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The analytical investigation of time-fractional multi-dimensional Navier–Stokes equation

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Abstract In the present research article, we implemented two well-known analytical techniques to solve fractional-order multi-dimensional Navier–Stokes equation. The proposed methods are the modification of Adomian decomposition method and variational iteration method by using natural transformation. Furthermore, some illustrative examples are presented to confirm the validity of the suggested methods. The solutions graphs and tables are constructed for both fractional and integer-order problems. It is investigated that the suggested techniques have the identical solutions of the problems. The solution comparison via graphs and tables have also supported the greater accuracy and higher rate of convergence of the present methods.

1. Introduction

In 1822 a popular governing equation of viscus fluid flow movement was obtained, called the Navier–Stokes (NS) equation. This equation can be termed as the second law of Newton's motion for fluid, and is a mixture of continuity equations, energy equations and moment equations. Navier–Stokes equations are useful in describing the physics of many scientific and engineering phenomena of interest. This equation identifies several physical things around the wings of the aircraft, such as liquid flow in pipes, blood flow and air flow [1–5]. The Navier–Stokes equation create the link between

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pressure and fluid-acting external forces to the fluid flow response [6–9]. The Navier–Stokes equation and classical fluid dynamics have been extremely effective in gaining quantitative knowledge of shock waves, turbulence, and solitons [10,11]. In many significant phenomena such as their thermodynamics, aeronautical sciences, geophysics, the petroleum industry, plasma physics and so on, Navier-Stoke's equations, provides a natural description of the interaction of a viscous fluid with a rigid body, and are considered as important computational tools for greater understanding of a number of real problems [12,13].

Fractional calculus is a general expansion of the calculus of integer order to arbitrary order and was described previously in a letter between Leibniz and L'Hospital mathematicians in 1695. Due to its distinctive capacity to explain anomalous behavior and memory impacts, which are the vital features of complicated phenomena, fractional calculus is increasingly placed to enhance current mathematical models [14–16]. The mathematical basis for fractional order derivatives was set by the combine attempts of researchers like Caputo, Riemann, Liouville, Ross and Miller, Podlubny, and others. Fractional-order calculus theory was connected to practical applications and applied to the theory of chaos, electrodynamics, signal processing, thermodynamics, economics and other fields [17–21].

In fractional calculus, we often model different physical phenomena rather in a sophisticated manner as compare to ordinary calculus. The present method is based on the direct implementation of the natural transformation on the Caputo defined fractional-order derivatives. At the end of the proposed algorithm, we get the solution of the fractional-order Navier–Stokes equation in terms of the given fractional order. As a result, we can get different solutions at different fractional-order of the Navier–Stokes equations. The contribution of the present methods is that, we can analyze different dynamics of the Navier–Stokes equations by using different fractional order derivative in the model. We can choose an optimal fractional-order to obtain a solution which is in close contact with the exact solution of the problem.

In this work, we consider a time-fractional NS equation for an incompressible fluid flow of density ρ, ϕ is density and kinematic viscosity $\upsilon = \frac{\phi}{\rho}$. It is indicated as

$$\begin{cases} D_\eta^\delta V + (V.\nabla)V = \rho\nabla^2 V - \frac{1}{\rho}\nabla p, \\ \nabla.V = 0, \\ V = 0, \quad on \quad \Omega \times (0, T) \end{cases}$$

Here, $V = (\mu, v, \omega), p$ and η represent fluid vector, pressure and time, respectively. (α, β, γ) represent the spatial components in Ω.

The above equations can also be defined as

$$D_\eta^\delta(\mu) + \mu\frac{\partial\mu}{\partial\alpha} + v\frac{\partial\mu}{\partial\beta} + \omega\frac{\partial\mu}{\partial\gamma} = \rho\left[\frac{\partial^2\mu}{\partial\alpha^2} + \frac{\partial^2\mu}{\partial\beta^2} + \frac{\partial^2\mu}{\partial\gamma^2}\right] - \frac{1}{\rho}\frac{\partial g}{\partial\alpha},$$
$$D_\eta^\delta(v) + \mu\frac{\partial v}{\partial\alpha} + v\frac{\partial v}{\partial\beta} + \omega\frac{\partial v}{\partial\gamma} = \rho\left[\frac{\partial^2 v}{\partial\alpha^2} + \frac{\partial^2 v}{\partial\beta^2} + \frac{\partial^2 v}{\partial\gamma^2}\right] - \frac{1}{\rho}\frac{\partial g}{\partial\beta},$$
$$D_\eta^\delta(\omega) + \mu\frac{\partial\omega}{\partial\alpha} + v\frac{\partial\omega}{\partial\beta} + \omega\frac{\partial\omega}{\partial\gamma} = \rho\left[\frac{\partial^2\omega}{\partial\alpha^2} + \frac{\partial^2\omega}{\partial\beta^2} + \frac{\partial^2\omega}{\partial\gamma^2}\right] - \frac{1}{\rho}\frac{\partial g}{\partial\gamma},$$

Here, some significant contribution from the researchers are discussed; see, for example, Herrmann [22] and Hilfer [23], particularly fractional partial differential equations such as time-fractional Navier–Stokes equations, which was derived by applications in various areas of science and engineering. El-Shahed and Salem first performed fractional modeling of NS equations in 2005 [24]. The authors [24] used Laplace transformation, finite Hankel transformation and finite Fourier Sine transformation to generalize the classical NS equations. Kumar et al. [25] analytically solved a nonlinear fractional model of NS equation by combining HPM and LTA. Ganji et al. [26] and Ragab et al. [27] have resolved the non-linear time fractional NS equation by implementing homotopy analysis method. Odibat [28], Momani, and Birajdar [29] have introduced adomian decomposition method (ADM) for the numerical algorithms of the time fractional NS equation. Analytical solution of time-fractional NS equation is achieved by Kumar et al. [30] using combination of ADM and Laplace transform while Chaurasia and Kumar [31] have solved the same equation by combining finite Hankel transform and Laplace transform.

In 2014, M. Rawashdeh and S.Maitama first introduced the Natural Decomposition Method (NDM) [32,33] to solve linear and nonlinear ODEs and PDEs. A large number of physical problems have been studied by using NDM, such as the study of the fractional telegraph equation [34], fractional-order Whitham-Broer-Kaup equations [35] fractional-order heat and wave equations [36], non-linear PDEs [37,38], the fractional uncertain flow of a system of polytropic gas [39], fractional physical models [40], fractional-order PDE's with proportional delay [41] and fractional-order diffusion equations [42].

The Natural variational iteration method (NVIM) is a combination of the variational iteration technique and the Natural transformation method (NTM). This technique enables us to achieve the problems that arise in the identification of the general Lagrange multiplier [43]. The Lagrange multiplier technique [44] has been commonly used to address a number of nonlinear issues that occur in mathematical physics and other related fields and has been developed into a strong analytical method, i.e. the method of variational iteration [45,46] to solve differential equations. Non-linear time-fractional wave-like equations with variable coefficients using NVIM [47].

The present manuscript is concerned with the analytical solution of fractional-order Navier–Stokes equations. The solution of the classical Navier–Stokes equations is a topic for the researchers since long. Recently the analytical solutions of fractional-order Navier–Stokes equation is the main focus of the researchers and mathematicians. This was the challenging work to extend or develop the existing techniques for the solutions of fractional-order Navier–Stokes equations. Many of them have got success and developed innovative techniques to solve fractional-order Navier–Stokes. In this regard, the current research work is a novel contribution towards the analytical solution of fractional-order Navier–Stokes equations. In this work, we not only implemented two analytical techniques namely NVIM and NDM, but also done their comparison and confirmed the applicability of the proposed algorithms. The present research work is conducted in a very simple and straightforward manner to achieve the analytical solutions of the targeted problems with a small amount of numerical calculations. The convergence of the proposed methods is trivial. In conclusion the proposed techniques are considered to be the sophisticated contribution towards the analytical solution of fractional-order partial differential equations which are frequently arising in science and engineering.

2. Preliminaries concepts

2.1. Definition

The natural transform of the function $\bar{f}(\eta)$ is defined by $N[\bar{f}(\eta)]$ for $\eta \in R$ and is denoted by the function $\bar{f}(\eta) \in R$.

$$N[\bar{f}(\eta)] = \overline{\mathcal{G}}(s,u) = \int_{-\infty}^{\infty} e^{-s\eta}\bar{f}(\eta)d\eta; \quad s,u \in (-\infty,\infty),$$

where the natural transform variables are s and u. If $\bar{f}(\eta)Q(\eta)$ is expressed on the actual positive axis, the natural transform is defined as

$$N[\bar{f}(\eta)Q(\eta)] = N^{+}[\bar{f}(\eta)] = \overline{\mathcal{G}}^{+}(s,u) = \int_{0}^{\infty} e^{-s\eta}\bar{f}(\eta)d\eta;$$
$$s,u \in (0,\infty), \quad and \quad \eta \in R$$

where $Q(\eta)$ describes the Heaviside function. Actually, for $u = 1$,the equation is reduced to the transform of Laplace, and for $s = 1$,is the Sumud transform.

2.2. Theorem

Let $\overline{\mathcal{G}}(s,u)$ be the natural transform of the $\bar{f}(\eta)$ variable, then the natural transform $\overline{\mathcal{G}}_{\delta}(s,u)$ of the Riemann–Liouville fractional derivative of $\bar{f}(\eta)$ is described by $D_{\delta}\bar{f}(\eta)$ and is represented as

$$N^{+}\left[D^{\delta}\bar{f}(\eta)\right] = \overline{\mathcal{G}}_{\delta}(s,u) = \frac{s^{\delta}}{u^{\delta}}\overline{\mathcal{G}}(s,u) - \sum_{j=0}^{m-1}\frac{s^{j}}{u^{\delta-j}}\left[D^{\delta-j-1}\bar{f}(\eta)\right]_{\eta=0}$$
$$m-1 \leqslant \delta < m$$

2.3. Theorem

Let $\overline{\mathcal{G}}(s,u)$ be the natural transformation of $\bar{f}(\eta)$, then the natural transformation $\overline{\mathcal{G}}_{\delta}(s,u)$ of the Caputo fractional derivative of $\bar{f}(\eta)$ is represented by ${}^{0}cD_{\delta}\bar{f}(\eta)$ and described as

$$N^{+}\left[{}^{c}D^{\delta}\bar{f}(\eta)\right] = \overline{\mathcal{G}}_{\delta}^{c}(s,u) = \frac{s^{\delta}}{u^{\delta}}\overline{\mathcal{G}}(s,u) - \sum_{j=0}^{m-1}\frac{s^{\delta-(j+1)}}{u^{\delta-j}}\left[D^{j}\bar{f}(\eta)\right]_{\eta=0}$$
$$m-1 \leqslant \delta < m$$

2.4. Definition

Caputo operator of fractional partial derivative

$$D_{\eta}^{\delta}\bar{f}(\eta) = \begin{cases} \frac{\partial^{m}\bar{f}(\eta)}{\partial\eta^{m}}, & \delta = m \in N, \\ \frac{1}{\Gamma(m-\delta)}\int_{0}^{\eta}(\eta-\phi)^{m-\delta-1}g^{m}(\phi)\partial\phi, \\ m-1 < \gamma < m \end{cases}$$

2.5. Definition

Function of Mittag–Leffler, $E_{\delta}(z)$ for $\delta > 0$ is defined as

$$E_{\delta}(z) = \sum_{m=0}^{\infty}\frac{z^{m}}{\Gamma(\delta m+1)}\delta > 0 z \in \mathbb{C}$$

3. The procedure of NVIM

This section describes the NVIM solution system for fractional partial differential equations.

$$\begin{aligned} &D_{\eta}^{\delta}\mu(\alpha,\eta) + \overline{\mathcal{G}}_{1}(\mu,\nu) + \mathcal{N}_{1}(\mu,\nu) - \mathcal{P}_{1}(\alpha,\eta) = 0, \\ &D_{\eta}^{\delta}\nu(\alpha,\eta) + \overline{\mathcal{G}}_{2}(\mu,\nu) + \mathcal{N}_{2}(\mu,\nu) - \mathcal{P}_{2}(\alpha,\eta) = 0, \\ &m-1 < \delta \leqslant m, \end{aligned} \tag{1}$$

with initial conditions

$$\mu(\alpha,0) = g_{1}(\alpha), \quad \nu(\alpha,0) = g_{2}(\alpha). \tag{2}$$

where is $D_{\eta}^{\delta} = \frac{\partial^{\delta}}{\partial\eta^{\delta}}$ the Caputo fractional derivative of order $\delta, \overline{\mathcal{G}}_{1}, \overline{\mathcal{G}}_{2}$ and $\mathcal{N}_{1}, \mathcal{N}_{2}$ are linear and non-linear functions, respectively, and $\mathcal{P}_{1}, \mathcal{P}_{2}$ are source operators.

