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Stability data dependency and errors estimation for a general iteration method

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Abstract In this paper, we present a result of stability, data Dependency and errors estimation for D Iteration Method. We also prove that errors in D iterative process is controllable. Especially stability, data dependence, controllability, error accumulation of such iterative methods are being studied.

1. Introduction

In many fields of mathematics and other sciences, a problem can be translated into an equation for a suitable operator. In addition, the existence of a solution to this equation can be reduced to the existence of a fixed point of the mentioned operator. The theory of a fixed point itself is a perfect combination of functional analysis, topology and geometry. Reducing the real-life or theoretical problem into the fixed point problem is a great step to finding the corresponding solution. That is the reason why the fixed point theory plays an indispensable role in almost all quantitative sciences, in particular, economics, game theory, theatrical computer science, biology, chemistry, engineering, and physics, see e.g. [2,3–9,12,13,26–30].

Although, proving the existence of a fixed point is a crucial step on finding a solution, the main and probably the final step is to find the exact value of the desired fixed point. One of the best method to calculate the desired fixed point is to use an iterative process. For this reason, a number of interesting iterative processes have been developed. Indeed, the well-known Banach contraction theorem approximates fixed-point using Picard's iterative process. After then, Mann iteration and Ishikawa iteration appeared, for details on these iteration and some others, see e.g. [1,9–11,14–19,21–25].

Two qualities "Fastness" and "stability" play an important role so that an iteration process is preferred to another iteration process. In [1], the author uses numerical examples to show that for non-expansive mapping, the convergence rate of the Picard-S iterative process is faster than that of Picard, Mann, Ishikawa, Noor, SP, Agarwal, CR, S*, Abbas and Normal-s. The speed was fast. In [18], the authors demonstrate

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that the convergence of the iterative process M * is better than the iterative processes Agarwal and Picard-S. Recently, in [19], another iterative, the iterative process M, was introduced and its speed of convergence was compared to the iterative processes Agarwal and Picard. In [10], another iterative process called k iterative process was introduced, which proves that convergence is faster than the existing iterative process. They also proved that their iterative process "K" was T-stable. In [16], a new iterative process called "K^*" was developed, they prove the speed of convergence and the stability of their iterative process. Based on the above reasons, recently in [10], we introduced a new iteration method D defined as

$$\begin{cases} \xi_0 \in C \\ \omega_n = F((1-\vartheta_n)\xi_n + \vartheta_n F\xi_n) \\ \eta_n = F((1-\theta_n)F\xi_n + \theta_n F\omega_n) \\ \xi_{n+1} = F\eta_n \end{cases} \quad (1)$$

In, numerically compare the convergence rate of the new iterative process with the iterative process of Agarwal, Picards, M, M * and K, and prove the weak and strong convergence theorems of the generalized non-expansive map of Suzuki for "D" iteration. This article demonstrates the "stability" and "data dependence" of the D-iteration method. We also estimate the error of the "D" iteration, and prove that the error accumulation in (1) is bounded.

2. Preliminaries

In this section, some basic definitions are recalled.

Definition 1 [14]. A Banach space X is called uniformly convex if for each $\epsilon \in (0,2]$ there exists $\delta > 0$ such that for r,s $\in X$ with $\|r\| \leqslant 1$ and $\|s\| \leqslant 1, \|r-s\| > \epsilon$ implies $\|\frac{r+s}{2}\| \leqslant \delta$.

Definition 2 [16]. Let $\{u_n\}_{n=0}^{\infty}$ be an arbitrary sequence in M. Then an iteration procedure $r_{n+1} = f(F, r_n)$ converging to a fixed point p is said to be F-stable or stable with respect to F.

If for $\epsilon_n = \|t_n + 1 - f(F : u_n)\|, n \in N$, we have $lim_{n\to\infty}\epsilon = 0$ if and only if $lim_{n\to\infty} u_n = p$.

Definition 3 [9]. Let F, $\widetilde{F} : X \to X$ be two operators. We say that $\widetilde{F}$ is an approximate operator for F if for some $\epsilon > 0$ we have $\|Fx - \widetilde{F}x\| \leqslant \epsilon$ for all $x \in X$.

Lemma 1 [22]. Let $\{r_n\}_{n=0}^{\infty}$ and $\{t_n\}_{n=0}^{\infty}$ be nonnegative real sequences satisfying the relation.

$r_{n+1} \leqslant (1-t_n)r_n + t_n$, where $t_n \in (0,1)$ for all $n \in N, \Sigma_{n=0}^{\infty} t_n = \infty$ and $\frac{r_n}{t_n} \to 0$ as $n \to \infty$. Then $lim_{n\to\infty} r_n = 0$.

Lemma 2 [23]. Let $\{r_n\}_{n=0}^{\infty}$ be nonnegative real sequences for which one assumes there exists $n_0 \in N$ such that for all $n \geqslant n_o$ satisfying the relation.

$r_{n+1} \leqslant (1-t_n)r_n + t_n t_n$, where $t_n \in (0,1)$ for all $n \in N, \Sigma_{n=0}^{\infty} t_n = \infty$ and $t_n \geqslant 0$, for all n $\in$ N, then

$$0 \leqslant \limsup_{n\to\infty} r_n \leqslant \limsup_{n\to\infty} t_n.$$

3. Stability for D iteration process

In this section we first prove that D iteration Process is strongly convergent. Then we prove that D iteration Process is T-stable. Furthermore we also discuss about Data dependency.

Theorem 2.1. Let C be a nonempty closed convex subset of a Banach space X and F: C $\to$ C be a contraction mapping. Let $\{\xi_n\}_{n=0}^{\infty}$ be an iterative sequence generated by D iteration process with real sequences $\{\theta_n\}_{n=0}^{\infty}$ and $\{\vartheta_n\}_{n=0}^{\infty} \in$[0 1] satisfying $\Sigma_{n=0}^{\infty}\theta_n = \infty$ or $\Sigma_{n=0}^{\infty}\vartheta_n = \infty$. Then $\{\xi_n\}_{n=0}^{\infty}$ converge strongly to a unique fixed point of F.

