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Research Article

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On surrounding quasi-contractions on non-triangular metric spaces

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Abstract: The aim of this paper is to establish some fixed point results for surrounding quasi-contractions in non-triangular metric spaces. Also, we prove the Banach principle of contraction in non-triangular metric spaces. As applications of our theorems, we deduce certain well-known results in b-metric spaces as corollaries.

Keywords: non-triangular metric spaces, quasi-contraction, b-metric spaces

MSC 2020: 47H10, 54H25

1 Introduction and preliminaries

In this paper [1], Banach opened up a new way in non-linear analysis, upon which various applications in a variety of sciences have appeared. After this interesting principle, several authors generalized this principle by introducing the various contractions on metric spaces (see [2–8]). In 2014, the notion of manageable function was introduced by Du and Khojasteh [9,10] to generalize and unify the several existing fixed point results in the literature. After that, Jleli and Samet [11] introduced a generalization of metric spaces that recovers a large class of topological spaces including standard metric spaces, b-metric spaces, dislocated metric spaces and modular spaces called JS-metric spaces. In this paper, we establish some fixed point theorems for surrounding quasi-contractions in non-triangular metric spaces. Also, we prove the Banach principle of contraction in non-triangular metric spaces. As applications of our theorems, we deduce certain well-known results in b-metric spaces as corollaries.

Here, we preliminarily provide some auxiliary facts which will be needed later.

The concept of b-metric space was introduced by Bakhtin [12] and Czerwik [13], which is an interesting generalization of usual metric space (see [14–31]). A b-metric space (see [12,13]) (X, d) is a space defined on a non-empty set X with a mapping $d : X \times X \to [0, +\infty)$ and constant $s \geq 1$ satisfying the following conditions:
(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

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In this case, d is called a b-metric on X. Regarding the concept of b-convergent sequence, b-Cauchy sequence and b-completeness, the reader may refer to [19] and references therein.

Let X be a non-empty set and let $\varrho : X \times X \to [0, +\infty]$ be a given mapping. For every $x \in X$, define the sets:

$$\mathcal{M}(\varrho, X, x) = \{\{x_n\} \subset X : \lim_{n\to\infty} \varrho(x_n, x) = 0\}.$$

Definition 1.1. [11] We say that ϱ is a JS-metric on X if it satisfies the following conditions:

(a_1) for each pair $(x, y) \in X \times X$, we have

$$\varrho(x, y) = 0 \quad \text{implies} \;\; x = y,$$

(a_2) for each pair $(x, y) \in X \times X$, we have

$$\varrho(x, y) = \varrho(y, x),$$

(a_3) there exists $\kappa > 0$ such that for all $x, y \in X$, if $\{x_n\} \in \mathcal{M}(\varrho, X, x)$

$$\varrho(x, y) \leq \kappa \limsup_{n\to\infty} \varrho(x_n, y).$$

In this case, we say the pair (X, ϱ) is a JS-metric space by modulus κ.

Definition 1.2. [11] Let (X, ϱ) be a JS-metric space.

(b_1) We say that $\{x_n\}$ ϱ-converges to x if $\{x_n\} \in \mathcal{M}(\varrho, X, x)$,

(b_2) if $\{x_n\}$ ϱ-converges to x and ϱ-converges to y, then $x = y$,

(b_3) $\{x_n\}$ is a ϱ-Cauchy sequence if $\lim_{m,n\to\infty} \varrho(x_n, x_m) = 0$,

(b_4) (X, ϱ) is said to be ϱ-complete if every ϱ-Cauchy sequence in X is convergent to some element in X.

Very recently, Khojasteh and Khandani [32] introduced the concept of non-triangular metric space and obtained some fixed point results which are the generalization of some new recent results in the literature.

Definition 1.3. Let X be a non-empty set and let $\rho : X \times X \to \mathbb{R}^+$ be a mapping. We say that ρ is a non-triangular metric on X if it satisfies the following conditions:

(c_1) $\rho(x, x) = 0$ for all $x \in X$;

(c_2) If $\{x_n\} \in \mathcal{M}(\rho, X, x) \cap \mathcal{M}(\rho, X, y)$, then $x = y$ for all $x, y \in X$.