The natural transformation is applied to Eq. (1),

$$\begin{aligned} &N^{+}\left[D_{\eta}^{\delta}\mu(\alpha,\eta)\right] + N^{+}\left[\overline{\mathcal{G}}_{1}(\mu,\nu) + \mathcal{N}_{1}(\mu,\nu) - \mathcal{P}_{1}(\alpha,\eta)\right] = 0, \\ &N^{+}\left[D_{\eta}^{\delta}\nu(\alpha,\eta)\right] + N^{+}\left[\overline{\mathcal{G}}_{2}(\mu,\nu) + \mathcal{N}_{2}(\mu,\nu) - \mathcal{P}_{2}(\alpha,\eta)\right] = 0, \end{aligned} \tag{3}$$

Using the Natural Transform differentiation property, we get

$$\begin{aligned} &N^{+}[\mu(\alpha,\eta)] - \sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^{k}\mu(\alpha,\eta)}{\partial^{k}\eta}|_{\eta=0} \\ &= -N^{+}\left[\overline{\mathcal{G}}_{1}(\mu,\nu) + \mathcal{N}_{1}(\mu,\nu) - \mathcal{P}_{1}(\alpha,\eta)\right], \\ &N^{+}[\nu(\alpha,\eta)] - \sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^{k}\nu(\alpha,\eta)}{\partial^{k}\eta}|_{\eta=0} \\ &= -N^{+}\left[\overline{\mathcal{G}}_{2}(\mu,\nu) + \mathcal{N}_{2}(\mu,\nu) - \mathcal{P}_{2}(\alpha,\eta)\right], \end{aligned} \tag{4}$$

The iteration technique for the Eq. (4) may be used to indicate the major iterative system requiring the Lagrange multiplier as

$$\begin{aligned} &N^{+}[\mu_{m+1}(\alpha,\eta)] = N^{+}[\mu_{m}(\alpha,\eta)] \\ &+\lambda(s)\left[\frac{s^{\delta}}{u^{\delta}}\mu_{m}(\alpha,\eta) - \sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^{k}\mu(\alpha,\eta)}{\partial^{k}\eta}|_{\eta=0} - N^{+}[\mathcal{P}_{1}(\alpha,\eta)] - N^{+}\left\{\overline{\mathcal{G}}_{1}(\mu,\nu) + \mathcal{N}_{1}(\mu,\nu)\right\}\right], \\ &N^{+}[\nu_{m+1}(\alpha,\eta)] = N^{+}[\nu_{m}(\alpha,\eta)] \\ &+\lambda(s)\left[\frac{s^{\delta}}{u^{\delta}}\nu_{m}(\alpha,\eta) - \sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^{k}\nu(\alpha,\eta)}{\partial^{k}\eta}|_{\eta=0} - N^{+}[\mathcal{P}_{2}(\alpha,\eta)] - N^{+}\left\{\overline{\mathcal{G}}_{2}(\mu,\nu) + \mathcal{N}_{2}(\mu,\nu)\right\}\right], \end{aligned} \tag{5}$$

A Lagrange multiplier as

$$\lambda(s) = -\frac{u^{\delta}}{s^{\delta}}, \tag{6}$$

using inverse Natural transformation N^{-}, Eq. (5) can be written as

$$\begin{aligned} &\mu_{m+1}(\alpha,\eta) = \mu_{m}(\alpha,\eta) \\ &-N^{-}\left[\frac{u^{\delta}}{s^{\delta}}\left[\sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^{k}\mu(\alpha,\eta)}{\partial^{k}\eta}|_{\eta=0} - N^{+}[\mathcal{P}_{1}(\alpha,\eta)] - N^{+}\left\{\overline{\mathcal{G}}_{1}(\mu,\nu) + \mathcal{N}_{1}(\mu,\nu)\right\}\right]\right], \\ &\nu_{m+1}(\alpha,\eta) = \nu_{m}(\alpha,\eta) \\ &-N^{-}\left[\frac{u^{\delta}}{s^{\delta}}\left[\sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^{k}\nu(\alpha,\eta)}{\partial^{k}\eta}|_{\eta=0} - N^{+}[\mathcal{P}_{2}(\alpha,\eta)] - N^{+}\left\{\overline{\mathcal{G}}_{2}(\mu,\nu) + \mathcal{N}_{2}(\mu,\nu)\right\}\right]\right], \end{aligned} \tag{7}$$

the initial iteration can be find as

$$\mu_0(\alpha,\eta)=N^-\left[\frac{u^\delta}{s^\delta}\left\{\sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^k\mu(\alpha,\eta)}{\partial^k\eta}|_{\eta=0}\right\}\right],$$
$$v_0(\alpha,\eta)=N^-\left[\frac{u^\delta}{s^\delta}\left\{\sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^k v(\alpha,\eta)}{\partial^k\eta}|_{\eta=0}\right\}\right]. \quad (8)$$

The converges of this technique is shown in [48,49].

4. The procedure of NDM

We describe in this section the NDM solution for system of partial differential fractional equations.

$$D_\eta^\delta\mu(\alpha,\eta)+\overline{\mathcal{G}}_1(\mu,v)+\mathcal{N}_1(\mu,v)-\mathcal{P}_1(\alpha,\eta)=0,$$
$$D_\eta^\delta v(\alpha,\eta)+\overline{\mathcal{G}}_2(\mu,v)+\mathcal{N}_2(\mu,v)-\mathcal{P}_2(\alpha,\eta)=0, \quad (9)$$
$$m-1<\delta\leqslant m,$$

with initial conditions

$$\mu(\alpha,0)=g_1(\alpha),\quad v(\alpha,0)=g_2(\alpha). \quad (10)$$

where is $D_\eta^\delta=\frac{\partial^\delta}{\partial\eta^\delta}$ the Caputo fractional derivative of order δ, $\overline{\mathcal{G}}_1,\overline{\mathcal{G}}_2$ and $\mathcal{N}_1,\mathcal{N}_2$ are linear and non-linear functions, respectively, and $\mathcal{P}_1,\mathcal{P}_2$ are source operators.

The natural transformation is applied to Eq. (9),

$$N^+\left[D_\eta^\delta\mu(\alpha,\eta)\right]+N^+\left[\overline{\mathcal{G}}_1(\mu,v)+\mathcal{N}_1(\mu,v)-\mathcal{P}_1(\alpha,\eta)\right]=0,$$
$$N^+\left[D_\eta^\delta v(\alpha,\eta)\right]+N^+\left[\overline{\mathcal{G}}_2(\mu,v)+\mathcal{N}_2(\mu,v)-\mathcal{P}_2(\alpha,\eta)\right]=0. \quad (11)$$

Using the Natural transform differentiation property, we get

$$N^+[\mu(\alpha,\eta)]=\frac{u^\delta}{s^\delta}\sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^k\mu(\alpha,\eta)}{\partial^k\eta}|_{\eta=0}+\frac{u^\delta}{s^\delta}N^+[\mathcal{P}_1(\alpha,\eta)]-\frac{u^\delta}{s^\delta}N^+\{\overline{\mathcal{G}}_1(\mu,v)+\mathcal{N}_1(\mu,v)\}],$$
$$N^+[v(\alpha,\eta)]=\frac{u^\delta}{s^\delta}\sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^k v(\alpha,\eta)}{\partial^k\eta}|_{\eta=0}+\frac{u^\delta}{s^\delta}N^+[\mathcal{P}_2(\alpha,\eta)]-\frac{u^\delta}{s^\delta}N^+\{\overline{\mathcal{G}}_2(\mu,v)+\mathcal{N}_2(\mu,v)\}], \quad (12)$$

NDM describes the solution of infinite series $\mu(\alpha,\eta)$ and $v(\alpha,\eta)$,

$$\mu(\alpha,\eta)=\sum_{m=0}^{\infty}\mu_m(\alpha,\eta),\quad v(\alpha,\eta)=\sum_{m=0}^{\infty}v_m(\alpha,\eta) \quad (13)$$

Adomian polynomials decomposition of nonlinear terms of $\mathcal{N}_1$ and $\mathcal{N}_2$ are described as

$$\mathcal{N}_1(\mu,v)=\sum_{m=0}^{\infty}\mathcal{A}_m,\quad \mathcal{N}_2(\mu,v)=\sum_{m=0}^{\infty}\mathcal{B}_m, \quad (14)$$

All forms of nonlinearity the Adomian polynomials can be represented as

$$\mathcal{A}_m=\frac{1}{m!}\left[\frac{\partial^m}{\partial\lambda^m}\left\{\mathcal{N}_1\left(\sum_{k=0}^{\infty}\lambda^k\mu_k,\sum_{k=0}^{\infty}\lambda^k v_k\right)\right\}\right]_{\lambda=0},$$
$$\mathcal{B}_m=\frac{1}{m!}\left[\frac{\partial^m}{\partial\lambda^m}\left\{\mathcal{N}_2\left(\sum_{k=0}^{\infty}\lambda^k\mu_k,\sum_{k=0}^{\infty}\lambda^k v_k\right)\right\}\right]_{\lambda=0}. \quad (15)$$

Substituting Eq. (13) and Eq. (14) into (12), gives

$$N^+\left[\sum_{m=0}^{\infty}\mu_m(\alpha,\eta)\right]=\frac{u^\delta}{s^\delta}\sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^k\mu(\alpha,\eta)}{\partial^k\eta}|_{\eta=0}+\frac{u^\delta}{s^\delta}N^+\{\mathcal{P}_1(\alpha,\eta)\}-\frac{u^\delta}{s^\delta}N^+\left\{\overline{\mathcal{G}}_1\left(\sum_{m=0}^{\infty}\mu_m,\sum_{m=0}^{\infty}v_m\right)+\sum_{m=0}^{\infty}\mathcal{A}_m\right\},$$
$$N^+\left[\sum_{m=0}^{\infty}v_m(\alpha,\eta)\right]=\frac{u^\delta}{s^\delta}\sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^k v(\alpha,\eta)}{\partial^k\eta}|_{\eta=0}+\frac{u^\delta}{s^\delta}N^+\{\mathcal{P}_2(\alpha,\eta)\}-\frac{u^\delta}{s^\delta}N^+\left\{\overline{\mathcal{G}}_2\left(\sum_{m=0}^{\infty}\mu_m,\sum_{m=0}^{\infty}v_m\right)+\sum_{m=0}^{\infty}\mathcal{B}_m\right\}. \quad (16)$$

Applying the inverse Natural transformation to Eq. (16),

$$\sum_{m=0}^{\infty}\mu_m(\alpha,\eta)=N^-\left[\frac{u^\delta}{s^\delta}\sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^k\mu(\alpha,\eta)}{\partial^k\eta}|_{\eta=0}+\frac{u^\delta}{s^\delta}N^+\{\mathcal{P}_1(\alpha,\eta)\}-\frac{u^\delta}{s^\delta}N^+\left\{\overline{\mathcal{G}}_1\left(\sum_{m=0}^{\infty}\mu_m,\sum_{m=0}^{\infty}v_m\right)+\sum_{m=0}^{\infty}\mathcal{A}_m\right\}\right],$$
$$\sum_{m=0}^{\infty}v_m(\alpha,\eta)=N^-\left[\frac{u^\delta}{s^\delta}\sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^k v(\alpha,\eta)}{\partial^k\eta}|_{\eta=0}+\frac{u^\delta}{s^\delta}N^+\{\mathcal{P}_2(\alpha,\eta)\}-\frac{u^\delta}{s^\delta}N^+\left\{\overline{\mathcal{G}}_2\left(\sum_{m=0}^{\infty}\mu_m,\sum_{m=0}^{\infty}v_m\right)+\sum_{m=0}^{\infty}\mathcal{B}_m\right\}\right], \quad (17)$$

we define the following terms,

$$\mu_0(\alpha,\eta)=N^-\left[\frac{u^\delta}{s^\delta}\sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^k\mu(\alpha,\eta)}{\partial^k\eta}|_{\eta=0}+\frac{u^\delta}{s^\delta}N^+\{\mathcal{P}_1(\alpha,\eta)\}\right],$$
$$v_0(\alpha,\eta)=N^-\left[\frac{u^\delta}{s^\delta}\sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^k v(\alpha,\eta)}{\partial^k\eta}|_{\eta=0}+\frac{u^\delta}{s^\delta}N^+\{\mathcal{P}_2(\alpha,\eta)\}\right], \quad (18)$$

$$\mu_1(\alpha,\eta)=-N^-\left[\frac{u^\delta}{s^\delta}N^+\{\overline{\mathcal{G}}_1(\mu_0,v_0)+\mathcal{A}_0\}\right],$$
$$v_1(\alpha,\eta)=-N^-\left[\frac{u^\delta}{s^\delta}N^+\{\overline{\mathcal{G}}_2(\mu_0,v_0)+\mathcal{B}_0\}\right],$$

the general for $m\geqslant 1$, is given by

$$\mu_{m+1}(\alpha,\eta)=-N^-\left[\frac{u^\delta}{s^\delta}N^+\{\overline{\mathcal{G}}_1(\mu_m,v_m)+\mathcal{A}_m\}\right],$$
$$v_{m+1}(\alpha,\eta)=-N^-\left[\frac{u^\delta}{s^\delta}N^+\{\overline{\mathcal{G}}_2(\mu_m,v_m)+\mathcal{B}_m\}\right],$$

5. Numerical examples

5.1. Example 1

Consider fractional order system of NS equation

$$D_\eta^\delta(\mu)+\mu\frac{\partial\mu}{\partial\alpha}+v\frac{\partial\mu}{\partial\beta}=\rho\left[\frac{\partial^2\mu}{\partial\alpha^2}+\frac{\partial^2\mu}{\partial\beta^2}\right]+q,$$
$$D_\eta^\delta(v)+\mu\frac{\partial v}{\partial\alpha}+v\frac{\partial v}{\partial\beta}=\rho\left[\frac{\partial^2 v}{\partial\alpha^2}+\frac{\partial^2 v}{\partial\beta^2}\right]-q, \quad (19)$$

with initial conditions

$$\begin{cases}\mu(\alpha,\beta,0)=-\sin(\alpha+\beta),\\ v(\alpha,\beta,0)=\sin(\alpha+\beta).\end{cases} \quad (20)$$

we solve this with **NDM** first we will solve this scheme.