Proof. Since F is a contraction mapping in a Banach space, F has a unique fixed point in C. Let us suppose that p is a fixed point of F. From D iteration process, we get

$$\begin{aligned} \|\omega_n - p\| &= \|F((1-\vartheta_n)\xi_n + \vartheta_n F\xi_n) - Fp\| \\ &\leqslant k\|(1-\vartheta_n)\xi_n + \vartheta_n F\xi_n - p\| \\ &\leqslant k\|(1-\vartheta_n)(\xi_n - p) + \beta_n(F\xi_n - p)\| \\ &\leqslant k(1-\vartheta_n)\|\xi_n - p\| + \vartheta_n\|F\xi_n - p\| \\ &\leqslant k\{(1-\vartheta_n)\|\xi_n - p\| + k\vartheta_n\|\xi_n - p\|\} \\ &\leqslant k\{1 - \vartheta_n(1-k)\}\|\xi_n - p\|. \end{aligned}$$

Now,

$$\begin{aligned} \|\eta_n - p\| &= \|F((1-\theta_n)F\xi_n + \theta_n F\omega_n) - Fp\| \\ &\leqslant k[(1-\theta_n)\|F\xi_n - p\| + \theta_n\|F\omega_n - p\|] \\ &\leqslant k[(1-\theta_n)k\|\xi_n - p\| + \theta_n k\|\omega_n - p\|] \\ &\leqslant k^2[(1-\theta_n)\|(\xi_n - p)\| + \theta_n\|\omega_n - p\|] \\ &\leqslant k^2[(1-\theta_n)\|(\xi_n - p)\| + \theta_n(k\{1-\vartheta_n(1-k)\}\|\xi_n - p\|)] \\ &\leqslant k^2[1 - (\theta_n + k\theta_n\vartheta_n)(1-k)]\|(\xi_n - p)\|. \end{aligned}$$

Then,

$$\begin{aligned} \|\xi_{n+1} - p\| &= \|F\eta_n - Fp\| \\ &\leqslant k\|\eta_n - p\| \\ &\leqslant k^3[1 - (\theta_n + k\theta_n\vartheta_n)(1-k)]\|(\xi_n - p)\|. \end{aligned}$$

By repeating the above process, we get

$$\|\xi_n - p\| \leqslant k^3[1 - (\theta_{n-1} + k\theta_{n-1}\vartheta_{n-1})(1-k)]\|(\xi_{n-1} - p)\|$$

$$\|\xi_{n-1} - p\| \leqslant k^3[1 - (\theta_{n-2} + k\theta_{n-2}\vartheta_{n-2})(1-k)]\|(\xi_{n-2} - p)\|$$

$$\|\xi_{n-2} - p\| \leqslant k^3[1 - (\theta_{n-3} + k\theta_{n-3}\vartheta_{n-3})(1-k)]\|(\xi_{n-3} - p)\|$$

$$\vdots$$

$$\|\xi_1 - p\| \leqslant k^3[1 - (\theta_0 + k\theta_0\vartheta_0)(1-k)]\|(\xi_0 - p)\|.$$

Therefore, we obtain $\|\xi_{n+1} - p\| \leqslant k^{3(n+1)}\|(\xi_0 - p)\| \prod_{i=0}^{n}[1 - (\theta_i + k\theta_i\vartheta_i)(1-k)]$. Now, k <1 so (1-k)>0 and $\theta_n, \vartheta_n \leqslant 1$ for all n $\in$ N. Therefore, we get $[1 - (\theta_i + k\theta_i\vartheta_i)(1-k)] < 1$ for all n $\in$ N. After that, we know that 1-x $\leqslant e^{-x}$, for all $\xi \in$[0 1]. So we have.

$\|\xi_{n+1} - p\| \leqslant k^{3(n+1)}\|(\xi_0 - p)\|e^{-(1-k)\sum_{i=0}^{n}\{\theta_i + k\theta_i\vartheta_i\}}$.

Taking the limits n $\to \infty$ both sides we get $lim_{n\to\infty}\|\xi_n - p\| = 0$.

Theorem 2.2. Let C be a nonempty closed convex subset of a Banach space X and F: C $\to$ C be a contraction mapping. Let $\{t_n\}_{n=0}^{\infty}$ be an iterative sequence generated by D iteration process, with real sequences $\{\theta_n\}_{n=0}^{\infty}$ and $\{\vartheta_n\}_{n=0}^{\infty} \in$ [0 1] satisfying $\sum_{i=0}^{n}\{\theta_i + k\theta_i\vartheta_i\} = \infty$ and for all n $\in$ N. Then the D iterative process is T-stable.

Proof. Let $\{t_n\}_{n=0}^{\infty} \subset X$be an arbitratry sequence in C. Also, let the sequence generated by D iterative process be $t_{(n+1)} = f(T; tn)$ converging to unique fixed point p (follows from Theorem 2.1) and $\epsilon_n = \|t_{(n+1)-f(T;tn)}\|$. We will prove that

$lim_{n\to\infty}\epsilon_n = 0$ if and only if $lim_{n\to\infty}t_n = p$. Let $lim_{n\to\infty}\epsilon_n = 0$. Then, we have.

$\|t_{n+1} - p\| \leqslant \|t_{n+1} - f(T, t_n)\| + \|f(T, t_n) - p\| = \epsilon_n + \|t_{(n+1)-p}\|$.