Note that if $x, y \in X$ and $\rho(x, y) = 0$, then taking $x_n = x$ for each $n \in \mathbb{N}$ and applying conditions (c_1) and (c_2) it follows that $x = y$.

Definition 1.4. Let (X, ρ) be a non-triangular metric space.

(d_1) We say that $\{x_n\}$ ρ-converges to x if $\lim_{n\to\infty} \rho(x_n, x) = 0$,

(d_2) $\{x_n\}$ is a ρ-Cauchy sequence if $\limsup_{n\to\infty} \{\rho(x_n, x_m) : m \geq n\} = 0$,

(d_3) (X, ρ) is said to ρ-complete if every ρ-Cauchy sequence in X is ρ-convergent to some element in X.

Definition 1.5. Let (X, ρ) be a non-triangular metric space and $T : X \to X$ be a mapping. A mapping T is $\mathcal{M}$-continuous in $x \in X$ if

$$\{x_n\} \in \mathcal{M}(\rho, X, x) \, \text{implies} \;\; \{Tx_n\} \in \mathcal{M}(\rho, X, Tx).$$

Remark 1.6. Note that, if T is a contraction, *i.e.*, there exists $k \in [0, 1)$ such that

$$\rho(Tx, Ty) \leq k\rho(x, y)$$

for all $x, y \in X$, then T is $\mathcal{M}$-continuous at each point x in X.

The following example shows that non-triangular metric space is a real generalization of generalized metric space in sense a concept of Jleli and Samet [11].

Example 1.7. Let $X = [0, +\infty)$, define

$$\rho(x,y) = \begin{cases} \frac{(x+y)^2}{(x+y)^2+1}, & 0 \neq x \neq y \neq 0, \\ \frac{x}{2}, & y = 0, \\ \frac{y}{2}, & x = 0, \\ 0, & x = y. \end{cases}$$

Condition (c_1) is trivially satisfied. We need to verify condition (c_2). For this, let $x, y \in X$ and $\{x_n\}$ be a sequence in X such that $\rho(x_n, x) \to 0$ and $\rho(x_n, y) \to 0$ as $n \to \infty$. It implies that

$$\lim_{n\to\infty} \frac{(x_n + x)^2}{(x_n + x)^2 + 1} = \lim_{n\to\infty} \frac{(x_n + y)^2}{(x_n + y)^2 + 1} = 0$$

and these hold if and only if $\lim_{n\to\infty} x_n = -x = -y$ in $\mathbb{R}$ and so $x = y$. Hence, condition (c_2) is true. Therefore, (X, ρ) is a non-triangular metric space. On the other hand, condition (a_3) does not hold. For all $n \in \mathbb{N}$ and for each $y \in X$,

$$\rho(x_n, y) = \begin{cases} \frac{(x_n+y)^2}{(x_n+y)^2+1}, & \text{if } x_n \neq 0, \\ \frac{y}{2}, & \text{if } x_n = 0. \end{cases}$$

Since $\{x_n\}$ is a convergent sequence to zero. If there exists $C \geq 1$ such that

$$\frac{y}{2} = \rho(0, y) \leq C \limsup_{n\to\infty} \rho(x_n, y) = C \limsup_{n\to\infty} \frac{(x_n + y)^2}{(x_n + y)^2 + 1} = C\frac{y^2}{y^2 + 1},$$

we have $C \geq \frac{y^2+1}{2y} \geq \frac{y}{2}$. Therefore, there is no bound for C, by which,

$$\rho(y, 0) \leq C \limsup_{n\to\infty} \rho(y, x_n).$$

2 Main results

In this section, we prove that quasi-contraction in non-triangular metric space has a fixed point.

