infty} d\left(\nu_{n_*(k)}, \nu_{m_*(k)}\right) = \epsilon > 0. \quad (20)$$

Moreover, since

$$\epsilon = d\left(\nu_{n_*(k)}, \nu_{m_*(k)}\right) \leq d\left(\nu_{n_*(k)}, \nu_{m_*(k)+1}\right) + d\left(\nu_{m_*(k)+1}, \nu_{m_*(k)}\right), \quad \epsilon = d\left(\nu_{n_*(k)}, \nu_{m_*(k)}\right) \leq d\left(\nu_{n_*(k)}, \nu_{n_*(k)+1}\right) + d\left(\nu_{m_*(k)}, \nu_{n_*(k)+1}\right), \quad (21)$$

we have

$$\lim_{n\longrightarrow\infty} d\left(\nu_{n_*(k)}, \nu_{m_*(k)+1}\right) = \lim_{n\longrightarrow\infty} d\left(\nu_{m_*(k)}, \nu_{n_*(k)+1}\right) = \varepsilon. \quad (22)$$

So, the inequality

$$d\left(\mathcal{T}\nu_{n_*(k)}, \mathcal{T}\nu_{m_*(k)}\right) = d\left(\nu_{n_*(k)+1}, \nu_{m_*(k)+1}\right) > 0 \quad (23)$$

occurs for all $k \geq N$, and using (5), there exists $\tau > 0$ such that

$$\tau + G\left(d\left(\nu_{n_*(k)+1}, \nu_{m_*(k)+1}\right)\right) \leq G\left(\mathcal{A}_{\mathcal{T}}^{p}\left(\nu_{n_*(k)}, \nu_{m_*(k)}\right)\right), \quad (24)$$

where

$$\mathscr{A}_{\mathscr{T}}^{p}\left(\nu_{n_*(k)},\nu_{m_*(k)}\right)=\left[\kappa_1\left(d\left(\nu_{n_*(k)},\nu_{m_*(k)}\right)\right)^p+\kappa_2\left(d\left(\nu_{n_*(k)},\nu_{n_*(k)+1}\right)\right)^p+\kappa_3\left(d\left(\nu_{m_*(k)},\nu_{m_*(k)+1}\right)\right)^p\right.$$
$$\left.+\kappa_4\left(\frac{d\left(\nu_{n_*(k)},\nu_{m_*(k)+1}\right)+d\left(\nu_{m_*(k)},\nu_{n_*(k)+1}\right)}{2}\right)^p\right]^{1/p}. \quad (25)$$

Moreover, since the function G is increasing, we have

$$\begin{aligned}
\tau+\liminf_{k\longrightarrow\infty}G\left(\left(\kappa_3+\kappa_4\right)^{1/p}d\left(\nu_{n_*(k)+1},\nu_{m_*(k)+1}\right)\right)
&\le\tau+\liminf_{k\longrightarrow\infty}G\left(d\left(\mathscr{T}\nu_{n_*(k)},\mathscr{T}\nu_{m_*(k)}\right)\right)\\
&\le\liminf_{k\longrightarrow\infty}G\left(\mathscr{A}_{\mathscr{T}}^{p}\left(\nu_{n_*(k)},\nu_{m_*(k)}\right)\right)\\
&\le\limsup_{n\longrightarrow\infty}G\left(\mathscr{A}_{\mathscr{T}}^{p}\left(\nu_{n_*(k)},\nu_{m_*(k)}\right)\right).
\end{aligned} \quad (26)$$

And letting $k\longrightarrow\infty$,

$$\tau+G(\varepsilon+)\le G(\varepsilon+). \quad (27)$$

That is a contradiction, so $\varepsilon=0$ and then, $\varepsilon=0$. Consequently, the sequence $\{\nu_n\}$ is Cauchy and by completeness of $\mathscr{M}$, it converges to some point $\zeta\in\mathscr{M}$.