From Theorem 2.1 we get $\leqslant \epsilon_n + k^3[1 - (\theta_n + k\theta_n\vartheta_n)(1-k)]\|(t_n - p)\|$. Since $0 < k < 1$ and $0 \leqslant \theta_n \leqslant 1, 0 \leqslant \vartheta_n \leqslant 1$ for all $n \in$ N and $lim_{n\to\infty}\epsilon_n = 0$ and using Lemma 1, we get $lim_{n\to\infty}\|t_n - p\| = 0$. Hence $lim_{n\to\infty}t_n = p$. Conversly, let $lim_{n\to\infty}t_n = p$. Then we have

$$\epsilon_n = \|t_{(n+1)-f(T,t_n)}\| \leqslant \|t_{n+1} - p\| + \|f(T, t_n) - p\|. \leqslant \|t_{n+1} - p\| + k^3[1 - (\theta_n + k\theta_n\vartheta_n)(1-k)]\|(\xi_n - p)\|.$$

Therefore, we have $lim_{n\to\infty}\epsilon_n = 0$. Hence the D iteration process is T-stable.

Theorem 2.3. Let $\tilde{F}$ be an approximate operator of a contraction mapping F. Let $\{\xi_n\}_{n=0}^{\infty}$ be an iterative sequence generated by D iteration Process for F and define an iterative sequence $\{\tilde{\xi}_n\}_{n=0}^{\infty}$ as follows

$$\begin{cases} \{\tilde{\xi}_0\} \in C, \\ \tilde{\omega}_n = \tilde{F}\Big((1-\vartheta_n)\tilde{\xi}_n + \vartheta_n \tilde{F}\tilde{\xi}_n\Big), \\ \tilde{\eta}_n = \tilde{F}\Big((1-\theta_n)\tilde{F}\tilde{\xi}_n + \theta_n \tilde{F}\tilde{\omega}_n\Big), \\ \tilde{\xi_{n+1}} = \tilde{F}\tilde{\eta}_n. \end{cases} \quad (2)$$

with real sequences $\{\theta_n\}_{n=0}^{\infty}$ and $\{\vartheta_n\}_{n=0}^{\infty} \in [0\ 1]$ satisfying (i).$\frac{1}{2} \leqslant \theta_n + k\theta_n\vartheta_n$ for all n $\in$ N, and (ii).$\Sigma_{n=0}^{\infty}\theta_n + k\theta_n\vartheta_n = \infty$. If F (p) = p and $\tilde{F}\tilde{p} = \tilde{p}$ such that $lim_{n\to\infty}\tilde{\xi}_n = \tilde{p}$, then we have

$$\|p - \tilde{p}\| \leqslant \frac{7\epsilon}{1-k},$$

where $\epsilon > 0$ is a fixed number.

Proof. It follows from (1) and (2),

$$\begin{aligned} \|\omega_n - \tilde{\omega}_n\| &= \|F((1-\vartheta_n)\xi_n + \vartheta_n F\xi_n) - \tilde{F}\Big((1-\vartheta_n)\tilde{\xi}_n + \vartheta_n \tilde{F}\tilde{\xi}_n\Big)\| \\ &\leqslant \|F((1-\vartheta_n)\xi_n + \vartheta_n F\xi_n) - F(1-\vartheta_n)\tilde{\xi}_n + \vartheta_n \tilde{F}\tilde{\xi}_n)\| \\ &+ \|F(1-\vartheta_n)\tilde{\xi}_n + \vartheta_n \tilde{F}\tilde{\xi}_n) - \tilde{F}(1-\vartheta_n)\tilde{\xi}_n + \vartheta_n \tilde{F}\tilde{\xi}_n) \\ &\leqslant k\Big[(1-\vartheta_n)\|\xi_n - \tilde{\xi}_n\| + \vartheta_n\|F\xi_n - \tilde{F}\tilde{\xi}_n\|\Big] + \epsilon \\ &\leqslant k\Big[(1-\vartheta_n)\|\xi_n - \tilde{\xi}_n\| + \vartheta_n\Big[\|F\xi_n - F\tilde{\xi}_n\| + \|F\tilde{\xi}_n - \tilde{F}\tilde{\xi}_n\|\Big]\Big] + \epsilon \\ &\leqslant k\Big[(1-\vartheta_n)\|\xi_n - \tilde{\xi}_n\| + \vartheta_n\Big[k\|\xi_n - \tilde{\xi}_n\| + \epsilon\Big]\Big] + \epsilon \\ &\leqslant k\{1 - \vartheta_n(1-k)\}\|\xi_n - \tilde{\xi}_n\| + k\vartheta_n\epsilon + \epsilon. \end{aligned}$$

Now,

$$\begin{aligned} \|\eta_n - \tilde{\eta}_n\| &= \|F((1-\theta_n)F\xi_n + \theta_n F\omega_n) - \tilde{F}\Big((1-\theta_n)\tilde{F}\tilde{\xi}_n + \theta_n \tilde{F}\tilde{w}_n\Big)\| \\ &\leqslant \|F((1-\theta_n)F\xi_n + \theta_n F\omega_n) - F\Big((1-\theta_n)\tilde{F}\tilde{\xi}_n + \theta_n \tilde{F}\tilde{\omega}_n\Big)\| \\ &+ \|F\Big((1-\theta_n)\tilde{F}\tilde{\xi}_n + \theta_n \tilde{F}\tilde{\omega}_n\Big) - \tilde{F}\Big((1-\theta_n)\tilde{F}\tilde{\xi}_n + \theta_n \tilde{F}\tilde{\omega}_n\Big)\| \\ &\leqslant k\Big[(1-\theta_n)\|F\xi_n - \tilde{F}\tilde{\xi}_n\| + \alpha_n\|F\omega_n - \tilde{F}\tilde{\omega}_n\|\Big] + \epsilon \\ &\leqslant k\Big[(1-\theta_n)\Big(k\|\xi_n - \tilde{\xi}_n\| + \epsilon\Big) + \theta_n(k\|\omega_n - \tilde{\omega}_n\| + \epsilon)\Big] + \epsilon \\ &\leqslant k\Big[(1-\theta_n)k\|\xi_n - \tilde{\xi}_n\| + \alpha_n k\|\omega_n - \tilde{\omega}_n\|\Big] + k\epsilon + \epsilon \\ &\leqslant k\Big[(1-\theta_n)k\|\xi_n - \tilde{\xi}_n\| + \theta_n k\Big(k\{1 - \vartheta_n(1-k)\}\|\xi_n - \tilde{\xi}_n\| + k\vartheta_n\epsilon + \epsilon\Big)\Big] + k\epsilon + \epsilon \\ &\leqslant k^2[1 - (\theta_n + k\theta_n\vartheta_n)(1-k)]\|\Big(\xi_n - \tilde{\xi}_n\Big)\| \\ &+ k\epsilon(\theta_n + k\theta_n\vartheta_n + 1) + \epsilon. \end{aligned}$$

