OPEN ACCESS
Fractals, Vol. 28, No. 8 (2020) 2040011 (12 pages)

DOI: 10.1142/S0218348X20400113

ON THE WEIGHTED FRACTIONAL OPERATORS OF A FUNCTION WITH RESPECT TO ANOTHER FUNCTION

F. JARAD
Çankaya University, Department of Mathematics
06790 Etimesgut, Ankara, Turkey
fahd@cankaya.edu.tr

T. ABDELJAWAD*
Department of Mathematics and General Sciences
Prince Sultan University, P. O. Box 66833
Riyadh 11586, Saudi Arabia

Department of Medical Research, China Medical University
Taichung 40402, Taiwan

Department of Computer Science and Information Engineering
Asia University, Taichung, Taiwan
tabdeljawad@psu.edu.sa

K. SHAH
Department of Mathematics, University of Malakand
Khyber Pakhtunkhwa, Pakistan
kamalshah408@gmail.com

Received December 27, 2019
Accepted March 1, 2020
Published June 25, 2020

*Corresponding author.

Abstract

The primary goal of this study is to define the weighted fractional operators on some spaces. We first prove that the weighted integrals are bounded in certain spaces. Afterwards, we discuss the weighted fractional derivatives defined on absolute continuous-like spaces. At the end, we present a modified Laplace transform that can be applied perfectly to such operators.

Keywords: Weighted Fractional Integrals; Weighted Spaces of Summable Functions; Weighted Spaces of Absolute Continuous Functions; Weighted Generalized Laplace Transform.

1. INTRODUCTION AND PRELIMINARIES

In the last decades, the fractional calculus have acquired a largish significance on the account of diversity of applications in different fields of science and engineering.[1–6]

In most used fractional operators (with singular kernels), there exists the Riemann–Liouville and Caputo fractional derivatives. Nonetheless, there are more types of fractional operators that helped researchers in their attempts to understand the world surrounding us. We mention the ones in Refs. 7–14. It should be noted that all the operators mentioned in the aforementioned references are just special cases of the operators studied by Agrawal.[15,16] In these papers, the author presented some generalized fractional derivatives and listed some of their properties. However, the spaces on which these operators are defined were not mentioned.

It should be remarked that in Refs. 17 and 18, the authors discussed the weighted fractional operators associated with the Caputo–Fabrizio and Atangana–Baleanu fractional operators.[19,20]

Motivated by the works mentioned previously, we prove that the weighted fractional integrals are bounded in a space of Lebesgue measurable functions. We also show that the weighted fractional derivatives of functions defined on a certain space exist everywhere. We discuss the semigroup property of weighted fractional integrals, the action of integrals on derivatives and vice versa. At last, we propose an appropriately modified Laplace transform.

The nth, $n \in \mathbb{N}$ order weighted fractional integral of a function f with respect to the function g has the form

$$\begin{aligned}({}_{a^+}\mathfrak{I}_w^n f)(x) &= w^{-1}(x)\int_a^x g'(t_1)dt_1 \int_a^{t_1} g'(t_2)dt_2 \\ &\quad \cdots \int_a^{t_{n-1}} w(t_n)f(t_n)g'(t_n)dt_n \\ &= \frac{w^{-1}(x)}{(n-1)!}\int_a^x (g(x)-g(t))^{n-1} \\ &\quad \times w(t)f(t)g'(t)dt, \quad x>a,\end{aligned} \tag{1.1}$$

where $w(x) \neq 0$ is a weighted function, $w^{-1}(x) = \frac{1}{w(x)}$ and g is a strictly increasing differentiable function. The corresponding derivative is

$$\begin{aligned}(\mathfrak{D}_w^1 f)(x) &= w^{-1}(x)\frac{D_x}{g'(x)}(w(x)f(x)), \\ \mathfrak{D}_w^n f &= \mathfrak{D}_w^1(\mathfrak{D}_w^{n-1} f) \\ &= w^{-1}(x)\left(\frac{D_x}{g'(x)}\right)^n (w(x)f(x)),\end{aligned} \tag{1.2}$$

where $D_x = \frac{d}{dx}$. The fractional versions of the integral in (1.1) and the derivative (in Riemann–Liouville settings) in (1.2) are

$$\begin{aligned}({}_{a^+}\mathfrak{I}_w^\alpha f)(x) &= \frac{w^{-1}(x)}{\Gamma(\alpha)}\int_a^x (g(x)-g(t))^{\alpha-1} \\ &\quad \times w(t)f(t)g'(t)dt, \quad x>a, \ \alpha>0\end{aligned} \tag{1.3}$$

and

$$\begin{aligned}({}_{a^+}\mathfrak{D}_w^\alpha f)(x) &= (\mathfrak{D}_w^n \mathfrak{I}_w^{n-\alpha} f)(x) \\ &= \frac{1}{\Gamma(n-\alpha)}\mathfrak{D}_w^n\end{aligned}$$

$$\times \left(\int_a^x (g(x) - g(t))^{n-\alpha-1} \right.$$
$$\left. \times w(t) f(t) g'(t) dt \right), \quad x > a, \quad \alpha > 0, \tag{1.4}$$

respectively, where $n = [\alpha] + 1$, $[\alpha]$ being the integer part of α.

Before we go further, let's present the effect of the weighted integrals and derivatives of integer order on each other and the weighted fractional integrals and derivatives of a certain function. This will help us in the rest of this paper.

Lemma 1.1. *For $n \in \mathbb{N}$, we have*

$$(\mathfrak{D}^n_{w a^+} \mathfrak{I}^n_w) f = f. \tag{1.5}$$

Proof. The proof can be done by using (1.1), (1.2) and the Leibniz rule for integrals. □

Lemma 1.2. *For $n \in \mathbb{N}$, we have*

$$({}_{a^+}\mathfrak{I}^n_w \mathfrak{D}^n_w) f(x) = f(x) - w^{-1}(x) \sum_{k=0}^{n-1} \times \frac{(g(x) - g(a))^k}{k!} f_k(a), \tag{1.6}$$

where $f_m(x) = (\frac{D_x}{g'(x)})^m (w(x) f(x)), m = 0, 1, 2, \ldots.$

Proof. Using (1.1) and (1.2), we have

$$\begin{aligned}({}_{a^+}\mathfrak{I}^n_w \mathfrak{D}^n_w) f(x) &= w^{-1}(x) \int_a^x \frac{(g(x) - g(t))^{n-1}}{(n-1)!} \times f_n(t) g'(t) dt \\ &= w^{-1}(x) \int_a^x \frac{(g(x) - g(t))^{n-1}}{(n-1)!} w(t) \times w^{-1}(t) f'_{n-1}(t) dt.\end{aligned}$$