There exists a subsequence $\{\nu_{n_i}\}$ such that $\mathscr{T}\nu_{n_i}=\mathscr{T}\zeta$ for all $i\in\mathbb{N}$; then,

$$d(\zeta,\mathscr{T}\zeta)=\lim_{i\longrightarrow\infty}d\left(\nu_{n_i+1},\mathscr{T}\zeta\right)=\lim_{i\longrightarrow\infty}d\left(\mathscr{T}\nu_{n_i},\mathscr{T}\zeta\right)=0. \quad (28)$$

On the contrary, if there is a natural number N such that $\mathscr{T}\nu_n\ne\mathscr{T}\zeta$ for all $n\ge N$, applying (5), for $\nu=\nu_n$ and $\omega=\zeta$, we have

$$\begin{aligned}
\tau+G\left(\left(\kappa_3+\kappa_4\right)^{1/p}d\left(\nu_{n+1},\mathscr{T}\zeta\right)\right)&\le\tau+G\left(d\left(\nu_{n+1},\mathscr{T}\zeta\right)\right)\\
&=\tau+G\left(d\left(\mathscr{T}\nu_n,\mathscr{T}\zeta\right)\right)\le G\left(\mathscr{A}\left(\nu_n,\zeta\right)\right),
\end{aligned} \quad (29)$$

where

$$\mathscr{A}_{\mathscr{T}}^{p}\left(\nu_n,\zeta\right)=\left[\kappa_1\left(d\left(\nu_n,\zeta\right)\right)^p+\kappa_2\left(d\left(\nu_n,\mathscr{T}\nu_n\right)\right)^p+\kappa_3\left(d\left(\zeta,\mathscr{T}\zeta\right)\right)^p+\kappa_4\left(\frac{d\left(\nu_n,\mathscr{T}\zeta\right)+d\left(\zeta,\mathscr{T}\nu_n\right)}{2}\right)^p\right]^{1/p},$$
$$=G\left(\left[\kappa_1\left(d\left(\nu_n,\zeta\right)\right)^p+\kappa_2\left(d\left(\nu_n,\nu_{n+1}\right)\right)^p+\kappa_3\left(d\left(\zeta,\mathscr{T}\zeta\right)\right)^p++\kappa_4\left(\frac{d\left(\nu_n,\mathscr{T}\zeta\right)+d\left(\zeta,\mathscr{T}\nu_n\right)}{2}\right)^p\right]\right)^{1/p}. \quad (30)$$

We suppose that $\zeta\ne\mathscr{T}\zeta$. Inasmuch as

$$\lim_{n\longrightarrow\infty}d\left(\nu_n,T\zeta\right)=d(\zeta,T\zeta),$$

$$\begin{aligned}
d\lim_{n\longrightarrow\infty}\mathscr{A}\left(\nu_n,\zeta\right)
&=\lim_{n\longrightarrow\infty}\left[\kappa_1\left(d\left(\nu_n,\zeta\right)\right)^p+\kappa_2\left(d\left(\nu_n,\nu_{n+1}\right)\right)^p+\kappa_3\left(d\left(\zeta,\mathscr{T}\zeta\right)\right)^p\right.\\
&\quad\left.+\kappa_4\left(\frac{d\left(\nu_n,\mathscr{T}\zeta\right)+d\left(\zeta,\mathscr{T}\nu_n\right)}{2}\right)^p\right]\Bigg)^{1/p}\\
&=\left(\kappa_3+\kappa_4\right)^{1/p}d(\zeta,\mathscr{T}\zeta).
\end{aligned} \quad (31)$$

Letting $n\longrightarrow\infty$ in inequality (29), we find that

$$\begin{aligned}
\tau+\liminf_{t\longrightarrow d(\zeta,\mathscr{T}\zeta)}G\left(\left(\kappa_3+\kappa_4\right)^{1/p}t\right)&\le\tau+\liminf_{t\longrightarrow d(\zeta,\mathscr{T}\zeta)}G(t)\\
&<\liminf_{t\longrightarrow d(\zeta,\mathscr{T}\zeta)}G\left(\left(\kappa_3+\kappa_4\right)^{1/p}t\right)<\limsup_{t\longrightarrow d(\zeta,\mathscr{T}\zeta)}G\left(\left(\kappa_3+\kappa_4\right)^{1/p}t\right),
\end{aligned} \quad (32)$$

which contradicts G_b. Therefore, $\mathcal{T}\zeta = \zeta$.