Then,

$$\begin{aligned} \|\xi_{n+1} - \tilde{\xi_{n+1}}\| &= \|F\eta_n - \tilde{F}\tilde{\eta}_n\| \\ &\leqslant k\|\eta_n - \tilde{\eta}_n\| \\ &\leqslant \|\eta_n - \tilde{\eta}_n\| + \epsilon \\ &\leqslant k^3[1 - (\theta_n + k\theta_n\vartheta_n)(1-k)]\|\Big(\xi_n - \tilde{\xi}_n\Big)\| \\ &+ k^2\epsilon(\theta_n + k\theta_n\vartheta_n + 1) + k\epsilon + \epsilon \\ &\leqslant [1 - (\eta_n + k\eta_n\vartheta_n)(1-k)]\|\Big(\xi_n - \tilde{\xi}_n\Big)\| \\ &+ \epsilon(\theta_n + k\theta_n\vartheta_n) + 3\epsilon \\ &\leqslant [1 - (\theta_n + k\theta_n\vartheta_n)(1-k)]\|\Big(\xi_n - \tilde{\xi}_n\Big)\| \\ &+ \epsilon(\theta_n + k\theta_n\vartheta_n) + 3(1 - (\theta_n + k\theta_n\vartheta_n) + \theta_n + k\theta_n\vartheta_n)\epsilon. \end{aligned}$$

By using assumption (i) $\frac{1}{2} \leqslant \theta_n + k\theta_n\vartheta_n$. Then,

$$\begin{aligned} \|\xi_{n+1} - \tilde{\xi_{n+1}}\| &\leqslant [1 - (\theta_n + k\theta_n\vartheta_n)(1-k)]\|\Big(\xi_n - \tilde{\xi}_n\Big)\| + 7(\theta_n + k\theta_n\vartheta_n)\epsilon \\ &= [1 - (\theta_n + k\theta_n\vartheta_n)(1-k)]\|\Big(\xi_n - \tilde{\xi}_n\Big)\| \\ &\quad + \Big(\theta_n + k\theta_n\vartheta_n(1-k)\frac{7\epsilon}{(1-k)} \end{aligned}$$

Let $r_n = \|\Big(\xi_n - \tilde{\xi}_n\Big)\|, t_n = \Big(\theta_n + k\theta_n\vartheta_n(1-k), t_n = \frac{7\epsilon}{(1-k)}$, then by using Lemma 2, we get

$$0 \leqslant \limsup_{n\to\infty} \|\xi_n - \tilde{\xi}_n\| \leqslant \limsup_{n\to\infty} \frac{7\epsilon}{(1-k)}.$$

By using Theorem 2.1 and by using assumption $lim_{n\to\infty}\tilde{\xi}_n = \tilde{p}$, we have

$$\|p - \tilde{p}\| \leqslant \frac{7\epsilon}{(1-k)},$$

as required.

Now, by an example we show that initial guess doesn't effect the efficiency of D iteration process.

Example 2.4. Let us define a function $F : R \to R$ by $F(\xi) = (4\xi + 2)/5$. Then clearly F is a contraction mapping. Let $\theta_n = 2n/(3n+1)$ and $\vartheta_n = 3n/(4n+1)$. The iterative values for $\xi_0 = 3.5$ are given in Table 1. Fig. 1 shows the conver-

Table 1 equence generated by D, Picard-S, S iteration process with initial guess $x_0 = 3.5$ for mapping F of Example 2.4.

	S	Picard-S	D
ξ_0	3.5	3.5	3.5
ξ_1	3.2	2.96	2.768
ξ_2	2.9024	2.57754	2.33502
ξ_3	2.66692	2.34146	2.14147
ξ_4	2.48921	2.20038	2.05893
ξ_5	2.35737	2.1171	2.02436
ξ_6	2.26037	2.06825	2.01002
ξ_7	2.18935	2.03971	2.00028
ξ_8	2.13752	2.02307	2.00168
ξ_9	2.09977	2.01339	2.00005
ξ_{10}	2.07233	2.00777	2.00028
ξ_{11}	2.05248	2.00456	2.00011
ξ_{12}	2.03794	2.00261	2.00005
ξ_{13}	2.02746	2.00151	2.00002
ξ_{14}	2.01987	2.00087	2.00001
ξ_{15}	2.01437	2.00051	2
ξ_{16}	2.01039	2.00029	2
ξ_{17}	2.00751	2.00017	2
ξ_{18}	2.00543	2.0001	2
ξ_{19}	2.00392	2.00006	2
ξ_{20}	2.00283	2.00003	2

Figures. Convergence of D, Picard-S, S iteration processes to the fixed point 2 by using different initial guess for mapping F of Example.