Let (X, ϱ) be a metric space and let $T : X \to X$ be a mapping. For every $x_0 \in X$, let

$$\delta_n(T, x_0) = \sup\{\varrho(T^i(x_0), T^j(x_0)) : i, j \geq n\}.$$

Definition 2.1. Let (X, ϱ) be a non-triangular metric space. The mapping $T : X \to X$ is said to be a surrounding quasi-contraction with respect to Θ if there exists $\alpha \in [0, 1)$ such that for all $x, y \in X$,

$$\varrho(Tx, Ty) \leq \alpha M_{T,\Theta}(x, y), \tag{1}$$

where $\Theta : \mathbb{R}^4 \to \mathbb{R}$ is a mapping such that $\Theta(t, s, z, w) \leq \max\{z, w\}$ and

$$M_{T,\Theta}(x, y) = \max\{\varrho(x, y), \varrho(x, Tx), \varrho(y, Ty), \Theta(\varrho(x, Tx), \varrho(y, Ty), \varrho(x, Ty), \varrho(y, Tx))\}.$$

Theorem 2.2. *Let (X, ϱ) be a ϱ-complete non-triangular metric space and $T : X \to X$ be a surrounding quasi-contraction with respect to Θ such that $\delta_1(T, x_0) < \infty$ for some $x_0 \in X$. Then $\{T^n x_0\}$ ϱ converges to $\omega \in X$. Moreover, if T is $\mathcal{M}$-continuous in ω, then ω is a fixed point of T.*

Proof. Suppose that $\{x_n\}$ is a sequence defined by $x_{n+1} = Tx_n$, $n = 0, 1, \ldots$. Note that $0 \le \delta_{n+1}(T, x_0) \le \delta_n(T, x_0)$. Therefore, $\{\delta_n(T, x_0)\}$ is a monotone bounded sequence from below and so is convergent. Thus, there exists $\delta \ge 0$ such that $\lim_{n\to\infty} \delta_n(T, x_0) = \delta$. We shall show that $\delta = 0$. If $\delta > 0$, then by the definition of $\delta_n(T, x_0)$, for every $k \in \mathbb{N}$ there exists n_k, m_k such that $m_k > n_k \ge k$ and

$$\delta_k(T, x_0) - \frac{1}{k} < \varrho(T^{m_k}(x_0), T^{n_k}(x_0)) \le \delta_k(T, x_0). \tag{2}$$

Hence,

$$\lim_{k\to\infty} \varrho(T^{m_k}(x_0), T^{n_k}(x_0)) = \delta. \tag{3}$$

Also, we have

$$\begin{aligned}
\rho(T^{m_k}(x_0), T^{n_k}(x_0)) &\le \alpha \max\{\varrho(T^{m_k-1}(x_0), T^{n_k-1}(x_0)), \varrho(T^{m_k}(x_0), T^{m_k-1}(x_0)), \varrho(T^{n_k}(x_0), T^{n_k-1}(x_0)),\\
&\quad \Theta(\varrho(T^{m_k}(x_0), T^{m_k-1}(x_0)), \varrho(T^{n_k}(x_0), T^{n_k-1}(x_0)),\\
&\quad \varrho(T^{m_k}(x_0), T^{n_k-1}(x_0)), \varrho(T^{m_k-1}(x_0), T^{n_k}(x_0)))\}\\
&\le \alpha \max\{\delta_{k-1}(T, x_0), \delta_k(T, x_0)\}\\
&= \alpha\delta_{k-1}(T, x_0).
\end{aligned}$$

If we let $k \to \infty$ get it $\delta \le \alpha\delta$. Thus, $\alpha \ge 1$ and this is a contradiction. Therefore, we deduce that $\delta = 0$ and so $\{x_n\}$ is a ϱ-Cauchy sequence. Since (X, ϱ) is ϱ-complete, there exists some $\omega \in X$ such that $\{x_n\}$ is ϱ-convergent to ω. Since $\{x_n\} \in \mathcal{M}(\varrho, X, \omega)$ and T is a $\mathcal{M}$-continuous we have that $\{Tx_n\} \in \mathcal{M}(\varrho, X, T\omega)$, so we conclude that

$$\{x_n\} \in \mathcal{M}(\varrho, X, \omega) \cap \mathcal{M}(\varrho, X, T\omega).$$

From condition (c_2), we obtain that $\omega = T\omega$, so ω is a fixed point of T. Condition (1) implies that ω is a unique fixed point. □

From Theorem 2.2 and Remark 1.6 follow directly the Banach principle of contraction in non-triangular metric spaces.

