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Solutions of BVPs arising in hydrodynamic and magnetohydro-dynamic stability theory using polynomial and non-polynomial splines

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Abstract This paper describes the exceptionally precise results of 6^{th}-order and 8^{th}-order nonlinear boundary-value problems(BVPs). Cubic-Nonpolynomial spline(CNPS) and Cubic-polynomial spline(CNPS) are utilized to solve such types of BVPs. We develop the class of numerical techniques for a particular selection of the factors that are associated with nonpolynomial and polynomial splines. Using the developed class of numerical techniques, the problem is reduced to a new system that consists of 2^{nd}-order BVPs only. The end conditions associated with the BVPs are determined. For each problem, the results obtained by CNPS and CPS is compared with the exact solution. The absolute error(AE) for every iteration is calculated. To show that the suitable responses established by using CNPS and CPS have a higher level of preciseness, the absolute errors of the CNPS and CPS have been compared with different techniques such as DTM, ADM, Parametric septic splines, Variational-iteration method(VIM), Daftardar Jafari strategy, MDM, Cubic B-Spline, Homotopy method(HM), Quintic and Sextic B-spline and observed to be more accurate. Graphs that describe the graphical comparison of CNPS and CPS at $n = 10$ are also included in this paper.

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1. Introduction

Currently, higher-order BVPs obtain extended attention because they may be referred to in many mathematical physics applications. In [1], the author offered the theorems that listed the conditions for closeness and singularity of solutions of $6^{th}$and higher-order BVPs, anyway there were no numerical methodologies confined. Authors in [2] presented the solution of BVPs of 8^{th}-order utilizing kernel-space which was applied on both linear and nonlinear problems. The solution of sixth-order BVPs was assessed in [3] by non-polynomial spline system. They contended that this methodology passes on a convergence of order two. The quadratic-cubic nonlinearity in the existence of Hamiltonian perturbations is considered in [4]. The Jacobi elliptic ansatz method is applied to obtain the optical soliton solutions.

The investigations of 6^{th}-order boundary value problems (SBVPs) for ODEs emerge in various fields of applied mathematics, material science and numerous applications of scientific disciplines. For example, the deformations of a flexible beam often referenced to as the beam equations and are represented by a differential equation [5,17]. BVPs of 6^{th}-order has been solved in [6] by applying the global phase-integral strategy. Searching-least-value(SLV) technique was explored for nonlinear BVPs. A family of numerical techniques was acquired for solving specific non-linear 6^{th} and 8^{th} order BVPs in [7,9]. In [8], 8^{th}-order differential equations that occur in torsional vibration of uniform beams was investigated. This paper extends the theory to allow for warping of the beam cross-section, for this can make a large difference to results for thin-walled beams of the open section. Authors utilized non-polynomial spline technique to explain a time-dependent Lame Emden condition and a nonlinear system of 2^{nd} BVPs in [11,10].

As described in [12], when a horizontal level infinite layer of liquid is heated from below and is under the action of rotation, unsteadiness sets in. At the point where this situation acts as normal convection, the ODEs is established in 6^{th}-order and after the situation sets in as over stability, it is described by means of ODEs in 8^{th}-order. Galerkin method was introduced in [13] by the method of Legendre polynomial using as a basis function to explain the 8^{th}-order linear BVPs that contains 2-point boundary conditions(BCs). In [15], New chirped dark solitons, bright solitons, and trigonometric map solutions by using the auxiliary equation technique are obtained. Adams-Bashforth-Moulton scheme (ABMS) was used in [14] to determine the approximate solution of a variable-order fractional three-dimensional chaotic process. DTM and ADM were utilized in [16] to explain 6^{th}-order non-linear BVPs.

Estimated solutions of the 8^{th} order boundary-value DE were assessed in [18] utilizing the ADM. CPS and CNPS are utilized to solve $6^{th}$and 12^{th}-order BVPs in [39,40], 10^{th}-order BVPs in [19], for 14^{th}-order BVPs in [22], a system of 3^{rd}-order and 4^{th}-order BVPs in [25,42] and 2^{nd}-order BVPs in [41] for only linear cases. This motivates us to solve 6^{th} and 8^{th}-order nonlinear boundary value problems by using CPS and CNPS. The residual power series method (RPSM) is used in [26] to solve fractional variation of $(1 + 1)$-dimensional Biswas-Milovic equation that defines the long-space optical communications.

In [27], the system of fractional Burger differential equations are presented as a new fractional model for Atangana–Baleanu fractional derivative with Mittag-Leffler kernel. The extended Jacobi's elliptic function approach is used in [28] to solve the Biswas–Arshed equation in two different types. Cubic B-Spline was utilized in [20,21,23,44] to comprehend 6^{th}-order, 8^{th}-order and tenth-order BVPs. Parametric septic splines were utilized in [24] to solve 6^{th}-order BVPs. Authors in [29] utilized 6^{th} degree splines where analogous values of $2n$-order derivatives were likewise associated through consistency relations. MDM for the numerically solving 8^{th}-order BVPs was depicted in [30]. (see Figs. 1–6).

The numerical methodology was verified on linear & nonlinear problems. In [31,33,35,32,34] authors utilized Homotopy perturbation method(HPM) and variational iteration method(VIM) for resolving higher-order nonlinear BVPs. The suggested techniques were a mix of VIM and HPM by changing 6^{th}-order BVP to a new system of integral equations. They gained the results that were in the convergent series form. In [36] they utilized a variety of procedures for clarifying linear and nonlinear 6^{th}-order BVPs. VIM might be reflected as an option and compelling for the assessed solutions of the BVPs. They initiate a numerical solution of higher-order BVPs utilizing HPM, DTM and modified ADM in [37,38]. These were applied with no transformation or computation of adomian polynomials (see Tables 1–15).

8^{th}-order Polynomial and non-polynomial spline method was utilized to tackle 8^{th}-order BVPs in [43]. Nonpolynomial spline methods, which were comparable to 7^{th}-degree polynomial splines in [45], were utilized to improve a class of numerical methodologies for determining estimates to the solution of 6^{th}-order BVPs with 2-point boundary conditions(BCS). Numerical techniques for 8^{th}-, 10^{th}-and 12^{th}-order boundary value problems ascending in thermal unsteadiness were set up in [46]. Numerical solutions of 5^{th} and 6^{th}-order nonlinear BVPs were created in [47] by daftardar-jafari technique. Authors in [48–51] discovered strategies including collocation technique with Septic, Quintic and 6^{th} degree B-splines as basis functions to explain 8^{th}-order BVPs. Wazwaz in [52] initiate the numerical solution of 6^{th} order BVPs by MDM and notices that this strategy is very fastly convergent and additionally requires very less computational expense.

1.1. Fundamentals of cubic non-polynomial splines

Suppose $v(\jmath)$ be the accurate root and $v_\ell$be an estimation to $v(\jmath_\ell)$ earned by using non-polynomial-cubic spline $Ł_\ell(\jmath)$ considering the points $(\jmath_\ell,\ v_\ell)$ and $(\jmath_{\ell+1},\ v_{\ell+1})$. This is imposed on $Ł_\ell(\jmath)$ to satisfy the boundary conditions, the extrapolating conditions at points $\jmath_\ell$and $\jmath_{\ell+1}$and similarly the continuity of 1^{st}-order derivative at all nodes $(\jmath_\ell,\ v_\ell)$. For all parts $(\jmath_\ell,\ \jmath_{\ell+1})$ where $\ell = \{0,\ 1,\ 2, \ldots, n-2,\ n-1\}$, then spline $Ł_\ell(\jmath)$ admits this form

$$Ł_\ell(\jmath) = s_\ell + g_\ell\ (\jmath - \jmath_\ell) + h_\ell\ sin\rho\ (\jmath - \jmath_\ell) + k_\ell cos\rho\ (\jmath - \jmath_\ell), \tag{1}$$

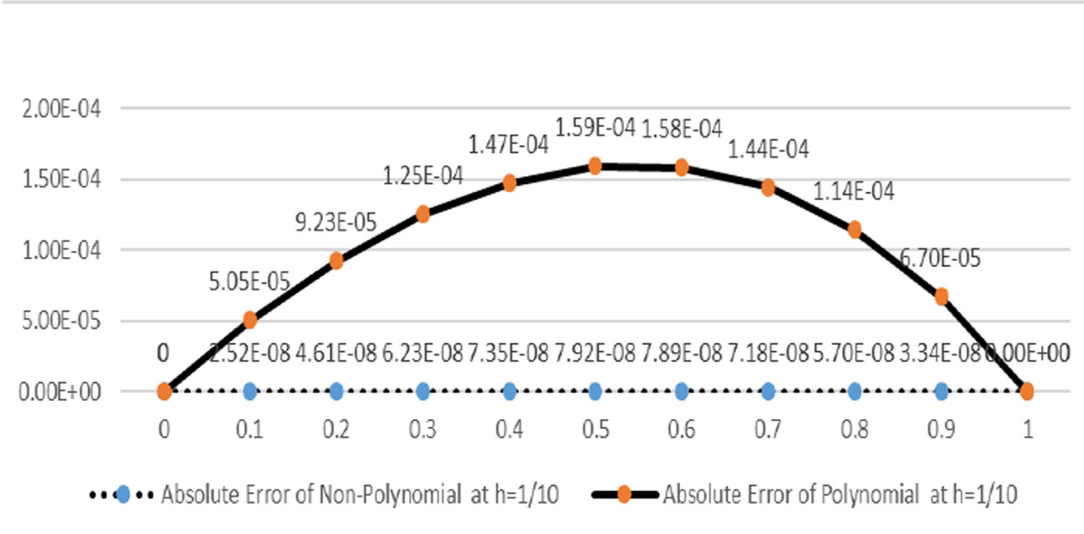

Fig. 1 Problem 4.1.1 at $h = \frac{1}{10}$.