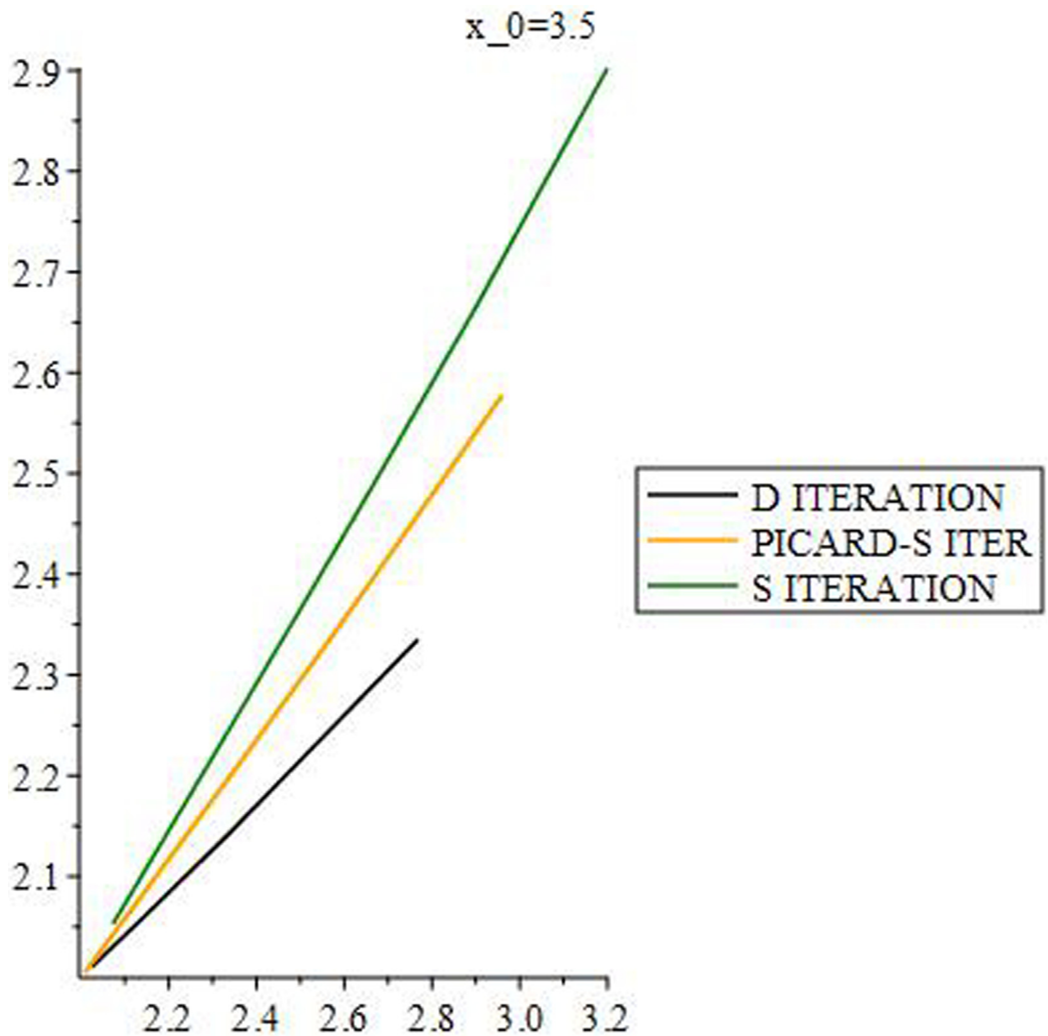

Fig. 1 Convergence of D iteration process when initial guess is 3.5.

gence graph. The efficiency of D iteration process is clear. (see also Figs. 2–4, for different initial values.).

4. Error estimation for D iteration process

Throughout this section, we assume that $(X, |.|)$ is an arbitrary real Banach space, C is a closed and convex non-empty space a subset of X, F: C → C is a non-expansive mapping, and $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty} \in [0\ 1]$ are sequences of parameters satisfying certain control conditions.

We basically want to evaluate the error estimation of the D iterative method in real Banach space, which is defined as

$$\begin{cases} x_0 \in C, \\ z_n = F((1-\beta_n)x_n + \beta_n F x_n), \\ y_n = F((1-\alpha_n)Fx_n + \alpha_n F z_n), \\ x_{n+1} = F y_n. \end{cases} \quad (3)$$

Many researchers have achieved this goal indirectly. As regards, their direct calculations (estimation, recently some papers have appeared in the literature (see, e.g., [14,15]. In this article, we have developed new ideas for direct estimation of the error of D iteration with regard to accumulation. It is point out that the direct error calculations for this method are much more complicated than those of the case of the iteration methods of Mann and Ishikawa (cf. [24,25]).

Define the errors of Fx_n, Fy_n and Fz_n by

$$p_n = Fx_n - \overline{Fx_n}, q_n = Fy_n - \overline{Fy_n}, r_n = Fz_n - \overline{Fz_n}$$

for all $n \in N$, where $\overline{Fx_n}, \overline{Fy_n}$ and $\overline{Fz_n}$ are the exact values of Fx_n, Fy_n and Fz_n, respectively, that is, Fx_n, Fy_n and Fz_n are approximate values of $\overline{Fx_n}, \overline{Fy_n}$ and $\overline{Fz_n}$, respectively. The theory of errors implies that $\{p_n\}_{n=0}^{\infty}, \{q_n\}_{n=0}^{\infty}$ and $\{r_n\}_{n=0}^{\infty}$ are bounded. Set

$$M = \max\{M_p, M_q, M_r\}$$

where $M_p = \sup_{n\in N}\|p_n\|, M_q = \sup_{n\in N}\|q_n\|$ and $M_r = \sup_{n\in N}\|r_n\|$ are the bounds on the absolute errors of $\{Fx_n\}_{n=0}^{\infty}, \{Fy_n\}_{n=0}^{\infty}$ and $\{Fz_n\}_{n=0}^{\infty}$ respectively.