Now, performing the integration by parts formula, we get

$$\begin{aligned}({}_{a^+}\mathfrak{I}^n_w \mathfrak{D}^n_w) f(x) &= -w^{-1}(x) \frac{(g(x) - g(t))^{n-1}}{(n-1)!} f_{n-1}(a) \\ &\quad + w^{-1}(x) \int_a^x \frac{(g(x) - g(t))^{n-1}}{(n-1)!} \times f'_{n-2}(t) dt.\end{aligned}$$

Repeating the same procedure $n-2$-times, we arrive at (1.6). □

Proposition 1.3. (1) *For $\alpha > 0$ and $\beta > 0$, we have*

$$({}_{a^+}\mathfrak{I}^\alpha_w (w^{-1}(t)(g(t) - g(a))^{\beta-1}))(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} w^{-1}(x)(g(x) - g(a))^{\beta+\alpha-1}. \tag{1.7}$$

(2) *For $\alpha < n$ and $\beta > 0$, we have*

$$({}_{a^+}\mathfrak{D}^\alpha_w (w^{-1}(t)(g(t) - g(a))^{\beta-1}))(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} w^{-1}(x)(g(x) - g(a))^{\beta-\alpha-1}. \tag{1.8}$$

Proof.

$$\begin{aligned}({}_{a^+}\mathfrak{I}^\alpha_w (w^{-1}(t)(g(t) - g(a))^{\beta-1}))(x) &= \frac{w^{-1}(x)}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} \times (g(t) - g(a))^{\alpha-1} g'(t) dt \\ &= \frac{w^{-1}(x)}{\Gamma(\alpha)} (g(x) - g(a))^{\beta+\alpha-1} \times \int_0^1 \delta^{\alpha-1}(1-\delta)^{\beta-1} d\delta \\ &= \frac{w^{-1}(x)}{\Gamma(\alpha)} (g(x) - g(a))^{\beta+\alpha-1} \times \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \\ &= \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} w^{-1}(x)(g(x) - g(a))^{\beta+\alpha-1},\end{aligned}$$

where $\delta = \frac{g(t)-g(a)}{g(x)-g(a)}$. Now,

$$\begin{aligned}({}_{a^+}\mathfrak{D}^\alpha_w (w^{-1}(t)(g(t) - g(a))^{\beta-1}))(x) &= (\mathfrak{D}^n_{w a^+} \mathfrak{I}^{n-\alpha}_w (w^{-1}(t)(g(t) - g(a))^{\beta-1}))(x) \\ &= \frac{\Gamma(\beta)}{\Gamma(\beta+n-\alpha)} (\mathfrak{D}^n_w (g(t) - g(a))^{\beta+n-\alpha-1})(x) \\ &= \frac{\Gamma(\beta)}{\Gamma(\beta+n-\alpha)} \frac{\Gamma(\beta+n-\alpha)}{\Gamma(\beta-\alpha)} \times (g(x) - g(a))^{\beta+\alpha-1} \\ &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} w^{-1}(x)(g(x) - g(a))^{\beta-\alpha-1}.\end{aligned}$$

□

2. THE WEIGHTED FRACTIONAL INTEGRALS

In this section, we define the space where the weighted fractional integrals are bounded and present some properties of these operators.

Definition 2.1. Let $w \neq 0$ be a function defined on $[a,b]$, g is a differentiable strictly increasing function on $[a,b]$. The space $X_w^p(a,b), 1 \leq p \leq \infty$ is the apace of all Lebesgue measurable functions f defined on $[a,b]$ for which $\|f\|_{X_w^p} < \infty$, where

$$\|f\|_{X_w^p} = \left(\int_a^b |w(x)f(x)|^p g'(x)dx\right)^{\frac{1}{p}}, \quad 1 < p < \infty \tag{2.1}$$

and

$$\|f\|_{X_w^\infty} = \text{ess sup}_{a \leq x \leq b}|w(x)f(x)| < \infty. \tag{2.2}$$

Remark 2.2. It should be noted that $f \in X_w^p(a,b) \Leftrightarrow w(x)f(x)(g'(x))^{\frac{1}{p}} \in L_p(a,b)$ for $1 \leq p < \infty$ and $f \in X_w^\infty(a,b) \Leftrightarrow w(x)f(x) \in L_\infty(a,b)$.

Theorem 2.3. *Let $\alpha > 0, 1 \leq p \leq \infty$ and $f \in X_w^p(a,b)$. Then ${}_{a^+}\mathfrak{I}_w^\alpha f$ is bounded in $X_w^p(a,b)$ and*

$$\|{}_{a^+}\mathfrak{I}_w^\alpha f\|_{X_w^p} \leq \frac{(g(b)-g(a))^\alpha}{\Gamma(\alpha+1)} \|f\|_{X_w^p}. \tag{2.3}$$

Proof. For $1 \leq p < \infty$, we have

$$\begin{aligned}
\|{}_{a^+}\mathfrak{I}_w^\alpha f\|_{X_w^p} &= \frac{1}{\Gamma(\alpha)}\left[\int_a^b \left|\int_a^x (g(x)-g(t))^{\alpha-1} \times w(t)f(t)g'(t)dt\right|^p g'(x)dx\right]^{\frac{1}{p}} \\
&= \frac{1}{\Gamma(\alpha)}\left[\int_{g(a)}^{g(b)} \left|\int_{g(a)}^v (v-u)^{\alpha-1} w(g^{-1}(u)) \times f(g^{-1}(u))du\right|^p dv\right]^{\frac{1}{p}},
\end{aligned}$$

where g^{-1} is the inverse function of g. Due to the generalized Minkowski inequality (1.33 in Ref. 6), we can write

$$\begin{aligned}
\|{}_{a^+}\mathfrak{I}_w^\alpha f\|_{X_w^p} &\leq \frac{1}{\Gamma(\alpha)}\int_{g(a)}^{g(b)} \left(|w(g^{-1}(u))f(g^{-1}(u))|^p \times \int_u^{g(b)} (v-u)^{(\alpha-1)p}dv\right)^{\frac{1}{p}} du \\
&= \frac{1}{\Gamma(\alpha)}\int_{g(a)}^{g(b)} |w(g^{-1}(u))f(g^{-1}(u))| \times \left(\frac{(g(b)-u)^{(\alpha-1)p+1}}{(\alpha-1)p+1}\right)^{\frac{1}{p}} du.
\end{aligned}$$

