Rashid et al. *Advances in Difference Equations* (2020) 2020:125
https://doi.org/10.1186/s13662-020-02583-3

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

On Pólya–Szegö and Čebyšev type inequalities via generalized k-fractional integrals

Saima Rashid[1], Fahd Jarad[2], Humaira Kalsoom[3] and Yu-Ming Chu[4,5*]

*Correspondence:
chuyuming2005@126.com
[4]Department of Mathematics, Huzhou University, Huzhou, China
[5]School of Mathematics and Statistic, Changsha University of Science & Technology, Changsha, China
Full list of author information is available at the end of the article

Abstract

In this paper, we introduce the generalized k-fractional integral in terms of a new parameter $k > 0$, present some new important inequalities of Pólya–Szegö and Čebyšev types by use of the generalized k-fractional integral. Our consequences with this new integral operator have the abilities to implement the evaluation of many mathematical problems related to real world applications.

MSC: 26D15; 26D10; 26A33

Keywords: Pólya–Szegö inequality; Čebyšev inequality; Generalized fractional integral

1 Introduction

There are numerous problems wherein fractional derivatives (non-integer order derivatives and integrals) attain a valuable position [1–25]. It must be emphasized that fractional derivatives exist in many technologies, especially they can be described in three different approaches, and any of these approaches can be used to solve many important problems in the real world. Every classical fractional operator is typically described in terms of a particular significance. There are many well-recognized definitions of fractional operators, we can also point out the Riemann–Liouville, Caputo, Grunwald–Letnikov, and Hadamard operators [26], whose formulations include integrals with singular kernels and which may be used to check, for example, issues involving the reminiscence effect [27]. However, within the year 2010, specific formulations of fractional operators appeared in the literature [28]. The new formulations diverge from the classical ones in numerous components. As an example, classical fractional derivatives are described in such a manner that in the limit wherein the order of the derivative is an integer, one recovers the classical derivatives in the sense of Newton and Leibniz. In addition, new fractional operators [29–31] with a corresponding integral whose kernel may be a non-singular mapping have been currently proposed; for instance, a Mittag-Leffler function [32]. In such instances, integer-order derivatives are rediscovered by supposing suitable limits for the values of their parameters.

On the other hand, there are numerous approaches to acquire a generalization of classical fractional integrals. Many authors introduce parameters in classical definitions or in some unique specific function [4], as we shall do in what follows. Moreover, in the present paper, we introduce a parameter and enunciate a generalization for fractional integrals on a selected space, which we call generalized k-fractional integral, and further advocate a Pólya–Szegö and Čebyšev type inequalities modification of this generalization.

Inequalities and their potential applications are of great significance in pure mathematics and applied mathematics, many remarkable inequalities and their applications can be found in the literature [33–46]. In view of the broader applications, integral inequalities have received large interest [47–60]. Presently, many authors have provided the unique versions of such inequalities which may be beneficial within the study of diverse classes of differential and integral equations. Those inequalities act as far-reaching tools to look at the classes of differential and integral equations [61–70].

Čebyšev [71] introduced the well-known celebrated functional as follows:

$$\begin{aligned}\mathfrak{T}(\mathcal{U},\mathcal{V}) &= \frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\mathcal{U}(\lambda)\mathcal{V}(\lambda)\,d\lambda \\ &\quad -\left(\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\mathcal{U}(\lambda)\,d\lambda\right)\left(\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\mathcal{V}(\lambda)\,d\lambda\right),\end{aligned} \tag{1.1}$$

where $\mathcal{U}$ and $\mathcal{V}$ are two integrable functions on $[\sigma_1,\sigma_2]$. If $\mathcal{U}$ and $\mathcal{V}$ are synchronous, that is,

$$\bigl(\mathcal{U}(\lambda)-\mathcal{U}(\omega)\bigr)\bigl(\mathcal{V}(\lambda)-\mathcal{V}(\omega)\bigr)\geq 0$$

for any $\lambda,\omega\in[\sigma_1,\sigma_2]$, then $\mathfrak{T}(\mathcal{U},\mathcal{V})\geq 0$.

Functional (1.1) has attracted the attention of many researchers due to its demonstrated applications in probability, numerical analysis, quantum theory, statistical and transform theory. Alongside facet with numerous applications, functional (1.1) has gained plenty of interest to yield a variety of fundamental inequalities [72–76].

Another interesting and fascinating aspect of the theory of inequalities is the Grüss type inequality [66] which states

$$\bigl|\mathfrak{T}(\mathcal{U},\mathcal{V})\bigr|\leq\frac{(Q-q)(R-r)}{4},$$

where the integrable functions $\mathcal{U}$ and $\mathcal{V}$ satisfy

$$q\leq\mathcal{U}(\lambda)\leq Q$$

and

$$r\leq\mathcal{V}(\lambda)\leq R$$

for all $\lambda\in[\sigma_1,\sigma_2]$ and for some $q,Q,r,R\in\mathbb{R}$.

Many famous versions mentioned in the literature are direct effects of the numerous applications in optimizations and transform theory. In this regard Pólya–Szegö integral inequality is one of the most intensively studied inequalities. This inequality was introduced by Pólya and Szegö [76]:

$$\frac{\int_{\sigma_1}^{\sigma_2}\mathcal{U}^2(\lambda)\,d\lambda\int_{\sigma_1}^{\sigma_2}\mathcal{V}^2(\lambda)\,d\lambda}{(\int_{\sigma_1}^{\sigma_2}\mathcal{U}(\lambda)\mathcal{V}(\lambda)\,d\lambda)^2}\leq\frac{1}{4}\left(\sqrt{\frac{QR}{qr}}+\sqrt{\frac{qr}{QR}}\right)^2. \tag{1.2}$$

The constant $\frac{1}{4}$ is a best possible constant such that inequality (1.2) holds, that is, it can't be replaced by a smaller constant.

By using the Pólya–Szegö inequality, Dragomir and Diamond [75] proved that the inequality

$$\left|\mathfrak{T}(\mathcal{U},\mathcal{V})\right|\leq\frac{(Q-q)(R-r)}{4(\sigma_2-\sigma_1)\sqrt{qrQR}}\int_{\sigma_1}^{\sigma_2}\mathcal{U}(\lambda)\,d\lambda\int_{\sigma_1}^{\sigma_2}\mathcal{V}(\lambda)\,d\lambda$$

holds for all $\lambda\in[\sigma_1,\sigma_2]$ if the functions $\mathcal{U}$ and $\mathcal{V}$ defined on $[\sigma_1,\sigma_2]$ satisfy

$$0<q\leq\mathcal{U}(\lambda)\leq Q<\infty,\qquad 0<r\leq\mathcal{V}(\lambda)\leq R<\infty.$$

It has been extensively discussed that Pólya–Szegö and Čebyšev type inequalities in continuous and discrete cases play a considerable role in examining the qualitative conduct of differential and difference equations. As a result of these studies, many new branches of mathematics have been opened up. Inspired by Pólya, Szegö, and Čebyšev [71, 76], we intend to show more general versions of Pólya–Szegö and Čebyšev type inequalities.

Our present paper has been inspired by the resource of the above-defined work. The principal aim of the present paper is to set up new Pólya–Szegö and Čebyšev types integral inequalities associated with generalized k-fractional integrals. We introduce parameter $k>0$ and generalize the results in such a way that the existing results can be explored too. Thus, the results provided in this research paper are more generalized as compared to the existing results.

2 Preliminaries

In this section, we demonstrate some important concepts from fractional calculus that will play a major role in proving the results of the present paper. The essential points of interest are exhibited in the monograph by Kilbas et al. [27].

Definition 2.1 (see [27, 77]) Let $p\geq 1$ and $r\in\mathbb{R}$. Then the function $\mathcal{U}(\zeta)$ is said to be in $L_{p,r}[\upsilon_1,\upsilon_2]$ space if

$$\|\mathcal{U}\|_{L_{p,r}[\upsilon_1,\upsilon_2]}=\left(\int_{\upsilon_1}^{\upsilon_2}\left|\mathcal{U}(\zeta)\right|^p\zeta^r\,d\zeta\right)^{\frac{1}{p}}<\infty.$$

In particular,

$$L_{p,0}[\upsilon_1,\upsilon_2]=L_p[\upsilon_1,\upsilon_2]=\left\{\mathcal{U}:\|\mathcal{U}\|_{L_p[\upsilon_1,\upsilon_2]}=\left(\int_{\upsilon_1}^{\upsilon_2}\left|\mathcal{U}(\zeta)\right|^p\,d\zeta\right)^{\frac{1}{p}}<\infty\right\}.$$

Definition 2.2 (see [78]) Let $p \geq 1$, $\mathcal{U} \in L_1[0,\infty)$ and Ψ be an increasing and positive monotone function defined on $[0,\infty)$ such that Ψ' is continuous on $[0,\infty)$ and $\Psi(0) = 0$. Then $\mathcal{U}$ is said to be in $\chi_\Psi^p[0,\infty)$ space if $\|\mathcal{U}\|_{\chi_\Psi^p} < \infty$, where $\|\mathcal{U}\|_{\chi_\Psi^p}$ is defined by

$$\|\mathcal{U}\|_{\chi_\Psi^p} = \left(\int_0^\infty \left|\mathcal{U}(\zeta)\right|^p \Psi'(\zeta)\,d\zeta\right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$ and

$$\|\mathcal{U}\|_{\chi_\Psi^\infty} = ess \sup_{0\leq\zeta<\infty}\left[\Psi'(\zeta)\mathcal{U}(\zeta)\right].$$

In particular, if $\Psi(\lambda) = \lambda$, then $\chi_\Psi^p[0,\infty)$ coincides with $L_p[0,\infty)$; if $\Psi(\lambda) = \log\lambda$, then $\chi_\Psi^p[0,\infty)$ becomes $L_{p,-1}[0,\infty)$.

Definition 2.3 (see [27, 77]) Let $\sigma_1 < \sigma_2$ and $\mathcal{U} \in L_1([\sigma_1,\sigma_2])$. Then the left and right Riemann–Liouville fractional integrals of order $\varrho > 0$ are defined by

$$\mathcal{J}_{\sigma_1^+}^{\varrho}\mathcal{U}(\lambda) = \frac{1}{\Gamma(\varrho)}\int_{\sigma_1}^{\lambda}(\lambda-\zeta)^{\varrho-1}\mathcal{U}(\zeta)\,d\zeta \quad (\lambda > \sigma_1)$$

and

$$\mathcal{J}_{\sigma_2^-}^{\varrho}\mathcal{U}(\lambda) = \frac{1}{\Gamma(\varrho)}\int_{\lambda}^{\sigma_2}(\zeta-\lambda)^{\varrho-1}\mathcal{U}(\zeta)\,d\zeta \quad (\lambda < \sigma_2),$$

respectively, where $\Gamma(\varrho) = \int_0^\infty t^{\varrho-1}e^{-t}\,dt$ is the gamma function [79–87].