The accumulated errors in (3) comes from p_n, q_n and r_n, hence we can set

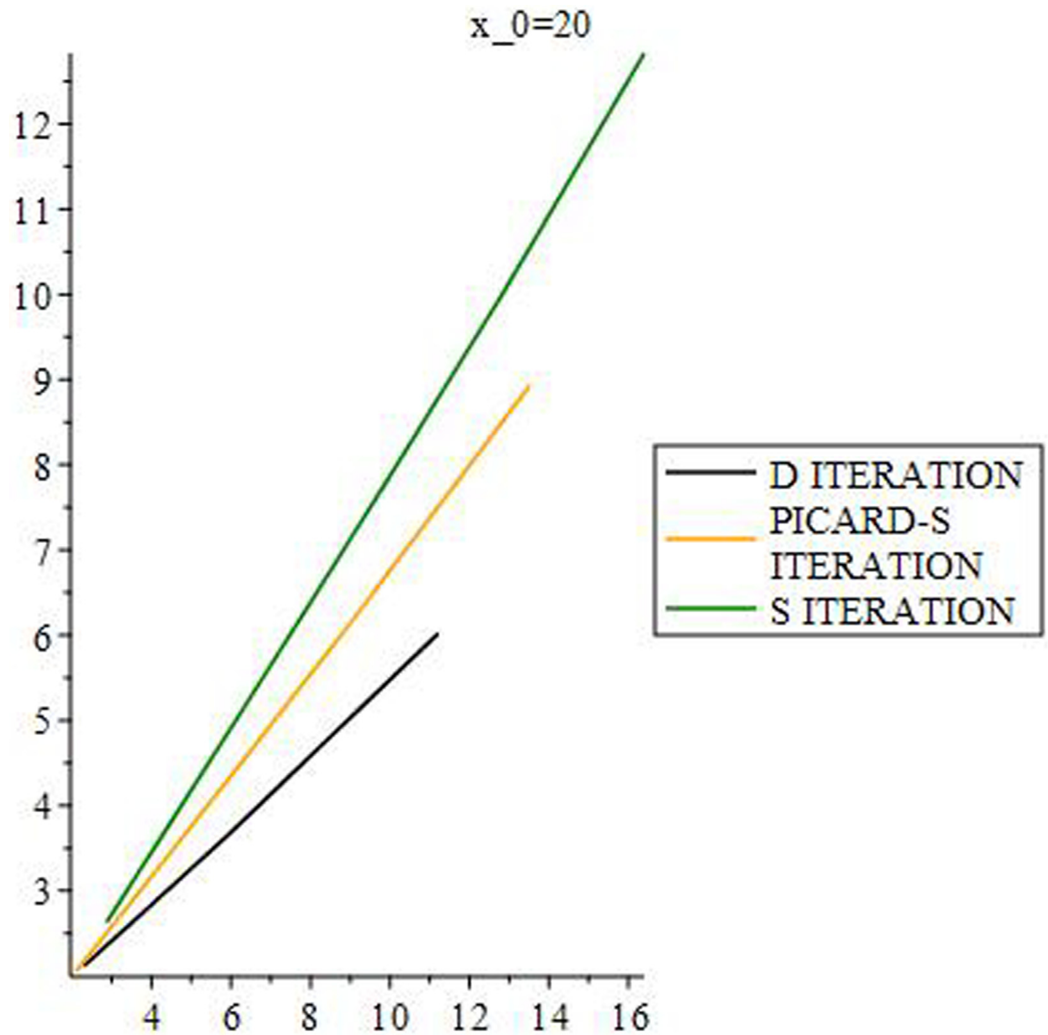

Fig. 2 Convergence of D iteration process when initial guess is 20.

$$\begin{cases} \overline{x_0} \in C, \\ \overline{z_n} = \overline{F}\big((1-\beta_n)\overline{x_n} + \beta_n\overline{Fx_n}\big), \\ \overline{y_n} = \overline{F}\big((1-\alpha_n)\overline{Fx_n} + \alpha_n\overline{Fz_n}\big), \\ \overline{x_{n+1}} = \overline{Fy_n}. \end{cases} \tag{4}$$

where $\overline{x_n}, \overline{y_n}$ and $\overline{z_n}$ are exact values of x_n, y_n and z_n, respectively. Obviously, error of an iteration will affect the next (n + 1) steps. So, we have

$$\|x_0\| = \|\overline{x_0}\|,$$

$$\|x_1 - \overline{x_1}\| = (1-\alpha_0)\|p_0\| + \alpha_0\|r_0\| + 2\acute{\epsilon},$$

$$\|x_1 - \overline{x_1}\| = (1-\alpha_0)\|p_0\| + \alpha_0\|r_0\| + \epsilon,$$

similarly

$$\|x_2 - \overline{x_2}\| = (1-\alpha_1)\|p_1\| + \alpha_1\|r_1\| + \epsilon,$$

$$\|x_3 - \overline{x_3}\| = (1-\alpha_2)\|p_2\| + \alpha_2\|r_2\| + \epsilon,$$

repeating on above process, we have

$$\|x_{n+1} - \overline{x_n + 1}\| = (1-\alpha_n)\|p_n\| + \alpha_n\|r_n\| + \epsilon$$

.