After the Natural transformation of Eq. (19), we get

$$N^{+}\left\{\frac{\partial^{\delta}\mu}{\partial\eta^{\delta}}\right\}=N^{+}\left[-\left(\mu\frac{\partial\mu}{\partial\alpha}+v\frac{\partial\mu}{\partial\beta}\right)+\rho\left\{\frac{\partial^2\mu}{\partial\alpha^2}+\frac{\partial^2\mu}{\partial\beta^2}\right\}+q\right],$$

$$N^{+}\left\{\frac{\partial^{\delta}v}{\partial\eta^{\delta}}\right\}=N^{+}\left[-\left(\mu\frac{\partial v}{\partial\alpha}+v\frac{\partial v}{\partial\beta}\right)+\rho\left\{\frac{\partial^2 v}{\partial\alpha^2}+\frac{\partial^2 v}{\partial\beta^2}\right\}-q\right],$$

$$\frac{s^{\delta}}{u^{\delta}}N^{+}\{\mu(\alpha,\beta,\eta)\}-\frac{s^{\delta-1}}{u^{\delta}}\mu(\alpha,\beta,0)$$

$$=N^{+}\left[-\left(\mu\frac{\partial\mu}{\partial\alpha}+v\frac{\partial\mu}{\partial\beta}\right)+\rho\left\{\frac{\partial^2\mu}{\partial\alpha^2}+\frac{\partial^2\mu}{\partial\beta^2}\right\}+q\right],$$

$$\frac{s^{\delta}}{u^{\delta}}N^{+}\{\mu(\alpha,\beta,\eta)\}-\frac{s^{\delta-1}}{u^{\delta}}v(\alpha,\beta,0)$$

$$=N^{+}\left[-\left(\mu\frac{\partial v}{\partial\alpha}+v\frac{\partial v}{\partial\beta}\right)+\rho\left\{\frac{\partial^2 v}{\partial\alpha^2}+\frac{\partial^2 v}{\partial\beta^2}\right\}-q\right].$$

The above algorithm is reduced to simplified

$$\begin{aligned}&N^{+}\{\mu(\alpha,\beta,\eta)\}=\frac{1}{s}\{\mu(\alpha,\beta,0)\}\\&+\frac{u^{\delta}}{s^{\delta}}N^{+}\left[-\left(\mu\frac{\partial\mu}{\partial\alpha}+v\frac{\partial\mu}{\partial\beta}\right)+\rho\left\{\frac{\partial^2\mu}{\partial\alpha^2}+\frac{\partial^2\mu}{\partial\beta^2}\right\}+q\right],\\&N^{+}\{v(\alpha,\beta,\eta)\}=\frac{1}{s}\{v(\alpha,\beta,0)\}\\&=\frac{u^{\delta}}{s^{\delta}}N^{+}\left[-\left(\mu\frac{\partial v}{\partial\alpha}+v\frac{\partial v}{\partial\beta}\right)+\rho\left\{\frac{\partial^2 v}{\partial\alpha^2}+\frac{\partial^2 v}{\partial\beta^2}\right\}-q\right].\end{aligned}\tag{21}$$

Applying inverse natural transformation, we get

$$\begin{aligned}&\mu(\alpha,\beta,\eta)=\mu(\alpha,\beta,0)+N^{-}\left[\frac{u^{\delta}}{s^{\delta}}N^{+}\{q\}\right]\\&+N^{-}\left[\frac{u^{\delta}}{s^{\delta}}N^{+}\left[-\left(\mu\frac{\partial\mu}{\partial\alpha}+v\frac{\partial\mu}{\partial\beta}\right)+\rho\left\{\frac{\partial^2\mu}{\partial\alpha^2}+\frac{\partial^2\mu}{\partial\beta^2}\right\}\right]\right],\\&v(\alpha,\beta,\eta)=v(\alpha,\beta,0)-N^{-}\left[\frac{u^{\delta}}{s^{\delta}}N^{+}\{q\}\right]\\&+N^{-}\left[\frac{u^{\delta}}{s^{\delta}}N^{+}\left[-\left(\mu\frac{\partial v}{\partial\alpha}+v\frac{\partial v}{\partial\beta}\right)+\rho\left\{\frac{\partial^2 v}{\partial\alpha^2}+\frac{\partial^2 v}{\partial\beta^2}\right\}\right]\right].\end{aligned}\tag{22}$$

Assume that the unknown functions $\mu(\alpha,\beta,\eta)$ and $v(\alpha,\beta,\eta)$ have infinite series solution as follows:

$$\mu(\alpha,\beta,\eta)=\sum_{m=0}^{\infty}\mu_m(\alpha,\beta,\eta),\quad and$$

$$v(\alpha,\beta,\eta)=\sum_{m=0}^{\infty}v_m(\alpha,\beta,\eta).$$

Remember that $\mu\mu_{\alpha}=\sum_{m=0}^{\infty}\mathcal{A}_m, v\mu_{\beta}=\sum_{m=0}^{\infty}\mathcal{B}_m, \mu v_{\alpha}=\sum_{m=0}^{\infty}\mathcal{C}_m$ and $vv_{\beta}=\sum_{m=0}^{\infty}\mathcal{D}_m$ are the Adomian polynomials and the nonlinear terms were characterized. Using such terms, Eq. (22) can be rewritten in the form

$$\sum_{m=0}^{\infty}\mu_m(\alpha,\beta,\eta)=\mu(\alpha,\beta,0)+N^{-}\left[\frac{u^{\delta}}{s^{\delta}}N^{+}\{q\}\right]$$

$$+N^{-}\left[\frac{u^{\delta}}{s^{\delta}}N^{+}\left[-\left(\sum_{m=0}^{\infty}\mathcal{A}_m+\sum_{m=0}^{\infty}\mathcal{B}_m\right)+\rho\left\{\frac{\partial^2\mu}{\partial\alpha^2}+\frac{\partial^2\mu}{\partial\beta^2}\right\}\right]\right],$$

$$\sum_{m=0}^{\infty}v_m(\alpha,\beta,\eta)=v(\alpha,\beta,0)-N^{-}\left[\frac{u^{\delta}}{s^{\delta}}N^{+}\{q\}\right]$$

$$+N^{-}\left[\frac{u^{\delta}}{s^{\delta}}N^{+}\left[-\left(\sum_{m=0}^{\infty}\mathcal{C}_m+\sum_{m=0}^{\infty}\mathcal{D}_m\right)+\rho\left\{\frac{\partial^2 v}{\partial\alpha^2}+\frac{\partial^2 v}{\partial\beta^2}\right\}\right]\right].$$

$$\begin{aligned}&\sum_{m=0}^{\infty}\mu_m(\alpha,\beta,\eta)=-\sin(\alpha+\beta)+\frac{q\eta^{\delta}}{\Gamma(\delta+1)}\\&+N^{-}\left[\frac{u^{\delta}}{s^{\delta}}N^{+}\left[-\left(\sum_{m=0}^{\infty}\mathcal{A}_m+\sum_{m=0}^{\infty}\mathcal{B}_m\right)\right]\right]\\&+N^{-}\left[\frac{u^{\delta}}{s^{\delta}}N^{+}\left[\rho\left\{\sum_{m=0}^{\infty}\frac{\partial^2\mu_m}{\partial\alpha^2}+\sum_{m=0}^{\infty}\frac{\partial^2\mu_m}{\partial\beta^2}\right\}\right]\right],\\&\sum_{m=0}^{\infty}v_m(\alpha,\beta,\eta)=\sin(\alpha+\beta)-\frac{q\eta^{\delta}}{\Gamma(\delta+1)}\\&N^{-}\left[\frac{u^{\delta}}{s^{\delta}}N^{+}\left[-\left(\sum_{m=0}^{\infty}\mathcal{C}_m+\sum_{m=0}^{\infty}\mathcal{D}_m\right)\right]\right]\\&+N^{-}\left[\frac{u^{\delta}}{s^{\delta}}N^{+}\left[\rho\left\{\sum_{m=0}^{\infty}\frac{\partial^2 v_m}{\partial\alpha^2}+\sum_{m=0}^{\infty}\frac{\partial^2 v_m}{\partial\beta^2}\right\}\right]\right].\end{aligned}\tag{23}$$

According to Eq. (15), all forms of non-linearity the Adomian polynomials can be defined as

$$\begin{aligned}&\mathcal{A}_0=\mu_0\frac{\partial\mu_0}{\partial\alpha},\quad \mathcal{A}_1=\mu_0\frac{\partial\mu_1}{\partial\alpha}+\mu_1\frac{\partial\mu_0}{\partial\alpha},\\&\mathcal{B}_0=v_0\frac{\partial\mu_0}{\partial\beta},\quad \mathcal{B}_1=v_0\frac{\partial\mu_1}{\partial\beta}+v_1\frac{\partial\mu_0}{\partial\beta},\\&\mathcal{C}_0=\mu_0\frac{\partial v_0}{\partial\alpha},\quad \mathcal{C}_1=\mu_0\frac{\partial v_1}{\partial\alpha}+\mu_1\frac{\partial v_0}{\partial\alpha},\\&\mathcal{D}_0=v_0\frac{\partial v_0}{\partial\alpha},\quad \mathcal{D}_1=v_0\frac{\partial v_1}{\partial\alpha}+v_1\frac{\partial v_0}{\partial\alpha}.\end{aligned}$$

Thus, we can easily obtain the recursive relationship by comparing two sides of Eq. (23)

$$\mu_0(\alpha,\beta,\eta)=-\sin(\alpha+\beta)+\frac{q\eta^{\delta}}{\Gamma(\delta+1)},$$
$$v_0(\alpha,\beta,\eta)=\sin(\alpha+\beta)-\frac{q\eta^{\delta}}{\Gamma(\delta+1)}.$$

For $m=0$

$$\mu_1(\alpha,\beta,\eta)=\sin(\alpha+\beta)\frac{2\rho\eta^{\delta}}{\Gamma(\delta+1)},$$
$$v_1(\alpha,\beta,\eta)=-\sin(\alpha+\beta)\frac{2\rho\eta^{\delta}}{\Gamma(\delta+1)}.$$

For $m=1$

$$\mu_2(\alpha,\beta,\eta)=-\sin(\alpha+\beta)\frac{(2\rho)^2\eta^{2\delta}}{\Gamma(2\delta+1)},$$
$$v_2(\alpha,\beta,\eta)=\sin(\alpha+\beta)\frac{(2\rho)^2\eta^{2\delta}}{\Gamma(2\delta+1)}.$$

For $m=2$

$$\mu_3(\alpha,\beta,\eta)=\sin(\alpha+\beta)\frac{(2\rho)^3\eta^{3\delta}}{\Gamma(3\delta+1)},$$
$$v_3(\alpha,\beta,\eta)=-\sin(\alpha+\beta)\frac{(2\rho)^3\eta^{3\delta}}{\Gamma(3\delta+1)}.$$

$$\vdots$$

In same technique, the remaining μ_m and v_m $(m\geqslant 3)$ elements of the NDM solution can be collected smoothly. Therefore, we define the series of alternatives as

$$\mu(\alpha,\beta,\eta)=\sum_{m=0}^{\infty}\mu_m(\alpha,\beta)=\mu_0(\alpha,\beta)+\mu_1(\alpha,\beta)+\mu_2(\alpha,\beta)+\mu_3(\alpha,\beta)+\cdots$$
$$v(\alpha,\beta,\eta)=\sum_{m=0}^{\infty}v_m(\alpha,\beta)=v_0(\alpha,\beta)+v_1(\alpha,\beta)+v_2(\alpha,\beta)+v_3(\alpha,\beta)+\cdots$$

$$\mu(\alpha,\beta,\eta)=-\sin(\alpha+\beta)+\frac{q\eta^\delta}{\Gamma(\delta+1)}+\sin(\alpha+\beta)\frac{2\rho\eta^\delta}{\Gamma(\delta+1)}-\sin(\alpha+\beta)\frac{(2\rho)^2\eta^{2\delta}}{\Gamma(2\delta+1)}+\sin(\alpha+\beta)\frac{(2\rho)^3\eta^{3\delta}}{\Gamma(3\delta+1)}-\cdots-\sin(\alpha+\beta)\sum_{m=0}^{\infty}\frac{(-2\rho)^m\eta^{m\delta}}{\Gamma(m\delta+1)}.$$
$$v(\alpha,\beta,\eta)=\sin(\alpha+\beta)-\frac{q\eta^\delta}{\Gamma(\delta+1)}-\sin(\alpha+\beta)\frac{2\rho\eta^\delta}{\Gamma(\delta+1)}$$
$$\sin(\alpha+\beta)\frac{(2\rho)^2\eta^{2\delta}}{\Gamma(2\delta+1)}-\sin(\alpha+\beta)\frac{(2\rho)^3\eta^{3\delta}}{\Gamma(3\delta+1)}+\cdots$$
$$\sin(\alpha+\beta)\sum_{m=0}^{\infty}\frac{(-2\rho)^m\eta^{m\delta}}{\Gamma(m\delta+1)}.$$

The exact solution of Eq. (19) at $\delta=1$ and $q=0$,

$$\mu(\alpha,\beta,\eta)=-e^{-2\rho\eta}\sin(\alpha+\beta),\quad v(\alpha,\beta,\eta)=e^{-2\rho\eta}\sin(\alpha+\beta). \tag{24}$$

The approximate solution by **NVIM**.

According to the Eqs. (7) the iteration formulas for system (19), we get

$$\mu_{m+1}(\alpha,\beta,\eta)=\mu_m(\alpha,\beta,\eta)-N^-\left[\frac{u^\delta}{s^\delta}N^+\left\{\frac{s^\delta}{u^\delta}\frac{\partial\mu_m}{\partial\eta}+\mu_m\frac{\partial\mu_m}{\partial\alpha}+v_m\frac{\partial\mu_m}{\partial\beta}-\rho\left(\frac{\partial^2\mu_m}{\partial\alpha^2}+\frac{\partial^2\mu_m}{\partial\beta^2}\right)-q\right\}\right],$$
$$v_{m+1}(\alpha,\beta,\eta)=v_m(\alpha,\beta,\eta)-N^-\left[\frac{u^\delta}{s^\delta}N^+\left\{\frac{s^\delta}{u^\delta}\frac{\partial v_m}{\partial\eta}+\mu_m\frac{\partial v_m}{\partial\alpha}+v_m\frac{\partial v_m}{\partial\beta}-\rho\left(\frac{\partial^2 v_m}{\partial\alpha^2}+\frac{\partial^2 v_m}{\partial\beta^2}\right)+q\right\}\right], \tag{25}$$

where

$$\mu_0(\alpha,\beta,\eta)=-\sin(\alpha+\beta),\quad v_0(\alpha,\beta,\eta)=\sin(\alpha+\beta). \tag{26}$$