Corollary 2.3. *Let (X, ϱ) be a ϱ-complete non-triangular metric space and $T : X \to X$ be a mapping. If there exists $k \in [0, 1)$ such that T satisfies*

$$\varrho(Tx, Ty) \le k\varrho(x, y),$$

for all $x, y \in X$ and $\delta_1(T, x_0) < \infty$ for some $x_0 \in X$, then T has a fixed point w in X.

Corollary 2.4. *Let (X, ϱ) be a ϱ-complete non-triangular metric space and $T : X \to X$ be a $\mathcal{M}$-continuous mapping. If there exists $k \in [0, 1)$ such that T satisfies*

$$\varrho(Tx, Ty) \le k \max\left\{\varrho(x, y), \varrho(x, Tx), \varrho(y, Ty), \frac{\max\{\varrho(x, Ty), \varrho(y, Tx)\}}{\varrho(x, Tx) + \varrho(y, Ty) + 1}\right\}$$

for all $x, y \in X$ and $\delta_1(T, x_0) < \infty$ for some $x_0 \in X$, then T has a fixed point w in X.

Proof. It suffices to consider $\Theta(t, s, z, w) = \frac{\max\{z, w\}}{t+s+1}$ and apply Theorem 2.2. □

Corollary 2.5. *Let (X, ϱ) be a ϱ-complete non-triangular metric space and $T : X \to X$ be a $\mathcal{M}$-continuous mapping. If there exists $k \in [0, 1)$ such that T satisfies*

$$\varrho(Tx, Ty) \le k \max\left\{\varrho(x, y), \varrho(x, Tx), \varrho(y, Ty), \frac{\varrho(x, Ty) + \varrho(y, Tx)}{2}\right\}$$

for all $x, y \in X$ and $\delta_1(T, x_0) < \infty$ for some $x_0 \in X$, then T has a fixed point w in X.

Proof. It suffices to consider $\Theta(t, s, z, w) = \frac{z+w}{2}$ and apply Theorem 2.2. □

Corollary 2.6. *Let (X, ϱ) be a ϱ-complete non-triangular metric space and $T : X \to X$ be a $\mathcal{M}$-continuous mapping. If there exists $k \in [0, 1)$ such that T satisfies*

$$\varrho(Tx, Ty) \leq k \max\{\varrho(x, y), \varrho(x, Tx), \varrho(y, Ty), \varrho(x, Ty), \varrho(y, Tx)\}$$

for all $x, y \in X$ and $\delta_1(T, x_0) < \infty$ for some $x_0 \in X$, then T has a fixed point w in X.

Proof. It suffices to consider $\Theta(t, s, z, w) = \max\{z, w\}$ and then apply Theorem 2.2. □

Example 2.7. Let $X = \left[0, \frac{1}{2}\right] \cup \left[\frac{2}{3}, 1\right]$ be endowed with the Euclidean metric. Obviously, (X, ρ) is a complete non-triangular metric space, where $\rho(x, y) = |x - y|$ for each $x, y \in X$. Let $T : X \to X$ be defined by

$$Tx = \begin{cases} 0, & 0 \leq x \leq \frac{1}{2}, \\ \frac{1}{4}, & \frac{2}{3} \leq x \leq 1. \end{cases}$$