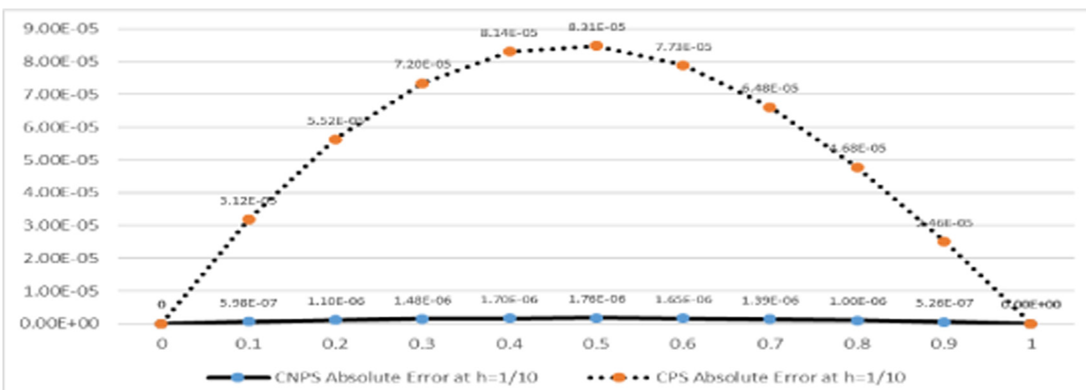

Fig. 2 Problem 4.1.2 at $h = \frac{1}{10}$.

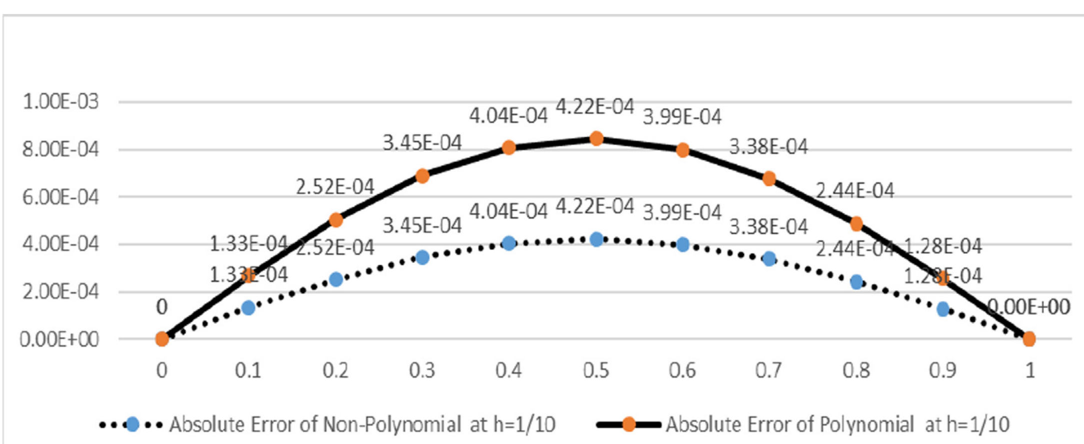

Fig. 3 Problem 4.1.3 at $h = \frac{1}{10}$.

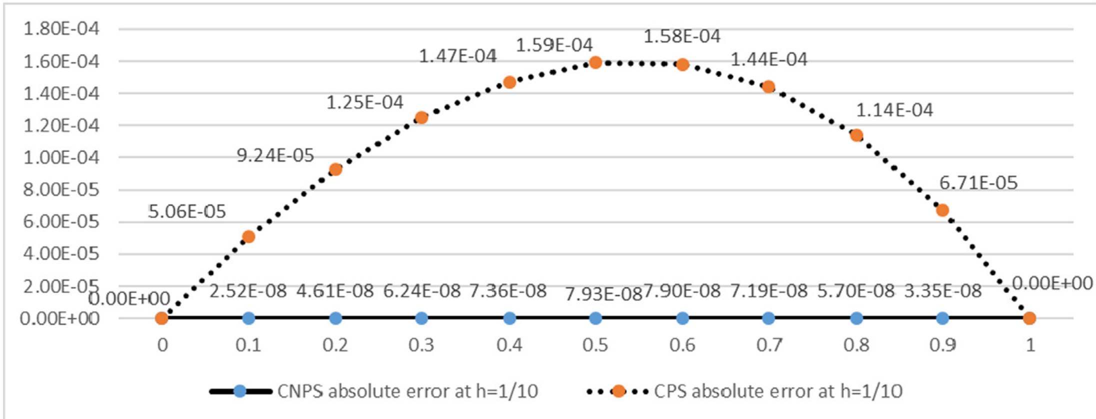

Fig. 4 Problem 4.2.1 at $h = \frac{1}{10}$.

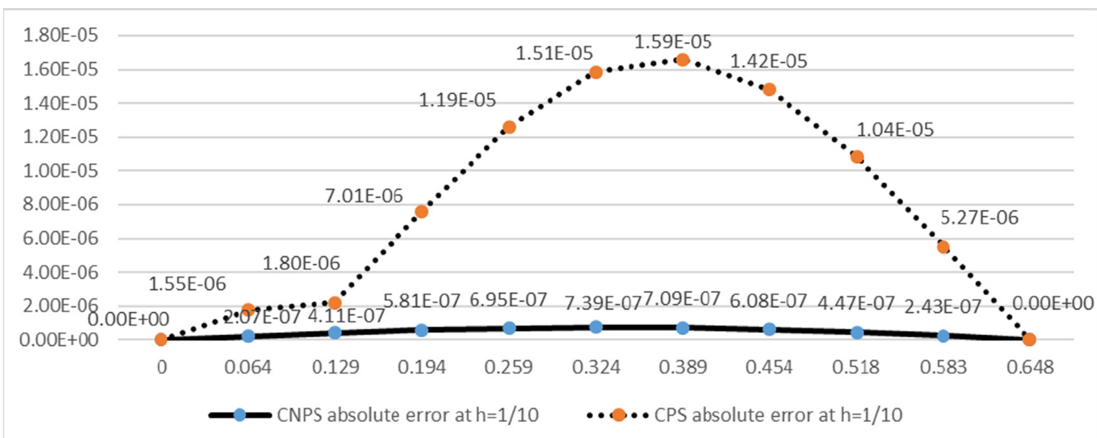

Fig. 5 Problem 4.2.2 at $h = 0.064872$.

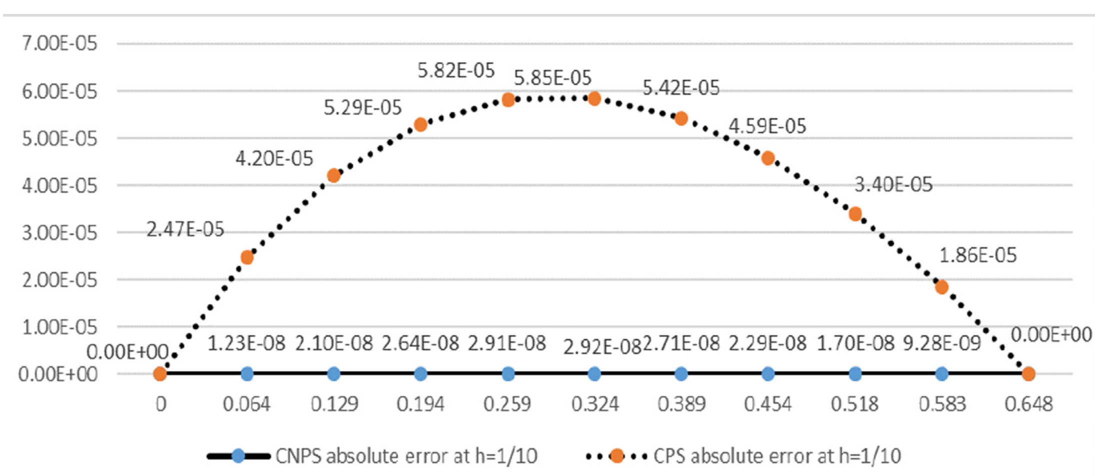

Fig. 6 Problem 4.2.3 at $h = \frac{1}{10}$.

where s_ℓ, g_ℓ, h_ℓ and $k_\ell$are constants and ρ is a exempt factor. A non-polynomial spline function Ł(ȷ) of class $C^2[a,b]$ that extrapolates $v(ȷ)$ at the grid points $ȷ_\ell$; $i = \{0, 1, 2, \dots, n-1, n\}$ be shown by a new factor ρ and cut down to a cubic spline Ł(ȷ) in $[a,b]$ as $\rho \to 0$.