Now, using Hölder's inequality, we get

$$\begin{aligned}
\|{}_{a^+}\mathfrak{I}_w^\alpha f\|_{X_w^p} &\leq \frac{1}{\Gamma(\alpha)}\left(\int_{g(a)}^{g(b)} \left|w(g^{-1}(u))f(g^{-1}(u))\right|^p du\right)^{\frac{1}{p}} \\
&\quad \times \left\{\int_{g(a)}^{g(b)} \left[\frac{(g(b)-u)^{(\alpha-1)p+1}}{(\alpha-1)p+1}\right]^{\frac{q}{p}} du\right\}^{\frac{1}{q}},
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q} = 1$. Thus, we have

$$\begin{aligned}
\|{}_{a^+}\mathfrak{I}_w^\alpha f\|_{X_w^p} &\leq \frac{1}{\Gamma(\alpha)}\left(\int_a^b |w(x)f(x)|^p g'(x)dx\right)^{\frac{1}{p}} \\
&\quad \times \left\{\int_{g(a)}^{g(b)} \left[\frac{(g(b)-u)^{(\alpha-1)p+1}}{(\alpha-1)p+1}\right]^{\frac{q}{p}} du\right\}^{\frac{1}{q}} \\
&\leq \frac{\|f\|_{X_w^p}}{\Gamma(\alpha)} \frac{(g(b)-g(a))^\alpha}{\alpha} \\
&= \frac{(g(b)-g(a))^\alpha}{\Gamma(\alpha+1)} \|f\|_{X_w^p}.
\end{aligned}$$

Now, for $p = \infty$, we have

$$\begin{aligned}
|w(x)\ {}_{a^+}\mathfrak{I}_w^\alpha f(x)| &= \frac{1}{\Gamma(\alpha)}\int_a^x (g(x)-g(t))^{\alpha-1} \times w(t)f(t)g'(t)dt \\
&\leq \frac{1}{\Gamma(\alpha)}\int_a^x (g(x)-g(t))^{\alpha-1}| \times w(t)f(t)|g'(t)dt \\
&\leq \frac{\|f\|_{X_w^\infty}}{\Gamma(\alpha)}\int_a^x (g(x)-g(t))^{\alpha-1}dt \\
&= \frac{\|f\|_{X_w^\infty}}{\Gamma(\alpha)} \frac{(g(x)-g(a))^\alpha}{\alpha} \\
&\leq \frac{(g(b)-g(a))^\alpha}{\Gamma(\alpha+1)} \|f\|_{X_w^\infty}.
\end{aligned}$$

□

Next, we present the semi-group property for integrals.

Theorem 2.4. *Let $f \in X_w^p(a,b), 1 \le p \le \infty, \alpha > 0, \beta > 0$. Then,*

$$({}_{a^+}\mathfrak{I}_w^{\alpha}\ {}_{a^+}\mathfrak{I}_w^{\beta})f = ({}_{a^+}\mathfrak{I}_w^{\alpha+\beta})f. \tag{2.4}$$

Proof.

$$\begin{aligned}
&({}_{a^+}\mathfrak{I}_w^{\alpha}\ {}_{a^+}\mathfrak{I}_w^{\beta}f)(x)\\
&\quad= \frac{w^{-1}(x)}{\Gamma(\alpha)}\int_a^x (g(x)-g(t))^{\alpha-1}w(t)\\
&\qquad\times ({}_{a^+}\mathfrak{I}_w^{\beta}f)(t)g'(t)dt\\
&\quad= \frac{w^{-1}(x)}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_a^t (g(x)-g(t))^{\alpha-1}\\
&\qquad\times (g(t)-g(\tau))^{\beta-1}w(\tau)f(\tau)g'(\tau)g'(t)d\tau dt\\
&\quad= \frac{w^{-1}(x)}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_\tau^x (g(x)-g(t))^{\alpha-1}\\
&\qquad\times (g(t)-g(\tau))^{\beta-1}w(\tau)f(\tau)g'(\tau)g'(t)dtd\tau.
\end{aligned}$$

Now, letting $\delta = \frac{g(t)-g(a)}{g(x)-g(a)}$, we get

$$\begin{aligned}
&({}_{a^+}\mathfrak{I}_w^{\alpha}\ {}_{a^+}\mathfrak{I}_w^{\beta}f)(x)\\
&\quad= \frac{w^{-1}(x)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 (1-\delta)^{\alpha-1}\delta^{\beta-1}d\delta\int_a\\
&\qquad\times \int_\tau^x (g(x)-g(\tau))^{\alpha+\beta-1}w(\tau)f(\tau)g'(\tau)d\tau\\
&\quad= \frac{w^{-1}(x)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\\
&\qquad\times \int_a^x\int_\tau^x (g(x)-g(\tau))^{\alpha+\beta-1}\\
&\qquad\times w(\tau)f(\tau)g'(\tau)d\tau\\
&\quad= \frac{w^{-1}(x)}{\Gamma(\alpha+\beta)}\int_a^x (g(x)-g(\tau))^{\alpha+\beta-1}\\
&\qquad\times w(\tau)f(\tau)g'(\tau)d\tau\\
&\quad= ({}_{a^+}\mathfrak{I}_w^{\alpha+\beta}f)(x).
\end{aligned}$$

□

In what follows, we discuss the combination of the weighted fractional integrals with the weighted differential operator (1.2).

Theorem 2.5. *Let $\alpha > m, m \in \mathbb{N}$. Then*

$$\mathfrak{D}_w^m({}_{a^+}\mathfrak{I}_w^{\alpha}f) = {}_{a^+}\mathfrak{I}_w^{\alpha-m}f. \tag{2.5}$$

Proof. The proof can be adhibited by using (1.2) and using the Leibniz rule for integrals. □

Theorem 2.6. *Let $\alpha > 1$ and $\mathfrak{D}_w^1 f \in X_w^p(a,b)$. Then*

$$\begin{aligned}
({}_{a^+}\mathfrak{I}_w^{\alpha}\mathfrak{D}_w^1 f)(x) &= (\mathfrak{D}_w^1\ {}_{a^+}\mathfrak{I}_w^{\alpha}f)(x) - \frac{w^{-1}(x)}{\Gamma(\alpha)}\\
&\quad\times (g(x)-g(a))^{\alpha-1}w(a)f(a).
\end{aligned} \tag{2.6}$$

Proof.