Note that for each $x \in \left[0, \frac{1}{2}\right]$ and for each $y \in \left[\frac{2}{3}, 1\right]$, we have $|Tx - Ty| = \frac{1}{4}$. Since

$$\frac{8}{27} \leq \frac{|x - Ty| + |y - Tx|}{|x - Tx| + |y - Ty| + 1} \leq \frac{15}{17},$$

thus, if we take $1 > \alpha \geq \frac{27}{32}$ (for example, $\alpha = \frac{7}{8}$), we have

$$|Tx - Ty| \leq \alpha \frac{|x - Ty| + |y - Tx|}{|x - Tx| + |y - Ty| + 1}.$$

Therefore,

$$|Tx - Ty| \leq \alpha \max\left\{|x - y|, |x - Tx|, |y - Ty|, \left(\frac{|x - Ty| + |y - Tx|}{|x - Tx| + |y - Ty| + 1}\right)\right\}.$$

Similar argument holds for the other cases with the same α. It is easy to see that T is $\mathcal{M}$-continuous and $\delta_1(T, 0) < 1$. Therefore, T is satisfied in the conditions of Corollary 2.4 and so it has a fixed point in X.

Theorem 2.8. *Let (X, ϱ) be a ϱ-complete non-triangular metric space and $T : X \to X$ be a mapping. Let there exists $\alpha \in \left[0, \frac{1}{2}\right)$ such that*

$$\varrho(Tx, Ty) \leq \alpha U_{T,\Theta}(x, y), \tag{4}$$

for all $x, y \in X$, where $\Theta : \mathbb{R}^4 \to \mathbb{R}$ is a mapping such that $\Theta(t, s, z, w) \leq \max\{z, w\}$ and

$$U_{T,\Theta}(x, y) = \max\{\varrho(x, y), \varrho(x, Tx) + \varrho(y, Ty), \Theta(\varrho(x, Tx), \varrho(y, Ty), \varrho(x, Ty), \varrho(y, Tx))\}.$$

If $\delta_1(T, x_0) < \infty$ for some $x_0 \in X$, then $\{T^n x_0\}$ converges to some $\omega \in X$. Also, if T $\mathcal{M}$-continuous in ω, then ω is a fixed point of T.

Proof. We will use the same technique as the proof of Theorem 2.2. Let $\{x_n\}$ be the Picard sequence based at x_0. We show that $\{x_n\}$ is a Cauchy sequence. Note that $0 \leq \delta_{n+1}(T, x_0) \leq \delta_n(T, x_0)$. Therefore, $\{\delta_n(T, x_0)\}$ is a monotone bounded sequence and so is convergent. Thus, there exists $\delta \geq 0$ such that $\lim_{n\to\infty} \delta_n(T, x_0) = \delta$. We shall show that δ=0. If δ>0, then by the definition of $\delta_n(T, x_0)$ for every $k \in \mathbb{N}$ there exist n_k, m_k such that $m_k > n_k \geq k$ and

$$\delta_k(T, x_0) - \frac{1}{k} < \rho(T^{m_k}(x_0), T^{n_k}(x_0)) \leq \delta_k(T, x_0). \tag{5}$$

Hence,

$$\lim_{k\to\infty} \rho(T^{m_k}(x_0), T^{n_k}(x_0)) = \delta. \tag{6}$$

Also, we have

$$\begin{aligned}
\varrho(T^{m_k}(x_0), T^{n_k}(x_0)) &\leq \alpha \max\{\varrho(T^{m_k-1}(x_0), T^{n_k-1}(x_0)), \varrho(T^{m_k}(x_0), \\
&\quad T^{m_k-1}(x_0)) + \varrho(T^{n_k}(x_0), T^{n_k-1}(x_0)), \\
&\quad \Theta(\varrho(T^{m_k}(x_0), T^{m_k-1}(x_0)), \varrho(T^{n_k}(x_0), T^{n_k-1}(x_0)), \\
&\quad \varrho(T^{m_k}(x_0), T^{n_k-1}(x_0)), \varrho(T^{m_k-1}(x_0), T^{n_k}(x_0)))\} \\
&\leq \alpha \max\{2\delta_{k-1}(T, x_0), \delta_k(T, x_0)\} \\
&= 2\alpha\delta_{k-1}(T, x_0).
\end{aligned}$$

Taking limit as $k \to \infty$ we get $\delta \leq 2\alpha\delta$. Thus, $\alpha \geq \frac{1}{2}$ and this is a contradiction. Therefore, we deduce that $\delta = 0$ and so $\{x_n\}$ is a ϱ-Cauchy sequence. Since (X, ϱ) is ϱ-complete, there exists some $\omega \in X$ such that $\{x_n\}$ is ϱ-convergent to ω. Proof that ω is a unique fixed point of T is similar to that in the proof of Theorem 2.2. □

From Theorem 2.8 we obtain version of Kannan's result on fixed point (see [33]).