Table 1 Analyzing precise solution with CNPS solution, CPS solution, [36,47] [52], and [21] of Problem 4.1.1 at $h = \frac{1}{10}$

j	Precise solution	CNPS solution	CNPS | *error* |	CPS solution	CPS | *error* |	In [36,47,52]	In [21]
0	1	1	0	1	0	0	0
0.1	1.10517091	1.10517089	2.52E−08	1.10512041	5.05E−05	1.23E−04	3.95E−06
0.2	1.22140275	1.22140271	4.61E−08	1.22131048	9.23E−05	2.35E−04	1.43E−05
0.3	1.34985880	1.34985874	6.23E−08	1.34973403	1.25E−04	3.25E−04	2.79E−05
0.4	1.49182469	1.49182462	7.35E−08	1.49167744	1.47E−04	3.85E−04	4.07E−05
0.5	1.64872127	1.64872119	7.92E−08	1.64856250	1.59E−04	4.08E−04	4.88E−05
0.6	1.82211880	1.82211872	7.89E−08	1.82196070	1.58E−04	3.91E−04	4.92E−05
0.7	2.01375270	2.01375263	7.18E−08	2.01360890	1.44E−04	3.36E−04	4.09E−05
0.8	2.22554092	2.22554087	5.70E−08	2.22542680	1.14E−04	2.45E−04	2.56E−05
0.9	2.45960311	2.45960307	3.34E−08	2.45953610	6.70E−05	1.29E−04	8.63E−06
1	2.71828182	2.71828182	0	2.71828182	0	0	0

Table 2 Analyzing accurate solution, CPS solution and CNPS solution of Problem 4.1.1 at $h = \frac{1}{5}$

j	Precise solution	CNPS solution	CNPS | *error* |	CPS solution	CPS | *error* |
0	1	1	0	1	0
0.2	1.2214027581602	1.2214020220905	7.36E−07	1.221402758160	3.71E−04
0.4	1.4918246976413	1.4918235229803	1.17E−06	1.491824697641	5.91E−04
0.6	1.8221188003905	1.8221175392783	1.26E−06	1.822118800391	6.35E−04
0.8	2.2255409284925	2.2255400181297	9.10E−07	2.225540928492	4.58E−04
1	2.7182818284590	2.7182818284590	0	2.7182818284590	0

Table 3 Analyzing precise solution, CPS solution and CNPS solution of Problem 4.1.2 at $h = \frac{1}{5}$

j	Precise solution	CNPS solution	CNPS Absolute Error	CPS solution	CPS Absolute Error
0	0	0	0	0	0
0.2	0.030386926	0.030403056	1.61E−05	0.030607921	2.21E−04
0.4	0.056078706	0.056103635	2.49E−05	0.056403919	3.25E−04
0.6	0.078333938	0.078358133	2.42E−05	0.078643219	3.09E−04
0.8	0.097964444	0.097979137	1.47E−05	0.098151650	1.87E−04
1	0.115524530	0.115524530	0	0.115524530	0

Table 4 Analyzing accurate solution, CPS solution and CNPS solution of Problem 4.1.2 at $h = \frac{1}{10}$

j	Precise solution	CNPS solution	CNPS Absolute Error	CPS solution	CPS Absolute Error	[21]
0	0	0	0	0	0	0
0.1	0.015885029967	0.015885627621	5.98E−07	0.01591622560	3.12E−05	2.95E−06
0.2	0.030386926132	0.030388025780	1.10E−06	0.03044209648	5.52E−05	8.70E−06
0.3	0.043727377411	0.043728854172	1.48E−06	0.04379937781	7.20E−05	1.35E−05
0.4	0.056078706103	0.056080408310	1.70E−06	0.05616006044	8.14E−05	1.55E−05
0.5	0.067577518018	0.067579279019	1.76E−06	0.06766057189	8.31E−05	1.43E−05
0.6	0.078333938207	0.078335590995	1.65E−06	0.07841125726	7.73E−05	1.09E−05
0.7	0.088438041843	0.088439433391	1.39E−06	0.08850288291	6.48E−05	6.49E−06
0.8	0.097964444150	0.097965447886	1.00E−06	0.09801120054	4.68E−05	2.68E−06
0.9	0.106975647695	0.106976173288	5.26E−07	0.10700021048	2.46E−05	5.03E−07
1	0.115524530093	0.115524530093	0	0.11552453009	0	0

Table 5 Maximum absolute errors discussed in [24,45]

	$h = 1/8$	$h = 1/16$	$h = 1/32$
Second order Method [24]	2.27 $\times 10^{-4}$	7.31×10^{-6}	1.87×10^{-6}
Second order Method [45]	2.3×10^{-3}	3.4×10^{-4}	1.54×10^{-4}

For driving all the constants of the above Eq. (1) in the forms of $v_\ell,\ v_{\ell+1},\ \aleph_\ell,\ \aleph_{\ell+1}$, we first define

$$Ł_\ell(\mathrm{j}_\ell) = v_\ell,\quad Ł_\ell(\mathrm{j}_{\ell+1}) = v_{\ell+1},\quad Ł''_\ell(v_\ell) = \aleph_\ell,\ Ł''_\ell(v_{\ell+1}) = \aleph_{\ell+1}. \tag{2}$$

By substance of the elementary algebraical handling we achieve the resulting expressions for all the constants of Eq. (1) as

$$s_\ell = v_\ell + \frac{\aleph_\ell}{\rho^2},\ g_\ell = \frac{v_{\ell+1} - v_\ell}{h} + \frac{\aleph_{\ell+1} - \aleph_\ell}{\rho\psi},$$

$$h_\ell = \frac{\aleph_\ell \cos\psi - \aleph_{\ell+1}}{\rho^2 \sin\psi},\ k_\ell = -\frac{\aleph_\ell}{\rho^2},$$

where $\psi = \rho h, \ell = \{0,\ 1,\ 2, \ldots, n-2,\ n-1\}$

Utilizing the conditions of continuity of the 1^{st}-order derivative at all grid points $(\mathrm{j}_\ell,\ v_\ell)$.

i.e.,$Ł'_{\ell-1}(\mathrm{j}_\ell) = Ł'_\ell(\mathrm{j}_\ell)$ is consistence relation for $\ell = \{0,\ 1,\ 2, \ldots, n-2,\ n-1\}$,

$$\gamma\,(\aleph_{\ell+1} + \aleph_{\ell-1}) + 2\xi\,\aleph_\ell = \frac{1}{h^2}(v_{\ell-1} + v_{\ell+1} - 2v_\ell), \tag{3}$$

where we have substituted

$$\gamma = \frac{1}{\psi\sin\psi} - \frac{1}{\psi^2}, \xi = -\frac{1}{\psi^2} - \frac{\cos\psi}{\psi}, \qquad v'' = \aleph \text{ and } \psi = \rho h.$$

1.2. Fundamentals of cubic polynomial splines

Suppose $v(\mathrm{j})$ be the accurate root and $v_\ell$be an estimation to $v(\mathrm{j}_\ell)$ earned by using non-polynomial-cubic spline $Ł_\ell(\mathrm{j})$ considering the points $(\mathrm{j}_\ell,\ v_\ell)$ and $(\mathrm{j}_{\ell+1},\ v_{\ell+1})$. This is imposed on $Ł_\ell(\mathrm{j})$ to satisfy the boundary conditions, the extrapolating conditions at points $\mathrm{j}_\ell$and $\mathrm{j}_{\ell+1}$and similarly the continuity of 1^{st}-order derivative at all nodes $(\mathrm{j}_\ell,\ v_\ell)$. For all parts $(\mathrm{j}_\ell,\ \mathrm{j}_{\ell+1})$ where $\ell = \{0,\ 1,\ 2, \ldots, n-2,\ n-1\}$, then spline $Ł_\ell(\mathrm{j})$ admits this form

$$Ł_\ell(\mathrm{j}) = s_\ell + g_\ell\ (\mathrm{j} - \mathrm{j}_\ell) + h_\ell\ (\mathrm{j} - \mathrm{j}_\ell)^2 + k_\ell\,(\mathrm{j} - \mathrm{j}_\ell)^3, \tag{4}$$

where $s_\ell,\ g_\ell,\ h_\ell$ and $k_\ell$are described here as constants and ρ is an unrestricted factor. A non-polynomial function $Ł(\mathrm{j})$ of family $C^2[a,b]$ which extrapolates $v(\mathrm{j})$ at the grid points j_ℓ *where* $i = \{0,\ 1,\ 2, \ldots, n-1,\ n\}$ be determined by a factor ρ and cut down to a cubic-spline $Ł(\mathrm{j})$ in $[a,b]$ as $\rho \to 0$.