Now, we recall the definition of k-fractional integral [88].

Definition 2.4 (see [88]) Let $\sigma_1 < \sigma_2$, $k > 0$, and $\mathcal{U} \in L_1([\sigma_1,\sigma_2])$. Then the left and right k-fractional integrals of order ϱ are defined by

$$\mathcal{J}_{\sigma_1^+}^{\varrho,k}\mathcal{U}(\lambda) = \frac{1}{k\Gamma_k(\varrho)}\int_{\sigma_1}^{\lambda}(\lambda-\zeta)^{\frac{\varrho}{k}-1}\mathcal{U}(\zeta)\,d\zeta \quad (\lambda > \sigma_1)$$

and

$$\mathcal{J}_{\sigma_2^-}^{\varrho,k}\mathcal{U}(\lambda) = \frac{1}{k\Gamma_k(\varrho)}\int_{\lambda}^{\sigma_2}(\zeta-\lambda)^{\frac{\varrho}{k}-1}\mathcal{U}(\zeta)\,d\zeta \quad (\lambda < \sigma_2),$$

respectively, where $\Gamma_k(\varrho) = \int_0^\infty t^{\varrho-1}e^{-\frac{t^k}{k}}\,dt$ is the k-gamma function [89].

A generalization of the Riemann–Liouville fractional integrals with respect to another function is given in [27] as follows.

Definition 2.5 (see [27]) Let $\sigma_1 < \sigma_2$, $\varrho > 0$, and $\Psi(\zeta)$ be an increasing and positive monotone function defined on $(\sigma_1,\sigma_2]$. Then the left and right generalized Riemann–Liouville fractional integrals of the function $\mathcal{U}$ with respect the function Ψ of order ϱ are defined by

$$\mathcal{J}^{\varrho}_{\Psi,\sigma_1^+}\mathcal{U}(\lambda) = \frac{1}{\Gamma(\varrho)}\int_{\sigma_1}^{\lambda} \Psi'(\zeta)\big(\Psi(\lambda)-\Psi(\zeta)\big)^{\varrho-1}\mathcal{U}(\zeta)\,d\zeta \tag{2.1}$$

and

$$\mathcal{J}^{\varrho}_{\Psi,\sigma_2^-}\mathcal{U}(\lambda) = \frac{1}{\Gamma(\varrho)}\int_{\lambda}^{\sigma_2} \Psi'(\zeta)\big(\Psi(\zeta)-\Psi(\lambda)\big)^{\varrho-1}\mathcal{U}(\zeta)\,d\zeta, \tag{2.2}$$

respectively.

Next, we present a new fractional integral operator which is known as the generalized k-fractional integral operator of a function with respect to another function.

Definition 2.6 Let $\sigma_1 < \sigma_2$, $\varrho, k > 0$, and $\Psi(\zeta)$ be an increasing and positive monotone function defined on $(\sigma_1, \sigma_2]$. Then the left and right generalized k-fractional integrals of the function $\mathcal{U}$ with respect to the function Ψ of order ϱ are defined by

$$\mathcal{J}^{\varrho,k}_{\Psi,\sigma_1^+}\mathcal{U}(\lambda) = \frac{1}{k\Gamma_k(\varrho)}\int_{\sigma_1}^{\lambda} \Psi'(\zeta)\big(\Psi(\lambda)-\Psi(\zeta)\big)^{\frac{\varrho}{k}-1}\mathcal{U}(\zeta)\,d\zeta \tag{2.3}$$

and

$$\mathcal{J}^{\varrho,k}_{\Psi,\sigma_2^-}\mathcal{U}(\lambda) = \frac{1}{k\Gamma_k(\varrho)}\int_{\lambda}^{\sigma_2} \Psi'(\zeta)\big(\Psi(\zeta)-\Psi(\lambda)\big)^{\frac{\varrho}{k}-1}\mathcal{U}(\zeta)\,d\zeta, \tag{2.4}$$

respectively.

Remark 2.7 Several existing fractional operators are the special cases of Definition 2.6. For example:

1. Let $k = 1$. Then Definition 2.6 reduces to Definition 2.5.
2. Let $\Psi(\lambda) = \lambda$. Then Definition 2.6 reduces to Definition 2.4.
3. Let $\Psi(\lambda) = \lambda$ and $k = 1$. Then Definition 2.6 reduces to 2.3.
4. Let $\Psi(\lambda) = \log\lambda$ and $k = 1$. Then Definition 2.6 leads to the Hadamard fractional integral operators given in [27, 77].
5. Let $\beta > 0$, $\Psi(\lambda) = \frac{\lambda^{\beta}}{\beta}$, and $k = 1$. Then Definition 2.6 leads to the Katugampola fractional integral operators in the literature [90].
6. Let $\beta > 0$, $\Psi(\lambda) = \frac{(\lambda-a)^{\beta}}{\beta}$, and $k = 1$. Then Definition 2.6 becomes the conformable fractional integral operators defined by Jarad et al. in [91].
7. Let $\Psi(\lambda) = \frac{\lambda^{u+v}}{u+v}$ and $k = 1$. Then Definition 2.6 becomes the generalized conformable fractional integrals defined by Khan et al. in [92].

Throughout this paper, we suppose that $\Psi(\zeta)$ is a strictly increasing function on $(0,\infty)$ and $\Psi'(\zeta)$ is continuous, $0 \le \sigma_1 < \sigma_2$ with the condition that at any point $\sigma_3 \in [\sigma_1, \sigma_2]$, we have $\Psi(\sigma_3) = 0$.

3 Pólya–Szegö type inequalities involving the generalized $\mathcal{K}$-fractional integrals

In this section, we derive certain Pólya–Szegö type integral inequalities for real-valued integrable functions via generalized Riemann–Liouville k-fractional integral defined in (2.3) and (2.4). Throughout this paper, we assume that $\Psi(\zeta)$ is an increasing, positive, and monotone function defined on $[0,\infty)$ such that $\Psi(0)=0$, and $\Psi'(\zeta)$ is continuous on $[0,\infty)$.

Lemma 3.1 *Let $k,\lambda,\varrho>0$, $\mathcal{U}$ and $\mathcal{V}$, ρ_1, ρ_2, χ_1, and χ_2 be six positive integrable functions defined on $[0,\infty)$ such that*

$$0<\rho_1(\zeta)\le\mathcal{U}(\zeta)\le\rho_2(\zeta),\quad 0<\chi_1(\zeta)\le\mathcal{V}(\zeta)\le\chi_2(\zeta) \tag{3.1}$$

for all $\zeta\in[0,\lambda]$. Then one has

$$\frac{1}{4}\bigl(\mathcal{J}_{\Psi}^{\varrho,k}\bigl[(\rho_1\chi_1+\rho_2\chi_2)\mathcal{U}\mathcal{V}\bigr](\lambda)\bigr)^2\ge\mathcal{J}_{\Psi}^{\varrho,k}\bigl[\chi_1\chi_2\mathcal{U}^2\bigr](\lambda)\mathcal{J}_{\Psi}^{\varrho,k}\bigl[\rho_1\rho_2\mathcal{V}^2\bigr](\lambda). \tag{3.2}$$

Proof It follows from (3.1) that

$$\frac{\rho_2(\zeta)}{\chi_1(\zeta)}-\frac{\mathcal{U}(\zeta)}{\mathcal{V}(\zeta)}\ge 0 \tag{3.3}$$

and

$$\frac{\mathcal{U}(\zeta)}{\mathcal{V}(\zeta)}-\frac{\rho_1(\zeta)}{\chi_2(\zeta)}\ge 0 \tag{3.4}$$

for all $\zeta\in[0,\lambda]$.

Multiplying (3.3) and (3.4), we obtain

$$\bigl[\rho_1(\zeta)\chi_1(\zeta)+\rho_2(\zeta)\chi_2(\zeta)\bigr]\mathcal{U}(\zeta)\mathcal{V}(\zeta)\ge\chi_1(\zeta)\chi_2(\zeta)\mathcal{U}^2(\zeta)+\rho_1(\zeta)\rho_2(\zeta)\mathcal{V}^2(\zeta). \tag{3.5}$$

Multiplying both sides of inequality (3.5) by

$$\frac{1}{k\Gamma_k(\varrho)}\Psi'(\zeta)\bigl(\Psi(\lambda)-\Psi(\zeta)\bigr)^{\frac{\varrho}{k}-1}$$

and integrating the obtained result with respect to ζ to $(0,\lambda)$, we get

$$\mathcal{J}_{\Psi}^{\varrho,k}\bigl[(\rho_1\chi_1+\rho_2\chi_2)\mathcal{U}\mathcal{V}\bigr](\lambda)\ge\mathcal{J}_{\Psi}^{\varrho,k}\bigl[\chi_1\chi_2\mathcal{U}^2\bigr](\lambda)+\mathcal{J}_{\Psi}^{\varrho,k}\bigl[\rho_1\rho_2\mathcal{V}^2\bigr](\lambda).$$

Applying the arithmetic-geometric inequality, we have

$$\mathcal{J}_{\Psi}^{\varrho,k}\bigl[(\rho_1\chi_1+\rho_2\chi_2)\mathcal{U}\mathcal{V}\bigr](\lambda)\ge 2\sqrt{\mathcal{J}_{\Psi}^{\varrho,k}\bigl[\chi_1\chi_2\mathcal{U}^2\bigr](\lambda)\mathcal{J}_{\Psi}^{\varrho,k}\bigl[\rho_1\rho_2\mathcal{V}^2\bigr](\lambda)},$$

which leads to

$$\frac{1}{4}\bigl(\mathcal{J}_{\Psi}^{\varrho,k}\bigl[(\rho_1\chi_1+\rho_2\chi_2)\mathcal{U}\mathcal{V}\bigr](\lambda)\bigr)^2\ge\mathcal{J}_{\Psi}^{\varrho,k}\bigl[\chi_1\chi_2\mathcal{U}^2\bigr](\lambda)\mathcal{J}_{\Psi}^{\varrho,k}\bigl[\rho_1\rho_2\mathcal{V}^2\bigr](\lambda).$$