For $m=0,1,2,\cdots$

$$\mu_1(\alpha,\beta,\eta)=\mu_0(\alpha,\beta,\eta)-N^-\left[\frac{u^\delta}{s^\delta}N^+\left\{\frac{s^\delta}{u^\delta}\frac{\partial\mu_0}{\partial\eta}+\mu_0\frac{\partial\mu_0}{\partial\alpha}+v_0\frac{\partial\mu_0}{\partial\beta}-\rho\left(\frac{\partial^2\mu_0}{\partial\alpha^2}+\frac{\partial^2\mu_0}{\partial\beta^2}\right)-q\right\}\right],$$
$$v_1(\alpha,\beta,\eta)=v_0(\alpha,\beta,\eta)-N^-\left[\frac{u^\delta}{s^\delta}N^+\left\{\frac{s^\delta}{u^\delta}\frac{\partial v_0}{\partial\eta}+\mu_0\frac{\partial v_0}{\partial\alpha}+v_0\frac{\partial v_0}{\partial\beta}-\rho\left[\frac{\partial^2 v_0}{\partial\alpha^2}+\frac{\partial^2 v_0}{\partial\beta^2}\right]+q\right\}\right],$$

$$\mu_1(\alpha,\beta,\eta)=-\sin(\alpha+\beta)+\sin(\alpha+\beta)\frac{2\rho\eta^\delta}{\Gamma(\delta+1)}+\frac{q\eta^\delta}{\Gamma(\delta+1)},$$
$$v_1(\alpha,\beta,\eta)=\sin(\alpha+\beta)-\sin(\alpha+\beta)\frac{2\rho\eta^\delta}{\Gamma(\delta+1)}-\frac{q\eta^\delta}{\Gamma(\delta+1)},$$

$$\mu_2(\alpha,\beta,\eta)=\mu_1(\alpha,\beta,\eta)-N^-\left[\frac{u^\delta}{s^\delta}N^+\left\{\frac{s^\delta}{u^\delta}\frac{\partial\mu_1}{\partial\eta}+\mu_1\frac{\partial\mu_1}{\partial\alpha}+v_1\frac{\partial\mu_1}{\partial\beta}-\rho\left(\frac{\partial^2\mu_1}{\partial\alpha^2}+\frac{\partial^2\mu_1}{\partial\beta^2}\right)-q\right\}\right],$$
$$v_2(\alpha,\beta,\eta)=v_1(\alpha,\beta,\eta)-N^-\left[\frac{u^\delta}{s^\delta}N^+\left\{\frac{s^\delta}{u^\delta}\frac{\partial v_1}{\partial\eta}+\mu_1\frac{\partial v_1}{\partial\alpha}+v_1\frac{\partial v_1}{\partial\beta}-\rho\left(\frac{\partial^2 v_1}{\partial\alpha^2}+\frac{\partial^2 v_1}{\partial\beta^2}\right)+q\right\}\right],$$

$$\mu_2(\alpha,\beta,\eta)=-\sin(\alpha+\beta)+\sin(\alpha+\beta)\frac{2\rho\eta^\delta}{\Gamma(\delta+1)}+\frac{q\eta^\delta}{\Gamma(\delta+1)}-\sin(\alpha+\beta)\frac{(2\rho)^2\eta^{2\delta}}{\Gamma(2\delta+1)},$$
$$v_2(\alpha,\beta,\eta)=\sin(\alpha+\beta)-\sin(\alpha+\beta)\frac{2\rho\eta^\delta}{\Gamma(\delta+1)}-\frac{q\eta^\delta}{\Gamma(\delta+1)}+\sin(\alpha+\beta)\frac{(2\rho)^2\eta^{2\delta}}{\Gamma(2\delta+1)},$$

$$\mu_3(\alpha,\beta,\eta)=\mu_2(\alpha,\beta,\eta)-N^-\left[\frac{u^\delta}{s^\delta}N^+\left\{\frac{s^\delta}{u^\delta}\frac{\partial\mu_2}{\partial\eta}+\mu_2\frac{\partial\mu_2}{\partial\alpha}+v_2\frac{\partial\mu_2}{\partial\beta}-\rho\left[\frac{\partial^2\mu_2}{\partial\alpha^2}+\frac{\partial^2\mu_2}{\partial\beta^2}\right]-q\right\}\right],$$
$$v_3(\alpha,\beta,\eta)=v_2(\alpha,\beta,\eta)-N^-\left[\frac{u^\delta}{s^\delta}N^+\left\{\frac{s^\delta}{u^\delta}\frac{\partial v_2}{\partial\eta}+\mu_2\frac{\partial v_2}{\partial\alpha}+v_2\frac{\partial v_2}{\partial\beta}-\rho\left[\frac{\partial^2 v_2}{\partial\alpha^2}+\frac{\partial^2 v_2}{\partial\beta^2}\right]+q\right\}\right],$$

$$\mu_3(\alpha,\beta,\eta)=-\sin(\alpha+\beta)+\sin(\alpha+\beta)\frac{2\rho\eta^\delta}{\Gamma(\delta+1)}+\frac{q\eta^\delta}{\Gamma(\delta+1)}-\sin(\alpha+\beta)\frac{(2\rho)^2\eta^{2\delta}}{\Gamma(2\delta+1)}+\sin(\alpha+\beta)\frac{(2\rho)^3\eta^{3\delta}}{\Gamma(3\delta+1)},$$
$$v_3(\alpha,\beta,\eta)=\sin(\alpha+\beta)-\sin(\alpha+\beta)\frac{2\rho\eta^\delta}{\Gamma(\delta+1)}-\frac{q\eta^\delta}{\Gamma(\delta+1)}+\sin(\alpha+\beta)\frac{(2\rho)^2\eta^{2\delta}}{\Gamma(2\delta+1)}-\sin(\alpha+\beta)\frac{(2\rho)^3\eta^{3\delta}}{\Gamma(3\delta+1)},$$

$$\mu(\alpha,\beta,\eta)=\sum_{m=0}^{\infty}\mu_m(\alpha,\beta)=-\sin(\alpha+\beta)\sum_{m=0}^{\infty}\frac{(-2\rho)^m\eta^{m\delta}}{\Gamma(m\delta+1)}+\frac{q\eta^\delta}{\Gamma(\delta+1)},$$
$$v(\alpha,\beta,\eta)=\sum_{m=0}^{\infty}\mu_m(\alpha,\beta)=\sin(\alpha+\beta)\sum_{m=0}^{\infty}\frac{(-2\rho)^m\eta^{m\delta}}{\Gamma(m\delta+1)}-\frac{q\eta^\delta}{\Gamma(\delta+1)}. \tag{27}$$

The exact solution of Eq. (19) at $\delta=1$ and $q=0$,

$$\mu(\alpha,\beta,\eta)=-e^{-2\rho\eta}\sin(\alpha+\beta),\quad v(\alpha,\beta,\eta)=e^{-2\rho\eta}\sin(\alpha+\beta). \tag{28}$$

5.2. Example 2

Consider two-dimensional fractional order system of NS equation

$$D_\eta^\delta(\mu)+\mu\frac{\partial\mu}{\partial\alpha}+v\frac{\partial\mu}{\partial\beta}=\rho\left[\frac{\partial^2\mu}{\partial\alpha^2}+\frac{\partial^2\mu}{\partial\beta^2}\right]+q,$$
$$D_\eta^\delta(v)+\mu\frac{\partial v}{\partial\alpha}+v\frac{\partial v}{\partial\beta}=\rho\left[\frac{\partial^2 v}{\partial\alpha^2}+\frac{\partial^2 v}{\partial\beta^2}\right]-q, \tag{29}$$

with the initial conditions

$$\begin{cases}\mu(\alpha,\beta,0)=-e^{\alpha+\beta},\\ v(\alpha,\beta,0)=e^{\alpha+\beta}.\end{cases} \tag{30}$$

First, by using **NDM**, we will solve this scheme.

After the Natural transformation of Eq. (29), we get

$$N^+\left\{\frac{\partial^\delta\mu}{\partial\eta^\delta}\right\}=N^+\left[-\left(\mu\frac{\partial\mu}{\partial\alpha}+v\frac{\partial\mu}{\partial\beta}\right)+\rho\left\{\frac{\partial^2\mu}{\partial\alpha^2}+\frac{\partial^2\mu}{\partial\beta^2}\right\}+q\right],$$
$$N^+\left\{\frac{\partial^\delta v}{\partial\eta^\delta}\right\}=N^+\left[-\left(\mu\frac{\partial v}{\partial\alpha}+v\frac{\partial v}{\partial\beta}\right)+\rho\left\{\frac{\partial^2 v}{\partial\alpha^2}+\frac{\partial^2 v}{\partial\beta^2}\right\}-q\right],$$

$$\frac{s^\delta}{u^\delta}N^+\{\mu(\alpha,\beta,\eta)\}-\frac{s^{\delta-1}}{u^\delta}\mu(\alpha,\beta,0)$$
$$=N^+\left[-\left(\mu\frac{\partial\mu}{\partial\alpha}+v\frac{\partial\mu}{\partial\beta}\right)+\rho\left\{\frac{\partial^2\mu}{\partial\alpha^2}+\frac{\partial^2\mu}{\partial\beta^2}\right\}+q\right],$$

$$\frac{s^\delta}{u^\delta}N^+\{\mu(\alpha,\beta,\eta)\}-\frac{s^{\delta-1}}{u^\delta}v(\alpha,\beta,0)$$
$$=N^+\left[-\left(\mu\frac{\partial v}{\partial\alpha}+v\frac{\partial v}{\partial\beta}\right)+\rho\left\{\frac{\partial^2 v}{\partial\alpha^2}+\frac{\partial^2 v}{\partial\beta^2}\right\}-q\right].$$

The above algorithm is reduced to simplified

$$N^+\{\mu(\alpha,\beta,\eta)\}=\frac{1}{s}\{\mu(\alpha,\beta,0)\}+\frac{u^\delta}{s^\delta}N^+\left[-\left(\mu\frac{\partial\mu}{\partial\alpha}+v\frac{\partial\mu}{\partial\beta}\right)+\rho\left\{\frac{\partial^2\mu}{\partial\alpha^2}+\frac{\partial^2\mu}{\partial\beta^2}\right\}+q\right],$$
$$N^+\{v(\alpha,\beta,\eta)\}=\frac{1}{s}\{v(\alpha,\beta,0)\}=\frac{u^\delta}{s^\delta}N^+\left[-\left(\mu\frac{\partial v}{\partial\alpha}+v\frac{\partial v}{\partial\beta}\right)+\rho\left\{\frac{\partial^2 v}{\partial\alpha^2}+\frac{\partial^2 v}{\partial\beta^2}\right\}-q\right]. \tag{31}$$

Applying inverse natural transformation, we get

$$\mu(\alpha,\beta,\eta)=\mu(\alpha,\beta,0)+N^{-}\left[\frac{u^\delta}{s^\delta}N^{+}\{q\}\right]+N^{-}\left[\frac{u^\delta}{s^\delta}N^{+}\left[-\left(\mu\frac{\partial\mu}{\partial\alpha}+v\frac{\partial\mu}{\partial\beta}\right)+\rho\left\{\frac{\partial^2\mu}{\partial\alpha^2}+\frac{\partial^2\mu}{\partial\beta^2}\right\}\right]\right],$$
$$v(\alpha,\beta,\eta)=v(\alpha,\beta,0)-N^{-}\left[\frac{u^\delta}{s^\delta}N^{+}\{q\}\right]+N^{-}\left[\frac{u^\delta}{s^\delta}N^{+}\left[-\left(\mu\frac{\partial v}{\partial\alpha}+v\frac{\partial v}{\partial\beta}\right)+\rho\left\{\frac{\partial^2 v}{\partial\alpha^2}+\frac{\partial^2 v}{\partial\beta^2}\right\}\right]\right]. \quad (32)$$

Assume that the unknown functions $\mu(\alpha,\beta,\eta)$ and $v(\alpha,\beta,\eta)$ have infinite series solution as follows:

$$\mu(\alpha,\beta,\eta)=\sum_{m=0}^{\infty}\mu_m(\alpha,\beta,\eta),\quad \textit{and}$$
$$v(\alpha,\beta,\eta)=\sum_{m=0}^{\infty}v_m(\alpha,\beta,\eta).$$

Remember that $\mu\mu_\alpha=\sum_{m=0}^{\infty}\mathcal{A}_m$, $v\mu_\beta=\sum_{m=0}^{\infty}\mathcal{B}_m$, $\mu v_\alpha=\sum_{m=0}^{\infty}\mathcal{C}_m$ and $vv_\beta=\sum_{m=0}^{\infty}\mathcal{D}_m$ are the Adomian polynomials and the nonlinear terms were characterized. Using such terms, Eq. (32) can be rewritten in the form

$$\sum_{m=0}^{\infty}\mu_m(\alpha,\beta,\eta)=\mu(\alpha,\beta,0)+N^{-}\left[\frac{u^\delta}{s^\delta}N^{+}\{q\}\right]+N^{-}\left[\frac{u^\delta}{s^\delta}N^{+}\left[-\left(\sum_{m=0}^{\infty}\mathcal{A}_m+\sum_{m=0}^{\infty}\mathcal{B}_m\right)+\rho\left\{\frac{\partial^2\mu}{\partial\alpha^2}+\frac{\partial^2\mu}{\partial\beta^2}\right\}\right]\right],$$
$$\sum_{m=0}^{\infty}v_m(\alpha,\beta,\eta)=v(\alpha,\beta,0)-N^{-}\left[\frac{u^\delta}{s^\delta}N^{+}\{q\}\right]+N^{-}\left[\frac{u^\delta}{s^\delta}N^{+}\left[-\left(\sum_{m=0}^{\infty}\mathcal{C}_m+\sum_{m=0}^{\infty}\mathcal{D}_m\right)+\rho\left\{\frac{\partial^2 v}{\partial\alpha^2}+\frac{\partial^2 v}{\partial\beta^2}\right\}\right]\right].$$

For $m=1$

$$\mu_2(\alpha,\beta,\eta)=-e^{\alpha+\beta}\frac{(2\rho)^2\eta^{2\delta}}{\Gamma(2\delta+1)},$$
$$v_2(\alpha,\beta,\eta)=e^{\alpha+\beta}\frac{(2\rho)^2\eta^{2\delta}}{\Gamma(2\delta+1)}.$$