$$\begin{aligned}
&({}_{a^+}\mathfrak{I}_w^{\alpha}\mathfrak{D}_w^1 f)(x)\\
&\quad= \frac{w^{-1}(x)}{\Gamma(\alpha)}\int_a^x (g(x)-g(t))^{\alpha-1}w(t)w^{-1}(t)\\
&\qquad\times \left(\frac{D_t}{g'(t)}w(t)f(t)\right)g'(t)dt\\
&\qquad= \frac{w^{-1}(x)}{\Gamma(\alpha)}\int_a^x (g(x)-g(t))^{\alpha-1}(w(t)f(t))'dt\\
&\quad= \frac{w^{-1}(x)}{\Gamma(\alpha)}\Bigg[(g(x)-g(t))^{\alpha-1}w(t)f(t)|_a^x\\
&\qquad+(\alpha-1)\int_a^x (g(x)-g(t))^{\alpha-2}\\
&\qquad\times w(t)f(t)g'(t)dt\Bigg]\\
&\quad= -\frac{w^{-1}(x)}{\Gamma(\alpha)}(g(x)-g(a))^{\alpha-1}w(a)f(a)\\
&\qquad+\frac{w^{-1}(x)(\alpha-1)}{\Gamma(\alpha)}\int_a^x (g(x)-g(t))^{\alpha-2}\\
&\qquad\times w(t)f(t)g'(t)dt\\
&\quad= -\frac{w^{-1}(x)}{\Gamma(\alpha)}(g(x)-g(a))^{\alpha-1}w(a)f(a)\\
&\quad+w^{-1}(x)\left(\frac{D_x}{g'(x)}w(x)\right)\frac{w^{-1}(x)}{\Gamma(\alpha)}\int_a^x\\
&\qquad\times (g(x)-g(t))^{\alpha-1}w(t)f(t)g'(t)dt\\
&\qquad= (\mathfrak{D}_w^1\ {}_{a^+}\mathfrak{I}_w^{\alpha}f)(x) - \frac{w^{-1}(x)}{\Gamma(\alpha)}\\
&\qquad\times (g(x)-g(a))^{\alpha-1}w(a)f(a).
\end{aligned}$$

□

By induction on m, one can prove the following corollary.

Corollary 2.7. *For $\alpha > m$, $\mathfrak{D}_w^m f \in X_w^p(a,b)$, we have*

$$({}_{a^+}\mathfrak{I}_w^\alpha \mathfrak{D}_w^m f)(x) = (\mathfrak{D}_w^m \; {}_{a^+}\mathfrak{I}_w^\alpha f)(x) - w^{-1}(x) \times \sum_{k=0}^{m-1} \frac{(g(x)-g(a))^{\alpha-m+k}}{\Gamma(\alpha-m+k+1)} f_k(a). \tag{2.7}$$

3. THE WEIGHTED FRACTIONAL DERIVATIVES

In this section, we define the weighted fractional derivatives on a certain weighted absolute continuous functions space and discuss their interaction with the weighted fractional integrals.

Definition 3.1. The space $AC_w^n[a,b]$ is defined as follows:

$$AC_w^n[a,b] := \{f : [a,b] \to \mathbb{R} \text{ such that } f_{n-1} \in AC[a,b]\}, \tag{3.1}$$

where $AC[a,b]$ is the set of absolute continuous functions on the interval $[a,b]$.

In what follows, we propose the form of functions in the space $AC_w^n[a,b]$.

Theorem 3.2. *The space $AC_w^n[a,b]$ consists of those functions which has the form*

$$f(x) = w^{-1}(x)\left[\int_a^x \frac{(g(x)-g(t))^{n-1}}{(n-1)!}\xi(t)dt + \sum_{k=0}^{n-1} \frac{c_k}{k!}(g(x)-g(a))^k\right]. \tag{3.2}$$

Proof. Let a function $f \in AC_w^n[a,b]$. Then, by Definition 3.1, we have $f_{n-1}(x) = (\frac{D_x}{g'(x)})^{n-1} f(x) \in AC[a,b]$. Thus, there exists a function $\xi \in L_1[a,b]$ such that

$$f_{n-1}(x) = c_{n-1} + \int_a^x \xi(t)dt.$$

Hence,

$$f'_{n-2}(x) = c_{n-1}g'(x) + g'(x)\int_a^x \xi(t)dt.$$

Therefore,

$$f_{n-2}(x) = c_{n-2} + c_{n-1}\int_a^x g'(t)dt + \int_a^x g'(\tau) \times \int_a^\tau \xi(t)dtd\tau$$

$$= c_{n-2} + c_{n-1}(g(x)-g(a)) + \int_a^x (g(x) - g(t)\xi(t)dt.$$

Taking a step further, we will have

$$f_{n-3}(x) = c_{n-3} + c_{n-2}(g(x)-g(a)) + c_{n-2}\frac{(g(x)-g(a))^2}{2!} + \int_a^x \frac{(g(x)-g(t))^2}{2!}\xi(t)dt.$$

Repeating this procedure, we reach at

$$w(x)f(x) = \sum_{k=0}^{n-1} c_k \frac{(g(x)-g(a))^k}{k!} + \int_a^x \frac{(g(x)-g(t))^{(n-1)}}{(n-1)!}\xi(t)dt.$$

The result is found then by multiplying the above equation by $w^{-1}(x)$. □

By construction, it can be obviously seen that

$$c_k = f_k(a), \quad k = 0,1,\ldots,n-1 \quad \text{and} \quad \xi(x) = g'(x)f_n(x). \tag{3.3}$$

Theorem 3.3. *If $f \in AC_w^n[a,b]$, then the weighted fractional derivative of f exists almost everywhere and can be written as*

$$({}_{a^+}\mathfrak{D}_w^\alpha f)(x) = w^{-1}(x)\left[\frac{1}{\Gamma(n-\alpha)}\int_a^x \times (g(x)-g(t))^{n-\alpha-1} f_n(t)g'(t)dt + \sum_{k=0}^{n-1}\frac{(g(x)-g(a))^{k-\alpha} f_k(a)}{\Gamma(k-\alpha+1)}\right]. \tag{3.4}$$

Proof. Since $f \in AC_w^n[a,b]$, by Theorem 3.2 and Eq. (3.3), f can be written as

$$f(x) = w^{-1}(x)\int_a^x \frac{(g(x)-g(t))^{(n-1)}}{(n-1)!} f_n(t)g'(t)dt + \sum_{k=0}^{n-1} w^{-1}(x)\frac{(g(x)-g(a))^k f_k(a)}{k!}$$

$$= w^{-1}(x)\int_a^x \frac{(g(x)-g(t))^{(n-1)}}{(n-1)!}w(t)(\mathfrak{D}_w^n f)(t) \times g'(t)dt + \sum_{k=0}^{n-1} w^{-1}(x)\frac{(g(x)-g(a))^k f_k(a)}{k!}$$
$$= ({}_{a^+}\mathfrak{I}_w^n \mathfrak{D}_w^n f)(x) + \sum_{k=0}^{n-1} w^{-1}(x) \times \frac{(g(x)-g(a))^k f_k(a)}{k!}.$$