Therefore, we obtain the desired inequality (3.1). □

Corollary 3.2 *Let $k,\lambda,q,r,\varrho,Q,R>0$ with $q\leq Q$ and $r\leq R$, and $\mathcal{U}$ and $\mathcal{V}$ be two positive integrable functions defined on $[0,\infty)$ such that*

$$0<q\leq\mathcal{U}(\zeta)\leq Q<\infty,\qquad 0<r\leq\mathcal{U}(\zeta)\leq R<\infty \tag{3.6}$$

for all $\zeta\in[0,\lambda]$. Then one has

$$\frac{\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}^2(\lambda)\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{V}^2(\lambda)}{(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}\mathcal{V}(\lambda))^2}\leq\frac{1}{4}\left(\sqrt{\frac{qr}{QR}}+\sqrt{\frac{QR}{qr}}\right)^2.$$

Corollary 3.3 *Let $k=1$. Then Lemma 3.1 reduces to the inequality for generalized Riemann–Liouville fractional integrals as follows*:

$$\frac{1}{4}\left(\mathcal{J}_{\Psi}^{\varrho}\left[(\rho_1\chi_1+\rho_2\chi_2)\mathcal{U}\mathcal{V}\right](\lambda)\right)^2\geq\mathcal{J}_{\Psi}^{\varrho}\left[\chi_1\chi_2\mathcal{U}^2\right](\lambda)\mathcal{J}_{\Psi}^{\varrho}\left[\rho_1\rho_2\mathcal{V}^2\right](\lambda). \tag{3.7}$$

Corollary 3.4 *Let $\Psi(\lambda)=\lambda$. Then Lemma 3.1 leads to the inequality for k-fractional integral as follows*:

$$\frac{1}{4}\left(\mathcal{J}^{\varrho,k}\left[(\rho_1\chi_1+\rho_2\chi_2)\mathcal{U}\mathcal{V}\right](\lambda)\right)^2\geq\mathcal{J}^{\varrho,k}\left[\chi_1\chi_2\mathcal{U}^2\right](\lambda)\mathcal{J}^{\varrho,k}\left[\rho_1\rho_2\mathcal{V}^2\right](\lambda).$$

Remark 3.5 Let $\Psi(\lambda)=\lambda$ and $k=1$. Then Lemma 3.1 becomes Lemma 3.1 of [67].

Lemma 3.6 *Let $k,\lambda,\varrho,\delta>0$ and $\mathcal{U}$, $\mathcal{V}$, ρ_1, ρ_2, χ_1, and χ_2 be six positive integrable functions defined on $[0,\infty)$ such that (3.1) holds for all $\zeta\in[0,\lambda]$. Then we have*

$$\frac{\mathcal{J}_{\Psi}^{\varrho,k}\rho_1\rho_2(\lambda)\mathcal{J}_{\Psi}^{\delta,k}\chi_1\chi_2(\lambda)\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}^2(\lambda)\mathcal{J}_{\Psi}^{\delta,k}\mathcal{V}^2(\lambda)}{(\mathcal{J}_{\Psi}^{\varrho,k}\rho_1\mathcal{U}(\lambda)\mathcal{J}_{\Psi}^{\delta,k}\chi_1\mathcal{V}(\lambda)+\mathcal{J}_{\Psi}^{\varrho,k}\rho_2\mathcal{U}(\lambda)\mathcal{J}_{\Psi}^{\delta,k}\chi_2\mathcal{V}(\lambda))^2}\leq\frac{1}{4}. \tag{3.8}$$

Proof From (3.1) we clearly see that

$$\frac{\rho_2(\zeta)}{\chi_1(\eta)}-\frac{\mathcal{U}(\zeta)}{\mathcal{V}(\eta)}\geq 0$$

and

$$\frac{\mathcal{U}(\zeta)}{\mathcal{V}(\eta)}-\frac{\rho_1(\zeta)}{\chi_2(\eta)}\geq 0,$$

which imply that

$$\left(\frac{\rho_1(\zeta)}{\chi_2(\eta)}+\frac{\rho_2(\zeta)}{\chi_1(\eta)}\right)\frac{\mathcal{U}(\zeta)}{\mathcal{V}(\eta)}\geq\frac{\mathcal{U}^2(\zeta)}{\mathcal{V}^2(\eta)}+\frac{\rho_1(\zeta)\rho_2(\zeta)}{\chi_1(\eta)\chi_2(\eta)}. \tag{3.9}$$

Multiplying both sides of inequality (3.9) by $\chi_1(\eta)\chi_2(\eta)\mathcal{V}^2(\eta)$, we have

$$\begin{aligned}&\rho_1(\zeta)\mathcal{U}(\zeta)\chi_1(\eta)\mathcal{V}(\eta)+\rho_2(\zeta)\mathcal{U}(\zeta)\chi_2(\eta)\mathcal{V}(\eta)\\&\quad\geq\chi_1(\eta)\chi_2(\eta)\mathcal{U}^2(\zeta)+\rho_1(\zeta)\rho_2(\zeta)\mathcal{V}^2(\eta).\end{aligned} \tag{3.10}$$

Multiplying both sides of inequality (3.10) by

$$\frac{1}{k\Gamma_k(\varrho)(k\Gamma_k(\delta))}\Psi'(\zeta)\big(\Psi(\lambda)-\Psi(\zeta)\big)^{\frac{\varrho}{k}-1}\Psi'(\eta)\big(\Psi(\lambda)-\Psi(\eta)\big)^{\frac{\delta}{k}-1}$$

and then integrating the obtained inequality with respect to ζ and η from 0 to λ, one has

$$\begin{aligned}&\big(\mathcal{J}_{\Psi}^{\varrho,k}\rho_1\mathcal{U}\big)(\lambda)\big(\mathcal{J}_{\Psi}^{\delta,k}\chi_1\mathcal{V}\big)(\lambda)+\big(\mathcal{J}_{\Psi}^{\varrho,k}\rho_2\mathcal{U}\big)(\lambda)\big(\mathcal{J}_{\Psi}^{\delta,k}\chi_2\mathcal{V}\big)(\lambda)\\&\quad\geq\big(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}^2\big)(\lambda)\big(\mathcal{J}_{\Psi}^{\delta,k}\chi_1\chi_2\big)(\lambda)+\big(\mathcal{J}_{\Psi}^{\delta,k}\mathcal{V}^2\big)(\lambda)\big(\mathcal{J}_{\Psi}^{\varrho,k}\rho_1\rho_2\big)(\lambda).\end{aligned}$$

Making use of the arithmetic-geometric mean inequality, we obtain

$$\begin{aligned}&\big(\mathcal{J}_{\Psi}^{\varrho,k}\rho_1\mathcal{U}\big)(\lambda)\big(\mathcal{J}_{\Psi}^{\delta,k}\chi_1\mathcal{V}\big)(\lambda)+\big(\mathcal{J}_{\Psi}^{\varrho,k}\rho_2\mathcal{U}\big)(\lambda)\big(\mathcal{J}_{\Psi}^{\delta,k}\chi_2\mathcal{V}\big)(\lambda)\\&\quad\geq 2\sqrt{\big(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}^2\big)(\lambda)\big(\mathcal{J}_{\Psi}^{\delta,k}\chi_1\chi_2\big)(\lambda)\big(\mathcal{J}_{\Psi}^{\delta,k}\mathcal{V}^2\big)(\lambda)\big(\mathcal{J}_{\Psi}^{\varrho,k}\rho_1\rho_2\big)(\lambda)},\end{aligned}$$

which leads to the desired inequality (3.8). □

Corollary 3.7 *For $k,\lambda,\varrho,\delta>0$, and $\mathcal{U}$ and $\mathcal{V}$ being two positive integrable functions defined on $[0,\infty)$ such that inequality (3.6) holds for $\zeta\in[0,\lambda]$, we have*

$$\frac{\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}^2(\lambda)\mathcal{J}_{\Psi}^{\delta,k}\mathcal{V}^2(\lambda)}{(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}(\lambda)\mathcal{J}_{\Psi}^{\delta,k}\mathcal{V}(\lambda))^2}\leq\frac{\Gamma_k(\varrho+k)\Gamma_k(\delta+k)}{4(\Psi(\lambda))^{\frac{\varrho+\delta}{k}}}\bigg(\sqrt{\frac{qr}{QR}}+\sqrt{\frac{QR}{qr}}\bigg)^2.$$

Corollary 3.8 *Let $k=1$. Then Lemma 3.6 leads to a new inequality for generalized Riemann–Liouville fractional integral as follows*:

$$\frac{\mathcal{J}_{\Psi}^{\varrho}\rho_1\rho_2(\lambda)\mathcal{J}_{\Psi}^{\delta}\chi_1\chi_2(\lambda)\mathcal{J}_{\Psi}^{\varrho}\mathcal{U}^2(\lambda)\mathcal{J}_{\Psi}^{\delta}\mathcal{V}^2(\lambda)}{(\mathcal{J}_{\Psi}^{\varrho}\rho_1\mathcal{U}(\lambda)\mathcal{J}_{\Psi}^{\delta}\chi_1\mathcal{V}(\lambda)+\mathcal{J}_{\Psi}^{\varrho}\rho_2\mathcal{U}(\lambda)\mathcal{J}_{\Psi}^{\delta}\chi_2\mathcal{V}(\lambda))^2}\leq\frac{1}{4}. \tag{3.11}$$

Corollary 3.9 *Let $\Psi(\lambda)=\lambda$. Then Lemma 3.6 leads to a new inequality for k-fractional integral as follows*:

$$\frac{\mathcal{J}^{\varrho,k}\rho_1\rho_2(\lambda)\mathcal{J}^{\delta,k}\chi_1\chi_2(\lambda)\mathcal{J}^{\varrho,k}\mathcal{U}^2(\lambda)\mathcal{J}^{\delta,k}\mathcal{V}^2(\lambda)}{(\mathcal{J}^{\varrho,k}\rho_1\mathcal{U}(\lambda)\mathcal{J}^{\delta,k}\chi_1\mathcal{V}(\lambda)+\mathcal{J}^{\varrho,k}\rho_2\mathcal{U}(\lambda)\mathcal{J}^{\delta,k}\chi_2\mathcal{V}(\lambda))^2}\leq\frac{1}{4}. \tag{3.12}$$

Remark 3.10 If $\Psi(\lambda)=\lambda$ and $k=1$, then Lemma 3.6 reduces to Lemma 3.3 of [67].