For $m=2$

$$\mu_3(\alpha,\beta,\eta)=e^{\alpha+\beta}\frac{(2\rho)^3\eta^{3\delta}}{\Gamma(3\delta+1)},$$
$$v_3(\alpha,\beta,\eta)=-e^{\alpha+\beta}\frac{(2\rho)^3\eta^{3\delta}}{\Gamma(3\delta+1)}.$$
$$\vdots$$

using the same technique, the remaining μ_m and v_m $(m\geqslant 3)$ elements of the NDM solution can be obtained smoothly. Therefore, we define the series of alternatives as

$$\mu(\alpha,\beta,\eta)=\sum_{m=0}^{\infty}\mu_m(\alpha,\beta)=\mu_0(\alpha,\beta)+\mu_1(\alpha,\beta)+\mu_2(\alpha,\beta)+\mu_3(\alpha,\beta)+\cdots$$
$$v(\alpha,\beta,\eta)=\sum_{m=0}^{\infty}v_m(\alpha,\beta)=v_0(\alpha,\beta)+v_1(\alpha,\beta)+v_2(\alpha,\beta)+v_3(\alpha,\beta)+\cdots$$

$$\sum_{m=0}^{\infty}\mu_m(\alpha,\beta,\eta)=-e^{(\alpha+\beta)+\frac{q\eta^\delta}{\Gamma(\delta+1)}+N^{-}\left[\frac{u^\delta}{s^\delta}N^{+}\left[-\left(\sum_{m=0}^{\infty}\mathcal{A}_m+\sum_{m=0}^{\infty}\mathcal{B}_m\right)\right]\right]+N^{-}\left[\frac{u^\delta}{s^\delta}N^{+}\left[\rho\left\{\sum_{m=0}^{\infty}\frac{\partial^2\mu_m}{\partial\alpha^2}+\sum_{m=0}^{\infty}\frac{\partial^2\mu_m}{\partial\beta^2}\right\}\right]\right]},\ \sum_{m=0}^{\infty}v_m(\alpha,\beta,\eta)=e^{(\alpha+\beta)-\frac{q\eta^\delta}{\Gamma(\delta+1)}-N^{-}\left[\frac{u^\delta}{s^\delta}N^{+}\left[-\left(\sum_{m=0}^{\infty}\mathcal{C}_m+\sum_{m=0}^{\infty}\mathcal{D}_m\right)\right]\right]+N^{-}\left[\frac{u^\delta}{s^\delta}N^{+}\left[\rho\left\{\sum_{m=0}^{\infty}\frac{\partial^2 v_m}{\partial\alpha^2}+\sum_{m=0}^{\infty}\frac{\partial^2 v_m}{\partial\beta^2}\right\}\right]\right]}. \quad (33)$$

According to Eq. (15), all forms of non-linearity the Adomian polynomials can be defined as

$$\mathcal{A}_0=\mu_0\frac{\partial\mu_0}{\partial\alpha},\quad \mathcal{A}_1=\mu_0\frac{\partial\mu_1}{\partial\alpha}+\mu_1\frac{\partial\mu_0}{\partial\alpha},$$
$$\mathcal{B}_0=v_0\frac{\partial\mu_0}{\partial\beta},\quad \mathcal{B}_1=v_0\frac{\partial\mu_1}{\partial\beta}+v_1\frac{\partial\mu_0}{\partial\beta},$$
$$\mathcal{C}_0=\mu_0\frac{\partial v_0}{\partial\alpha},\quad \mathcal{C}_1=\mu_0\frac{\partial v_1}{\partial\alpha}+\mu_1\frac{\partial v_0}{\partial\alpha},$$
$$\mathcal{D}_0=v_0\frac{\partial v_0}{\partial\alpha},\quad \mathcal{D}_1=v_0\frac{\partial v_1}{\partial\alpha}+v_1\frac{\partial v_0}{\partial\alpha}.$$

Thus, we can easily obtain the recursive relationship by comparing two sides of Eq. (33)

$$\mu_0(\alpha,\beta,\eta)=-e^{\alpha+\beta}+\frac{q\eta^\delta}{\Gamma(\delta+1)},$$
$$v_0(\alpha,\beta,\eta)=e^{\alpha+\beta}-\frac{q\eta^\delta}{\Gamma(\delta+1)}.$$

For $m=0$

$$\mu_1(\alpha,\beta,\eta)=e^{\alpha+\beta}\frac{2\rho\eta^\delta}{\Gamma(\delta+1)},$$
$$v_1(\alpha,\beta,\eta)=-e^{\alpha+\beta}\frac{2\rho\eta^\delta}{\Gamma(\delta+1)}.$$

$$\mu(\alpha,\beta,\eta)=-e^{\alpha+\beta}+\frac{q\eta^\delta}{\Gamma(\delta+1)}+e^{\alpha+\beta}\frac{2\rho\eta^\delta}{\Gamma(\delta+1)}-e^{\alpha+\beta}\frac{(2\rho)^2\eta^{2\delta}}{\Gamma(2\delta+1)}+e^{\alpha+\beta}\frac{(2\rho)^3\eta^{3\delta}}{\Gamma(3\delta+1)}-\cdots$$
$$-e^{\alpha+\beta}\sum_{m=0}^{\infty}\frac{(-2\rho)^m\eta^{m\delta}}{\Gamma(m\delta+1)}.$$
$$v(\alpha,\beta,\eta)=e^{\alpha+\beta}-\frac{q\eta^\delta}{\Gamma(\delta+1)}-e^{\alpha+\beta}\frac{2\rho\eta^\delta}{\Gamma(\delta+1)}$$
$$e^{\alpha+\beta}\frac{(2\rho)^2\eta^{2\delta}}{\Gamma(2\delta+1)}-e^{\alpha+\beta}\frac{(2\rho)^3\eta^{3\delta}}{\Gamma(3\delta+1)}+\cdots$$
$$e^{\alpha+\beta}\sum_{m=0}^{\infty}\frac{(-2\rho)^m\eta^{m\delta}}{\Gamma(m\delta+1)}.$$

The exact solution of Eq. (29) at $\delta=1$ and $q=0$,

$$\mu(\alpha,\beta,\eta)=-e^{\alpha+\beta+2\rho\eta},$$
$$v(\alpha,\beta,\eta)=e^{\alpha+\beta+2\rho\eta}. \quad (34)$$

The approximate solution by **NVIM**.

According to the Eqs. (7) the iteration formulas for system (29), after the Natural transformation of Eq. (29), we get

$$\mu_{m+1}(\alpha,\beta,\eta)=\mu_m(\alpha,\beta,\eta)-N^{-}\left[\frac{u^\delta}{s^\delta}N^{+}\left\{\frac{s^\delta}{u^\delta}\frac{\partial\mu_m}{\partial\eta}+\mu_m\frac{\partial\mu_m}{\partial\alpha}+v_m\frac{\partial\mu_m}{\partial\beta}-\rho\left(\frac{\partial^2\mu_m}{\partial\alpha^2}+\frac{\partial^2\mu_m}{\partial\beta^2}\right)-q\right\}\right],$$
$$v_{m+1}(\alpha,\beta,\eta)=v_m(\alpha,\beta,\eta)-N^{-}\left[\frac{u^\delta}{s^\delta}N^{+}\left\{\frac{s^\delta}{u^\delta}\frac{\partial v_m}{\partial\eta}+\mu_m\frac{\partial v_m}{\partial\alpha}+v_m\frac{\partial v_m}{\partial\beta}-\rho\left(\frac{\partial^2 v_m}{\partial\alpha^2}+\frac{\partial^2 v_m}{\partial\beta^2}\right)+q\right\}\right]. \quad (35)$$

where

$$\mu_0(\alpha,\beta,\eta) = -e^{\alpha+\beta},$$
$$v_0(\alpha,\beta,\eta) = e^{\alpha+\beta}. \tag{36}$$

For $m = 0, 1, 2, \cdots$

$$\mu_1(\alpha,\beta,\eta) = \mu_0(\alpha,\beta,\eta) - N^-\left[\frac{u^\delta}{s^\delta}N^+\left\{\frac{s^\delta}{u^\delta}\frac{\partial\mu_0}{\partial\eta} + \mu_0\frac{\partial\mu_0}{\partial\alpha} + v_0\frac{\partial\mu_0}{\partial\beta} - \rho\left(\frac{\partial^2\mu_0}{\partial\alpha^2} + \frac{\partial^2\mu_0}{\partial\beta^2}\right) - q\right\}\right],$$
$$v_1(\alpha,\beta,\eta) = v_0(\alpha,\beta,\eta) - N^-\left[\frac{u^\delta}{s^\delta}N^+\left\{\frac{s^\delta}{u^\delta}\frac{\partial v_0}{\partial\eta} + \mu_0\frac{\partial v_0}{\partial\alpha} + v_0\frac{\partial v_0}{\partial\beta} - \rho\left(\frac{\partial^2 v_0}{\partial\alpha^2} + \frac{\partial^2 v_0}{\partial\beta^2}\right) + q\right\}\right],$$

$$\mu_1(\alpha,\beta,\eta) = -e^{\alpha+\beta} + e^{\alpha+\beta}\frac{2\rho\eta^\delta}{\Gamma(\delta+1)} + \frac{q\eta^\delta}{\Gamma(\delta+1)},$$
$$v_1(\alpha,\beta,\eta) = e^{\alpha+\beta} - e^{\alpha+\beta}\frac{2\rho\eta^\delta}{\Gamma(\delta+1)} - \frac{q\eta^\delta}{\Gamma(\delta+1)},$$

$$\mu_2(\alpha,\beta,\eta) = \mu_1(\alpha,\beta,\eta) - N^-\left[\frac{u^\delta}{s^\delta}N^+\left\{\frac{s^\delta}{u^\delta}\frac{\partial\mu_1}{\partial\eta} + \mu_1\frac{\partial\mu_1}{\partial\alpha} + v_1\frac{\partial\mu_1}{\partial\beta} - \rho\left(\frac{\partial^2\mu_1}{\partial\alpha^2} + \frac{\partial^2\mu_1}{\partial\beta^2}\right) - q\right\}\right],$$
$$v_2(\alpha,\beta,\eta) = v_1(\alpha,\beta,\eta) - N^-\left[\frac{u^\delta}{s^\delta}N^+\left\{\frac{s^\delta}{u^\delta}\frac{\partial v_1}{\partial\eta} + \mu_1\frac{\partial v_1}{\partial\alpha} + v_1\frac{\partial v_1}{\partial\beta} - \rho\left(\frac{\partial^2 v_1}{\partial\alpha^2} + \frac{\partial^2 v_1}{\partial\beta^2}\right) + q\right\}\right],$$

$$\mu_2(\alpha,\beta,\eta) = -e^{\alpha+\beta} + e^{\alpha+\beta}\frac{2\rho\eta^\delta}{\Gamma(\delta+1)} + \frac{q\eta^\delta}{\Gamma(\delta+1)} - e^{\alpha+\beta}\frac{(2\rho)^2\eta^{2\delta}}{\Gamma(2\delta+1)},$$
$$v_2(\alpha,\beta,\eta) = e^{\alpha+\beta} - e^{\alpha+\beta}\frac{2\rho\eta^\delta}{\Gamma(\delta+1)} - \frac{q\eta^\delta}{\Gamma(\delta+1)} + e^{\alpha+\beta}\frac{(2\rho)^2\eta^{2\delta}}{\Gamma(2\delta+1)},$$

$$\mu_3(\alpha,\beta,\eta) = \mu_2(\alpha,\beta,\eta) - N^-\left[\frac{u^\delta}{s^\delta}N^+\left\{\frac{s^\delta}{u^\delta}\frac{\partial\mu_2}{\partial\eta} + \mu_2\frac{\partial\mu_2}{\partial\alpha} + v_2\frac{\partial\mu_2}{\partial\beta} - \rho\left(\frac{\partial^2\mu_2}{\partial\alpha^2} + \frac{\partial^2\mu_2}{\partial\beta^2}\right) - q\right\}\right],$$
$$v_3(\alpha,\beta,\eta) = v_2(\alpha,\beta,\eta) - N^-\left[\frac{u^\delta}{s^\delta}N^+\left\{\frac{s^\delta}{u^\delta}\frac{\partial v_2}{\partial\eta} + \mu_2\frac{\partial v_2}{\partial\alpha} + v_2\frac{\partial v_2}{\partial\beta} - \rho\left(\frac{\partial^2 v_2}{\partial\alpha^2} + \frac{\partial^2 v_2}{\partial\beta^2}\right) + q\right\}\right],$$

$$\mu_3(\alpha,\beta,\eta) = -e^{\alpha+\beta} + e^{\alpha+\beta}\frac{2\rho\eta^\delta}{\Gamma(\delta+1)} + \frac{q\eta^\delta}{\Gamma(\delta+1)} - e^{\alpha+\beta}\frac{(2\rho)^2\eta^{2\delta}}{\Gamma(2\delta+1)} + e^{\alpha+\beta}\frac{(2\rho)^3\eta^{3\delta}}{\Gamma(3\delta+1)},$$
$$v_3(\alpha,\beta,\eta) = e^{\alpha+\beta} - e^{\alpha+\beta}\frac{2\rho\eta^\delta}{\Gamma(\delta+1)} - \frac{q\eta^\delta}{\Gamma(\delta+1)} + e^{\alpha+\beta}\frac{(2\rho)^2\eta^{2\delta}}{\Gamma(2\delta+1)} - e^{\alpha+\beta}\frac{(2\rho)^3\eta^{3\delta}}{\Gamma(3\delta+1)},$$

$$\mu(\alpha,\beta,\eta) = \sum_{m=0}^{\infty}\mu_m(\alpha,\beta) = -e^{\alpha+\beta}\sum_{m=0}^{\infty}\frac{(-2\rho)^m\eta^{m\delta}}{\Gamma(m\delta+1)} + \frac{q\eta^\delta}{\Gamma(\delta+1)},$$
$$v(\alpha,\beta,\eta) = \sum_{m=0}^{\infty}v_m(\alpha,\beta) = e^{\alpha+\beta}\sum_{m=0}^{\infty}\frac{(-2\rho)^m\eta^{m\delta}}{\Gamma(m\delta+1)} - \frac{q\eta^\delta}{\Gamma(\delta+1)}, \tag{37}$$

The exact solution of Eq. (29) at $\delta = 1$ and $q = 0$,

$$\mu(\alpha,\beta,\eta) = -e^{\alpha+\beta+2\rho\eta},$$
$$v(\alpha,\beta,\eta) = e^{\alpha+\beta+2\rho\eta}. \tag{38}$$

5.3. Example 3

Consider two-dimensional NS equation with time-fractional order

$$D_\eta^\delta(\mu) + \mu\frac{\partial\mu}{\partial\alpha} + v\frac{\partial\mu}{\partial\beta} + \omega\frac{\partial\mu}{\partial\gamma} = \rho\left[\frac{\partial^2\mu}{\partial\alpha^2} + \frac{\partial^2\mu}{\partial\beta^2} + \frac{\partial^2\mu}{\partial\gamma^2}\right] + q_1,$$
$$D_\eta^\delta(v) + \mu\frac{\partial v}{\partial\alpha} + v\frac{\partial v}{\partial\beta} + \omega\frac{\partial v}{\partial\gamma} = \rho\left[\frac{\partial^2 v}{\partial\alpha^2} + \frac{\partial^2 v}{\partial\beta^2} + \frac{\partial^2 v}{\partial\gamma^2}\right] + q_2, \tag{39}$$
$$D_\eta^\delta(\omega) + \mu\frac{\partial\omega}{\partial\alpha} + v\frac{\partial\omega}{\partial\beta} + \omega\frac{\partial\omega}{\partial\gamma} = \rho\left[\frac{\partial^2\omega}{\partial\alpha^2} + \frac{\partial^2\omega}{\partial\beta^2} + \frac{\partial^2\omega}{\partial\gamma^2}\right] + q_3,$$

with initial conditions

$$\begin{cases}\mu(\alpha,\beta,\gamma,0) = -0.5\alpha + \beta + \gamma,\\ v(\alpha,\beta,\gamma,0) = \alpha - 0.5\beta + \gamma,\\ \omega(\alpha,\beta,\gamma,0) = \alpha + \beta - 0.5\gamma.\end{cases} \tag{40}$$

Further, if ρ is known, then $q_1 = -\frac{1}{\rho}\frac{\partial g}{\partial\alpha}, q_2 = -\frac{1}{\rho}\frac{\partial g}{\partial\beta}$ and $q_3 = -\frac{1}{\rho}\frac{\partial g}{\partial\gamma}$ can be determined.