Applying the weighted fractional derivative to both sides and using Proposition 1.3, we get

$$({}_{a^+}\mathfrak{D}_w^\alpha f)(x)$$
$$= ({}_{a^+}\mathfrak{D}_w^\alpha\ {}_{a^+}\mathfrak{I}_w^n \mathfrak{D}_w^n f)(x) + w^{-1}(x) \times \sum_{k=0}^{n-1}\frac{(g(x)-g(a))^{k-\alpha} f_k(a)}{k!} \times \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}$$
$$= (\mathfrak{D}_w^n\ {}_{a^+}\mathfrak{I}_w^{n-\alpha}\ {}_{a^+}\mathfrak{I}_w^n \mathfrak{D}_w^n f)(x) + w^{-1}(x) \times \sum_{k=0}^{n-1}\frac{(g(x)-g(a))^{k-\alpha} f_k(a)}{\Gamma(k+1-\alpha)}$$
$$= (\mathfrak{D}_w^n\ {}_{a^+}\mathfrak{I}_w^{2n-\alpha} \mathfrak{D}_w^n f)(x) + w^{-1}(x) \times \sum_{k=0}^{n-1}\frac{(g(x)-g(a))^{k-\alpha} f_k(a)}{\Gamma(k+1-\alpha)}$$
$$= (\mathfrak{I}_w^{n-\alpha} \mathfrak{D}_w^n f)(x) + w^{-1}(x) \times \sum_{k=0}^{n-1}\frac{(g(x)-g(a))^{k-\alpha} f_k(a)}{\Gamma(k+1-\alpha)}$$
$$= w^{-1}(x)\left[\frac{1}{\Gamma(n-\alpha)}\int_a^x (g(x)-g(t))^{n-\alpha-1} \times f_n(t)g'(t)dt + \sum_{k=0}^{n-1}\frac{(g(x)-g(a))^{k-\alpha} f_k(a)}{\Gamma(k-\alpha+1)}\right],$$

where the last two steps before the final step are obtained by using Theorems 2.4 and 2.5. □

In what follows, we consider the combination of weighted fractional derivatives and weighted fractional integrals.

Theorem 3.4. *Let $\alpha > \beta > 0$, where $m = [\beta] + 1$. Then,*

$$({}_{a^+}\mathfrak{D}_w^\beta\ {}_{a^+}\mathfrak{I}_w^\alpha) f = ({}_{a^+}\mathfrak{I}_w^{\alpha-\beta}) f. \tag{3.5}$$

Proof.

$$({}_{a^+}\mathfrak{D}_w^\beta\ {}_{a^+}\mathfrak{I}_w^\alpha) f$$
$$= (\mathfrak{D}_w^m\ {}_{a^+}\mathfrak{I}_w^{m-\beta}\mathfrak{I}_w^\alpha) f$$
$$= (\mathfrak{D}_w^m\ {}_{a^+}\mathfrak{I}_w^{\alpha+m-\beta}) f \quad \text{by using Theorem 2.4}$$
$$= ({}_{a^+}\mathfrak{I}_w^{\alpha-\beta}) f$$

by using Theorem 2.5. □

In a similar way, one can prove the following theorem.

Theorem 3.5. *Let $\alpha > 0$. Then, we have*

$$({}_{a^+}\mathfrak{D}_w^\alpha\ {}_{a^+}\mathfrak{I}_w^\alpha) f = f. \tag{3.6}$$

Theorem 3.6. *Let $\alpha > 0, n = -[-\alpha], f \in X_w^p(a,b)$ and ${}_{a^+}\mathfrak{I}_w^\alpha f \in AC_w^n[a,b]$. Then*

$$({}_{a^+}\mathfrak{I}_w^\alpha\ {}_{a^+}\mathfrak{D}_w^\alpha\ f)(x) = f(x) - w^{-1}(x)\sum_{k=1}^{n}\frac{(g(x)-g(a))^{\alpha-k}}{\Gamma(\alpha-k+1)} \times ({}_{a^+}\mathfrak{I}_w^{n-\alpha} f)_{n-k}(a^+), \tag{3.7}$$

where

$$({}_{a^+}\mathfrak{I}_w^{n-\alpha} f)_k(a^+) = \left(\frac{D_x}{g'(x)}\right)^k (w(x)_{a^+}\mathfrak{I}_w^{n-\alpha} f)(a^+).$$

Proof. Since

$$({}_{a^+}\mathfrak{I}_w^\alpha\ {}_{a^+}\mathfrak{D}_w^\alpha) f = ({}_{a^+}\mathfrak{I}_w^\alpha\ {}_{a^+}\mathfrak{D}_w^n)\ {}_{a^+}\mathfrak{I}_w^{n-\alpha} f,$$

we have

$$({}_{a^+}\mathfrak{I}_w^\alpha\ {}_{a^+}\mathfrak{D}_w^\alpha f)(x)$$
$$= ({}_{a^+}\mathfrak{I}_w^\alpha \mathfrak{D}_w^n\ {}_{a^+}\mathfrak{I}_w^{n-\alpha} f)(x) - w^{-1}(x) \times \sum_{k=0}^{n-1}\frac{(g(x)-g(a))^{\alpha-n+k}}{\Gamma(\alpha-n+k+1)}({}_{a^+}\mathfrak{I}_w^{n-\alpha} f)_k(a^+)$$
$$= (\mathfrak{D}_w^n\ {}_{a^+}\mathfrak{I}_w^\alpha\ {}_{a^+}\mathfrak{I}_w^{n-\alpha} f)(x) - w^{-1}(x) \times \sum_{k=0}^{n-1}\frac{(g(x)-g(a))^{\alpha-n+k}}{\Gamma(\alpha-n+k+1)} \times ({}_{a^+}\mathfrak{I}_w^{n-\alpha} f)_k(a^+)$$

$$= (\mathfrak{D}_w^n\ {}_{a^+}\mathfrak{I}_w^n f)(x) - w^{-1}(x) \times \sum_{k=1}^{n} \frac{(g(x)-g(a))^{\alpha-k}}{\Gamma(\alpha-k+1)} ({}_{a^+}\mathfrak{I}_w^{n-\alpha} f)_{n-k}(a^+)$$

$$= f(x) - w^{-1}(x) \sum_{k=1}^{n} \frac{(g(x)-g(a))^{\alpha-k}}{\Gamma(\alpha-k+1)} \times ({}_{a^+}\mathfrak{I}_w^{n-\alpha} f)_{n-k}(a^+),$$

where in the last two steps we used Corollary 2.7 and Lemma 1.1. □

4. WEIGHTED FRACTIONAL DERIVATIVES IN THE CAPUTO SETTINGS

In this section, we define the weighted Caputo fractional derivatives and present some of its properties.

Definition 4.1. Let $\alpha \geq 0$ and $n = [\alpha] + 1$. The weighted Caputo fractional derivative of order α of f is defined by

$$({}^C_{a^+}\mathfrak{D}_w^\alpha f)(x) = {}_{a^+}\mathfrak{D}_w^\alpha \left(f(t) - w^{-1}(t) \sum_{k=0}^{n-1} \times \frac{(g(t)-g(a))^k}{k!} f_k(a) \right)(x). \tag{4.1}$$

Now, we give an explicit expression of the weighted Caputo fractional derivatives in the space of weighted absolute continuous functions.