We claim now that T admits only one fixed point. If there exists another point $\xi \in \mathcal{M}$, $\xi \neq \zeta$, such that $\xi = \mathcal{T}\xi$, then $d(\xi,\zeta) = d(\mathcal{T}\xi,\mathcal{T}\zeta) > 0$ and we have

$$\begin{aligned}\tau + G(d(\xi,\zeta)) &= \tau + G(d(\mathcal{T}\xi,\mathcal{T}\zeta)) \leq G\left(\mathcal{A}_{\mathcal{T}}^{p}(\xi,\zeta)\right)\\ &= G\left(\left[\kappa_1(d(\xi,\zeta))^p + \kappa_2(d(\xi,\mathcal{T}\xi))^p + \kappa_3(d(\zeta,\mathcal{T}\zeta))^p + \kappa_4\left(\frac{d(\xi,\mathcal{T}\zeta)+d(\zeta,\mathcal{T}\xi)}{2}\right)^p\right]^{1/p}\right),\\ &= G\left(\left[\kappa_1(d(\xi,\zeta))^p + \kappa_2(d(\xi,\xi))^p + \kappa_3(d(\zeta,\zeta))^p + \kappa_4\left(\frac{d(\xi,\zeta)+d(\zeta,\xi)}{2}\right)^p\right]^{1/p}\right),\\ &= G\left((\kappa_1+\kappa_4)^{1/p} d(\xi,\zeta)\right),\\ &\leq G(d(\xi,\zeta)),\end{aligned} \tag{33}$$

which is a contradiction. □

Example 2. Let $\mathcal{M} = [0,1]$ be endowed with the standard metric $d(\nu,\omega) = |\nu - \omega|$. Let the mapping $\mathcal{T}: \mathcal{M} \longrightarrow \mathcal{M}$ be defined by $\mathcal{T} = \begin{cases} x/8 & \text{for } x \in [0,1) \\ 1/4 & \text{for } x = 1 \end{cases}$. Take $p = 2$, $\tau = \ln 4/3$, $\kappa_1 = 1/9$, $\kappa_2 = \kappa_4 = 6/81$, $\kappa_3 = 60/81$, and $G(t) = \ln t$. Then, we have the following:

For $x, y \in [0,1)$,

$$\ln\frac{4}{3} + \ln G(d(x,y)) \ln\frac{4|x-y|}{24} < \ln\frac{|x-y|}{3} = \ln\left(\frac{|x-y|^2}{9}\right)^{1/2} = \ln\left(\kappa_1 d(x,y)^2\right)^{1/2} < \ln \mathcal{A}_{\mathcal{T}}^{2}(x,y). \tag{34}$$

For $x \in [0,1)$ and $y = 1$,

$$\ln\frac{4}{3} + \ln G(d(x,1)) = \ln\frac{4|x-2|}{24} < \ln\left(\frac{7}{9}\cdot\frac{3}{4}\right) = \ln\left(\frac{49}{81} d\left(1,\frac{1}{4}\right)^2\right)^{1/2} < \ln \mathcal{A}_{\mathcal{T}}^{2}(x,1). \tag{35}$$

Thus, all assumptions of Theorem 4 hold, and $\mathcal{T}$ has a unique fixed point. On the other hand, for $x = 7/8$ and $y = 1$, we have

$$d\left(\mathcal{T}\frac{7}{8},\mathcal{T}1\right) = d\left(\frac{7}{64},\frac{1}{4}\right) = \frac{9}{64} > \frac{1}{8} = d\left(\frac{7}{8},1\right). \tag{36}$$

Thus, it is not a Wardowski contraction, since for every function $G \in \mathcal{B}$ and $\tau > 0$

$$\tau + G\left(d\left(\mathcal{T}\frac{7}{8},\mathcal{T}1\right)\right) > G\left(d\left(\frac{7}{8},1\right)\right). \tag{37}$$

Theorem 5. *A 0-hybrid Wardowski contraction self-mapping on a complete metric space admits a fixed point in $\mathcal{M}$ provided that for each sequence $\{\eta_n\}$ in $(0,\infty)$, $\lim_{n\longrightarrow\infty}\eta_n = 0$ iff $\lim_{n\longrightarrow\infty}G(\eta_n) = -\infty$.*

Proof. Following the same reasoning from the proof of the previous theorem, we can assume that for all $n \in \mathbb{N}_0$,

$$\nu_{n+1} \neq \nu_n \Longleftrightarrow d(\nu_{n+1},\nu_n) > 0. \tag{38}$$

On account of (4), for $\nu = \nu_n$ and $\omega = \nu_{n-1}$, we have that

Table 1: Definition of metric d.