Theorem 3.11 *Let $k,\lambda,\varrho,\delta>0$, and $\mathcal{U}$, $\mathcal{V}$, ρ_1, ρ_2, χ_1, and χ_2 be six positive integrable functions defined on $[0,\infty)$ such that (3.1) holds for all $\zeta\in[0,\lambda]$. Then we have*

$$\mathcal{J}_{\Psi}^{\varrho,k}\bigg(\frac{\rho_2\mathcal{U}\mathcal{V}}{\chi_1}\bigg)(\lambda)\mathcal{J}_{\Psi}^{\delta,k}\bigg(\frac{\chi_2\mathcal{U}\mathcal{V}}{\rho_1}\bigg)(\lambda)\geq\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}^2(\lambda)\mathcal{J}_{\Psi}^{\delta,k}\mathcal{V}^2(\lambda). \tag{3.13}$$

Proof It follows from (3.1) that

$$\frac{1}{k\Gamma_k(\varrho)}\int_0^{\lambda}\Psi'(\zeta)\big(\Psi(\lambda)-\Psi(\zeta)\big)^{\frac{\varrho}{k}-1}\frac{\rho_2(\zeta)}{\chi_1(\zeta)}\mathcal{U}(\zeta)\mathcal{V}(\zeta)\,d\zeta$$

$$\geq \frac{1}{k\Gamma_k(\varrho)} \int_0^{\lambda} \Psi'(\zeta)\big(\Psi(\lambda)-\Psi(\zeta)\big)^{\frac{\varrho}{k}-1} \mathcal{U}^2(\zeta)\,d\zeta,$$

which implies

$$\mathcal{J}_{\Psi}^{\varrho,k}\left(\frac{\rho_2 \mathcal{U}\mathcal{V}}{\chi_1}\right)(\lambda) \geq \mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}^2(\lambda). \tag{3.14}$$

Analogously, we obtain

$$\frac{1}{k\Gamma_k(\delta)} \int_0^{\lambda} \Psi'(\eta)\big(\Psi(\lambda)-\Psi(\eta)\big)^{\frac{\delta}{k}-1} \frac{\chi_2(\eta)}{\rho_1(\eta)} \mathcal{U}\mathcal{V}\,d\eta$$
$$\geq \frac{1}{k\Gamma_k(\delta)} \int_0^{\lambda} \Psi'(\eta)\big(\Psi(\lambda)-\Psi(\eta)\big)^{\frac{\delta}{k}-1} \mathcal{V}^2(\eta)\,d\eta,$$

from which one has

$$\mathcal{J}_{\Psi}^{\delta,k}\left(\frac{\chi_2 \mathcal{U}\mathcal{V}}{\rho_1}\right)(\lambda) \geq \mathcal{J}_{\Psi}^{\delta,k}\mathcal{V}^2(\lambda). \tag{3.15}$$

Multiplying (3.14) and (3.15), we get the desired inequality (3.13). □

Corollary 3.12 *For $k,\lambda,\varrho,\delta>0$, and $\mathcal{U}$ and $\mathcal{V}$ being two positive integrable functions defined on $[0,\infty)$ such that (3.6) holds for all $\zeta\in[0,\lambda]$, we have*

$$\frac{\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}^2(\lambda)\mathcal{J}_{\Psi}^{\delta,k}\mathcal{V}^2(\lambda)}{\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}\mathcal{V}(\lambda)\mathcal{J}_{\Psi}^{\delta,k}\mathcal{U}\mathcal{V}(\lambda)} \leq \frac{QR}{qr}.$$

Corollary 3.13 *If $k=1$, then Theorem 3.11 gives the following new result for generalized Riemann–Liouville fractional integral*:

$$\mathcal{J}_{\Psi}^{\varrho}\left(\frac{\rho_2 \mathcal{U}\mathcal{V}}{\chi_1}\right)(\lambda)\mathcal{J}_{\Psi}^{\delta}\left(\frac{\chi_2 \mathcal{U}\mathcal{V}}{\rho_1}\right)(\lambda) \geq \mathcal{J}_{\Psi}^{\varrho}\mathcal{U}^2(\lambda)\mathcal{J}_{\Psi}^{\delta}\mathcal{V}^2(\lambda).$$

Corollary 3.14 *Let $\Psi(\lambda)=\lambda$. Then Theorem 3.11 leads to the following new result for Riemann–Liouville k-fractional integral*:

$$\mathcal{J}^{\varrho,k}\left(\frac{\rho_2 \mathcal{U}\mathcal{V}}{\chi_1}\right)(\lambda)\mathcal{J}^{\delta,k}\left(\frac{\chi_2 \mathcal{U}\mathcal{V}}{\rho_1}\right)(\lambda) \geq \mathcal{J}^{\varrho,k}\mathcal{U}^2(\lambda)\mathcal{J}^{\delta,k}\mathcal{V}^2(\lambda).$$

Remark 3.15 If $\Psi(\lambda)=\lambda$ and $\mathcal{K}=1$, then Theorem 3.11 reduces to Lemma 3.4 of [67].

4 Pólya–Szegö type inequalities involving the generalized *k*-fractional integrals

In this section, we present several Čebyšev type inequalities for generalized k-fractional integrals defined in (2.3) and (2.4).

Theorem 4.1 *Let $k,\lambda,\varrho>0$, and $\mathcal{U}$ and $\mathcal{V}$ be two integrable and synchronous functions on $[0,\infty)$. Then one has*

$$\big(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}\mathcal{V}\big)(\lambda) \geq \frac{\Gamma_k(\varrho+k)}{(\Psi(\lambda))^{\frac{\varrho}{k}}}\big(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}\big)(\lambda)\big(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{V}\big)(\lambda).$$

Proof It follows from the synchronism of the functions $\mathcal{U}$ and $\mathcal{V}$ on the interval $[0,\infty)$ that

$$\mathcal{U}(r)\mathcal{V}(r)+\mathcal{U}(s)\mathcal{V}(s)\geq\mathcal{U}(r)\mathcal{V}(s)+\mathcal{U}(s)\mathcal{V}(r). \tag{4.1}$$

Multiplying both sides of inequality (4.1) by

$$\frac{1}{k\Gamma_k(\varrho)}\Psi'(r)\bigl(\Psi(\lambda)-\Psi(r)\bigr)^{\frac{\varrho}{k}-1}$$

for $\lambda\in\mathbb{R}$ gives

$$\begin{aligned}&\frac{1}{k\Gamma_k(\varrho)}\Psi'(r)\bigl(\Psi(\lambda)-\Psi(r)\bigr)^{\frac{\varrho}{k}-1}\mathcal{U}(r)\mathcal{V}(r)+\mathcal{U}(s)\mathcal{V}(s)\frac{1}{k\Gamma_k(\varrho)}\Psi'(r)\bigl(\Psi(\lambda)-\Psi(r)\bigr)^{\frac{\varrho}{k}-1}\\&\quad\geq\mathcal{V}(s)\frac{1}{k\Gamma_k(\varrho)}\Psi'(r)\bigl(\Psi(\lambda)-\Psi(r)\bigr)^{\frac{\varrho}{k}-1}\mathcal{U}(r)\\&\qquad+\mathcal{U}(s)\frac{1}{k\Gamma_k(\varrho)}\Psi'(r)\bigl(\Psi(\lambda)-\Psi(r)\bigr)^{\frac{\varrho}{k}-1}\mathcal{V}(r).\end{aligned}$$

Integrating the above inequality with respect to r over $(0,\lambda)$ leads to

$$\begin{aligned}&\frac{1}{k\Gamma_k(\varrho)}\int_0^{\lambda}\Psi'(r)\bigl(\Psi(\lambda)-\Psi(r)\bigr)^{\frac{\varrho}{k}-1}\mathcal{U}(r)\mathcal{V}(r)\,dr\\&\qquad+\mathcal{U}(s)\mathcal{V}(s)\frac{1}{k\Gamma_k(\varrho)}\int_0^{\lambda}\Psi'(r)\bigl(\Psi(\lambda)-\Psi(r)\bigr)^{\frac{\varrho}{k}-1}\,dr\\&\quad\geq\mathcal{V}(s)\frac{1}{k\Gamma_k(\varrho)}\int_0^{\lambda}\Psi'(r)\bigl(\Psi(\lambda)-\Psi(r)\bigr)^{\frac{\varrho}{k}-1}\mathcal{U}(r)\,dr\\&\qquad+\mathcal{U}(s)\frac{1}{k\Gamma_k(\varrho)}\int_0^{\lambda}\Psi'(r)\bigl(\Psi(\lambda)-\Psi(r)\bigr)^{\frac{\varrho}{k}-1}\mathcal{V}(r)\,dr.\end{aligned}$$

Therefore, we get

$$\begin{aligned}&\bigl(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}\mathcal{V}\bigr)(\lambda)+\mathcal{U}(s)\mathcal{V}(s)\frac{1}{k\Gamma_k(\varrho)}\int_0^{\lambda}\Psi'(r)\bigl(\Psi(\lambda)-\Psi(r)\bigr)^{\frac{\varrho}{k}-1}\,dr\\&\quad\geq\mathcal{V}(s)\bigl(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}\bigr)(\lambda)+\mathcal{U}(s)\bigl(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{V}\bigr)(\lambda)\end{aligned}$$

and

$$\begin{aligned}&\bigl(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}\mathcal{V}\bigr)(\lambda)+\mathcal{U}(s)\mathcal{V}(s)\frac{(\Psi(\lambda))^{\frac{\varrho}{k}}}{\Gamma_k(\varrho+k)}\\&\quad\geq\mathcal{V}(s)\bigl(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}\bigr)(\lambda)+\mathcal{U}(s)\bigl(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{V}\bigr)(\lambda),\end{aligned} \tag{4.2}$$

where

$$\frac{1}{k\Gamma_k(\varrho)}\int_0^{\lambda}\Psi'(r)\bigl(\Psi(\lambda)-\Psi(r)\bigr)^{\frac{\varrho}{k}-1}\,dr=\frac{(\Psi(\lambda))^{\frac{\varrho}{k}}}{\Gamma_k(\varrho+k)}.$$