First, by using **NDM**, we will solve this scheme.

After the Natural transformation of Eq. (39), we get

$$N^+\left\{\frac{\partial^\delta\mu}{\partial\eta^\delta}\right\} = N^+\left[-\left(\mu\frac{\partial\mu}{\partial\alpha} + v\frac{\partial\mu}{\partial\beta} + \omega\frac{\partial\mu}{\partial\gamma}\right)\right] + N^+\left\{\rho\left\{\frac{\partial^2\mu}{\partial\alpha^2} + \frac{\partial^2\mu}{\partial\beta^2} + \frac{\partial^2\mu}{\partial\gamma^2}\right\} + q_1\right],$$
$$N^+\left\{\frac{\partial^\delta v}{\partial\eta^\delta}\right\} = N^+\left[-\left(\mu\frac{\partial v}{\partial\alpha} + v\frac{\partial v}{\partial\beta} + \omega\frac{\partial v}{\partial\gamma}\right)\right] + N^+\left\{\rho\left\{\frac{\partial^2 v}{\partial\alpha^2} + \frac{\partial^2 v}{\partial\beta^2} + \frac{\partial^2 v}{\partial\gamma^2}\right\} + q_2\right],$$
$$N^+\left\{\frac{\partial^\delta\omega}{\partial\eta^\delta}\right\} = N^+\left[-\left(\mu\frac{\partial\omega}{\partial\alpha} + v\frac{\partial\omega}{\partial\beta} + \omega\frac{\partial\omega}{\partial\gamma}\right)\right] + N^+\left[\rho\left\{\frac{\partial^2\omega}{\partial\alpha^2} + \frac{\partial^2\omega}{\partial\beta^2} + \frac{\partial^2\omega}{\partial\gamma^2}\right\} + q_3\right],$$

$$\frac{s^\delta}{u^\delta}N^+\{\mu(\alpha,\beta,\gamma,\eta)\} - \frac{s^{\delta-1}}{u^\delta}\mu(\alpha,\beta,\gamma,0) = N^+\left[-\left(\mu\frac{\partial\mu}{\partial\alpha} + v\frac{\partial\mu}{\partial\beta} + \omega\frac{\partial\mu}{\partial\gamma}\right)\right] + N^+\left[\rho\left\{\frac{\partial^2\mu}{\partial\alpha^2} + \frac{\partial^2\mu}{\partial\beta^2} + \frac{\partial^2\mu}{\partial\gamma^2}\right\} + q_1\right],$$
$$\frac{s^\delta}{u^\delta}N^+\{v(\alpha,\beta,\gamma,\eta)\} - \frac{s^{\delta-1}}{u^\delta}v(\alpha,\beta,\gamma,0) = N^+\left[-\left(\mu\frac{\partial v}{\partial\alpha} + v\frac{\partial v}{\partial\beta} + \omega\frac{\partial v}{\partial\gamma}\right)\right] + N^+\left\{\rho\left\{\frac{\partial^2 v}{\partial\alpha^2} + \frac{\partial^2 v}{\partial\beta^2} + \frac{\partial^2 v}{\partial\gamma^2}\right\} + q_2\right],$$
$$\frac{s^\delta}{u^\delta}N^+\{\omega(\alpha,\beta,\gamma,\eta)\} - \frac{s^{\delta-1}}{u^\delta}\omega(\alpha,\beta,\gamma,0) = N^+\left[-\left(\mu\frac{\partial\omega}{\partial\alpha} + v\frac{\partial\omega}{\partial\beta} + \omega\frac{\partial\omega}{\partial\gamma}\right)\right] + N^+\left[\rho\left\{\frac{\partial^2\omega}{\partial\alpha^2} + \frac{\partial^2\omega}{\partial\beta^2} + \frac{\partial^2\omega}{\partial\gamma^2}\right\} + q_3\right],$$

The above algorithm is reduced to simplified

$$N^+\{\mu(\alpha,\beta,\gamma,\eta)\} = \frac{1}{s}\mu(\alpha,\beta,\gamma,0) + \frac{u^\delta}{s^\delta}N^+\left[-\left(\mu\frac{\partial\mu}{\partial\alpha} + v\frac{\partial\mu}{\partial\beta} + \omega\frac{\partial\mu}{\partial\gamma}\right)\right] + \frac{u^\delta}{s^\delta}N^+\left\{\rho\left\{\frac{\partial^2\mu}{\partial\alpha^2} + \frac{\partial^2\mu}{\partial\beta^2} + \frac{\partial^2\mu}{\partial\gamma^2}\right\} + q_1\right],$$
$$N^+\{v(\alpha,\beta,\gamma,\eta)\} = \frac{1}{s}v(\alpha,\beta,\gamma,0) + \frac{u^\delta}{s^\delta}N^+\left[-\left(\mu\frac{\partial v}{\partial\alpha} + v\frac{\partial v}{\partial\beta} + \omega\frac{\partial v}{\partial\gamma}\right)\right] + \frac{u^\delta}{s^\delta}N^+\left[\rho\left\{\frac{\partial^2 v}{\partial\alpha^2} + \frac{\partial^2 v}{\partial\beta^2} + \frac{\partial^2 v}{\partial\gamma^2}\right\} + q_2\right],$$
$$N^+\{\omega(\alpha,\beta,\gamma,\eta)\} = \frac{1}{s}\omega(\alpha,\beta,\gamma,0) + \frac{u^\delta}{s^\delta}N^+\left[-\left(\mu\frac{\partial\omega}{\partial\alpha} + v\frac{\partial\omega}{\partial\beta} + \omega\frac{\partial\omega}{\partial\gamma}\right)\right] + \frac{u^\delta}{s^\delta}N^+\left[\rho\left\{\frac{\partial^2\omega}{\partial\alpha^2} + \frac{\partial^2\omega}{\partial\beta^2} + \frac{\partial^2\omega}{\partial\gamma^2}\right\} + q_3\right], \tag{41}$$

Applying inverse natural transformation, we get

$$\mu(\alpha,\beta,\gamma,\eta) = \mu(\alpha,\beta,\gamma,0) + N^-\left[\frac{u^\delta}{s^\delta}N^+\left[-\left(\mu\frac{\partial\mu}{\partial\alpha} + v\frac{\partial\mu}{\partial\beta} + \omega\frac{\partial\mu}{\partial\gamma}\right)\right]\right] + N^-\left[\frac{u^\delta}{s^\delta}N^+\left\{\rho\left\{\frac{\partial^2\mu}{\partial\alpha^2} + \frac{\partial^2\mu}{\partial\beta^2} + \frac{\partial^2\mu}{\partial\gamma^2}\right\} + q_1\right]\right]$$
$$v(\alpha,\beta,\gamma,\eta) = v(\alpha,\beta,\gamma,0) + N^-\left[\frac{u^\delta}{s^\delta}N^+\left[-\left(\mu\frac{\partial v}{\partial\alpha} + v\frac{\partial v}{\partial\beta} + \omega\frac{\partial v}{\partial\gamma}\right)\right]\right] + N^-\left[\frac{u^\delta}{s^\delta}N^+\left\{\rho\left\{\frac{\partial^2 v}{\partial\alpha^2} + \frac{\partial^2 v}{\partial\beta^2} + \frac{\partial^2 v}{\partial\gamma^2}\right\} + q_2\right]\right]$$
$$\omega(\alpha,\beta,\gamma,\eta) = \omega(\alpha,\beta,\gamma,0) + N^-\left[\frac{u^\delta}{s^\delta}N^+\left[-\left(\mu\frac{\partial\omega}{\partial\alpha} + v\frac{\partial\omega}{\partial\beta} + \omega\frac{\partial\omega}{\partial\gamma}\right)\right]\right] + N^-\left[\frac{u^\delta}{s^\delta}N^+\left\{\rho\left\{\frac{\partial^2\omega}{\partial\alpha^2} + \frac{\partial^2\omega}{\partial\beta^2} + \frac{\partial^2\omega}{\partial\gamma^2}\right\} + q_3\right]\right] \tag{42}$$

Assume that the unknown functions $\mu(\alpha,\beta,\gamma,\eta), v(\alpha,\beta,\gamma,\eta)$ and $\omega(\alpha,\beta,\gamma,\eta)$ have infinite series solution as follows:

$$\mu(\alpha,\beta,\gamma,\eta) = \sum_{m=0}^{\infty}\mu_m(\alpha,\beta,\gamma,\eta),$$
$$v(\alpha,\beta,\gamma,\eta) = \sum_{m=0}^{\infty}v_m(\alpha,\beta,\gamma,\eta), \quad \textit{and}$$
$$\omega(\alpha,\beta,\gamma,\eta) = \sum_{m=0}^{\infty}\omega_m(\alpha,\beta,\gamma,\eta)$$

The Adomian polynomials of non-linear terms as, using such terms, Eq. (42) can be rewritten in the form

$$\mu\mu_\alpha=\sum_{m=0}^{\infty}\mathcal{A}_m,\quad \nu\mu_\beta=\sum_{m=0}^{\infty}\mathcal{B}_m,$$
$$\omega\mu_\gamma=\sum_{m=0}^{\infty}\mathcal{C}_m,\quad \mu\nu_\alpha=\sum_{m=0}^{\infty}\mathcal{D}_m,$$
$$\nu\nu_\beta=\sum_{m=0}^{\infty}\mathcal{E}_m,\quad \omega\nu_\beta=\sum_{m=0}^{\infty}\mathcal{F}_m,$$
$$\mu\omega_\alpha=\sum_{m=0}^{\infty}\mathcal{G}_m,\quad \nu\omega_\beta=\sum_{m=0}^{\infty}\mathcal{H}_m,\quad and$$
$$\omega\omega_\beta=\sum_{m=0}^{\infty}\mathcal{I}_m,$$

$$\begin{aligned}\sum_{m=0}^{\infty}\mu_m(\alpha,\beta,\gamma,\eta)&=\mu(\alpha,\beta,\gamma,0)+N^-\left[\tfrac{u^\delta}{s^\delta}N^+\{q_1\}\right]\\&+N^-\left[\tfrac{u^\delta}{s^\delta}N^+\left[-\left(\sum_{m=0}^{\infty}\mathcal{A}_m+\sum_{m=0}^{\infty}\mathcal{B}_m+\sum_{m=0}^{\infty}\mathcal{C}_m\right)\right]\right]\\&+N^-\left[\tfrac{u^\delta}{s^\delta}N^+\left\{\rho\left\{\tfrac{\partial^2\mu_m}{\partial\alpha^2}+\tfrac{\partial^2\mu_m}{\partial\beta^2}+\tfrac{\partial^2\mu_m}{\partial\gamma^2}\right\}+q_1\right]\right],\\ \sum_{m=0}^{\infty}\nu_m(\alpha,\beta,\gamma,\eta)&=\nu(\alpha,\beta,\gamma,0)-N^-\left[\tfrac{u^\delta}{s^\delta}N^+\{q_2\}\right]\\&+N^-\left[\tfrac{u^\delta}{s^\delta}N^+\left[-\left(\sum_{m=0}^{\infty}\mathcal{D}_m+\sum_{m=0}^{\infty}\mathcal{E}_m+\sum_{m=0}^{\infty}\mathcal{F}_m\right)\right]\right]\\&+N^-\left[\tfrac{u^\delta}{s^\delta}N^+\left\{\rho\left\{\tfrac{\partial^2\nu_m}{\partial\alpha^2}+\tfrac{\partial^2\nu_m}{\partial\beta^2}+\tfrac{\partial^2\nu_m}{\partial\gamma^2}\right\}+q_2\right]\right],\\ \sum_{m=0}^{\infty}\omega_m(\alpha,\beta,\gamma,\eta)&=\omega(\alpha,\beta,\gamma,0)-N^-\left[\tfrac{u^\delta}{s^\delta}N^+\{q_3\}\right]\\&+N^-\left[\tfrac{u^\delta}{s^\delta}N^+\left[-\left(\sum_{m=0}^{\infty}\mathcal{G}_m+\sum_{m=0}^{\infty}\mathcal{H}_m+\sum_{m=0}^{\infty}\mathcal{I}_m\right)\right]\right]\\&+N^-\left[\tfrac{u^\delta}{s^\delta}N^+\left\{\tfrac{\partial^2\omega}{\partial\alpha^2}+\tfrac{\partial^2\omega}{\partial\beta^2}+\tfrac{\partial^2\omega}{\partial\gamma^2}\right\}+q_3\right].\end{aligned}\tag{43}$$