Theorem 4.2. *If $f \in AC_w^n[a,b]$, then the weighted Caputo fractional derivative, of order $\alpha > 0, n = [\alpha] + 1$ of f can be written as*

$$({}^C_{a^+}\mathfrak{D}_w^\alpha f)(x) = ({}^C_{a^+}\mathfrak{I}_w^{n-\alpha} \mathfrak{D}_w^n f)(x)$$
$$= \frac{w^{-1}(x)}{\Gamma(n-\alpha)} \int_a^x (g(x)-g(t)^{n-\alpha-1} \times w(t) \mathfrak{D}_w^n f(t) g'(t) dt$$
$$= \frac{w^{-1}(x)}{\Gamma(n-\alpha)} \int_a^x (g(x)-g(t)^{n-\alpha-1} \times f_n(t) g'(t) dt. \tag{4.2}$$

Proof. Using Definition 4.1 and Proposition 1.3, we can write

$$({}^C_{a^+}\mathfrak{D}_w^\alpha f)(x) = ({}_{a^+}\mathfrak{D}_w^\alpha f)(x) - w^{-1}(x) \times \sum_{k=0}^{n-1} \frac{(g(x)-g(a))^{k-\alpha}}{\Gamma(k-\alpha+1)} f_k(a).$$

Now, by Theorem 3.3, we have

$$({}^C_{a^+}\mathfrak{D}_w^\alpha f)(x) = \frac{w^{-1}(x)}{\Gamma(n-\alpha)} \int_a^x (g(x)-g(t)^{n-\alpha-1} f_n(t) g'(t) dt$$
$$= \frac{w^{-1}(x)}{\Gamma(n-\alpha)} \int_a^x (g(x)-g(t)^{n-\alpha-1} w(t) \times \mathfrak{D}_w^n f(t) g'(t) dt$$
$$= ({}^C_{a^+}\mathfrak{I}_w^{n-\alpha} \mathfrak{D}_w^n f)(x).$$ □

The following proposition can be proved very easily.

Proposition 4.3. *For $\alpha \geq 0$ and $\beta > n$, where $n = [\alpha] + 1$, we have*

$$({}^C_{a^+}\mathfrak{D}_w^\alpha w^{-1}(t)(g(t)-g(a))^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (g(x)-g(a))^{\beta-\alpha-1}. \tag{4.3}$$

Remark 4.4. It can be observed that

$$[{}^C_{a^+}\mathfrak{D}_w^\alpha (w^{-1}(t)(g(t)-g(a))^k)](x) = 0, \quad k = 0, 1, \ldots, n-1. \tag{4.4}$$

In what follows, we combine the weighted fractional integral with the weighted Caputo fractional derivative

Theorem 4.5.

$$({}^C_{a^+}\mathfrak{I}_w^\alpha\ {}^C_{a^+}\mathfrak{D}_w^\alpha f)(x) = f(x) - w^{-1}(x) \times \sum_{k=0}^{n-1} \frac{(g(x)-g(a))^k}{k!} f_k(a). \tag{4.5}$$

Proof.

$$\begin{aligned}&({}_{a^+}\mathfrak{I}_w^{\alpha}\ {}_{a^+}^{C}\mathfrak{D}_w^{\alpha} f)(x)\\&\quad = ({}_{a^+}^{C}\mathfrak{I}_w^{\alpha}\ {}_{a^+}^{C}\mathfrak{I}_w^{n-\alpha}\mathfrak{D}_w^{n} f)(x)\\&\quad = ({}_{a^+}^{C}\mathfrak{I}_w^{n}\mathfrak{D}_w^{n} f)(x) \quad \text{using Theorem 2.4}\\&\quad = f(x) - w^{-1}(x)\sum_{k=0}^{n-1}\frac{(g(x)-g(a))^k}{k!}f_k(a),\end{aligned}$$

where the last step is implemented by using Lemma 1.2. □

5. THE WEIGHTED LAPLACE TRANSFORM

The classical Laplace transform is difficult to utilize for the weighted fractional order. However, a small modification of the conventional Laplace transform would be more effective. For this sake, we propose the following modification of the Laplace transform which is slightly different than the one proposed in Ref. 13.

Definition 5.1. Let f and w be functions defined on the interval $[a,\infty)$ and g be a strictly increasing function on $[a,\infty)$. Then the weighted Laplace transform of f is defined by

$$\mathcal{L}_g^w\{f\}(s) = \int_a^{\infty} e^{-s(g(x)-g(a))}w(x)f(x)g'(x)dx \tag{5.1}$$

for all values of s for which the integral in (5.1) is valid.

The assertions in the following proposition can be easily obtained.

Proposition 5.2.

(1) $$\mathcal{L}_g^w\{w^{-1}(x)e^{\lambda(g(x)-g(a))}\}(s) = \frac{1}{s-\lambda}, \quad \lambda\in\mathbb{R},\ s>\lambda. \tag{5.2}$$

(2) $$\mathcal{L}_g^w\{w^{-1}(x)(g(x)-g(a))^{\beta-1}\}(s) = \frac{\Gamma(\beta)}{s^{\beta}}, \quad \beta>-1,\ s>0. \tag{5.3}$$

Definition 5.3. Let $f, w(\neq 0) : [a,\infty) \to \mathbb{R}$. f is called w-weighted g-exponential function if there exist constants M, c and X such that

$$|w(x)f(x)| \le Me^{cg(x)} \quad \text{for } x > X. \tag{5.4}$$

The next theorem presents the sufficient conditions for the existence of the weighted Laplace transform.

Theorem 5.4. *Let $f, w(\neq 0) : [a,\infty) \to \mathbb{R}$ be functions such that wf is piecewise continuous such that f is w-weighted g-exponential function. Then, the weighted Laplace transform of f exists for $s > c$.*

Proof. The proof is direct. □

Theorem 5.5. *Let $f \in AC_w[a,x)$ and of w-weighted g-exponential order. Let $\mathfrak{D}_w f$ be piecewise continuous on every interval $[a,X)$. Then, the weighted Laplace transform of $\mathfrak{D}_w f$ exists and*

$$\mathcal{L}_g^w\{\mathfrak{D}_w f\}(s) = s\mathcal{L}_g^w\{f\}(s) - w(a)f(a). \tag{5.5}$$

Proof. The proof is similar to the proof of Theorem 3.7 in Ref. 13. □

Theorem 5.5 can be generalized as follows.