For driving all the constants of the above Eq. (4) in forms of $v_\ell,\ v_{\ell+1},\ \aleph_\ell,\ \aleph_{\ell+1}$, we first define

$$Ł_\ell(\mathrm{j}_\ell) = v_\ell,\quad Ł_\ell(\mathrm{j}_{\ell+1}) = v_{\ell+1},\quad Ł''_\ell(v_\ell) = \aleph_\ell,\ Ł''_\ell(v_{\ell+1}) = \aleph_{\ell+1}. \tag{5}$$

By substance of the elementary algebraical handling we achieve the resulting expressions for all the constants of Eq. (4) as

$$s_\ell = v_\ell + \frac{\aleph_\ell}{\rho^2}, g_\ell = \frac{v_{\ell+1} - v_\ell}{h} + \frac{\aleph_{\ell+1} - \aleph_\ell}{\rho\psi},$$

$$h_\ell = \frac{\aleph_\ell \cos\psi - \aleph_{\ell+1}}{\rho^2 \sin\psi},\quad k_\ell = -\frac{\aleph_\ell}{\rho^2},$$

where $\psi = \rho h, \ell = \{0,\ 1,\ 2, \ldots, n-2,\ n-1\}$

Utilizing the conditions of continuity of the 1^{st}-order derivative at all grid points $(\mathrm{j}_\ell,\ v_\ell)$. iti.e.,$Ł'_{\ell-1}(\mathrm{j}_\ell) = Ł'_\ell(\mathrm{j}_\ell)$ is consistence relation for $\ell = \{0,\ 1,\ 2, \ldots, n-2,\ n-1\}$

$$\aleph_{\ell+1} + \aleph_{\ell-1} + 4\,\aleph_\ell = \frac{6}{h^2}(v_{\ell-1} + v_{\ell+1} - 2v_\ell), \tag{6}$$

where we have substituted

$v'' = \aleph.$

The residue of this paper is sorted out as follows. The development of CPS and CNPS and usage for 6^{th}-order non-linear BVPs is introduced in Section 2. In Section 3, the development of CPS and CNPS and use for 8^{th}-order non-linear BVPs are introduced. Results and discussions are presented in Section 4. Likewise, a few problems are reasoned right now to reveal the effectiveness of the CPS and CNPS schemes. At last, the concluding comments are given in Section 5.

2. Sixth order non-linear BVPs

A new numerical technique with CPS and CNPS is developed here for locating an assessed solution of a 6^{th} non-linear BVP

$$w^{(6)}(\mathrm{j}) = f(\mathrm{j},\ w(\mathrm{j}),\ w^{(1)}(\mathrm{j}),\ w^{(2)}(\mathrm{j}),\ w^{(3)}(\mathrm{j}),\ w^{(4)}(\mathrm{j}),\ w^{(5)}(\mathrm{j})),\quad \mathrm{j} \in [a,\ b] \tag{7}$$

Table 6 Analyzing precise solution, CPS solution and CNPS solution of Problem 4.1.3 at $h = \frac{1}{5}$

j	Precise solution	CNPS solution	CNPS | *error* |	CPS solution	CPS Absolute Error
0	1	1	0	1	0
0.2	0.904837418	0.904236181	6.01E−04	0.904261586	5.76E−04
0.4	0.818730753	0.817760938	9.70E−04	0.817803346	9.27E−04
0.6	0.740818220	0.739850739	9.67E−04	0.739893242	9.25E−04
0.8	0.670320046	0.669722654	5.97E−04	0.669748414	5.72E−04
1	0.606530659	0.606530659	0	0.606530659	0

Table 7 Analyzing accurate solution, CPS solution, [16] and CNPS solution of Problem 4.1.3 at $h = \frac{1}{10}$

ј	Precise solution	CNPS solution	CPS solution	| *error* | of presented splines	DTM and ADM [16]
0	1	1	1	0	0
0.1	0.951229424	0.951096339	0.951096047	1.33E−04	1.89E−03
0.2	0.904837418	0.904585198	0.904584861	2.52E−04	1.30E−02
0.3	0.860707976	0.860362638	0.860362325	3.45E−04	3.52E−02
0.4	0.818730753	0.818327220	0.818326916	4.04E−04	6.49E−02
0.5	0.778800783	0.778379070	0.778378728	4.22E−04	9.44E−02
0.6	0.740818220	0.740419405	0.740418991	3.99E−04	1.15E−01
0.7	0.704688089	0.704350443	0.704349963	3.38E−04	3.28E−04
0.8	0.670320046	0.670075602	0.670075117	2.44E−04	2.41E−04
0.9	0.637628151	0.637499920	0.637499567	1.28E−04	1.28E−04
1	0.606530659	0.606530659	0.606530659	0	0

Table 8 Analyzing precise solution with CNPS and CPS of Problem 4.2.1 at $h = \frac{1}{10}$

ј	Precise solution	CNPS solution	CPS solution
0	1	1	1
0.1	1.1051708	1.1051709	1.1051204
0.2	1.2214027	1.2214027	1.2213104
0.3	1.3498587	1.3498588	1.3497339
0.4	1.4918246	1.4918246	1.4916773
0.5	1.6487211	1.6487212	1.6485623
0.6	1.8221187	1.8221188	1.8219605
0.7	2.0137526	2.0137527	2.0136088
0.8	2.2255408	2.2255409	2.2254267
0.9	2.4596030	2.4596031	1.1051204
1	2.7182818	2.7182818	2.7182818

where the boundary conditions are

$$w^{(2\ell)}(a) = \gamma_\ell, \qquad w^{(2\ell)}(b) = \xi_\ell, \tag{8}$$

where ($\gamma_\ell,\ \xi_\ell$; $\ell = 0$ to 2) are arbitrary real quantities, ($a_\ell; i = 0$ to 6) and f are the continuous functions on interval [a, b].

Now Eq. (7) is comprehended here with the given boundary conditions (8) by the process of reducing the Eq. (7) such a way that this method squeeze the set of the equations of 2^{nd}-order BVPs in a form that it is changed into such a new system that has linear algebraical equations with the given BCs

$$v^{(2)}(\mathrm{j}) = f(\mathrm{j},\ w(\mathrm{j}),\ w^{(1)}(\mathrm{j}),\ y(\mathrm{j}),\ y^{(1)}(\mathrm{j}),\ v(\mathrm{j}),\ v^{(1)}(\mathrm{j}))\ ; \quad \mathrm{j} \in [a,\ b] \tag{9}$$

$$w''(\mathrm{j}) = y(\mathrm{j}), \qquad y''(\mathrm{j}) = v(\mathrm{j}), \tag{10}$$

alongside boundary conditions

$$\begin{array}{lll} w(a) = \gamma_0, & w(b) = \xi_0, & y(a) = \gamma_1, \\ y(b) = \xi_1, & v(a) = \gamma_2, & v(b) = \xi_2. \end{array} \tag{11}$$

2.1. Cubic non-polynomial spline

On similar lines from Eq. (6), Using the continuity condition of the 1^{st}-order derivative we obtain a new relations for y and w respectively as;

$$\left.\begin{array}{l} \gamma\,(\chi_{\ell+1} + \chi_{\ell-1}) + 2\xi\,\chi_\ell = \frac{1}{h^2}(y_{\ell-1} + y_{\ell+1} - 2y_\ell), \\ \gamma\,(U_{\ell+1} + U_{\ell-1}) + 2\xi\,U_\ell = \frac{1}{h^2}(w_{\ell-1} + w_{\ell+1} - 2w_\ell), \end{array}\right\} \tag{12}$$

where we have substituted

$$\begin{array}{ll} w'' = U, & y'' = \chi, \\ v'' = \aleph, & \end{array} \tag{13}$$

Now we discretize Eq. (9) at the grid points $(\mathrm{j}_\ell,\ v_\ell)$, $(\mathrm{j}_\ell,\ y_\ell)$ and $(\mathrm{j}_\ell,\ w_\ell)$

$$v^{(2)}(\mathrm{j}_\ell) = f(\mathrm{j}_\ell,\ w(\mathrm{j}_\ell),\ w^{(1)}(\mathrm{j}_\ell),\ y(\mathrm{j}_\ell),\ y^{(1)}(\mathrm{j}_\ell),\ v(\mathrm{j}_\ell),\ v^{(1)}(\mathrm{j}_\ell)) \tag{14}$$

$$w''(\mathrm{j}_\ell) = y(\mathrm{j}_\ell) = y_\ell, \qquad y''(\mathrm{j}_\ell) = v(\mathrm{j}_\ell) = v_\ell. \tag{15}$$

Now substituting

$$w''_\ell = U_\ell, \qquad y''_\ell = \chi_\ell, \qquad v''_\ell = \aleph_\ell. \tag{16}$$

the Eq. 14 becomes

$$\aleph_\ell = f(\mathrm{j}_\ell, w_\ell, w^{(1)}_\ell, y_\ell, y^{(1)}_\ell, v_\ell, v^{(1)}_\ell), \tag{17}$$

$$U_\ell = y_\ell, \qquad \chi_\ell = v_\ell. \tag{18}$$

Table 9 Analyzing accurate solution, CPS and CNPS of Problem 4.2.1 at $h = \frac{1}{5}$

ј	Precise solution	CNPS solution	CPS solution	CNPS | *error* |	CPS | *error* |
0	1	1	1	0	0
0.2	1.22140275816	1.22140202118	1.2210318333	7.37E−07	3.71E−04
0.4	1.49182469764	1.49182352151	1.4912327490	1.18E−06	5.92E−04
0.6	1.82211880039	1.82211753781	1.8214833301	1.26E−06	6.35E−04
0.8	2.22554092849	2.22554001722	2.2250822577	9.11E−07	4.59E−04
1	2.7182818284	2.7182818284	2.7182818284	0	0