Multiplying both sides of inequality (4.2) by

$$\frac{1}{k\Gamma_k(\varrho)}\Psi'(s)\bigl(\Psi(\lambda)-\Psi(s)\bigr)^{\frac{\varrho}{k}-1}\quad(\lambda\in\mathbb{R})$$

leads to the conclusion that

$$\begin{aligned}
&\big(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}\mathcal{V}\big)(\lambda)\frac{1}{k\Gamma_k(\varrho)}\Psi'(s)\big(\Psi(\lambda)-\Psi(s)\big)^{\frac{\varrho}{k}-1} \\
&\quad+\frac{1}{k\Gamma_k(\varrho)}\Psi'(s)\big(\Psi(\lambda)-\Psi(s)\big)^{\frac{\varrho}{k}-1}\mathcal{U}(s)\mathcal{V}(s)\frac{(\Psi(\lambda))^{\frac{\varrho}{k}}}{\Gamma_k(\varrho+k)} \\
&\geq\big(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}\big)(\lambda)\frac{1}{k\Gamma_k(\varrho)}\Psi'(s)\big(\Psi(\lambda)-\Psi(s)\big)^{\frac{\varrho}{k}-1}\mathcal{V}(s) \\
&\quad+\big(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{V}\big)(\lambda)\frac{1}{k\Gamma_k(\varrho)}\Psi'(s)\big(\Psi(\lambda)-\Psi(s)\big)^{\frac{\varrho}{k}-1}\mathcal{U}(s).
\end{aligned}$$

Integrating the above inequality over $(0,\lambda)$ reveals

$$\begin{aligned}
&\big(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}\mathcal{V}\big)(\lambda)\frac{1}{k\Gamma_k(\varrho)}\int_0^{\lambda}\Psi'(s)\big(\Psi(\lambda)-\Psi(s)\big)^{\frac{\varrho}{k}-1}\,ds \\
&\quad+\frac{1}{k\Gamma_k(\varrho)}\int_0^{\lambda}\Psi'(s)\big(\Psi(\lambda)-\Psi(s)\big)^{\frac{\varrho}{k}-1}\mathcal{U}(s)\mathcal{V}(s)\,ds\frac{(\Psi(\lambda))^{\frac{\varrho}{k}}}{\Gamma_k(\varrho+k)} \\
&\geq\big(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}\big)(\lambda)\frac{1}{k\Gamma_k(\varrho)}\int_0^{\lambda}\Psi'(s)\big(\Psi(\lambda)-\Psi(s)\big)^{\frac{\varrho}{k}-1}\mathcal{V}(s)\,ds \\
&\quad+\big(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{V}\big)(\lambda)\frac{1}{k\Gamma_k(\varrho)}\int_0^{\lambda}\Psi'(s)\big(\Psi(\lambda)-\Psi(s)\big)^{\frac{\varrho}{k}-1}\mathcal{U}(s)\,ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{(\Psi(\lambda))^{\frac{\varrho}{k}}}{\Gamma_k(\varrho+k)}\big(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}\mathcal{V}\big)(\lambda)+\big(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}\mathcal{V}\big)(\lambda)\frac{(\Psi(\lambda))^{\frac{\varrho}{k}}}{\Gamma_k(\varrho+k)} \\
&\quad\geq\big(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}\big)(\lambda)\big(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{V}\big)(\lambda)+\big(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{V}\big)(\lambda)\big(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}\big)(\lambda).
\end{aligned}$$

This completes the proof of Theorem 4.1. □

Corollary 4.2 *Let $k=1$. Then Theorem 4.1 leads to a new result for generalized Riemann–Liouville fractional integrals as follows*:

$$\big(\mathcal{J}_{\Psi}^{\varrho}\mathcal{U}\mathcal{V}\big)(\lambda)\geq\frac{\Gamma(\varrho+1)}{(\Psi(\lambda))^{\varrho}}\big(\mathcal{J}_{\Psi}^{\varrho}\mathcal{U}\big)(\lambda)\big(\mathcal{J}_{\Psi}^{\varrho}\mathcal{V}\big)(\lambda).$$

Corollary 4.3 *If $\Psi(\lambda)=\lambda$, then Theorem 4.1 provides a new inequality for k-fractional integral as follows*:

$$\big(\mathcal{J}^{\varrho,k}\mathcal{U}\mathcal{V}\big)(\lambda)\geq\frac{\Gamma_k(\varrho+k)}{\lambda^{\frac{\varrho}{k}}}\big(\mathcal{J}^{\varrho,k}\mathcal{U}\big)(\lambda)\big(\mathcal{J}^{\varrho,k}\mathcal{V}\big)(\lambda).$$

Corollary 4.4 *Let $\Psi(\lambda)=\lambda$ and $k=1$. Then Theorem 4.1 leads to a new result for Riemann–Liouville fractional integral as follows*:

$$\big(\mathcal{J}^{\varrho}\mathcal{U}\mathcal{V}\big)(\lambda)\geq\frac{\Gamma(\varrho+1)}{\lambda^{\varrho}}\big(\mathcal{J}^{\varrho}\mathcal{U}\big)(\lambda)\big(\mathcal{J}^{\varrho}\mathcal{V}\big)(\lambda).$$

Theorem 4.5 *Let $k, \lambda, \varrho, \delta > 0$, and $\mathcal{U}$ and $\mathcal{V}$ be two integrable and synchronous functions on $[0,\infty)$. Then*

$$\frac{(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}\mathcal{V})(\lambda)(\Psi(\lambda))^{\frac{\delta}{k}}}{\Gamma_k(\delta+k)} + \frac{(\Psi(\lambda))^{\frac{\varrho}{k}}(\mathcal{J}_{\Psi}^{\delta,k}\mathcal{U}\mathcal{V})(\lambda)}{\Gamma_k(\varrho+k)} \geq (\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U})(\lambda)(\mathcal{J}_{\Psi}^{\delta,k}\mathcal{V})(\lambda) + (\mathcal{J}_{\psi}^{\varrho,k}\mathcal{V})(\lambda)(\mathcal{J}_{\Psi}^{\delta,k}\mathcal{U})(\lambda).$$

Proof Multiplying both sides of inequality (4.2) by

$$\frac{1}{k\Gamma_k(\delta)}\Psi'(s)\big(\Psi(\lambda)-\Psi(s)\big)^{\frac{\delta}{k}-1} \quad (\lambda \in \mathbb{R})$$

gives

$$\begin{aligned}&\big(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}\mathcal{V}\big)(\lambda)\frac{1}{k\Gamma_k(\delta)}\Psi'(s)\big(\Psi(\lambda)-\Psi(s)\big)^{\frac{\delta}{k}-1}\\ &\quad+\frac{1}{k\Gamma_k(\delta)}\Psi'(s)\big(\Psi(\lambda)-\Psi(s)\big)^{\frac{\delta}{k}-1}\mathcal{U}(s)\mathcal{V}(s)\frac{(\Psi(\lambda))^{\frac{\varrho}{k}}}{\Gamma_k(\varrho+k)}\\ &\geq \big(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}\big)(\lambda)\frac{1}{k\Gamma_k(\delta)}\Psi'(s)\big(\Psi(\lambda)-\Psi(s)\big)^{\frac{\delta}{k}-1}\mathcal{V}(s)\\ &\quad+\big(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{V}\big)(\lambda)\frac{1}{k\Gamma_k(\delta)}\Psi'(s)\big(\Psi(\lambda)-\Psi(s)\big)^{\frac{\delta}{k}-1}\mathcal{U}(s).\end{aligned}$$

Integrating both sides of the above inequality with respect to s over $(0,\lambda)$ leads to

$$\begin{aligned}&\frac{(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}\mathcal{V})(\lambda)}{\Gamma_k(\delta+k)}\int_0^{\lambda}\Psi'(s)\big(\Psi(\lambda)-\Psi(s)\big)^{\frac{\delta}{k}-1}\,ds\\ &\quad+\frac{(\Psi(\lambda))^{\frac{\varrho}{k}}}{\Gamma_k(\varrho+k)}\frac{1}{k\Gamma_k(\delta)}\int_0^{\lambda}\Psi'(s)\big(\Psi(\lambda)-\Psi(s)\big)^{\frac{\delta}{k}-1}\mathcal{U}(s)\mathcal{V}(s)\,ds\\ &\geq \frac{(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U})(\lambda)}{k\Gamma_k(\delta)}\int_0^{\lambda}\Psi'(s)\big(\Psi(\lambda)-\Psi(s)\big)^{\frac{\delta}{k}-1}\mathcal{V}(s)\,ds\\ &\quad+\frac{(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{V})(\lambda)}{k\Gamma_k(\delta)}\int_0^{\lambda}\Psi'(s)\big(\Psi(\lambda)-\Psi(s)\big)^{\frac{\delta}{k}-1}\mathcal{U}(s)\,ds.\end{aligned}$$

Therefore,

$$\frac{(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}\mathcal{V})(\lambda)(\Psi(\lambda))^{\frac{\delta}{k}}}{\Gamma_k(\delta+k)} + \frac{(\Psi(\lambda))^{\frac{\varrho}{k}}(\mathcal{J}_{\Psi}^{\delta,k}\mathcal{U}\mathcal{V})(\lambda)}{\Gamma_k(\varrho+k)} \geq (\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U})(\lambda)(\mathcal{J}_{\Psi}^{\delta,k}\mathcal{V})(\lambda) + (\mathcal{J}_{\psi}^{\varrho,k}\mathcal{V})(\lambda)(\mathcal{J}_{\Psi}^{\delta,k}\mathcal{U})(\lambda),$$

which is the proof of Theorem 4.5. □

Remark 4.6 Let $\varrho = \delta$. Then Theorem 4.5 becomes Theorem 4.1.