Now,

$$\|y_0 - \overline{y_0}\| = (1-\alpha_0)\|p_0\| + \alpha_0\|r_0\| + \epsilon,$$

$$\|y_1 - \overline{y_1}\| = (1-\alpha_1)\|p_1\| + \alpha_1\|r_1\| + \epsilon,$$

$$\|y_2 - \overline{y_2}\| = (1-\alpha_2)\|p_2\| + \alpha_2\|r_2\| + \epsilon,$$

$$\|y_3 - \overline{y_3}\| = (1-\alpha_3)\|p_3\| + \alpha_3\|r_3\| + \epsilon,$$

repeating on above process, we have

$$\|y_n - \overline{y_n}\| = (1-\alpha_n)\|p_n\| + \alpha_n\|r_n\| + \epsilon.$$

Now,

$$\|z_0 - \overline{z_0}\| = \beta_0\|p_0\| + \epsilon,$$

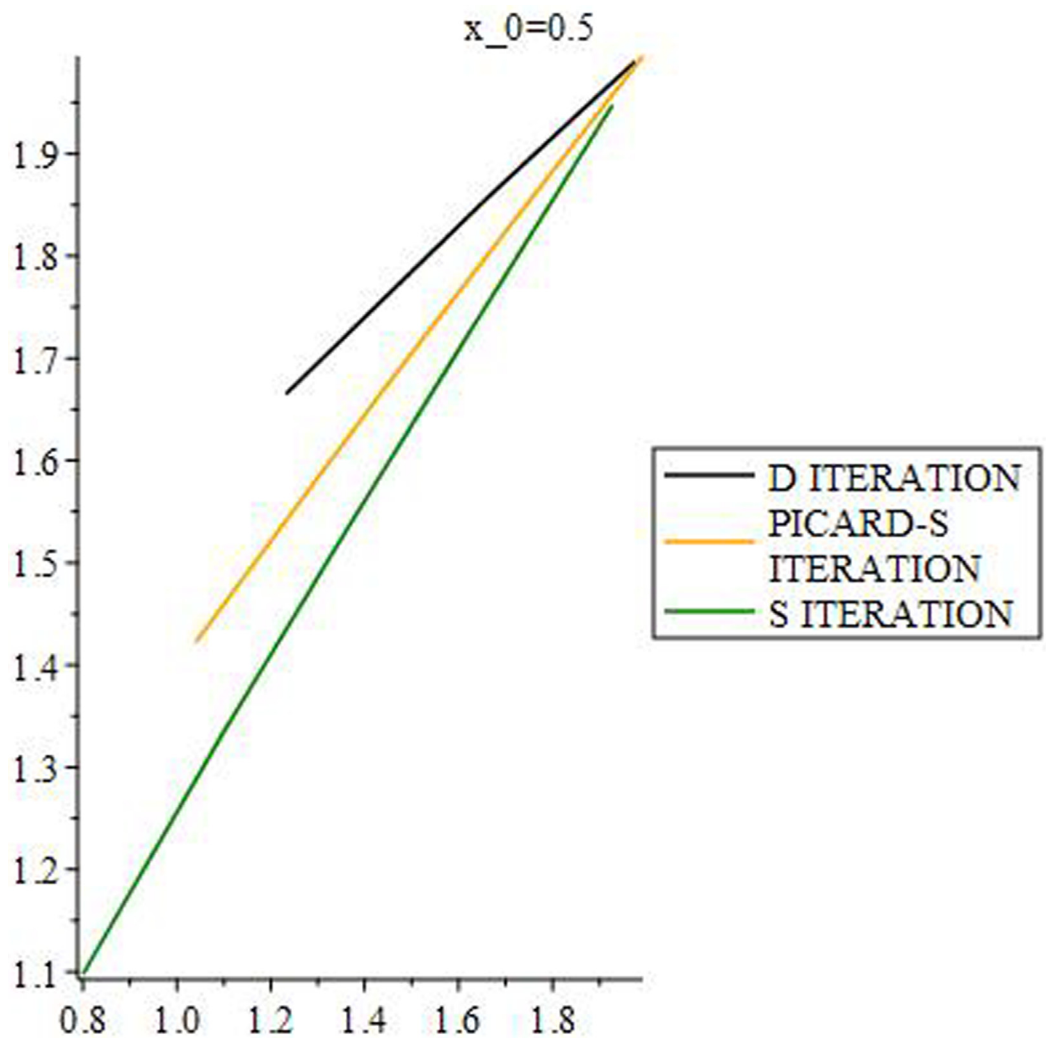

Fig. 3 Convergence of D iteration process when initial guess is 0.5.

$$\|z_1 - \overline{z_1}\| = (1-\beta_1)[(1-\alpha_0)\|p_0\| + \alpha_0\|r_0\| + \epsilon] + \beta_1\|p_1\| + \epsilon,$$

$$\|z_2 - \overline{z_2}\| = (1-\beta_2)[(1-\alpha_1)\|p_1\| + \alpha_1\|r_1\| + \epsilon] + \beta_2\|p_2\| + \epsilon,$$

$$\|z_3 - \overline{z_3}\| = (1-\beta_3)[(1-\alpha_2)\|p_2\| + \alpha_2\|r_2\| + \epsilon] + \beta_3\|p_3\| + \epsilon,$$

repeating on above process, we have

$$\|z_n - \overline{z_n}\| = (1-\beta_n)\|x_n - \overline{x_n}\| + \beta_n\|p_n\| + \epsilon.$$

Define

$$E_n^{(1)} := \|x_{n+1} - \overline{x_n + 1}\| = (1-\alpha_n)\|p_n\| + \alpha_n\|r_n\| + \epsilon,$$

$$E_n^{(1)} := \|y_n - \overline{y_n}\| = (1-\alpha_n)\|p_n\| + \alpha_n\|r_n\| + \epsilon,$$

and

$E_n^{(3)} := \|z_n - \overline{z_n}\| = (1-\beta_n)\|x_n - \overline{x_n}\| + \beta_n\|p_n\| + \epsilon$ for all $n \in N$.

We noticed that after (n + 1) iterations, the error of the iterative method accumulated to $E_n^{(1)}, E_n^{(2)}$ and $E_n^{(3)}$.

Now, we can give the following results.

Theorem 3.1. Let C, F, M, $E_n^{(1)}, E_n^{(2)}$ and $E_n^{(3)}$ be as above, and ϵ be a fixed value.

If $\sum_{i=0}^{\infty}\alpha_i = +\infty$(or $\sum_{i=0}^{\infty}\beta_i = +\infty$) then the accumulation of errors in (3) is bounded and does not exceed the number N.