Table 10 Comparison of absolute error of CNPS and CPS with other methods of Problem 4.2.1

j	Absolute error of CNPS	Absolute error of CPS	[31,33] and [35]	[48]	[49]
0	0	0	0	0	0
0.1	2.52E−08	5.06E−05	1.27E−05	1.19E−07	2.26E−06
0.2	4.61E−08	9.24E−05	2.43E−05	1.25E−05	6.08E−06
0.3	6.24E−08	1.25E−04	3.35E−05	4.52E−05	3.48E−05
0.4	7.36E−08	1.47E−04	3.94E−05	8.22E−05	4.65E−05
0.5	7.93E−08	1.59E−04	4.16E−05	1.04E−04	3.53E−05
0.6	7.90E−08	1.58E−04	3.96E−05	9.79E−05	5.34E−05
0.7	7.19E−08	1.44E−04	3.38E−05	6.62E−05	1.98E−05
0.8	5.70E−08	1.14E−04	2.45E−05	3.14E−05	5.01E−06
0.9	3.35E−08	6.71E−05	1.29E−05	1.16E−05	3.12E−05
1	0	0	0	0	0

Table 11 Analyzing precise solution with CNPS and CPS of Problem 4.2.2 at $h = 0.064872$

j	Precise solution	CNPS solution	CPS solution
0	0	0	0
0.064	0.06285472	0.06285451	0.06285628
0.129	0.12199128	0.12199087	0.12198949
0.194	0.17782512	0.17782454	0.17781811
0.259	0.23070570	0.23070500	0.23069378
0.324	0.28092981	0.28092907	0.28091467
0.389	0.32875163	0.32875092	0.32873569
0.454	0.37439052	0.37438992	0.37437630
0.518	0.41803710	0.41803666	0.41802671
0.583	0.45985806	0.45985782	0.45985279
0.648	0.5	0.5	0.5

From Eq. (17) and Eq. (18), we obtain

$$\aleph_{\ell+1} = f(\mathrm{j}_{\ell+1}, w_{\ell+1}, w^{(1)}_{\ell+1}, y_{\ell+1}, y^{(1)}_{\ell+1}, v_{\ell+1}, v^{(1)}_{\ell+1}), \tag{19}$$

$$U_{\ell+1} = y_{\ell+1}, \qquad \chi_{\ell+1} = v_{\ell+1}. \tag{20}$$

Similarly,

$$\aleph_{\ell-1} = f(\mathrm{j}_{\ell-1}, w_{\ell-1}, w^{(1)}_{\ell-1}, y_{\ell-1}, y^{(1)}_{\ell-1}, v_{\ell-1}, v^{(1)}_{\ell-1}), \tag{21}$$

$$U_{\ell-1} = y_{\ell-1}, \qquad \chi_{\ell-1} = v_{\ell-1}. \tag{22}$$

The pursuit estimation of $O(h^2)$ for the 1^{st}-order derivative of w, y and v in Eqs. (17), (19) and (21) can be used;

$$\left.\begin{array}{ll} v^{(1)}_{\ell} \cong \frac{v_{\ell+1}-v_{\ell-1}}{2h}, & v^{(1)}_{\ell+1} \cong \frac{3v_{\ell+1}-4v_{\ell}+v_{\ell-1}}{2h}, \\ v^{(1)}_{\ell-1} \cong \frac{-v_{\ell+1}+4v_{\ell}-3v_{\ell-1}}{2h}, & \\ y^{(1)}_{\ell} \cong \frac{y_{\ell+1}-y_{\ell-1}}{2h}, & y(1)_{\ell+1} \cong \frac{3y_{\ell+1}-4y_{\ell}+y_{\ell-1}}{2h}, \\ y^{(1)}_{\ell-1} \cong \frac{-y_{\ell+1}+4y_{\ell}-3y_{\ell-1}}{2h}, & \\ w^{(1)}_{\ell} \cong \frac{w_{\ell+1}-w_{\ell-1}}{2h}, & w^{(1)}_{\ell+1} \cong \frac{3w_{\ell+1}-4w_{\ell}+w_{\ell-1}}{2h}, \\ w^{(1)}_{\ell-1} \cong \frac{-w_{\ell+1}+4w_{\ell}-3w_{\ell-1}}{2h}. & \end{array}\right\} \tag{23}$$

using Eqs. (17)–(23) in Eqs. (6) and (12), where
$\ell = \{0, 1, 2, \ldots, n-2, n-1\}$

$$\left.\begin{array}{l} \gamma(y_{\ell-1} + y_{\ell+1}) + 2\xi y_{\ell} = \frac{1}{h^2}(w_{\ell+1} - 2w_{\ell} + w_{\ell-1}), \\ \gamma(v_{\ell-1} + v_{\ell+1}) + 2\xi v_{\ell} = \frac{1}{h^2}(y_{\ell+1} - 2y_{\ell} + y_{\ell-1}), \end{array}\right\} \tag{24}$$

Which can be written as,

$$\gamma\Big(f(\mathrm{j}_{\ell+1}, w_{\ell+1}, w^{(1)}_{\ell+1}, y_{\ell+1}, y^{(1)}_{\ell+1}, v_{\ell+1}, v^{(1)}_{\ell+1})\Big) + 2\xi\Big(f(\mathrm{j}_{\ell}, w_{\ell}, w^{(1)}_{\ell}, y_{\ell}, y^{(1)}_{\ell}, v_{\ell}, v^{(1)}_{\ell})\Big) + \gamma\Big(f(\mathrm{j}_{\ell-1}, w_{\ell-1}, w^{(1)}_{\ell-1}, y_{\ell-1}, y^{(1)}_{\ell-1}, v_{\ell-1}, v^{(1)}_{\ell-1})\Big) = \frac{1}{h^2}(v_{\ell-1} + v_{\ell+1} - 2v_{\ell}), \tag{25}$$

Then the above equation becomes,

$$\gamma f\Big(\mathrm{j}_{\ell+1}, w_{\ell+1}, \frac{3w_{\ell+1}-4w_{\ell}+w_{\ell-1}}{2h}, y_{\ell+1}, \frac{3y_{\ell+1}-4y_{\ell}+y_{\ell-1}}{2h}, v_{\ell+1}, \frac{3v_{\ell+1}-4v_{\ell}+v_{\ell-1}}{2h}\Big) + 2\xi f\Big(\mathrm{j}_{\ell}, w_{\ell}, \frac{w_{\ell+1}-w_{\ell-1}}{2h}, y_{\ell}, \frac{y_{\ell+1}-y_{\ell-1}}{2h}, v_{\ell}, \frac{v_{\ell+1}-v_{\ell-1}}{2h}\Big) + \gamma f\Big(\mathrm{j}_{\ell-1}, w_{\ell-1}, \frac{-w_{\ell+1}+4w_{\ell}-3w_{\ell-1}}{2h}, y_{\ell-1}, \frac{-y_{\ell+1}+4y_{\ell}-3y_{\ell-1}}{2h}, v_{\ell-1}, \frac{-v_{\ell+1}+4v_{\ell}-3v_{\ell-1}}{2h}\Big) = \frac{1}{h^2}(v_{\ell-1} + v_{\ell+1} - 2v_{\ell}), \tag{26}$$

Table 12 Analyzing accurate solution, CPS and CNPS of Problem 4.2.2 at $h = 0.12974426$

j	Precise solution	CNPS solution	CPS solution	CNPS | *error* |	CPS | *error* |
0	0	0	0	0	0
0.129	0.1219912885	0.1219851900	0.1219890578	6.10E−06	2.23E−06
0.259	0.2307057020	0.2306953802	0.2306649648	1.03E−05	4.07E−05
0.389	0.3287516379	0.3287411404	0.3286942842	1.05E−05	5.74E−05
0.518	0.4180371083	0.4180305882	0.4179994001	6.52E−06	3.77E−05
0.648	0.5	0.5	0.5	0	0

Table 13 Comparison of absolute error of CNPS and CPS with other methods of Problem 4.2.2

j	| *Error* | of CNPS	| *error* | of CPS	[51]	[48]	[50]	[23]
0	0	0	0	0	0	0
0.064	2.07E−07	1.55E−06	2.94E−06	1.42E−07	2.01E−07	3.74E−06
0.129	4.11E−07	1.80E−06	1.58E−05	1.07E−06	4.54E−07	1.27E−05
0.194	5.81E−07	7.01E−06	2.91E−05	3.37E−06	1.52E−06	2.14E−05
0.259	6.95E−07	1.19E−05	4.91E−05	7.02E−06	4.07E−06	2.56E−05
0.324	7.39E−07	1.51E−05	7.34E−05	9.54E−06	6.71E−06	2.40E−05
0.389	7.09E−07	1.59E−05	8.51E−05	1.07E−05	9.06E−06	1.79E−05
0.454	6.08E−07	1.42E−05	6.54E−05	1.03E−05	1.00E−05	1.03E−05
0.518	4.47E−07	1.04E−05	4.38E−05	5.22E−06	5.45E−06	4.15E−06
0.583	2.43E−07	5.27E−06	2.31E−05	2.41E−06	2.59E−06	8.59E−07
0.648	0	0	0	0	0	0