Corollary 4.7 *Let $k = 1$. Then Theorem 4.5 provides a new result for generalized Riemann–Liouville fractional integrals as follows*:

$$\begin{aligned}&\frac{(\mathcal{J}_{\Psi}^{\varrho}\mathcal{U}\mathcal{V})(\lambda)(\Psi(\lambda))^{\delta}}{\Gamma(\delta+1)}+\frac{(\Psi(\lambda))^{\varrho}(\mathcal{J}_{\Psi}^{\delta}\mathcal{U}\mathcal{V})(\lambda)}{\Gamma(\varrho+1)}\\&\quad\geq(\mathcal{J}_{\Psi}^{\varrho}\mathcal{U})(\lambda)(\mathcal{J}_{\Psi}^{\delta}\mathcal{V})(\lambda)+(\mathcal{J}_{\Psi}^{\varrho}\mathcal{V})(\lambda)(\mathcal{J}_{\Psi}^{\delta}\mathcal{U})(\lambda).\end{aligned}$$

Corollary 4.8 *If $\Psi(\lambda) = \lambda$ and $k = 1$, then Theorem 4.5 gives a new result for Riemann–Liouville fractional integral as follows*:

$$\begin{aligned}&\frac{\lambda^{\delta}(\mathcal{J}^{\varrho}\mathcal{U}\mathcal{V})(\lambda)}{\Gamma(\delta+1)}+\frac{\lambda^{\varrho}(\mathcal{J}^{\delta}\mathcal{U}\mathcal{V})(\lambda)}{\Gamma(\varrho+1)}\\&\quad\geq(\mathcal{J}^{\varrho}\mathcal{U})(\lambda)(\mathcal{J}^{\delta}\mathcal{V})(\lambda)+(\mathcal{J}^{\varrho}\mathcal{V})(\lambda)(\mathcal{J}^{\delta}\mathcal{U})(\lambda).\end{aligned}$$

Theorem 4.9 *Let $k,\lambda,\varrho > 0$, $\sigma_1,\sigma_2 \in \mathbb{R}$ with $\sigma_1 < \sigma_2$, and $\mathcal{U}_j$ $(1 \leq j \leq \gamma)$ be a real-valued increasing function on $[\sigma_1,\sigma_2]$. Then*

$$\left(\mathcal{J}_{\Psi}^{\varrho,k}\prod_{j=1}^{\gamma}\mathcal{U}_j\right)(\lambda)\geq\left[\frac{\Gamma_k(\varrho+k)}{(\Psi(\lambda))^{\frac{\varrho}{k}}}\right]^{\gamma-1}\prod_{j=1}^{\gamma}(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}_j)(\lambda).\tag{4.3}$$

Proof We use mathematical induction on $\gamma \in \mathbb{N}$ to prove Theorem 4.9. We clearly see that inequality (4.3) holds for $\gamma = 1$.

For $\gamma = 2$, since $\mathcal{U}_1,\mathcal{U}_2$ are increasing, we have

$$\langle\mathcal{U}_1(\lambda)-\mathcal{U}_1(\omega),\mathcal{U}_2(\lambda)-\mathcal{U}_2(\omega)\rangle\geq 0.$$

Note that the left-hand side of inequality (4.3) for $\gamma = 2$ is the same as that of Theorem 4.1. Therefore, inequality (4.3) also holds for $\gamma = 2$.

Suppose that inequality (4.3) holds for some $\gamma \geq 2$. We observe that $\mathcal{U} = \prod_{j=1}^{\gamma}\mathcal{U}_j$ is increasing due to $\mathcal{U}_j$ is increasing. Let $\mathcal{V} = \mathcal{U}_{\gamma+1}$. Then applying the case $\gamma = 2$ to the functions $\mathcal{U}$ and $\mathcal{V}$ produces

$$\begin{aligned}\left(\mathcal{J}_{\Psi}^{\varrho,k}\prod_{j=1}^{\gamma}\mathcal{U}_j\mathcal{U}_{\gamma+1}\right)(\lambda)&\geq\left[\frac{\Gamma_k(\varrho+k)}{(\Psi(\lambda))^{\frac{\varrho}{k}}}\right]\left(\mathcal{J}_{\Psi}^{\varrho,k}\prod_{j=1}^{\gamma}\mathcal{U}_j\right)(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}_{\gamma+1})(\lambda)\\&\geq\left[\frac{\Gamma_k(\varrho+k)}{(\Psi(\lambda))^{\frac{\varrho}{k}}}\right]^{\gamma}\prod_{j=1}^{\gamma+1}(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}_j)(\lambda),\end{aligned}$$

in which the induction hypothesis for γ is used inside the deduction of second inequality. The proof of Theorem 4.9 is completed. □

Corollary 4.10 *Let $k = 1$. Then Theorem 4.9 leads to the following new result for generalized Riemann–Liouville fractional integral*:

$$\left(\mathcal{J}_{\Psi}^{\varrho}\prod_{j=1}^{\gamma}\mathcal{U}_j\right)(\lambda)\geq\left[\frac{\Gamma(\varrho+1)}{(\Psi(\lambda))^{\varrho}}\right]^{\gamma-1}\prod_{j=1}^{\gamma}(\mathcal{J}_{\Psi}^{\varrho}\mathcal{U}_j)(\lambda).$$

Corollary 4.11 *If $\Psi(\lambda) = \lambda$, then Theorem 4.9 leads to a new result for k-fractional integral as follows*:

$$\left(\mathcal{J}^{\varrho,k}\prod_{j=1}^{\gamma}\mathcal{U}_j\right)(\lambda) \geq \left[\frac{\Gamma_k(\varrho+k)}{\lambda^{\frac{\varrho}{k}}}\right]^{\gamma-1}\prod_{j=1}^{\gamma}\left(\mathcal{J}^{\varrho,k}\mathcal{U}_j\right)(\lambda). \tag{4.4}$$

Corollary 4.12 *Let $\Psi(\lambda) = \lambda$ and $k = 1$. Then Theorem 4.9 provides a new result for Riemann–Liouville fractional integral as follows*:

$$\left(\mathcal{J}^{\varrho}\prod_{j=1}^{\gamma}\mathcal{U}_j\right)(\lambda) \geq \left[\frac{\Gamma(\varrho+1)}{\lambda^{\varrho}}\right]^{\gamma-1}\prod_{j=1}^{\gamma}\left(\mathcal{J}^{\varrho}\mathcal{U}_j\right)(\lambda). \tag{4.5}$$

Theorem 4.13 *Let $k, \lambda, \varrho > 0$, $\mathcal{U}$ and $\mathcal{V}$ be two positive functions defined on $[0,\infty)$ such that $\mathcal{U}$ is increasing and $\mathcal{V}$ is differentiable, and $\vartheta = \inf_{\mu\in[0,\infty)}\mathcal{V}'(\mu)$. Then one has*

$$\begin{aligned}\left(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}\mathcal{V}\right)(\lambda) &\geq \frac{\Gamma_k(\varrho+k)}{(\Psi(\lambda))^{\frac{\varrho}{k}}}\left(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}\right)(\lambda)\left(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{V}\right)(\lambda) \\ &\quad - \frac{\vartheta\lambda(\Psi(\lambda))^{\frac{\varrho}{k}}}{\Gamma_k(\varrho+k)}\left(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}\right)(\lambda) + \vartheta\left(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{I}\mathcal{U}\right)(\lambda),\end{aligned}$$

where $\mathcal{I}(\lambda)$ is the identity mapping.

Proof Let $\mathfrak{h}(\lambda) = \mathcal{V}(\lambda) - \vartheta\lambda$ and $\Upsilon(\lambda) = \vartheta\lambda$. Then we clearly see that $\mathfrak{h}$ is differentiable and increasing on $[0,\infty)$, and from the proof of Theorem 4.9 we know that

$$\begin{aligned}\left(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}(\mathcal{V}-\Upsilon)\right)(\lambda) &\geq \frac{\Gamma_k(\varrho+k)}{(\Psi(\lambda))^{\frac{\varrho}{k}}}\left(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}\right)(\lambda)\left(\mathcal{J}_{\Psi}^{\varrho,k}(\mathcal{V}-\Upsilon)\right)(\lambda) \\ &= \frac{\Gamma_k(\varrho+k)}{(\Psi(\lambda))^{\frac{\varrho}{k}}}\left(\mathcal{T}_{\Psi}^{\varrho,k}\mathcal{U}\right)(\lambda)\left(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{V}\right)(\lambda) \\ &\quad - \frac{\Gamma_k(\varrho+k)}{(\Psi(\lambda))^{\frac{\varrho}{k}}}\left(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}\right)(\lambda)\left(\mathcal{J}_{\Psi}^{\varrho,k}\Upsilon\right)(\lambda),\end{aligned} \tag{4.6}$$

where

$$\left(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}(\mathcal{V}-\Upsilon)\right)(\lambda) = \left(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{U}\mathcal{V}\right)(\lambda) - \vartheta\left(\mathcal{J}_{\Psi}^{\varrho,k}\mathcal{I}\mathcal{U}\right)(\lambda) \tag{4.7}$$

and

$$\left(\mathcal{J}_{\Psi}^{\varrho,k}\Upsilon\right)(\lambda) = \frac{\vartheta\lambda(\Psi(\lambda))^{\frac{\varrho}{k}}}{\Gamma_{\mathcal{K}}(\varrho+k)}. \tag{4.8}$$

Substituting (4.7) and (4.8) into (4.6) leads to the desired result. □

Corollary 4.14 *Let $k = 1$. Then Theorem 4.13 leads to a new result for generalized Riemann–Liouville fractional integral as follows*:

$$\left(\mathcal{J}_{\Psi}^{\varrho}\mathcal{U}\mathcal{V}\right)(\lambda) \geq \frac{\Gamma(\varrho+1)}{(\Psi(\lambda))^{\varrho}}\left(\mathcal{J}_{\Psi}^{\varrho}\mathcal{U}\right)(\lambda)\left(\mathcal{J}_{\Psi}^{\varrho}\mathcal{V}\right)(\lambda)$$

$$-\frac{\vartheta\lambda(\Psi(\lambda))^{\varrho}}{\Gamma(\varrho+1)}\big(\mathcal{J}_{\Psi}^{\varrho}\mathcal{U}\big)(\lambda)+\vartheta\big(\mathcal{J}_{\Psi}^{\varrho}\mathcal{I}\mathcal{U}\big)(\lambda).$$

Corollary 4.15 *If $\Psi(\lambda)=\lambda$, Theorem 4.13 provides the following new result for k-fractional integral*:

$$\begin{aligned}\big(\mathcal{J}^{\varrho,k}\mathcal{U}\mathcal{V}\big)(\lambda)\geq{}&\frac{\Gamma_k(\varrho+k)}{\lambda^{\frac{\varrho}{k}}}\big(\mathcal{J}^{\varrho,k}\mathcal{U}\big)(\lambda)\big(\mathcal{J}^{\varrho,k}\mathcal{V}\big)(\lambda)\\&-\frac{\vartheta\lambda^{\frac{\varrho}{k}+1}}{\Gamma_k(\varrho+k)}\big(\mathcal{J}^{\varrho,k}\mathcal{U}\big)(\lambda)+\vartheta\big(\mathcal{J}^{\varrho,k}\mathcal{I}\mathcal{U}\big)(\lambda).\end{aligned}$$

Corollary 4.16 *Let $\Psi(\lambda)=\lambda$ and $k=1$. Then Theorem 4.13 leads to a new inequality for Riemann–Liouville fractional integral as follows*:

$$\begin{aligned}\big(\mathcal{J}^{\varrho}\mathcal{U}\mathcal{V}\big)(\lambda)\geq{}&\frac{\Gamma(\varrho+1)}{\lambda^{\varrho}}\big(\mathcal{J}^{\varrho}\mathcal{U}\big)(\lambda)\big(\mathcal{J}^{\varrho}\mathcal{V}\big)(\lambda)\\&-\frac{\vartheta\lambda^{\varrho+1}}{\Gamma(\varrho+1)}\big(\mathcal{J}^{\varrho}\mathcal{U}\big)(\lambda)+\vartheta\big(\mathcal{J}^{\varrho}\mathcal{I}\mathcal{U}\big)(\lambda).\end{aligned}$$

5 Conclusion

In the article, we have established some new Pólya–Szegö and Čebyšev-type inequalities for two synchronous functions via generalized k-fractional integrals. Our obtained results are very general and can be specialized to discover numerous interesting fractional integral inequalities, and our approach may lead to a lot of follow-up research. Furthermore, they are expected to find some applications for establishing the uniqueness of solutions in fractional boundary value problems of the fractional partial differential equations.