Corollary 2.9. *Let (X, ϱ) be a ϱ-complete non-triangular metric space and $T : X \to X$ be a mapping. Let there exists $\alpha \in \left[0, \frac{1}{2}\right)$ such that*

$$\varrho(Tx, Ty) \leq \alpha[\varrho(x, Tx) + \varrho(y, Ty)] \tag{7}$$

for all $x, y \in X$. If $\delta_1(T, x_0) < \infty$ for some $x_0 \in X$, then $\{T^n x_0\}$ converges to some $\omega \in X$. Also, if T $\mathcal{M}$-continuous in ω, then ω is a fixed point of T.

3 Some applications in *b*-metric spaces

The next theorem is known, see for example [32, Theorem 12.2]. We give another proof here.

Theorem 3.1. *Let (X, d) be a complete b-metric space with coefficient s and $T : X \to X$ be a mapping satisfying:*

$$d(Tx, Ty) \leq \lambda d(x, y) \tag{8}$$

for all $x, y \in X$, where $0 \leq \lambda < 1$. Then T has a unique fixed point $x^\star$, and for every $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to $x^\star$.

Proof. In view of Corollary 2.3 it suffices to prove that

$$\delta_1(T, x_0) = \sup\{d(T^i x_0, T^j x_0) : i, j \geq 1\} < \infty \quad \text{for some } x_0 \in X.$$

Since for $j > i$ and for each x_0 we have

$$d(T^i x_0, T^j x_0) \leq \lambda^{j-i} d(x_0, T^{j-i} x_0) < d(x_0, T^{j-i} x_0),$$

so we need to show that there is a constant $C > 0$ such that $d(x_0, T^n x_0) \leq C$ for all $n \in \mathbb{N}$. We know that there exists $n_0 \in \mathbb{N}$ such that $\lambda^{n_0} < \frac{1}{s^2}$. Let $x_0 \in X$ be arbitrary. Define the sequence $\{x_n\}$ by $x_{n+1} = T^n x_0$ for all $n \geq 0$. Then (8) implies that

$$d(T^{n+n_0} x_0, T^n x_0) \leq \lambda^n d(T^{n_0} x_0, x_0) \tag{9}$$

and

$$d(T^{n+n_0}x_0, T^{n_0}x_0) \leq \lambda^{n_0} d(T^n x_0, x_0). \tag{10}$$

Applying the triangle-type inequality (3) for b-metric space to triples, we have

$$\begin{aligned} d(x_0, T^n x_0) &\leq s[d(x_0, T^{n_0}x_0) + d(T^{n_0}x_0, T^n x_0)] \\ &\leq sd(x_0, T^{n_0}x_0) + s^2[d(T^{n_0}x_0, T^{n+n_0}x_0) + d(T^{n+n_0}x_0, T^n x_0)] \\ &\leq sd(x_0, T^{n_0}x_0) + s^2\lambda^{n_0} d(T^n x_0, x_0) + s^2\lambda^n d(T^{n_0}x_0, x_0). \end{aligned}$$

Using (9) and (10) and the fact that $\lambda^n < 1$, for each $n \in \mathbb{N}$, we obtain

$$d(x_0, T^n x_0) \leq \frac{(s + s^2\lambda^{n_0}) d(T^{n_0}x_0, x_0)}{1 - s^2\lambda^n} = C_1.$$

Hence, we have

$$d(x_0, T^n x_0) \leq C = \max\{d(x_0, Tx_0), \ldots, d(x_0, T^{n-1}x_0), C_1\} < \infty.$$

Now, using Corollary 2.3 T has a unique fixed point in X. □

From Corollary 2.9, we obtain the following result.