Table 14 Analyzing precise solution, CPS solution and CNPS solution of Problem 4.2.3 at $h = \frac{1}{10}$

j	Precise solution	CNPS solution	CPS solution	Absolute error of CNPS	Absolute error of CPS	[50]	[23]
0	1	1	1	0	0	0	0
0.1	0.9048374180	0.9048374057	0.9048127534	1.23E−08	2.47E−05	3.58E−07	3.60E−06
0.2	0.8187307531	0.8187307321	0.8186887394	2.10E−08	4.20E−05	6.32E−06	1.21E−05
0.3	0.7408182207	0.7408181943	0.7407652797	2.64E−08	5.29E−05	1.90E−05	2.09E−05
0.4	0.6703200460	0.6703200170	0.6702618385	2.91E−08	5.82E−05	3.10E−05	2.59E−05
0.5	0.6065306597	0.6065306305	0.6064722042	2.92E−08	5.85E−05	3.64E−05	2.56E−05
0.6	0.5488116361	0.5488116090	0.5487574155	2.71E−08	5.42E−05	3.17E−05	2.07E−05
0.7	0.4965853038	0.4965852809	0.4965393610	2.29E−08	4.59E−05	1.93E−05	1.33E−05
0.8	0.4493289641	0.4493289472	0.4492949886	1.70E−08	3.40E−05	7.18E−06	6.30E−06
0.9	0.4065696597	0.4065696505	0.4065510662	9.28E−09	1.86E−05	1.46E−06	1.59E−06
1	0.3678794411	0.3678794411	0.3678794411	0	0	0	0

Table 15 Analyzing accurate solution, CPS solution and CNPS solution of Problem 4.2.3 at $h = \frac{1}{5}$

j	Precise solution	CNPS solution	CPS solution	CNPS | *error* |	CPS | *error* |
0	1	1	1	0	0
0.2	0.8187307531	0.8187304179	0.8185620547	3.35E−07	1.69E−04
0.4	0.6703200460	0.6703195817	0.6700863296	4.64E−07	2.34E−04
0.6	0.5488116361	0.5488112036	0.5485939304	4.33E−07	2.18E−04
0.8	0.4493289641	0.4493286931	0.4491925456	2.71E−07	1.36E−04
1	0.3678794411	0.3678794411	0.3678794411	0	0

Eqs. (24) and (26) along the boundary conditions Eq. (11) provide an entire system of $(3n+3)$ linear equations in $(3n+3)$ unknowns.

2.2. Cubic polynomial spline

On similar lines from Eq. (6), Using the given continuity condition of the 1^{st}-order derivative we then obtain a new relations for y and w respectively as;

$$\left.\begin{array}{l} \chi_{\ell+1} + \chi_{\ell-1} + 4\chi_{\ell} = \frac{6}{h^2}(y_{\ell-1} + y_{\ell+1} - 2y_{\ell}), \\ U_{\ell+1} + U_{\ell-1} + 4U_{\ell} = \frac{6}{h^2}(w_{\ell-1} + w_{\ell+1} - 2w_{\ell}), \end{array}\right\} \quad (27)$$

After substitution, we obtain

$$\left.\begin{array}{l} (y_{\ell-1} + y_{\ell+1}) + 4y_{\ell} = \frac{6}{h^2}(w_{\ell+1} - 2w_{\ell} + w_{\ell-1}), \\ (v_{\ell-1} + v_{\ell+1}) + 4v_{\ell} = \frac{6}{h^2}(y_{\ell+1} - 2y_{\ell} + y_{\ell-1}), \end{array}\right\} \quad (28)$$

In a parallel way as was established in the non-polynomial-cubic spline section, the polynomial- cubic spline strategy for $\ell = \{0,\ 1,\ 2, \ldots, n-2,\ n-1\}$

$$\left(f(\mathrm{j}_{\ell+1}, w_{\ell+1}, w^{(1)}_{\ell+1}, y_{\ell+1}, y^{(1)}_{\ell+1}, v_{\ell+1}, v^{(1)}_{\ell+1})\right) + 4\left(f(\mathrm{j}_{\ell}, w_{\ell}, w^{(1)}_{\ell}, y_{\ell}, y^{(1)}_{\ell}, v_{\ell}, v^{(1)}_{\ell})\right) + \left(f(\mathrm{j}_{\ell-1}, w_{\ell-1}, w^{(1)}_{\ell-1}, y_{\ell-1}, y^{(1)}_{\ell-1}, v_{\ell-1}, v^{(1)}_{\ell-1})\right) = \frac{1}{h^2}(v_{\ell-1} + v_{\ell+1} - 2v_{\ell}), \quad (29)$$

The above equation admits form

$$f\Big(\mathrm{j}_{\ell+1}, w_{\ell+1}, \tfrac{3w_{\ell+1}-4w_\ell+w_{\ell-1}}{2h}, y_{\ell+1}, \tfrac{3y_{\ell+1}-4y_\ell+y_{\ell-1}}{2h}, v_{\ell+1}, \tfrac{3v_{\ell+1}-4v_\ell+v_{\ell-1}}{2h}\Big) + 4f\Big(\mathrm{j}_\ell, w_\ell, \tfrac{w_{\ell+1}-w_{\ell-1}}{2h}, y_\ell, \tfrac{y_{\ell+1}-y_{\ell-1}}{2h}, v_\ell, \tfrac{v_{\ell+1}-v_{\ell-1}}{2h}\Big) + f\Big(\mathrm{j}_{\ell-1}, w_{\ell-1}, \tfrac{-w_{\ell+1}+4w_\ell-3w_{\ell-1}}{2h}, y_{\ell-1}, \tfrac{-y_{\ell+1}+4y_\ell-3y_{\ell-1}}{2h}, v_{\ell-1}, \tfrac{-v_{\ell+1}+4v_\ell-3v_{\ell-1}}{2h}\Big) = \tfrac{1}{h^2}(v_{\ell-1}+v_{\ell+1}-2v_\ell). \quad (30)$$

Eqs. (28) and (30) along the boundary conditions Eq. (11) that provide an entire system of $(3n+3)$ linear equations in $(3n+3)$ unknowns.

3. Eighth order non-linear BVPs

A new numerical technique with CPS and CNPS is developed here for locating an assessed solution of a 8^{th}-order non-linear BVP

$$w^{(8)}(\mathrm{j}) = f(\mathrm{j}, w(\mathrm{j}), w^{(1)}(\mathrm{j}), w^{(2)}(\mathrm{j}), w^{(3)}(\mathrm{j}), w^{(4)}(\mathrm{j}), w^{(5)}(\mathrm{j}), w^{(6)}(\mathrm{j}), w^{(7)}(\mathrm{j})); \mathrm{j} \in [a, b] \quad (31)$$

where the BCs are

$$w^{(2\ell)}(a) = \gamma_\ell, \qquad w^{(2\ell)}(b) = \xi_\ell, \quad (32)$$

where (γ_ℓ, ξ_ℓ ; $\ell = 0$ to 3) are arbitrary real quantities, ($a_\ell; i = 0$ to 8) and f are the continuous functions on the interval [a, b].

Now Eq. (31) is comprehended here with the given boundary conditions (32) by the process of reducing Eq. (31) such a way that this method squeeze the set of the equations of 2^{nd}-order BVPs in a new form that it is changed into such a new system that has linear algebraical equations with the given BCs.

$$u^{(2)}(\mathrm{j}) = f(\mathrm{j}, w(\mathrm{j}), w^{(1)}(\mathrm{j}), y(\mathrm{j}), y^{(1)}(\mathrm{j}), v(\mathrm{j}), v^{(1)}(\mathrm{j}), u(\mathrm{j}), u^{(1)}(\mathrm{j})); \; \mathrm{j} \in [a, b] \quad (33)$$

$$w''(\mathrm{j}) = y(\mathrm{j}), \qquad y''(\mathrm{j}) = v(\mathrm{j}), \qquad v''(\mathrm{j}) = u(\mathrm{j}), \quad (34)$$

alongside boundary conditions

$$\begin{array}{llll} w(a) = \gamma_0, & w(b) = \xi_0, & y(a) = \gamma_1, & y(b) = \xi_1, \\ v(a) = \gamma_2, & v(b) = \xi_2, & u(a) = \gamma_3, & u(b) = \xi_3. \end{array} \quad (35)$$

3.1. Cubic non-polynomial spline

On similar lines from Eq. (6), Using the conditions of continuity of the 1^{st}-order derivative we obtain entirely new relations for y, w and u respectively as;

$$\left.\begin{array}{l} \gamma(\chi_{\ell+1} + \chi_{\ell-1}) + 2\xi\,\chi_\ell = \frac{1}{h^2}(y_{\ell-1} + y_{\ell+1} - 2y_\ell), \\ \gamma(U_{\ell+1} + U_{\ell-1}) + 2\xi\,U_\ell = \frac{1}{h^2}(w_{\ell-1} + w_{\ell+1} - 2w_\ell), \\ \gamma(P_{\ell+1} + P_{\ell-1}) + 2\xi\,P_\ell = \frac{1}{h^2}(u_{\ell-1} + u_{\ell+1} - 2u_\ell), \end{array}\right\} \quad (36)$$

where we have substituted

$$\begin{array}{ll} w'' = U, & y'' = \chi, \\ v'' = \aleph, & u'' = P. \end{array} \quad (37)$$