Acknowledgements
The authors would like to express their sincere thanks to the anonymous reviewers for their helpful comments and suggestions.

Funding
The work was supported by the Natural Science Foundation of China (Grant Nos. 61673169, 11301127, 11701176, 11626101, 11601485).

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan. [2]Department of Mathematics, Çankaya University, Ankara, Turkey. [3]School of Mathematical Sciences, Zhejiang University, Hangzhou, China. [4]Department of Mathematics, Huzhou University, Huzhou, China. [5]School of Mathematics and Statistic, Changsha University of Science & Technology, Changsha, China.

Publisher's Note

Received: 21 October 2019 Accepted: 10 March 2020 Published online: 18 March 2020

References

1. Adjabi, Y., Jarad, F., Baleanu, D., Abdeljawad, T.: On Cauchy problems with Caputo Hadamard fractional derivatives. Math. Methods Appl. Sci. **40**(11), 661–681 (2016)
2. Agarwal, P., Dragomir, S.S., Jleli, M., Bessem Samet, B.: Advances in Mathematical Inequalities and Applications. Springer, Singapore (2018)
3. Bhairat, S.P., Dhaigude, D.B.: Existence and stability of fractional differential equations involving generalized Katugampola derivative. arXiv:1709.08838 [math.CA]
4. Oliveira, D.S., Capelas de Oliveira, E.: Hilfer–Katugampola fractional derivatives. Comput. Appl. Math. **37**(3), 3672–3690 (2018)
5. Ruzhansky, M., Cho, Y.J., Agarwal, P., Area, I.: Advances in Real and Complex Analysis with Applications. Springer, Singapore (2017)
6. Cheng, J.-F., Chu, Y.-M.: Solution to the linear fractional differential equation using Adomian decomposition method. Math. Probl. Eng. **2011**, Article ID 587068 (2011)
7. Cheng, J.-F., Chu, Y.-M.: On the fractional difference equations of order $(2, q)$. Abstr. Appl. Anal. **2011**, Article ID 497259 (2011)
8. Cheng, J.-F., Chu, Y.-M.: Fractional difference equations with real variable. Abstr. Appl. Anal. **2012**, Article ID 918529 (2012)
9. Chu, Y.-M., Adil Khan, M., Ali, T., Dragomir, S.S.: Inequalities for α-fractional differentiable functions. J. Inequal. Appl. **2017**, Article ID 93 (2017)
10. Adil Khan, M., Begum, S., Khurshid, Y., Chu, Y.-M.: Ostrowski type inequalities involving conformable fractional integrals. J. Inequal. Appl. **2018**, Article ID 70 (2018)
11. Adil Khan, M., Chu, Y.-M., Kashuri, A., Liko, R., Ali, G.: Conformable fractional integrals versions of Hermite–Hadamard inequalities and their generalizations. J. Funct. Spaces **2018**, Article ID 6928130 (2018)
12. Adil Khan, M., Iqbal, A., Suleman, M., Chu, Y.-M.: Hermite–Hadamard type inequalities for fractional integrals via Green's function. J. Inequal. Appl. **2018**, Article ID 161 (2018)
13. Adil Khan, M., Khurshid, Y., Du, T.-S., Chu, Y.-M.: Generalization of Hermite–Hadamard type inequalities via conformable fractional integrals. J. Funct. Spaces **2018**, Article ID 5357463 (2018)
14. Khurshid, Y., Adil Khan, M., Chu, Y.-M., Khan, Z.A.: Hermite–Hadamard–Fejér inequalities for conformable fractional integrals via preinvex functions. J. Funct. Spaces **2019**, Article ID 3146210 (2019)
15. Tan, W., Jiang, F.-L., Huang, C.-X., Zhou, L.: Synchronization for a class of fractional-order hyperchaotic system and its application. J. Appl. Math. **2012**, Article ID 974639 (2012)
16. Wu, J., Liu, Y.-C.: Uniqueness results and convergence of successive approximations for fractional differential equations. Hacet. J. Math. Stat. **42**(2), 149–158 (2013)
17. Zhang, Q., Liu, L.-Z.: A good λ estimate for multilinear commutator of fractional integral on spaces of homogeneous type. J. Math. Inequal. **4**(3), 371–389 (2010)
18. Liu, L.-Z.: Endpoint estimates for multilinear fractional singular integral operators on some Hardy spaces. Math. Notes **88**(5–6), 701–716 (2010)
19. Huang, C.-X., Liu, L.-Z.: Sharp function inequalities and boundedness for Toeplitz type operator related to general fractional singular integral operator. Publ. Inst. Math. **92**(106), 165–176 (2012)
20. Zhou, X.-S., Huang, C.-X., Hu, H.-J., Liu, L.: Inequality estimates for the boundedness of multilinear singular and fractional integral operators. J. Inequal. Appl. **2013**, Article ID 303 (2013)
21. Liu, F.-W., Feng, L.-B., Anh, V., Li, J.: Unstructured-mesh Galerkin finite element method for the two-dimensional multi-term time-space fractional Bloch–Torrey equations on irregular convex domains. Comput. Math. Appl. **78**(5), 1637–1650 (2019)
22. Jiang, Y.-J., Xu, X.-J.: A monotone finite volume method for time fractional Fokker–Planck equations. Sci. China Math. **62**(4), 783–794 (2019)
23. Zhou, S.-H., Jiang, Y.-J.: Finite volume methods for *N*-dimensional time fractional Fokker–Planck equations. Bull. Malays. Math. Sci. Soc. **42**(6), 3167–3186 (2019)
24. Rafeeq, S., Kalsoom, H., Hussain, S., Rashid, S., Chu, Y.-M.: Delay dynamic double integral inequalities on time scales with applications. Adv. Differ. Equ. **2020**, Article ID 40 (2020)
25. Latif, M.A., Rashid, S., Dragomir, S.S., Chu, Y.-M.: Hermite–Hadamard type inequalities for co-ordinated convex and quasi-convex functions and their applications. J. Inequal. Appl. **2019**, Article ID 317 (2019)
26. Baleanu, D., Diethelm, K., Scalas, E., Trujillo, J.J.: Fractional Calculus: Models and Numerical Methods. World Scientific, Hackensack (2012)
27. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
28. Mainardi, F.: Fractional Calculus and Waves in Linear Viscoelasticity. Imperial College Press, London (2010)
29. Akman, T., Yıldız, B., Baleanu, D.: New discretization of Caputo–Fabrizio derivative. Comput. Appl. Math. **37**(3), 3307–3333 (2018)
30. Caputo, M., Fabrizio, M.: A new definition of fractional derivative without singular kernel. Prog. Fract. Differ. Appl. **1**(2), 73–85 (2015)
31. Losad, J., Nieto, J.J.: Properties of a new fractional derivative without singular kernel. Prog. Fract. Differ. Appl. **1**(2), 87–92 (2015)
32. Yang, X.J., Srivastava, H.M., Tenreiro Machado, J.A.: A new fractional derivative without singular kernel: application to the modelling of the steady heat flow. arXiv:1601.01623 [math.GM]
33. Li, J., Guo, B.-L.: The quasi-reversibility method to solve the Cauchy problems for parabolic equations. Acta Math. Sin. **29**(8), 1617–1628 (2013)
34. Huang, C.-X., Guo, S., Liu, L.-Z.: Boundedness on Morrey space for Toeplitz type operator associated to singular integral operator with variable Calderón–Zygmund kernel. J. Math. Inequal. **8**(3), 453–464 (2014)
35. Deng, Y.-J., Fang, X.-P., Li, J.: Numerical methods for reconstruction of the source term of heat equations from the final overdetermination. Bull. Korean Math. Soc. **52**(5), 1495–1515 (2015)
36. Fang, X.-P., Deng, Y.-J., Li, J.: Plasmon resonance and heat generation in nanostructures. Math. Methods Appl. Sci. **38**(18), 4663–4672 (2015)