Theorem 3.2. *Let (X, d) be a complete b-metric space with coefficient $s \geq 1$ and $T : X \to X$ be a mapping satisfying:*

$$d(Tx, Ty) \leq \lambda[d(x, Tx) + d(y, Ty)] \tag{11}$$

for all $x, y \in X$, where $0 \leq \lambda < \frac{1}{2}$ and $s\lambda < 1$. Then T has a unique fixed point x^*, and for every $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to x^*.

Proof. Take $\Theta(x, y, z, w) = \frac{z+w}{2}$. Obviously, T is a surrounding quasi-contraction with respect to Θ. In view of Theorem 2.2, it suffices to prove that $\delta_1(T, x_0) < 1$ for some $x_0 \in X$. Let $x_0 \in X$ be arbitrary. Define the sequence $\{x_n\}$ by $x_{n+1} = T^n x_0$ for all $n \geq 0$. Then (11) implies that

$$d(T^{n+1}x_0, T^n x_0) \leq \lambda [d(T^n x_0, T^{n+1}x_0) + d(T^{n-1}x_0, T^n x_0)]$$

and

$$d(T^{n+1}x_0, T^n x_0) \leq \frac{\lambda}{1-\lambda} d(T^n x_0, T^{n-1}x_0),$$

so,

$$d(T^{n+1}x_0, T^n x_0) \leq \left(\frac{\lambda}{1-\lambda}\right)^n d(Tx_0, x_0). \tag{12}$$

Applying the triangle-type inequality for b-metric space, and from (11) and (12), we have

$$\begin{aligned} d(x_0, T^n x_0) &\leq s[d(x_0, Tx_0) + d(Tx_0, T^n x_0)] \\ &\leq s[d(x_0, Tx_0) + \lambda(d(x_0, Tx_0) + d(T^{n-1}x_0, T^n x_0))] \\ &\leq s\left[d(x_0, Tx_0) + \lambda\left(d(x_0, Tx_0) + \left(\frac{\lambda}{1-\lambda}\right)^{n-1} d(x_0, Tx_0)\right)\right] \\ &\leq 3sd(x_0, Tx_0). \end{aligned}$$

So, we can put

$$C = 3sd(x_0, Tx_0).$$

Now we obtain that $x^\star$ is the unique fixed point of T. Let $n \in \mathbb{N}$ be arbitrary, we have

$$\begin{aligned} d(x^\star, Tx^\star) &\leq s[d(x^\star, x_{n+1}) + d(x_{n+1}, Tx^\star)] \\ &= s[d(x^\star, x_{n+1}) + d(Tx_n, Tx^\star)] \\ &\leq s[d(x^\star, x_{n+1}) + \lambda(d(x_n, x_{n+1}) + d(x^\star, Tx^\star))]. \end{aligned}$$

Since $\lim\limits_{n\to\infty} d(x_{n+1}, x_n) = 0$ and $\lim\limits_{n\to\infty} d(x^\star, x_{n+1}) = 0$, we have

$$d(x^\star, Tx^\star) \leq \lambda s d(x^\star, Tx^\star).$$

Since $\lambda s < 1$, then $d(x^\star, Tx^\star) = 0$, i.e., $Tx^\star = x^\star$.

For uniqueness, let $y^\star$ be another fixed point of T. Then it follows from (11) that

$$d(x^\star, y^\star) = d(Tx^\star, Ty^\star) \leq \lambda(d(x^\star, Tx^\star) + d(y^\star, Ty^\star)) = 0.$$

Therefore, we must have $d(x^\star, y^\star) = 0$, i.e., $x^\star = y^\star$. □

Remark 3.3. Using the technique from the proofs of Theorem 3.1 and Theorem 3.2 we can also get the main result in [22], as well as the main result in [23], also in this way we can also get a whole series of known results in b-metric, rectangular metric and b-rectangular metric spaces.

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