$d(\nu,\omega)$	x	y	z	t
x	0	3	3	2
y	3	0	3	1
z	3	3	0	2
t	2	1	2	0

$$\begin{aligned}\mathscr{A}^0_{\mathcal{T}}(\nu_n,\nu_{n-1}) &= [d(\nu_n,\nu_{n-1})]^{\kappa_1}[d(\nu_n,\mathcal{T}\nu_n)]^{\kappa_2}[d(\nu_{n-1},\mathcal{T}\nu_{n-1})]^{\kappa_3}\left[\frac{d(\nu_n,\mathcal{T}\nu_{n-1})+d(\nu_{n-1},\mathcal{T}\nu_n)}{2}\right]^{\kappa_4}\\ &= [d(\nu_n,\nu_{n-1})]^{\kappa_1}[d(\nu_n,\nu_{n+1})]^{\kappa_2}[d(\nu_{n-1},\nu_n)]^{\kappa_3}\left[\frac{d(\nu_n,\nu_n)+d(\nu_{n-1},\nu_{n+1})}{2}\right]^{\kappa_4},\\ &\le [d(\nu_n,\nu_{n-1})]^{\kappa_1}[d(\nu_n,\nu_{n+1})]^{\kappa_2}[d(\nu_{n-1},\nu_n)]^{\kappa_3}\left[\frac{d(\nu_{n-1},\nu_n)+d(\nu_n,\nu_{n+1})}{2}\right]^{\kappa_4}.\end{aligned} \tag{39}$$

Using the same notation, $\chi_n = d(\nu_{n-1},\nu_n)$, and taking into account (G_a), by (5), we have

$$\tau + G(\chi_{n+1}) \le G\left(\chi_n^{\kappa_1+\kappa_3}\chi_{n+1}^{\kappa_2}\left(\frac{\chi_n+\chi_{n+1}}{2}\right)^{\kappa_4}\right) - \tau. \tag{40}$$

We can remark that the case $\max\{\chi_n,\chi_{n+1}\} = \chi_{n+1}$, is not possible since the above inequality becomes

$$G(\chi_{n+1}) \le G\left(\chi_{n+1}^{\kappa_1+\kappa_2+\kappa_3+\kappa_4}\right) - \tau < G(\chi_{n+1}), \tag{41}$$

a contradiction. Therefore, $\chi_n > \chi_{n+1}$ for all $n\in\mathbb{N}$, and then, there exists $\chi \ge 0$ such that

$$\lim_{n\longrightarrow\infty}\chi_n = \lim_{n\longrightarrow\infty} d(\nu_{n-1},\nu_n) = \chi. \tag{42}$$

We claim that $\chi = 0$. Indeed, if we suppose that $\chi > 0$, taking the limit as $n\longrightarrow\infty$ in (40), we have

$$\tau + G(\chi+0) \le G(\chi+0), \tag{43}$$

which contradicts $(G_2.)$ We conclude that

$$\chi = \lim_{n\infty} d(\nu_{n-1},\nu_n) = 0. \tag{44}$$

Let $n\in\mathbb{N}$ and $j\ge 1$ now; we have

$$\begin{aligned}\mathscr{A}^0_{\mathcal{T}}(\nu_n,\nu_{n+j}) &= [d(\nu_n,\nu_{n+j})]^{\kappa_1}[d(\nu_n,\mathcal{T}\nu_n)]^{\kappa_2}[d(\nu_{n+j},\mathcal{T}\nu_{n+j})]^{\kappa_3}\left[\frac{d(\nu_n,\mathcal{T}\nu_{n+j})+d(\nu_{n+j},\mathcal{T}\nu_n)}{2}\right]^{\kappa_4}\\ &= [d(\nu_n,\nu_{n+j})]^{\kappa_1}[d(\nu_n,\nu_{n+1})]^{\kappa_2}[d(\nu_{n+j},\nu_{n+j+1})]^{\kappa_3}\left[\frac{d(\nu_n,\nu_{n+j+1})+d(\nu_{n+j},\nu_{n+1})}{2}\right]^{\kappa_4}\\ &= 0.\end{aligned} \tag{45}$$

And taking into account (44),

$$\lim_{n\longrightarrow\infty}\mathscr{A}^0_{\mathcal{T}}(\nu_n,\nu_{n+j}) = 0. \tag{46}$$