37. Cai, Z.-W., Huang, J.-H., Huang, L.-H.: Generalized Lyapunov–Razumikhin method for retarded differential inclusions: applications to discontinuous neural networks. Discrete Contin. Dyn. Syst. **22B**(9), 3591–3614 (2017)
38. Duan, L., Huang, L.-H., Guo, Z.-Y., Fang, X.-W.: Periodic attractor for reaction-diffusion high-order Hopfield neural networks with time-varying delays. Comput. Math. Appl. **73**(2), 233–245 (2017)
39. Tan, Y.-X., Liu, L.-Z.: Weighted boundedness of multilinear operator associated to singular integral operator with variable Calderón–Zygmund kernel. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **111**(4), 931–946 (2017)
40. Cai, Z.-W., Huang, J.-H., Huang, L.-H.: Periodic orbit analysis for the delayed Filippov system. Proc. Am. Math. Soc. **146**(11), 4667–4682 (2018)
41. Chen, T., Huang, L.-H., Yu, P., Huang, W.-T.: Bifurcation of limit cycles at infinity in piecewise polynomial systems. Nonlinear Anal., Real World Appl. **41**, 82–106 (2018)
42. Duan, L., Fang, X.-W., Huang, C.-X.: Global exponential convergence in a delayed almost periodic Nicholson's blowflies model with discontinuous harvesting. Math. Methods Appl. Sci. **41**(5), 1954–1965 (2018)
43. Tan, Y.-X., Huang, C.-X., Sun, B., Wang, T.: Dynamics of a class of delayed reaction-diffusion systems with Neumann boundary condition. J. Math. Anal. Appl. **458**(2), 1115–1130 (2018)
44. Wang, J.-F., Chen, X.-Y., Huang, L.-H.: The number and stability of limit cycles for planar piecewise linear systems of node-saddle type. J. Math. Anal. Appl. **469**(1), 405–427 (2019)
45. Wang, J.-F., Huang, C.-X., Huang, L.-H.: Discontinuity-induced limit cycles in a general planar piecewise linear system of saddle-focus type. Nonlinear Anal. Hybrid Syst. **33**, 162–178 (2019)
46. Huang, C.-X., Zhang, H., Huang, L.-H.: Almost periodicity analysis for a delayed Nicholson's blowflies model with nonlinear density-dependent mortality term. Commun. Pure Appl. Anal. **18**(6), 3337–3349 (2019)
47. Huang, C.-X., Yang, Z.-C., Yi, T.-S., Zou, X.-F.: On the basins of attraction for a class of delay differential equations with non-monotone bistable nonlinearities. J. Differ. Equ. **256**(7), 2101–2114 (2014)
48. Xie, Y.-Q., Li, Q.-S., Zhu, K.-X.: Attractors for nonclassical diffusion equations with arbitrary polynomial growth nonlinearity. Nonlinear Anal., Real World Appl. **31**, 23–37 (2016)
49. Huang, C.-X., Liu, L.-Z.: Boundedness of multilinear singular integral operator with a non-smooth kernel and mean oscillation. Quaest. Math. **40**(3), 295–312 (2017)
50. Duan, L., Huang, C.-X.: Existence and global attractivity of almost periodic solutions for a delayed differential neoclassical growth model. Math. Methods Appl. Sci. **40**(3), 814–822 (2017)
51. Hu, H.-J., Liu, L.-Z.: Weighted inequalities for a general commutator associated to a singular integral operator satisfying a variant of Hörmander's condition. Math. Notes **101**(5–6), 830–840 (2017)
52. Hu, H.-J., Zou, X.-F.: Existence of an extinction wave in the Fisher equation with a shifting habitat. Proc. Am. Math. Soc. **145**(11), 4763–4771 (2017)
53. Tang, W.-S., Sun, Y.-J.: Construction of Runge–Kutta type methods for solving ordinary differential equations. Appl. Math. Comput. **234**, 179–191 (2014)
54. Xie, D.-X., Li, J.: A new analysis of electrostatic free energy minimization and Poisson–Boltzmann equation for protein in ionic solvent. Nonlinear Anal., Real World Appl. **21**, 185–196 (2015)
55. Dai, Z.-F., Chen, X.-H., Wen, F.-H.: A modified Perry's conjugate gradient method-based derivative-free method for solving large-scale nonlinear monotone equations. Appl. Math. Comput. **270**, 378–386 (2015)
56. Feng, L.-B., Zhuang, P., Liu, F., Turner, I., Anh, V., Li, J.: A fast second-order accurate method for a two-sided space-fractional diffusion equation with variable coefficients. Comput. Math. Appl. **73**(6), 1155–1171 (2017)
57. Li, J., Liu, F., Fang, L., Turner, I.: A novel finite volume method for the Riesz space distributed-order diffusion equation. Comput. Math. Appl. **74**(4), 772–783 (2017)
58. Wang, W.-S.: Fully-geometric mesh one-leg methods for the generalized pantograph equation: approximating Lyapunov functional and asymptotic contractivity. Appl. Numer. Math. **117**, 50–68 (2017)
59. Liu, Z.-Y., Wu, N.-C., Qin, X.-R., Zhang, Y.-L.: Trigonometric transform splitting methods for real symmetric Toeplitz systems. Comput. Math. Appl. **75**(8), 2782–2794 (2018)
60. Li, J., Ying, J.-Y., Xie, D.-X.: On the analysis and application of an ion size-modified Poisson–Boltzmann equation. Nonlinear Anal., Real World Appl. **47**, 188–203 (2019)
61. Agarwal, P.: Some inequalities involving Hadamard-type k-fractional integral operators. Math. Methods Appl. Sci. **40**(11), 3882–3891 (2017)
62. Agarwal, P.: Certain properties of the generalized Gauss hypergeometric functions. Appl. Math. Inf. Sci. **8**(5), 2315–2320 (2014)
63. Agarwal, P., Choi, J.: Certain fractional integral inequalities associated with pathway fractional integral operators. Bull. Korean Math. Soc. **53**(1), 181–193 (2016)
64. Agarwal, P., Jain, S., Mansour, T.: Further extended Caputo fractional derivative operator and its applications. Russ. J. Math. Phys. **24**(4), 415–425 (2017)
65. Agarwal, P., Jleli, M., Tomar, M.: Certain Hermite–Hadamard type inequalities via generalized k-fractional integrals. J. Inequal. Appl. **2017**, Article ID 55 (2017)
66. Grüss, G.: Über das Maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f(x)\,dx\int_a^b g(x)\,dx$. Math. Z. **39**(1), 215–226 (1935)
67. Ntouyas, S., Agarwal, P., Tariboon, J.: On Pólya–Szegö and Chebyshev types inequalities involving the Riemann–Liouville fractional integral operators. J. Math. Inequal. **10**(2), 491–504 (2016)
68. Özdemir, M.E., Set, E., Akdemir, A.O., Sarıkaya, M.Z.: Some new Chebyshev type inequalities for functions whose derivatives belongs to L_p spaces. Afr. Math. **26**(7–8), 1609–1619 (2015)
69. Set, E., Akdemir, A.O., Mumcu, İ.: Hadamard's inequality and its extensions for conformable fractional integrals of any order $\alpha > 0$. Creative Math. Inform. **27**(2), 197–206 (2018)
70. Agarwal, P.: Fractional integration of the product of two multivariables H-function and a general class of polynomials. In: Advances in Applied Mathematics and Approximation Theory. Springer Proc. Math. Stat., vol. 41, pp. 359–374. Springer, New York (2013)
71. Chebyshev, P.L.: Sur les expressions approximatives des integrales definies par les autres prises entre les mêmes limites. Proc. Math. Soc. Charkov **2**, 93–98 (1882)

72. Belarbi, S., Dahmani, Z.: On some new fractional integral inequalities. JIPAM. J. Inequal. Pure Appl. Math. **10**(3), Article ID 86 (2009)
73. Dahmani, Z.: New inequalities in fractional integrals. Int. J. Nonlinear Sci. **9**(4), 493–497 (2010)
74. Dahmani, Z., Mechouar, O., Brahami, S.: Certain inequalities related to the Chebyshev's functional involving a Riemann–Liouville operator. Bull. Math. Anal. Appl. **3**(4), 38–44 (2011)
75. Dragomir, S.S., Diamond, N.T.: Integral inequalities of Grüss type via Pólya–Szegö and Shisha–Mond results. East Asian Math. J. **19**(1), 27–39 (2003)
76. Pólya, G., Szegö, G.: Aufgaben und Lehrsätze aus der Analysis i. Springer, New York (1964)
77. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives: Theory and Applications. Gordon & Breach, Yverdon (1993)
78. Kacar, E., Kacar, Z., Yildirim, H.: Integral inequalities for Riemann–Liouville fractional integrals of a function with respect to another function. Iran. J. Math. Sci. Inform. **13**(1), 1–13 (2018)
79. Zhao, T.-H., Chu, Y.-M., Wang, H.: Logarithmically complete monotonicity properties relating to the gamma function. Abstr. Appl. Anal. **2011**, Article ID 896483 (2011)
80. Yang, Z.-H., Qian, W.-M., Chu, Y.-M., Zhang, W.: On rational bounds for the gamma function. J. Inequal. Appl. **2017**, Article ID 210 (2017)
81. Huang, T.-R., Han, B.-W., Ma, X.-Y., Chu, Y.-M.: Optimal bounds for the generalized Euler–Mascheroni constant. J. Inequal. Appl. **2018**, Article ID 118 (2018)
82. Huang, T.-R., Tan, S.-Y., Ma, X.-Y., Chu, Y.-M.: Monotonicity properties and bounds for the complete *p*-elliptic integrals. J. Inequal. Appl. **2018**, Article ID 239 (2018)
83. Wang, M.-K., Chu, Y.-M., Zhang, W.: Monotonicity and inequalities involving zero-balanced hypergeometric function. Math. Inequal. Appl. **22**(2), 601–617 (2019)
84. Wang, M.-K., Zhang, W., Chu, Y.-M.: Monotonicity, convexity and inequalities involving the generalized elliptic integrals. Acta Math. Sci. **39B**(5), 1440–1450 (2019)
85. Wang, M.-K., Hong, M.-Y., Xu, Y.-F., Shen, Z.-H., Chu, Y.-M.: Inequalities for generalized trigonometric and hyperbolic functions with one parameter. J. Math. Inequal. **14**(1), 1–21 (2020)
86. Yang, Z.-H., Qian, W.-M., Zhang, W., Chu, Y.-M.: Notes on the complete elliptic integral of the first kind. Math. Inequal. Appl. **23**(1), 77–93 (2020)
87. Wang, M.-K., Chu, H.-H., Chu, Y.-M.: Precise bounds for the weighted Hölder mean of the complete *p*-elliptic integrals. J. Math. Anal. Appl. **480**(2), Article ID 123388 (2020)
88. Mubeen, S., Habibullah, G.M.: *k*-fractional integrals and application. Int. J. Contemp. Math. Sci. **7**(1–4), 89–94 (2012)
89. Díaz, R., Pariguan, E.: On hypergeometric functions and Pochhammer *k*-symbol. Divulg. Mat. **15**(2), 179–192 (2007)
90. Katugampola, U.N.: New fractional integral unifying six existing fractional integrals. arXiv:1612.08596 [math.CA]
91. Jarad, F., Uğurlu, E., Abdeljawad, T., Baleanu, D.: On a new class of fractional operators. Adv. Differ. Equ. **2017**, Article ID 247 (2017)
92. Khan, T.U., Adil Khan, M.: Generalized conformable fractional operators. J. Comput. Appl. Math. **346**, 378–389 (2019)