Corollary 5.6. *Let $f \in AC_w^{n-1}[a,x)$, such that $\mathfrak{D}_w^k f, k = 0,1,\ldots,n-1$ are of w-weighted g-exponential order. Let $\mathfrak{D}_w^n f$ be piecewise continuous on every interval $[a,X)$. Then, the weighted Laplace transform of $\mathfrak{D}_w^n f$ exists and*

$$\mathcal{L}_g^w\{\mathfrak{D}_w^n f\}(s) = s^n\mathcal{L}_g^w\{f\}(s) - \sum_{k=0}^{n-1}s^{n-1-k}f_k(a). \tag{5.6}$$

In what follows, we define the weighted convolution of two functions.

Definition 5.7. The weighted convolution of functions f and h is defined by

$$\begin{aligned}&f *_g^w h(x)\\&\quad = w^{-1}(x)\int_a^x w(g^{-1}(g(x)+g(a)-g(t)))f\\&\qquad \times (g^{-1}(g(x)+g(a)-g(t)))\\&\qquad \times w(t)h(t)g'(t)dt,\end{aligned} \tag{5.7}$$

where g^{-1} is the inverse of g.

Remark 5.8. Note that we have $f *_g^w h = h *_g^w g$.

Theorem 5.9. *Let the weighted Laplace transform of f and h exist for $s > c_1$ and $s > c_2$, respectively. Then,*

$$\mathcal{L}_g^w\{f *_g^w h(x)\}(s) = \mathcal{L}_g^w\{f(x)\}(s)\mathcal{L}_g^w\{h(x)\}(s), \quad s > \max\{c_1,c_2\}. \tag{5.8}$$

Proof.

$$\mathcal{L}_g^w\{f(x)\}(s)\mathcal{L}_g^w\{h(x)\}(s)$$
$$= \int_a^\infty e^{-s(g(x)-g(a))}w(x)f(x)g'(x)dx$$
$$\times \int_a^\infty e^{-s(g(\tau)-g(a))}w(\tau)h(\tau)g'(\tau)d\tau$$
$$= \int_a^\infty \int_a^\infty e^{-s(g(x)+g(\tau)-2g(a))}w(x)f(x)g'(x)$$
$$\times w(\tau)h(\tau)g'(\tau)d\tau.$$

Letting $g(x)+g(\tau)-g(a)=g(t)$, we obtain

$$\mathcal{L}_g^w\{f(x)\}(s)\mathcal{L}_g^w\{h(x)\}(s)$$
$$= \int_a^\infty \int_t^\infty e^{-s(g(t)-g(a))}w$$
$$\times (g^{-1}(g(t)+g(a)-g(\tau)))f(g^{-1}(g(t)$$
$$+g(a)-g(\tau)))w(\tau)h(\tau)g'(t)g'(\tau)dtd\tau.$$

Now, changing the order of integration, we get

$$\mathcal{L}_g^w\{f(x)\}(s)\mathcal{L}_g^w\{h(x)\}(s)$$
$$= \int_a^\infty e^{-s(g(t)-g(a))}\int_a^t w$$
$$\times (g^{-1}(g(t)+g(a)-g(\tau)))f(g^{-1}(g(t)$$
$$+g(a)-g(\tau)))w(\tau)h(\tau)g'(\tau)d\tau g'(t)dt$$
$$= \int_a^\infty e^{-s(g(t)-g(a))}w(t)w^{-1}(t)$$
$$\times \int_a^t w(g^{-1}(g(t)+g(a)-g(\tau)))$$
$$\times f(g^{-1}(g(t)+g(a)-g(\tau)))w(\tau)h(\tau)$$
$$\times g'(\tau)d\tau g'(t)dt$$
$$= \int_a^\infty e^{-s(g(t)-g(a))}w(t)f *_g^w h(t)g'(t)dt$$
$$= \mathcal{L}_g^w\{f *_g^w h(x)\}(s). \qquad \square$$

In what follows, we present the weighted Laplace transforms of the weighted fractional operators.

Theorem 5.10. *Let f be a piecewise continuous function on each interval $[a,x]$ and of w-weighted g-exponential order. Then,*

$$\mathcal{L}_g^w\{({}_{a^+}\mathfrak{I}_w^\alpha f)(x)\}(s) = \frac{\mathcal{L}_g^w\{f(x)\}(s)}{s^\alpha}. \tag{5.9}$$

Proof.

$$\mathcal{L}_g^w\{({}_{a^+}\mathfrak{I}_w^\alpha f)(x)\}(s)$$
$$= \mathcal{L}_g^w\left\{\frac{w^{-1}(x)}{\Gamma(\alpha)}\int_a^x (g(x)-g(t))^{\alpha-1}\right.$$
$$\left.\times w(t)f(t)g'(t)dt\right\}(s)$$
$$= \frac{1}{\Gamma(\alpha)}\mathcal{L}_g^w\{(g(x)-g(a))^{\alpha-1} *_g^w f(x)\}(s)$$
$$= \frac{1}{\Gamma(\alpha)}\mathcal{L}_g^w\{(g(x)-g(a))^{\alpha-1}\}(s)\mathcal{L}_g^w\{f(x)\}(s)$$
$$= \frac{1}{\Gamma(\alpha)}\frac{\Gamma(\alpha)}{s^\alpha}\mathcal{L}_g^w\{f(x)\}(s)$$
$$= \frac{\mathcal{L}_g^w\{f(x)\}(s)}{s^\alpha}. \qquad \square$$

Corollary 5.11. *Let $\alpha>0$, $f\in AC_w^n[a,b]$ for any $b>0, g\in C^n[a,b], g'(x)>0$ and $({}_{a^+}\mathfrak{I}_w^{n-\alpha})_k f, k=0,1,\ldots,n-1$ be w-weighted g-exponential order. Then,*

$$\mathcal{L}_g^w\{({}_{a^+}\mathfrak{D}_w^\alpha f)(x)\}(s)$$
$$= s^\alpha \mathcal{L}_g^w\{f(x)\}(s)$$
$$- \sum_{k=0}^{n-1} s^{n-k-1}({}_{a^+}\mathfrak{I}_w^{n-\alpha})_k f(a^+). \tag{5.10}$$

Proof. The proof can be implemented by using (1.4), Theorem 5.10 and Corollary 5.6. $\square$

Corollary 5.12. *Let $\alpha>0$, $f\in AC_w^n[a,b]$ for any $b>0, g\in C^n[a,b], g'(x)>0$ and $f_k, k=0,1,\ldots,n-1$ be w-weighted g-exponential order. Then,*

$$\mathcal{L}_g^w\{({}^C_{a^+}\mathfrak{D}_w^\alpha f)(x)\}(s)$$
$$= s^\alpha\left[\mathcal{L}_g^w\{f(x)\}(s) - \sum_{k=0}^{n-1} s^{-k-1}f_k(a)\right]. \tag{5.11}$$