Therefore, $\lim_{n\longrightarrow\infty} G(\mathscr{A}^0_{\mathcal{T}}(\nu_n,\nu_{n+j})) = -\infty$ and since

$$\tau + \lim_{n\longrightarrow\infty} G\left(d\left(\nu_{n+1},\nu_{n+j+1}\right)\right) \le \lim_{n\longrightarrow\infty} G\left(\mathscr{A}^0_{\mathcal{T}}(\nu_n,\nu_{n+j})\right), \tag{47}$$

we obtain that $\lim_{n\longrightarrow\infty} G(d(\nu_n,\nu_{n+j})) = -\infty$ and so $\lim_{n\longrightarrow\infty} d(\nu_n,\nu_{n+j}) = 0$. Thus, $\{\nu_n\}$ is a Cauchy sequence on a complete metric space $(\mathscr{M},d)$ and there exists ζ such that $\lim_{n\longrightarrow\infty}\nu_n = \zeta$. Of course, it easy to see that, for $\nu=\nu_n$ and $\omega=\zeta$, we have

$$\lim_{n\longrightarrow\infty}\mathscr{A}^0_{\mathcal{T}}(\nu_n,\zeta) = 0. \tag{48}$$

If we suppose that there is a subsequence $\{\nu_{n_s}\}$ such that $\mathcal{T}\nu_{n_s} = \mathcal{T}\zeta$, then we have

$$\begin{aligned}0 = \lim_{n\longrightarrow\infty} d\left(\mathcal{T}\nu_{n_s},\mathcal{T}\zeta\right) &= \lim_{n\longrightarrow\infty} d\left(\nu_{n_s+1},\mathcal{T}\zeta\right)\\ &= d(\zeta,\mathcal{T}\zeta),\end{aligned} \tag{49}$$

which means that ζ is a fixed point of $\mathcal{T}$. Therefore, we can assume that $d(\mathcal{T}\nu_n, \mathcal{T}\zeta) > 0$ for every $n \in \mathbb{N}$, and by (5), we obtain

$$\tau + G(d(\mathcal{T}\nu_n, \mathcal{T}\zeta)) \le G(\mathcal{A}^0_{\mathcal{T}}(\nu_n, \zeta)). \tag{50}$$

Letting $n \longrightarrow \infty$ and taking into account the previous considerations, we have $\lim_{n\longrightarrow\infty} G\, d(\mathcal{T}\nu_n, \mathcal{T}\zeta) = -\infty$ and then $d(\zeta, \mathcal{T}\zeta) = \lim_{n\longrightarrow\infty} d(\mathcal{T}\nu_n, \mathcal{T}\zeta) = 0$. Consequently, ζ is a fixed point of $\mathcal{T}$. □

Example 3. Let $\mathcal{M} = \{x, y, z, t\}$ be a set endowed with the metric $d\colon \mathcal{M} \times \mathcal{M} \longrightarrow [0, \infty)$ (Table 1).

And the mapping $\mathcal{T}\colon \mathcal{M} \longrightarrow \mathcal{M}$ is defined as $T\colon \begin{pmatrix} x & y & z & t \\ x & x & t & t \end{pmatrix}$.

First, we remark that Theorem 1 is not satisfied, since for $\nu = y$ and $\omega = t$,

$$d(\mathcal{T}y, \mathcal{T}t) = d(x, t) = 2 > 1 = d(y, t). \tag{51}$$

Hence, for any $\tau > 0$ and $G \in \mathcal{B}$, we can write

$$\tau + G(d(\mathcal{T}y, \mathcal{T}t)) > G(d(y, t)). \tag{52}$$

Choosing $\tau = \ln 4/3$, $\kappa_1 = \kappa_2 = 7/16$, $\kappa_3 = \kappa_4 = 1/16$, and $G(t) = \ln t$, for $\nu = y$ and $\omega = z$, we have

$$\begin{aligned} \ln\frac{4}{3} + \ln d(\mathcal{T}y, \mathcal{T}z) &= \ln\left(\frac{4}{3} d(x,t)\right) = \ln\frac{8}{3} = 0,980829253 < 1,04792915 = \ln\left(3^{7/16} 3^{7/16} 2^{1/16} 2^{1/16}\right), \\ &< \ln\left(d(y,z)^{5/16} d(y,x)^{5/16} d(z,t)^{5/16} \left(\frac{d(y,t) + d(z,x)}{2}\right)^{1/16}\right), \\ &= \ln\left(d(y,z)^{5/16} d(y,\mathcal{T}y)^{5/16} d(z,\mathcal{T}z)^{5/16} \left(\frac{d(y,\mathcal{T}z) + d(z,\mathcal{T}y)}{2}\right)^{1/16}\right), \\ &= \ln \mathcal{A}^0_{\mathcal{T}}(y,z). \end{aligned} \tag{53}$$

3. Consequences

($C1$) Considering $G(t) = \ln t$ in Theorem 5 and $\sigma_i = e^{-\tau}\kappa_i$, we obtain Theorem 2.