Now we discretize Eq. (33) at the grid points $(\mathrm{j}_\ell, v_\ell)$, $(\mathrm{j}_\ell, y_\ell)$, $(\mathrm{j}_\ell, w_\ell)$ and $(\mathrm{j}_\ell, u_\ell)$ we have

$$u_\ell^{(2)}(\mathrm{j}_\ell) = f_\ell(\mathrm{j}_\ell, w(\mathrm{j}_\ell), w^{(1)}(\mathrm{j}_\ell), y(\mathrm{j}_\ell), y^{(1)}(\mathrm{j}_\ell), v(\mathrm{j}_\ell), v^{(1)}(\mathrm{j}_\ell), u(\mathrm{j}_\ell), u^{(1)}(\mathrm{j}_\ell)), \quad (38)$$

$$w''(\mathrm{j}_\ell) = y(\mathrm{j}_\ell) = y_\ell, \qquad y''(\mathrm{j}_\ell) = v(\mathrm{j}_\ell) = v_\ell, \qquad v''(\mathrm{j}_\ell) = u(\mathrm{j}_\ell) = u_\ell, \quad (39)$$

After substitution, we obtain

$$\begin{array}{ll} w''_\ell = U_\ell, & y''_\ell = \chi_\ell, \\ v''_\ell = \aleph_\ell, & u''_\ell = P_\ell, \end{array} \quad (40)$$

Then the above equation admits form

$$P_\ell = f_\ell(\mathrm{j}_\ell, w_\ell, w_\ell^{(1)}, y_\ell, y_\ell^{(1)}, v_\ell, v_\ell^{(1)}, u_\ell, u_\ell^{(1)}), \quad (41)$$

$$U_\ell = y_\ell, \qquad \chi_\ell = v_\ell, \qquad \aleph_\ell = u_\ell. \quad (42)$$

From Eq. (41) and Eq. (42)

$$P_{\ell+1} = f_{\ell+1}(\mathrm{j}_{\ell+1}, w_{\ell+1}, w_{\ell+1}^{(1)}, y_{\ell+1}, y_{\ell+1}^{(1)}, v_{\ell+1}, v_{\ell+1}^{(1)}, u_{\ell+1}, u_{\ell+1}^{(1)}), \quad (43)$$

$$U_{\ell+1} = y_{\ell+1}, \qquad \chi_{\ell+1} = v_{\ell+1}, \qquad \aleph_{\ell+1} = u_{\ell+1}. \quad (44)$$

Similarly,

$$P_{\ell-1} = f_{\ell-1}(\mathrm{j}_{\ell-1}, w_{\ell-1}, w_{\ell-1}^{(1)}, y_{\ell-1}, y_{\ell-1}^{(1)}, v_{\ell-1}, v_{\ell-1}^{(1)}, u_{\ell-1}, u_{\ell-1}^{(1)}), \quad (45)$$

$$U_{\ell-1} = y_{\ell-1}, \qquad \chi_{\ell-1} = v_{\ell-1}, \qquad \aleph_{\ell-1} = u_{\ell-1}. \quad (46)$$

The pursuit estimation of $O(h^2)$ for 1^{st}-order derivative of w, y, v and u in Eqs. (41), (43) and (45) can be then used;

$$\left.\begin{array}{ll} u_\ell^{(1)} \cong \frac{u_{\ell+1}-u_{\ell-1}}{2h}, & u_{\ell+1}^{(1)} \cong \frac{3u_{\ell+1}-4u_\ell+u_{\ell-1}}{2h}, \\ u_{\ell-1}^{(1)} \cong \frac{-u_{\ell+1}+4u_\ell-3u_{\ell-1}}{2h}, & \end{array}\right\} \quad (47)$$

using Eqs. 41,42,43,44,45,46,47, (23) in Eqs. (6) and (36), where $\ell = 0, 1, \ldots, n-1$,

$$\left.\begin{array}{l} \gamma(y_{\ell-1} + y_{\ell+1}) + 2\xi y_\ell = \frac{1}{h^2}(w_{\ell+1} - 2w_\ell + w_{\ell-1}), \\ \gamma(v_{\ell-1} + v_{\ell+1}) + 2\xi v_\ell = \frac{1}{h^2}(y_{\ell+1} - 2y_\ell + y_{\ell-1}), \\ \gamma(u_{\ell-1} + u_{\ell+1}) + 2\xi u_\ell = \frac{1}{h^2}(v_{\ell+1} - 2v_\ell + v_{\ell-1}). \end{array}\right\} \quad (48)$$

We obtain,

$$\gamma\Big(f(\mathrm{j}_{\ell+1}, w_{\ell+1}, w_{\ell+1}^{(1)}, y_{\ell+1}, y_{\ell+1}^{(1)}, v_{\ell+1}, v_{\ell+1}^{(1)}, u_{\ell+1}, u_{\ell+1}^{(1)})\Big) + 2\xi\Big(f(\mathrm{j}_\ell, w_\ell, w_\ell^{(1)}, y_\ell, y_\ell^{(1)}, v_\ell, v_\ell^{(1)}, u_\ell, u_\ell^{(1)})\Big) + \gamma\Big(f(\mathrm{j}_{\ell-1}, w_{\ell-1}, w_{\ell-1}^{(1)}, y_{\ell-1}, y_{\ell-1}^{(1)}, v_{\ell-1}, v_{\ell-1}^{(1)}, u_{\ell-1}, u_{\ell-1}^{(1)})\Big), = \tfrac{1}{h^2}(u_{\ell-1} + u_{\ell+1} - 2u_\ell), \quad (49)$$

Then the above equation becomes,

$$\gamma f\Big(\mathrm{j}_{\ell+1}, w_{\ell+1}, \tfrac{3w_{\ell+1}-4w_\ell+w_{\ell-1}}{2h}, y_{\ell+1}, \tfrac{3y_{\ell+1}-4y_\ell+y_{\ell-1}}{2h}, v_{\ell+1}, \tfrac{3v_{\ell+1}-4v_\ell+v_{\ell-1}}{2h}, u_{\ell+1}, \tfrac{3u_{\ell+1}-4u_\ell+u_{\ell-1}}{2h}\Big) + 2\xi f\Big(\mathrm{j}_\ell, w_\ell, \tfrac{w_{\ell+1}-w_{\ell-1}}{2h}, y_\ell, \tfrac{y_{\ell+1}-y_{\ell-1}}{2h}, v_\ell, \tfrac{v_{\ell+1}-v_{\ell-1}}{2h}, u_\ell, \tfrac{u_{\ell+1}-u_{\ell-1}}{2h}\Big) + \gamma f\Big(\mathrm{j}_{\ell-1}, w_{\ell-1}, \tfrac{-w_{\ell+1}+4w_\ell-3w_{\ell-1}}{2h}, y_{\ell-1}, \tfrac{-y_{\ell+1}+4y_\ell-3y_{\ell-1}}{2h}, v_{\ell-1}, \tfrac{-v_{\ell+1}+4v_\ell-3v_{\ell-1}}{2h}\Big), u_{\ell-1}, \tfrac{-u_{\ell+1}+4u_\ell-3u_{\ell-1}}{2h}\Big) = \tfrac{1}{h^2}(u_{\ell-1} + u_{\ell+1} - 2u_\ell). \quad (50)$$

Eqs. (48) and (50) along the boundary conditions Eq. (35) provide a new system of $(4n+4)$ linear equations in $(4n+4)$ unknowns.

3.2. Cubic polynomial spline

On similar lines from Eq. (6), Using the continuity condition of 1^{st}-order derivative we then obtain a new relations for y, w and u respectively as

$$\left.\begin{aligned}(\chi_{\ell+1}+\chi_{\ell-1})+4\chi_\ell &= \tfrac{6}{h^2}(y_{\ell-1}+y_{\ell+1}-2y_\ell),\\(U_{\ell+1}+U_{\ell-1})+4U_\ell &= \tfrac{6}{h^2}(w_{\ell-1}+w_{\ell+1}-2w_\ell),\\(P_{\ell+1}+P_{\ell-1})+4P_\ell &= \tfrac{6}{h^2}(u_{\ell-1}+u_{\ell+1}-2u_\ell),\end{aligned}\right\} \quad (51)$$

after substitution

$$\left.\begin{aligned}(y_{\ell-1}+y_{\ell+1})+4y_\ell &= \tfrac{6}{h^2}(w_{\ell+1}-2w_\ell+w_{\ell-1}),\\(v_{\ell-1}+v_{\ell+1})+4v_\ell &= \tfrac{6}{h^2}(y_{\ell+1}-2y_\ell+y_{\ell-1}),\\(u_{\ell-1}+u_{\ell+1})+4u_\ell &= \tfrac{6}{h^2}(v_{\ell+1}-2v_\ell+v_{\ell-1}),\end{aligned}\right\} \quad (52)$$