According to Eq. (15), all forms of non-linearity the Adomian polynomials can be defined as

$$\mathcal{A}_0=\mu_0\tfrac{\partial\mu_0}{\partial\alpha},\quad \mathcal{A}_1=\mu_0\tfrac{\partial\mu_1}{\partial\alpha}+\mu_1\tfrac{\partial\mu_0}{\partial\alpha},$$
$$\mathcal{B}_0=\nu_0\tfrac{\partial\mu_0}{\partial\beta},\quad \mathcal{B}_1=\nu_0\tfrac{\partial\mu_1}{\partial\beta}+\nu_1\tfrac{\partial\mu_0}{\partial\beta},$$
$$\mathcal{C}_0=\omega_0\tfrac{\partial\mu_0}{\partial\beta},\quad \mathcal{C}_1=\omega_0\tfrac{\partial\mu_1}{\partial\beta}+\omega_1\tfrac{\partial\mu_0}{\partial\beta},$$
$$\mathcal{D}_0=\mu_0\tfrac{\partial\nu_0}{\partial\alpha},\quad \mathcal{D}_1=\mu_0\tfrac{\partial\nu_1}{\partial\alpha}+\mu_1\tfrac{\partial\nu_0}{\partial\alpha},$$
$$\mathcal{E}_0=\nu_0\tfrac{\partial\nu_0}{\partial\alpha},\quad \mathcal{E}_1=\nu_0\tfrac{\partial\nu_1}{\partial\alpha}+\nu_1\tfrac{\partial\nu_0}{\partial\alpha},$$
$$\mathcal{F}_0=\omega_0\tfrac{\partial\nu_0}{\partial\alpha},\quad \mathcal{F}_1=\omega_0\tfrac{\partial\nu_1}{\partial\alpha}+\omega_1\tfrac{\partial\nu_0}{\partial\alpha},$$
$$\mathcal{G}_0=\mu_0\tfrac{\partial\omega_0}{\partial\alpha},\quad \mathcal{G}_1=\mu_0\tfrac{\partial\omega_1}{\partial\alpha}+\mu_1\tfrac{\partial\omega_0}{\partial\alpha},$$
$$\mathcal{H}_0=\nu_0\tfrac{\partial\omega_0}{\partial\alpha},\quad \mathcal{H}_1=\nu_0\tfrac{\partial\omega_1}{\partial\alpha}+\nu_1\tfrac{\partial\omega_0}{\partial\alpha},$$
$$\mathcal{I}_0=\omega_0\tfrac{\partial\omega_0}{\partial\alpha},\quad \mathcal{I}_1=\omega_0\tfrac{\partial\omega_1}{\partial\alpha}+\omega_1\tfrac{\partial\omega_0}{\partial\alpha},$$

Thus, we can easily obtain the recursive relationship by comparing two sides of Eq. (43)

$$\mu_0(\alpha,\beta,\gamma,\eta)=-0.5\alpha+\beta+\gamma,$$
$$\nu_0(\alpha,\beta,\gamma,\eta)=\alpha-0.5\beta+\gamma,$$
$$\omega_0(\alpha,\beta,\gamma,\eta)=\alpha+\beta-0.5\gamma.$$

For $m=0$

$$\mu_1(\alpha,\beta,\gamma,\eta)=\tfrac{-2.25\alpha\eta^\delta}{\Gamma(\delta+1)},$$
$$\nu_1(\alpha,\beta,\gamma,\eta)=\tfrac{-2.25\beta\eta^\delta}{\Gamma(\delta+1)},$$
$$\omega_1(\alpha,\beta,\gamma,\eta)=\tfrac{-2.25\gamma\eta^\delta}{\Gamma(\delta+1)}.$$

For $m=1$

$$\mu_2(\alpha,\beta,\gamma,\eta)=\tfrac{2(2.25)\alpha\eta^{2\delta}}{\Gamma(2\delta+1)}(-0.5\alpha+\beta+\gamma),$$
$$\nu_2(\alpha,\beta,\gamma,\eta)=\tfrac{2(2.25)\beta\eta^{2\delta}}{\Gamma(2\delta+1)}(\alpha-0.5\beta+\gamma),$$
$$\omega_2(\alpha,\beta,\gamma,\eta)=\tfrac{2(2.25)\gamma\eta^{2\delta}}{\Gamma(2\delta+1)}(\alpha+\beta-0.5\gamma).$$

For $m=2$

$$\mu_3(\alpha,\beta,\gamma,\eta)=-\frac{(2.25)^2\alpha\left(4(\Gamma(\delta+1))^2+\Gamma(2\delta+1)\right)\eta^{3\delta}}{\Gamma(2\delta+1)(\Gamma(\delta+1))^2},$$
$$\nu_3(\alpha,\beta,\gamma,\eta)=-\frac{(2.25)^2\beta\left(4(\Gamma(\delta+1))^2+\Gamma(2\delta+1)\right)\eta^{3\delta}}{\Gamma(2\delta+1)(\Gamma(\delta+1))^2},$$
$$\omega_3(\alpha,\beta,\gamma,\eta)=-\frac{(2.25)^2\gamma\left(4(\Gamma(\delta+1))^2+\Gamma(2\delta+1)\right)\eta^{3\delta}}{\Gamma(2\delta+1)(\Gamma(\delta+1))^2},.$$

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using the same technique, the remaining μ_m,ν_m and ω_m $(m\geqslant 3)$ elements of the NDM solution can be obtained smoothly. Therefore, we define the series of alternatives as

$$\mu(\alpha,\beta,\gamma,\eta)=\sum_{m=0}^{\infty}\mu_m(\alpha,\beta)=\mu_0(\alpha,\beta)+\mu_1(\alpha,\beta)+\mu_2(\alpha,\beta)+\mu_3(\alpha,\beta)+\cdots,$$
$$\nu(\alpha,\beta,\gamma,\eta)=\sum_{m=0}^{\infty}\nu_m(\alpha,\beta)=\nu_0(\alpha,\beta)+\nu_1(\alpha,\beta)+\nu_2(\alpha,\beta)+\nu_3(\alpha,\beta)+\cdots,$$

$$\mu(\alpha,\beta,\gamma,\eta)=-0.5\alpha+\beta+\gamma-\tfrac{2.25\alpha\eta^\delta}{\Gamma(\delta+1)}+\tfrac{2(2.25)\alpha\eta^{2\delta}}{\Gamma(2\delta+1)}$$
$$(-0.5\alpha+\beta+\gamma)-\frac{(2.25)^2\alpha\left(4(\Gamma(\delta+1))^2+\Gamma(2\delta+1)\right)\eta^{3\delta}}{\Gamma(2\delta+1)(\Gamma(\delta+1))^2}+\cdots,$$
$$\nu(\alpha,\beta,\gamma,\eta)=\alpha-0.5\beta+\gamma-\tfrac{2.25\beta\eta^\delta}{\Gamma(\delta+1)}+\tfrac{2(2.25)\beta\eta^{2\delta}}{\Gamma(2\delta+1)}$$
$$(\alpha-0.5\beta+\gamma)-\frac{(2.25)^2\beta\left(4(\Gamma(\delta+1))^2+\Gamma(2\delta+1)\right)\eta^{3\delta}}{\Gamma(2\delta+1)(\Gamma(\delta+1))^2}+\cdots,$$
$$\omega(\alpha,\beta,\gamma,\eta)=\alpha+\beta-0.5\gamma-\tfrac{2.25\gamma\eta^\delta}{\Gamma(\delta+1)}+\tfrac{2(2.25)\gamma\eta^{2\delta}}{\Gamma(2\delta+1)}$$
$$(\alpha+\beta-0.5\gamma)-\frac{(2.25)^2\gamma\left(4(\Gamma(\delta+1))^2+\Gamma(2\delta+1)\right)\eta^{3\delta}}{\Gamma(2\delta+1)(\Gamma(\delta+1))^2}+\cdots,$$

The approximate solution by **NVIM**.

According to the Eqs. (7) the iteration formulas for system (39),

$$\begin{aligned}&\mu_{m+1}(\alpha,\beta,\gamma,\eta)=\mu_m(\alpha,\beta,\gamma,\eta)\\&-N^-\left[\tfrac{u^\delta}{s^\delta}N^+\left\{\tfrac{s^\delta}{u^\delta}\tfrac{\partial\mu_m}{\partial\eta}+\mu\tfrac{\partial\mu}{\partial\alpha}+\nu\tfrac{\partial\mu}{\partial\beta}+\omega\tfrac{\partial\mu}{\partial\gamma}+\rho\left(\tfrac{\partial^2\mu}{\partial\alpha^2}+\tfrac{\partial^2\mu}{\partial\beta^2}+\tfrac{\partial^2\mu}{\partial\gamma^2}\right)+q_1\right\}\right],\\&\nu_{m+1}(\alpha,\beta,\gamma,\eta)=\nu_m(\alpha,\beta,\gamma,\eta)\\&-N^-\left[\tfrac{u^\delta}{s^\delta}N^+\left\{\tfrac{s^\delta}{u^\delta}\tfrac{\partial\nu_m}{\partial\eta}+\mu\tfrac{\partial\nu}{\partial\alpha}+\nu\tfrac{\partial\nu}{\partial\beta}+\omega\tfrac{\partial\nu}{\partial\gamma}+\rho\left(\tfrac{\partial^2\nu}{\partial\alpha^2}+\tfrac{\partial^2\nu}{\partial\beta^2}+\tfrac{\partial^2\nu}{\partial\gamma^2}\right)+q_2\right\}\right],\\&\omega_{m+1}(\alpha,\beta,\gamma,\eta)=\omega_m(\alpha,\beta,\gamma,\eta)\\&-N^-\left[\tfrac{u^\delta}{s^\delta}N^+\left\{\tfrac{s^\delta}{u^\delta}\tfrac{\partial\nu_m}{\partial\eta}+\mu\tfrac{\partial\omega}{\partial\alpha}+\nu\tfrac{\partial\omega}{\partial\beta}+\omega\tfrac{\partial\omega}{\partial\gamma}+\rho\left(\tfrac{\partial^2\omega}{\partial\alpha^2}+\tfrac{\partial^2\omega}{\partial\beta^2}+\tfrac{\partial^2\omega}{\partial\gamma^2}\right)+q_3\right\}\right],\end{aligned}\tag{44}$$

with the initial conditions

$$\begin{cases}\mu(\alpha,\beta,\gamma,0)=-0.5\alpha+\beta+\gamma,\\ \nu(\alpha,\beta,\gamma,0)=\alpha-0.5\beta+\gamma,\\ \omega(\alpha,\beta,\gamma,0)=\alpha+\beta-0.5\gamma.\end{cases}\tag{45}$$

For $\quad m = 0, 1, 2, \cdots$

$$\mu_1(\alpha,\beta,\gamma,\eta) = \mu_0(\alpha,\beta,\gamma,\eta)$$
$$-N^{-}\left[\frac{u^{\delta}}{s^{\delta}}N^{+}\left\{\frac{s^{\delta}}{u^{\delta}}\frac{\partial\mu_0}{\partial\eta} + \mu_0\frac{\partial\mu_0}{\partial\alpha} + v_0\frac{\partial\mu_0}{\partial\beta} + \omega_0\frac{\partial\mu_0}{\partial\gamma} + \rho\left(\frac{\partial^2\mu_0}{\partial\alpha^2} + \frac{\partial^2\mu_0}{\partial\beta^2} + \frac{\partial^2\mu_0}{\partial\gamma^2}\right) + q_1\right\}\right],$$
$$v_1(\alpha,\beta,\gamma,\eta) = v_0(\alpha,\beta,\gamma,\eta)$$
$$-N^{-}\left[\frac{u^{\delta}}{s^{\delta}}N^{+}\left\{\frac{s^{\delta}}{u^{\delta}}\frac{\partial v_0}{\partial\eta} + \mu_0 \times \frac{\partial v_0}{\partial\alpha} + v_0\frac{\partial v_0}{\partial\beta} + \omega_0\frac{\partial v_0}{\partial\gamma} + \rho\left(\frac{\partial^2 v_0}{\partial\alpha^2} + \frac{\partial^2 v_0}{\partial\beta^2} + \frac{\partial^2 v_0}{\partial\gamma^2}\right) + q_2\right\}\right],$$
$$\omega_1(\alpha,\beta,\gamma,\eta) = \omega_0(\alpha,\beta,\gamma,\eta)$$
$$-N^{-}\left[\frac{u^{\delta}}{s^{\delta}}N^{+}\left\{\frac{s^{\delta}}{u^{\delta}}\frac{\partial v_0}{\partial\eta} + \mu_0\frac{\partial\omega_0}{\partial\alpha} + v_0\frac{\partial\omega_0}{\partial\beta} + \omega_0\frac{\partial\omega_0}{\partial\gamma} + \rho\left(\frac{\partial^2\omega_0}{\partial\alpha^2} + \frac{\partial^2\omega_0}{\partial\beta^2} + \frac{\partial^2\omega_0}{\partial\gamma^2}\right) + q_3\right\}\right],$$

$$\mu_1(\alpha,\beta,\gamma,\eta) = -0.5\alpha + \beta + \gamma - \frac{2.25\alpha\eta^{\delta}}{\Gamma(\delta+1)},$$
$$v_1(\alpha,\beta,\gamma,\eta) = \alpha - 0.5\beta + \gamma - \frac{2.25\beta\eta^{\delta}}{\Gamma(\delta+1)},$$
$$\omega_1(\alpha,\beta,\gamma,\eta) = \alpha + \beta - 0.5\gamma - \frac{2.25\gamma\eta^{\delta}}{\Gamma(\delta+1)},$$

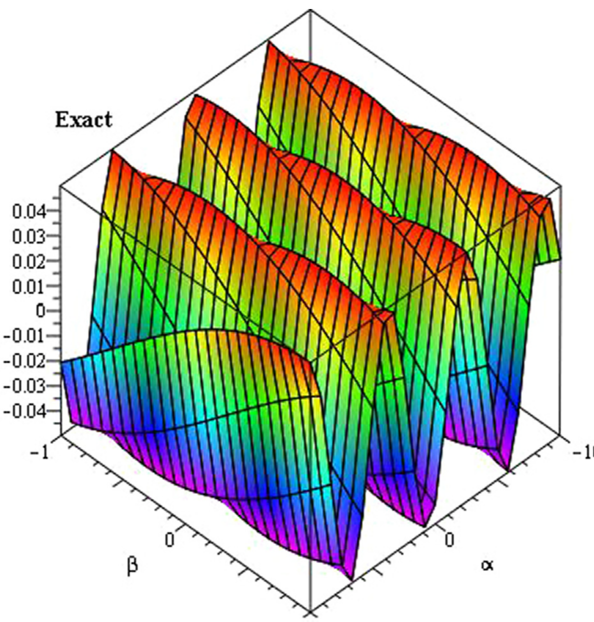

Fig. 1 Exact solution of $\mu(\alpha,\beta,\eta)$ of Example 1 at $\delta = 1$.