($C2$) Considering $G(t) = \ln t$ in Theorem 5 and $\lambda = e^{-\tau}$, we obtain Theorem 3.

($C3$) Considering $G(t) = \ln t$ in Theorem 4, $\lambda = e^{-\tau}$, and $p = 1$, we obtain Theorem 3.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] D. Wardowski, "Fixed points of a new type of contractive mappings in complete metric spaces," *Fixed Point Theory and Applications*, vol. 2012, Article ID 94, 2012.

[2] E. Ameer, H. Aydi, M. Arshad, and M. De la Sen, "Hybrid cirić type graphic ?,?-contraction mappings with applications to electric circuit and fractional differential equations," *Symmetry*, vol. 12, no. 3, p. 467, 2020.

[3] Z. Ma, A. Asif, H. Aydi, S. U. Khan, and M. Arshad, "Analysis of F-contractions in function weighted metric spaces with an application," *Open Mathematics*, vol. 18, no. 1, pp. 582–594, 2020.

[4] A. Nastasi and P. Vetro, "A generalization of Reich's fixed point theorem for multi-valued mappings," *Filomat*, vol. 31, no. 11, pp. 3295–3305, 2017.

[5] A. Nastasi and P. Vetro, "Existence and uniqueness for a first-order periodic differential problem via fixed point results," *Results in Mathematics*, vol. 71, no. 3-4, pp. 889–909, 2017.

[6] P. Patle, D. Patel, H. Aydi, and S. Radenović, "ON H[+]Type multivalued contractions and applications in symmetric and probabilistic spaces," *Mathematics*, vol. 7, no. 2, p. 144, 2019.

[7] W. Shatanawi, E. Karapinar, H. Aydi, and A. Fulga, "Wardowski type contractions with applications on Caputo type nonlinear fractional differential equations," *University Politechnica of Bucharest Scientific Bulletin-Series A-Applied*, vol. 82, no. 2, pp. 157–170, 2020.

[8] C. Vetro and F. Vetro, "The class of F-contraction mappings with a measure of noncompactness," in *Advances in Nonlinear Analysis via the Concept of Measure of Noncompactness*, J. Banaś, M. Jleli, M. Mursaleen, B. Samet, and C. Vetro, Eds., Springer, Singapore, Singapore, 2017.

[9] F. Vetro, "A generalization of Nadler fixed point theorem," *Carpathian Journal of Mathematics*, vol. 31, no. 3, pp. 403–410, 2015.

[10] F. Vetro, "F-contractions of Hardy-Rogers-type and application to multistage decision," *Nonlinear Analysis: Modelling and Control*, vol. 21, no. 4, pp. 531–546, 2016.

[11] G. E. Hardy and T. D. Rogers, "A generalization of a fixed point theorem of Reich," *Canadian Mathematical Bulletin*, vol. 16, no. 2, pp. 201–206, 1973.

[12] E. Karapınar, O. Algahtani, and H. Aydi, "On Interpolative Hardy-Rogers type contractions," *Symmetry*, vol. 11, no. 1, p. 8, 2018.

[13] H. Aydi, E. Karapinar, and A. F. Roldán López de Hierro, "ω-Interpolative cirić-reich-rus-type contractions," *Mathematics*, vol. 7, no. 1, p. 57, 2019.

[14] H. Aydi, C.-M. Chen, and E. Karapınar, "Interpolative cirić-reich-rus type contractions via the branciari distance," *Mathematics*, vol. 7, no. 1, p. 84, 2019.

[15] E. Karapinar, "Revisiting the Kannan type contractions via interpolation," *Advances in the Theory of Nonlinear Analysis and Its Application*, vol. 2, pp. 85–87, 2018.

[16] E. Karapinar, R. Agarwal, and H. Aydi, "Interpolative reich-rus-cirić type contractions on partial metric spaces," *Mathematics*, vol. 6, no. 11, p. 256, 2018.

[17] B. Mohammadi, V. Parvaneh, and H. Aydi, "On extended interpolative Ćirić–Reich–Rus type F-contractions and an application," *Journal of Inequalities and Applications*, vol. 2019, Article ID 290, 2019.