In a parallel way as was established in the non-polynomial-cubic spline section, the polynomial-cubic spline strategy for $\ell = 0, 1, \ldots, n-1$,

$$\Big(f(\mathrm{j}_{\ell+1}, w_{\ell+1}, w^{(1)}_{\ell+1}, y_{\ell+1}, y^{(1)}_{\ell+1}, v_{\ell+1}, v^{(1)}_{\ell+1}, u_{\ell+1}, u^{(1)}_{\ell+1})\Big) + 4\Big(f(\mathrm{j}_\ell, w_\ell, w^{(1)}_\ell, y_\ell, y^{(1)}_\ell, v_\ell, v^{(1)}_\ell, u_\ell, u^{(1)}_\ell)\Big) + \Big(f(\mathrm{j}_{\ell-1}, w_{\ell-1}, w^{(1)}_{\ell-1}, y_{\ell-1}, y^{(1)}_{\ell-1}, v_{\ell-1}, v^{(1)}_{\ell-1}, u_{\ell-1}, u^{(1)}_{\ell-1})\Big) = \tfrac{1}{h^2}(u_{\ell-1}+u_{\ell+1}-2u_\ell), \quad (53)$$

and

$$f\Big(\mathrm{j}_{\ell+1}, w_{\ell+1}, \tfrac{3w_{\ell+1}-4w_\ell+w_{\ell-1}}{2h}, y_{\ell+1}, \tfrac{3y_{\ell+1}-4y_\ell+y_{\ell-1}}{2h}, v_{\ell+1}, \tfrac{3v_{\ell+1}-4v_\ell+v_{\ell-1}}{2h}, u_{\ell+1}, \tfrac{3u_{\ell+1}-4u_\ell+u_{\ell-1}}{2h}\Big) + 4f\Big(\mathrm{j}_\ell, w_\ell, \tfrac{w_{\ell+1}-w_{\ell-1}}{2h}, y_\ell, \tfrac{y_{\ell+1}-y_{\ell-1}}{2h}, v_\ell, \tfrac{v_{\ell+1}-v_{\ell-1}}{2h}, u_\ell, \tfrac{u_{\ell+1}-u_{\ell-1}}{2h}\Big) + f\Big(\mathrm{j}_{\ell-1}, w_{\ell-1}, \tfrac{-w_{\ell+1}+4w_\ell-3w_{\ell-1}}{2h}, y_{\ell-1}, \tfrac{-y_{\ell+1}+4y_\ell-3y_{\ell-1}}{2h}, v_{\ell-1}, \tfrac{-v_{\ell+1}+4v_\ell-3v_{\ell-1}}{2h}), u_{\ell-1}, \tfrac{-u_{\ell+1}+4u_\ell-3u_{\ell-1}}{2h}\Big) = \tfrac{1}{h^2}(u_{\ell-1}+u_{\ell+1}-2u_\ell). \quad (54)$$

Eqs. (52) and (54) along the boundary conditions Eq. (35) provide a system of $4(n+1)$ linear equations in $4(n+1)$ unknowns.

4. Solution of BVPs using cubic polynomial and non-polynomial splines

The higher-order boundary value problems and especially non-linear problems have not been solved by using techniques of CNPS and CPS. CPS and CNPS are utilized to solve 2^{nd}, system of $3^{rd}, 4^{th}, 6^{th}, 10^{th}, 12^{th}, 14^{th}$-order BVPs in [19,22,25,39–42] only for linear cases. The investigations of 6^{th} and 8^{th}-order boundary value problems for ODEs emerge in changing in diverse fields of applied mathematics, material science and numerous applications of scientific disciplines. This motivates us to solve 6^{th} and 8^{th}-order nonlinear boundary value problems by using CPS and CNPS. In this section numerical application of CPS and CNPS on higher order linear and nonlinear BVPs is discussed in detail.

Here absolute error = | precise solution - prescribed method solution | = | *error* |

4.1. Sixth order non-linear BVPs

4.1.1. Problem 4.1.1

$$\frac{d^6 w(\mathrm{j})}{d\mathrm{j}^6} = e^{-\mathrm{j}}(w(\mathrm{j}))^2 \; ; \; 0 \leqslant \mathrm{j} \leqslant 1$$

subject to

$$w^{(2\ell)}(0) = 1, \qquad w^{(2\ell)}(1) = e,$$

where ($\ell = 0, 1, 2$) with precise solution $w(\mathrm{j}) = e^{\mathrm{j}}$.

4.1.2. Problem 4.1.2

$$\frac{d^6 w(\mathrm{j})}{d\mathrm{j}^6} = 20e^{-36w(\mathrm{j})} - 40(1+w(\mathrm{j}))^{-6} \; ; \; 0 \leqslant \mathrm{j} \leqslant 1$$

subject to

$$w(0) = 0, \quad w(1) = \frac{1}{6}\log 2,$$

$$w^{(2)}(0) = -\frac{1}{6}, \quad w^{(2)}(1) = -\frac{1}{24},$$

$$w^{(4)}(0) = -1, \quad w^{(4)}(1) = -\frac{1}{16}.$$

The precise solution for the above problem is specifically shown by $w(\mathrm{j}) = \frac{1}{6}\log(1+\mathrm{j})$.

4.1.3. Problem 4.1.3

$$\frac{d^6 w(\mathrm{j})}{d\mathrm{j}^6} = e^{\mathrm{j}}(w(\mathrm{j}))^3 \; ; \; 0 \leqslant \mathrm{j} \leqslant 1$$

subject to

$$w^{(2\ell)}(0) = 2^{-2\ell}, \qquad w^{(2\ell)}(1) = 2^{-2\ell}e^{-1/2},$$

where ($\ell = 0, 1, 2$). The precise solution is $w(\mathrm{j}) = e^{-\mathrm{j}/2}$.

4.2. Eighth order non-linear BVPs

4.2.1. Problem 4.2.1

$$w^{(8)}(\mathrm{j}) = e^{-\mathrm{j}}(w(\mathrm{j}))^2; \; 0 \leqslant \mathrm{j} \leqslant 1$$

subject to

$$w^{(2\ell)}(0) = 1, \qquad w^{(2\ell)}(1) = e,$$

where ($\ell = 0, 1, 2, 3$). The precise solution is $w(\mathrm{j}) = e^{\mathrm{j}}$.

4.2.2. Problem 4.2.2

$$w^{(8)}(\mathrm{j}) = 7!\left(e^{-8w(\mathrm{j})} - \frac{2}{(1+\mathrm{j})^8}\right); \; 0 \leqslant \mathrm{j} \leqslant e^{1/2-1}$$

subject to

$$w^{(2\ell)}(0) = -(2\ell - 1),$$
$$w^{(2\ell)}\left(e^{1/2-1}\right) = -(2\ell - 1)e^{(-\ell)},$$

where ($\ell = 0, 1, 2, 3$). The precise solution for the overhead problem is specified by $w(\mathrm{j}) = ln(1+\mathrm{j})$.

4.2.3. Problem 4.2.3

$$w^{(8)}(\mathrm{j}) + e^{-\mathrm{j}}(w(\mathrm{j}))^2 = e^{-3\mathrm{j}} + e^{-\mathrm{j}};\ 0 \leqslant \mathrm{j} \leqslant 1$$

subject to

$$w^{(2\ell)}(0) = 1,$$
$$w^{(2\ell)}(1) = e^{-1},$$

where ($\ell = 0, 1, 2, 3$). The precise solution for the overhead problem is specified by $w(\mathrm{j}) = e^{-\mathrm{j}}$.

5. Conclusions

In this paper, numerical methods are created to evaluate the 6^{th}-order and 8^{th}-order BVPs utilizing Polynomial and Non-polynomial Cubic splines. The schemes adopted in our work depend on CPS and CNPS techniques together with the decomposition method. Polynomial and non-polynomial cubic splines along the finite difference approximations are utilized to construct a new system of a 2^{nd}-order BVPs that consists of linear algebraical equations along with boundary conditions. For every example, the numerical results given by CNPS and CPS are compared with the exact solution and other methods. Absolute errors(AE) for every iteration is computed. These schemes can be extended to other higher-order problems, and authors of this paper are working on these problems. The authors have already solved tenth, twelfth, fourteenth and sixteenth order BVPs by using these splines and future work on odd-order nonlinear BVPs like fifth, seventh, ninth, eleventh, thirteenth and fifteenth is in progress.

To show that the solution given by the CNPS and CPS has higher level of accuracy, the absolute errors of the CNPS and CPS for 6^{th}-order BVP have been compared with the other methods such as DTM and ADM [16], Cubic B-Spline [21], Parametric septic splines [24], VIM [36], seven-degree Non-polynomial splines [45], Daftardar jafari method [47], modified decomposition method [52] and for 8^{th}-order BVP have been compared with other methods such as Cubic B-Spline [23], Homotopy-method(HM) [31], HPM [33], VIM [35], Quintic B-spline [48], Sextic B-spline [49], Galerkin-method with quintic and septic B-splines [50,51] as well and observed to be better. These constructed splines are applied on six problems and the results accomplished are appropriately exact up to 8-decimal places as appeared in tables and graphs. It has also been observed that numerical approximations obtained by CNPS scheme have better numerical accuracy compared with that for numerical approximation obtained by CPS for both 6^{th}-order and 8^{th}-order BVPs. For 8^{th}-order BVPs Polynomial and Non-polynomial Cubic splines give better results than 6^{th}-order BVPs.

Abbreviations

Cubic-nonpolynomial spline (CNPS), boundary-value problems (BVPs) Cubic-polynomial spline (CPS), Differential Transform Method (DTM), Modified Decomposition Method (MDM), Adomian Decomposition Method (ADM), Variational-iteration method (VIM), Homotopy perturbation method (HPM), Homotopy method(HM).

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Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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