Proof. It follows from above definitions and conditions

$$\begin{aligned}\|E_n^{(1)}\| &= \|(1-\alpha_n)\|p_n\| + \alpha_n\|r_n\| + 2\epsilon\|,\\ &\leqslant (1-\alpha_n)\|p_n\| + \alpha_n\|r_n\| + 2\epsilon,\\ &\leqslant (1-\alpha_n)M + \alpha_n M + 2\epsilon,\\ &\leqslant M + \epsilon \leqslant N'.\end{aligned}$$

Also,

$$\begin{aligned}\|E_n^{(2)}\| &= \|(1-\alpha_n)\|p_n\| + \alpha_n\|r_n\| + 2\epsilon\|,\\ &\leqslant (1-\alpha_n)\|p_n\| + \alpha_n\|r_n\| + 2\epsilon,\\ &\leqslant (1-\alpha_n)M + \alpha_n M + 2\epsilon,\\ &\leqslant M + \epsilon\\ &\leqslant N'\end{aligned}$$

and

$$\|E_n^{(3)}\| = \|(1-\beta_n)\|x_n - \overline{x_n}\| + \beta_n\|p_n\| + \epsilon\|$$

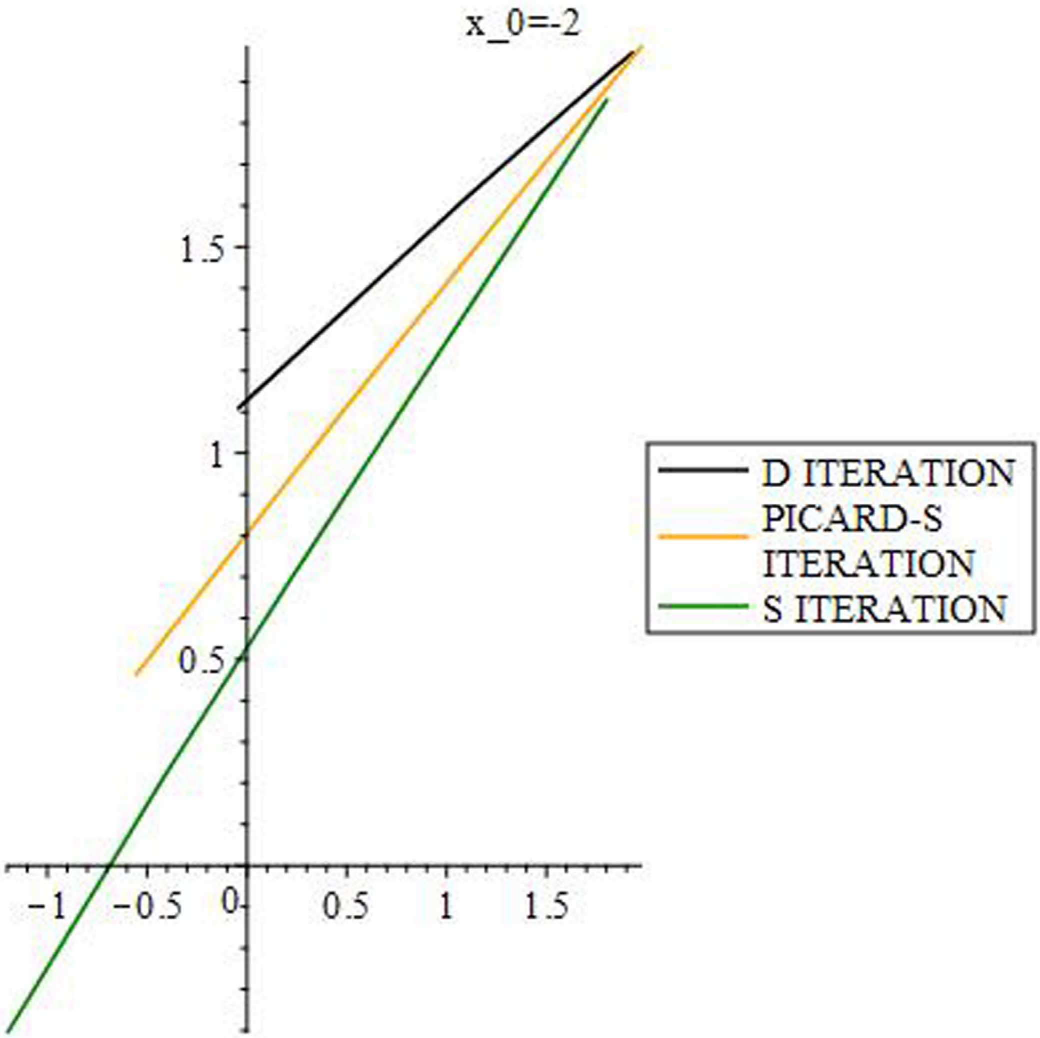

Fig. 4 Convergence of D iteration process when initial guess is -2.

$$\|E_n^{(3)}\| \leqslant (1-\beta_n)\|\|E_{n-1}^{(1)}\| + \beta_n\|p_n\| + \epsilon$$

$$\|E_n^{(3)}\| \leqslant (1-\beta_n)N' + \beta_n M + \epsilon$$

$$\|E_n^{(3)}\| \leqslant N' + \beta_n(N' + M) + \epsilon$$

$\|E_n^{(3)}\| \leqslant N$ Hence, we have $\max_{n\in N}\left[\|E_n^{(1)}\|, \|E_n^{(2)}\|, \|E_n^{(3)}\|\right] \leqslant N$.

5. Conclusion

For the iteration method define in [14–19,21–25], imposing specific conditions on parametric sequences is a common practice $\{\eta_n\}_{n=0}^{\infty}, \{\theta_n\}_{n=0}^{\infty}$ and $\{\vartheta_n\}_{n=0}^{\infty}$, like $\sum_{i=0}^{\infty}\{\eta_n\}_{n=0}^{\infty} = \infty, \sum_{i=0}^{\infty}\{\theta_n\}_{n=0}^{\infty} = \infty$ and $\sum_{i=0}^{\infty}\{\vartheta_n\}_{n=0}^{\infty} = \infty$ for all n $\in$ N to obtain convergence, stability, data dependency results and direct error estimation for general iteration methods. None of these conditions has been used in our corresponding results. Therefore, Our result is effective or efficient result in terms of all the above references. Furthermore, we also proved that any choice of initial guess does not effect the efficiency of D iteration process.

Author Contributions

All authors contributed equally and significantly in writing this article. All authors have read and agreed to the published version of the manuscript.

Declaration of Competing Interest

The authors declare that they have no competing interests.

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