Proof. The proof can be implemented by using Theorems 4.2, 5.10 and Corollary 5.6. $\square$

Remark 5.13. We can find weighted Laplace transform of a weighted type of Mittag–Leffler function[3,6] as follows:

$$\mathcal{L}_g^w\{w^{-1}(x)(g(x)-g(a))^{\beta-1}E_{\alpha,\beta}$$
$$\times (\lambda(g(x)-g(a))^\alpha)\}(s)$$

$$
\begin{aligned}
&= \mathcal{L}_g^w\left\{\sum_{k=0}^{\infty}\frac{\lambda^k w^{-1}(x)(g(x)-g(a))^{k\alpha+\beta-1}}{\Gamma(k\alpha+\beta)}\right\}(s)\\
&= \sum_{k=0}^{\infty}\frac{\lambda^k\mathcal{L}_g^w\{w^{-1}(x)(g(x)-g(a))^{k\alpha+\beta-1}\}(s)}{\Gamma(k\alpha+\beta)}\\
&= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(k\alpha+\beta)}\frac{\Gamma(k\alpha+\beta)}{s^{k\alpha+\beta}}\\
&= \frac{1}{s^\beta}\sum_{k=0}^{\infty}\left(\frac{\lambda}{s^\alpha}\right)^k\\
&= \frac{s^{\alpha-\beta}}{s^\alpha-\lambda}.
\end{aligned} \tag{5.12}
$$

Next, we find the solution of the following differential equation:

$$ ({}^{C}_{a^+}\mathfrak{D}^{\alpha}_{w}y)(x) - \lambda\, y = f(x); \quad 0<\alpha<1. \tag{5.13} $$

Taking the weighted Laplace transforms of both sides of (5.13), the Laplace transform of the unknown function y reads

$$ \mathcal{L}_g^w\{y(x)\} = w(a)f(a)\frac{s^{\alpha-1}}{s^\alpha-\lambda} + \frac{1}{s^\alpha-\lambda}\mathcal{L}_g^w\{f(x)\}. \tag{5.14} $$

Now, we get the solution using the inverse transform as

$$
\begin{aligned}
y(x) &= w(a)f(a)w^{-1}(x)E_{\alpha,1}(\lambda(g(x)-g(a))^\alpha)\\
&\quad + (g(x)-g(a))^{\alpha-1}E_{\alpha,\alpha}\\
&\quad \times(\lambda(g(x)-g(a))^\alpha) *_g^w f(x)\\
&= w^{-1}(x)\left[w(a)f(a)E_{\alpha,1}(\lambda(g(x)-g(a))^\alpha)\right.\\
&\quad + \int_a^x (g(x)-g(t))^{\alpha-1}E_{\alpha,\alpha}(\lambda(g(x)-g(t))^\alpha)\\
&\quad \left.\times w(g^{-1}(g(x)-g(t)+g(a)w(t)f(t)g'(t)dt\right].
\end{aligned} \tag{5.15}
$$

6. CONLUSION

In this paper, we defined the weighted fractional integrals and derivatives with respect to another function and discussed some of their properties. Above this, we proposed a modification of the Laplace transform suitable for such types of operators. It is worth mentioning that these operators cover many well-known fractional operators in the literature. In fact, when one considers $w(x) = 1$ and $g(x) = x$, these operators reduce to Reimann–Liouville fractional integrals, Riemann–Liouville fractional derivatives and the Caputo fractional derivative. When one considers $w(x) = x^\mu$ and $g(x) = \ln x$, the Hadamard fractional operators are obtained. Other fractional operators such as the Erdelyi–Kober fractional operators and the fractional operators proposed by Katugampola appear as special cases of these weighted fractional operators with certain choices of w and g.

REFERENCES

1. R. Hilfer, *Applications of Fractional Calculus in Physics* (Word Scientific, Singapore, 2000).
2. L. Debnath, Recent applications of fractional calculus to science and engineering, *Int. J. Math. Math. Sci.* **54** (2003) 3413–3442.
3. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Application of Fractional Differential Equations*, North Holland Mathematics Studies, Vol. 204, (Elsevier, 2006).
4. R. L. Magin, *Fractional Calculus in Bioengineering* (Begell House Publishers, 2006).
5. I. Podlubny, *Fractional Differential Equations, Mathematics in Science and Engineering* (Academic Press, New York, 1999).
6. S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications* (Gordon and Breach, Yverdon 1993).
7. A. A. Kilbas, Hadamard-type fractional calculus, *J. Korean Math. Soc.* **38**(6) (2001) 1191–1204.
8. F. Jarad, T. Abdeljawad and D. Baleanu, Caputo-type modification of the Hadamard fractional derivative, *Adv. Differ. Equ.* **2012** (2012) 142.
9. U. N. Katugampola, New approach to generalized fractional integral, *Appl. Math. Comput.* **218** (2011) 860–865.
10. U. N. Katugampola, A new approach to generalized fractional derivatives, *Bul. Math. Anal. Appl.* **6** (2014) 1–15.
11. F. Jarad, T. Abdeljawad and D. Baleanu, On the generalized fractional derivatives and their Caputo modification, *J. Nonlinear Sci. Appl.* **10**(5) (2017) 2607–2619.
12. F. Jarad, E. Uğurlu, T. Abdeljawad and D. Baleanu, On a new class of fractional operators, *Adv. Differ. Equ.* **2018** (2018) 142.
13. F. Jarad and T. Abdeljawad, Generalized fractional derivatives and Laplace transform, *Discret. Contin. Dyn. S* **13**(3) (2020) 709–722.
14. F. Jarad, T. Abdeljawad and J. Alzabut, Generalized fractional derivatives generated by a class of local proportional derivatives, *Eur. Phys. J. Special Top.* **226** (2017) 3457–3471.

15. O. M. Agrawal, Generalized multiparameters fractional variational calculus, *Int. J. Differ. Equ.* **2012** (2012) 1–38, doi:10.1155/2012/521750.
16. O. M. Agrawal, Some generalized fractional calculus operators and their applications in integral equations, *Fractals Calculas Appl. Anal.* **15**(4) (2012) 700-711.
17. M. Al-Refai and A. M. Jarrah, Fundamental results on weigted Caputo–Fabrizio fractional derivative, *Chaos Solitons Fractals* **126** (2019) 7–11.
18. M. Al-Refai, On weighted Atangana–Baleanu fractional operators, *Adv. Differ. Equ.* **2020** (2020) 3.
19. M. Caputo and M. Fabrizio, A new definition of fractional derivative without singular kernel, *Fractals Differ. Appl.* **1** (2015) 73.
20. A. Atangana and D. Baleanu, New fractional derivative with non-local and non–singular kernel, *Thermal Sci.* **20** (2016) 757.