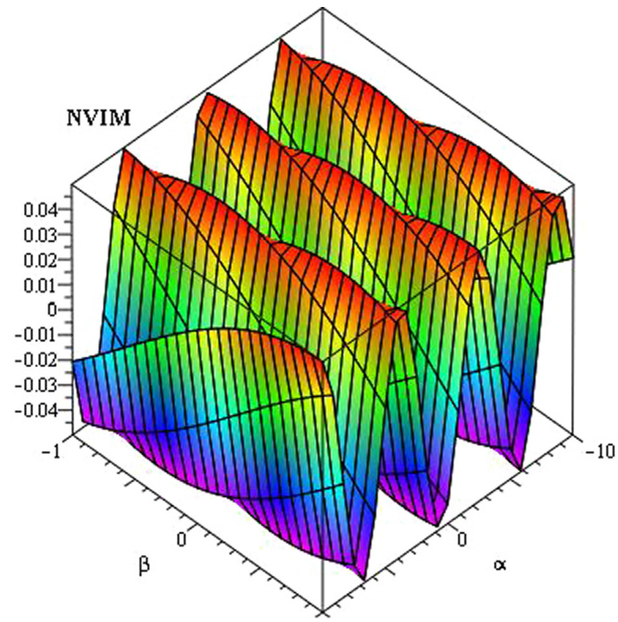

Fig. 3 NVIM solution $\mu(\alpha,\beta,\eta)$ of Example 1 at $\delta = 1$.

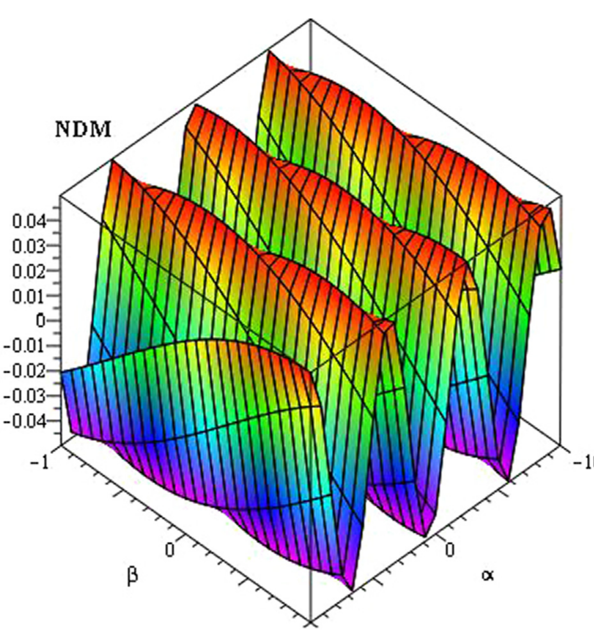

Fig. 2 NDM solution $\mu(\alpha,\beta,\eta)$ of Example 1 at $\delta = 1$.

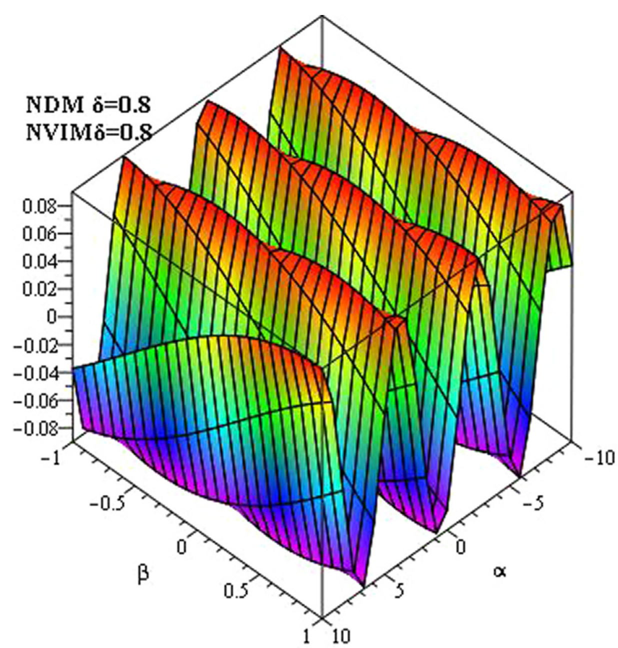

Fig. 4 NDM and NVIM solutions $\mu(\alpha,\beta,\eta)$ of Example 1 at $\delta = 0.8$.

$$\mu_2(\alpha,\beta,\gamma,\eta) = \mu_1(\alpha,\beta,\gamma,\eta)$$
$$-N^{-}\left[\frac{u^\delta}{s^\delta}N^{+}\left\{\frac{s^\delta}{u^\delta}\frac{\partial\mu_1}{\partial\eta} + \mu_1\times\frac{\partial\mu_1}{\partial\alpha} + v_1\frac{\partial\mu_1}{\partial\beta} + \omega_1\frac{\partial\mu_1}{\partial\gamma} + \rho\left(\frac{\partial^2\mu_1}{\partial\alpha^2} + \frac{\partial^2\mu_1}{\partial\beta^2} + \frac{\partial^2\mu_1}{\partial\gamma^2}\right) + q_1\right\}\right],$$
$$v_2(\alpha,\beta,\gamma,\eta) = v_1(\alpha,\beta,\gamma,\eta)$$
$$-N^{-}\left[\frac{u^\delta}{s^\delta}N^{+}\left\{\frac{s^\delta}{u^\delta}\frac{\partial v_1}{\partial\eta} + \mu_1\times\frac{\partial v_1}{\partial\alpha} + v_1\frac{\partial v_1}{\partial\beta} + \omega_1\frac{\partial v_1}{\partial\gamma} + \rho\left(\frac{\partial^2 v_1}{\partial\alpha^2} + \frac{\partial^2 v_1}{\partial\beta^2} + \frac{\partial^2 v_1}{\partial\gamma^2}\right) + q_2\right\}\right],$$
$$\omega_2(\alpha,\beta,\gamma,\eta) = \omega_1(\alpha,\beta,\gamma,\eta)$$
$$-N^{-}\left[\frac{u^\delta}{s^\delta}N^{+}\left\{\frac{s^\delta}{u^\delta}\frac{\partial v_1}{\partial\eta} + \mu_1\times\frac{\partial\omega_1}{\partial\alpha} + v_1\frac{\partial\omega_1}{\partial\beta} + \omega_1\frac{\partial\omega_1}{\partial\gamma} + \rho\left(\frac{\partial^2\omega_1}{\partial\alpha^2} + \frac{\partial^2\omega_1}{\partial\beta^2} + \frac{\partial^2\omega_1}{\partial\gamma^2}\right) + q_3\right\}\right],$$

$$\mu_2(\alpha,\beta,\gamma,\eta) = -0.5\alpha + \beta + \gamma - \frac{2.25\alpha\eta^\delta}{\Gamma(\delta+1)} + \frac{2(2.25)\alpha\eta^{2\delta}}{\Gamma(2\delta+1)}$$
$$(-0.5\alpha + \beta + \gamma,)$$
$$v_2(\alpha,\beta,\gamma,\eta) = \alpha - 0.5\beta + \gamma - \frac{2.25\beta\eta^\delta}{\Gamma(\delta+1)} + \frac{2(2.25)\beta\eta^{2\delta}}{\Gamma(2\delta+1)}$$
$$(\alpha - 0.5\beta + \gamma),$$
$$\omega_2(\alpha,\beta,\gamma,\eta) = \alpha + \beta - 0.5\gamma - \frac{2.25\gamma\eta^\delta}{\Gamma(\delta+1)} + \frac{2(2.25)\gamma\eta^{2\delta}}{\Gamma(2\delta+1)}$$
$$(\alpha + \beta - 0.5\gamma),$$

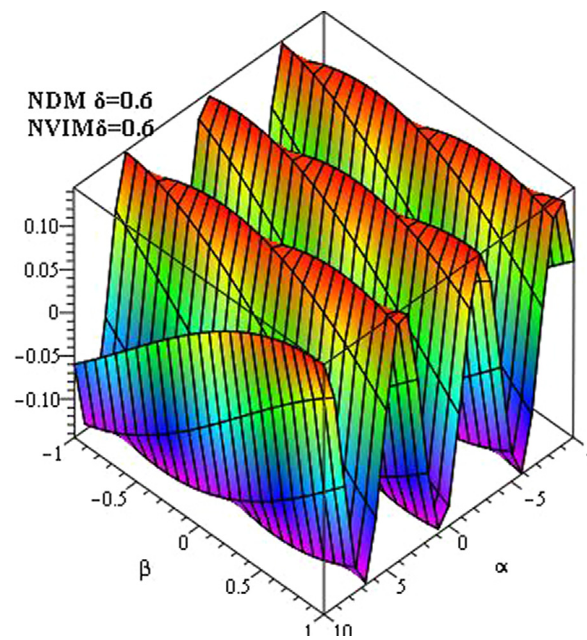

Fig. 5 NDM and NVIM solutions $\mu(\alpha,\beta,\eta)$ of Example 1 at $\delta = 0.6$.

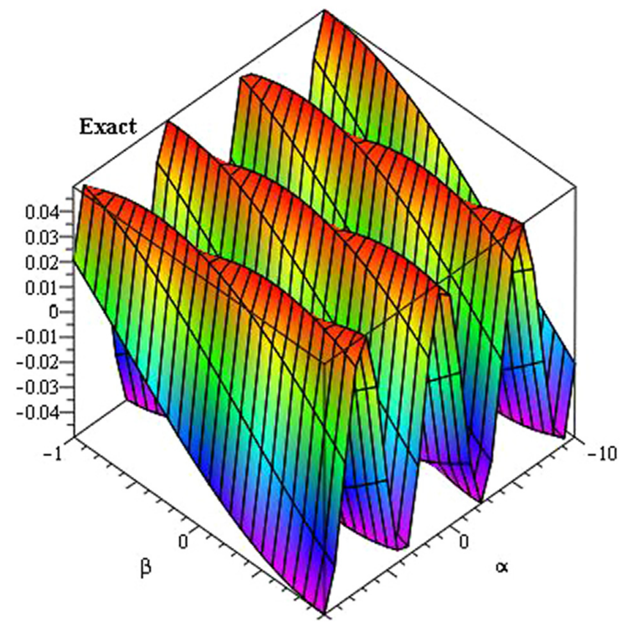

Fig. 7 Exact solution of $v(\alpha,\beta,\eta)$ of Example 1 at $\delta = 1$.

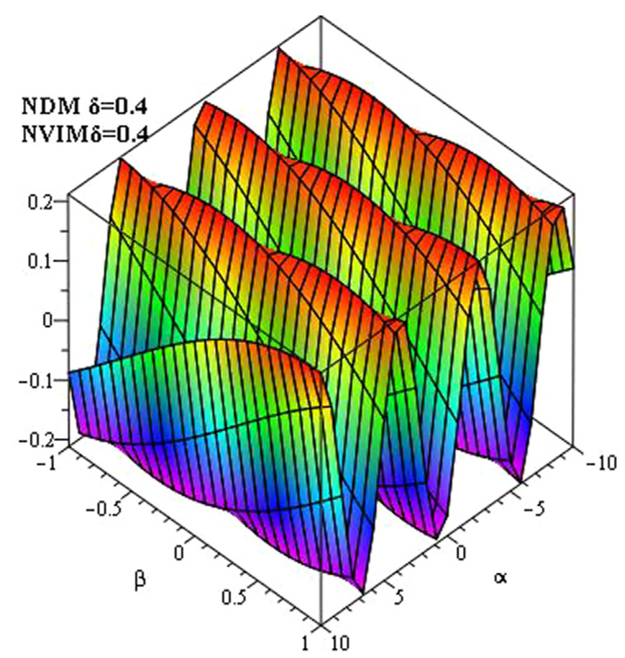

Fig. 6 NDM and NVIM solutions $\mu(\alpha,\beta,\eta)$ of Example 1 at $\delta = 0.4$.

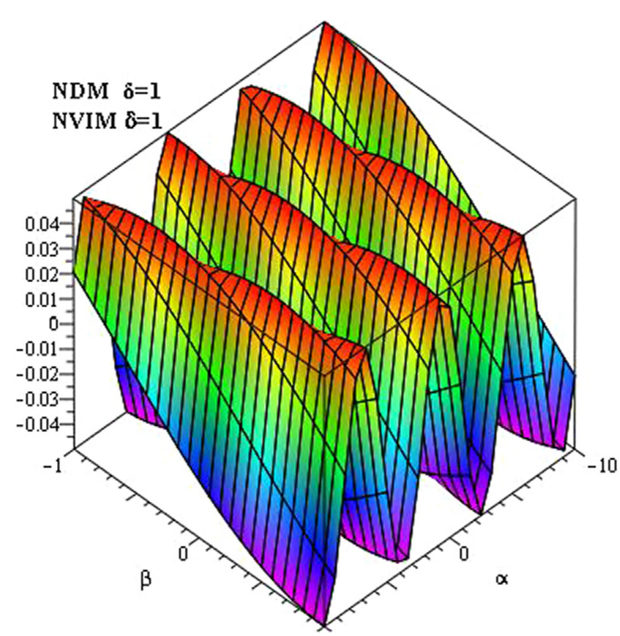

Fig. 8 NDM and NVIM solutions $v(\alpha,\beta,\eta)$ of Example 1 at $\delta = 1$.

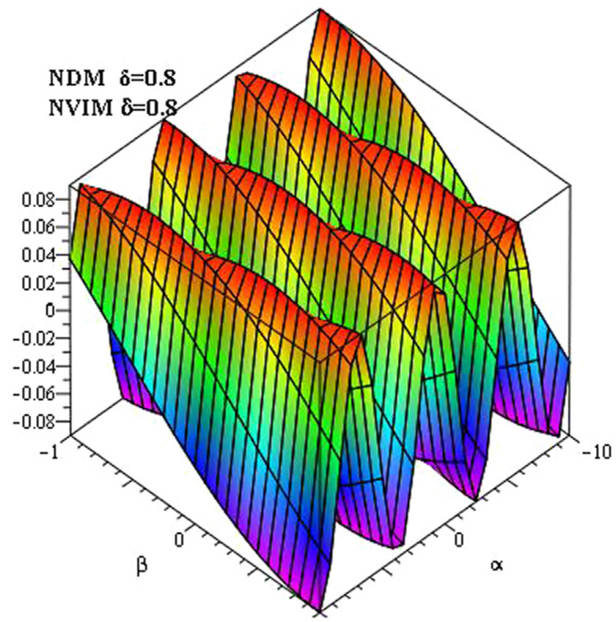

Fig. 9 NDM and NVIM solutions $v(\alpha, \beta, \eta)$ of Example 1 at $\delta = 0.8$.

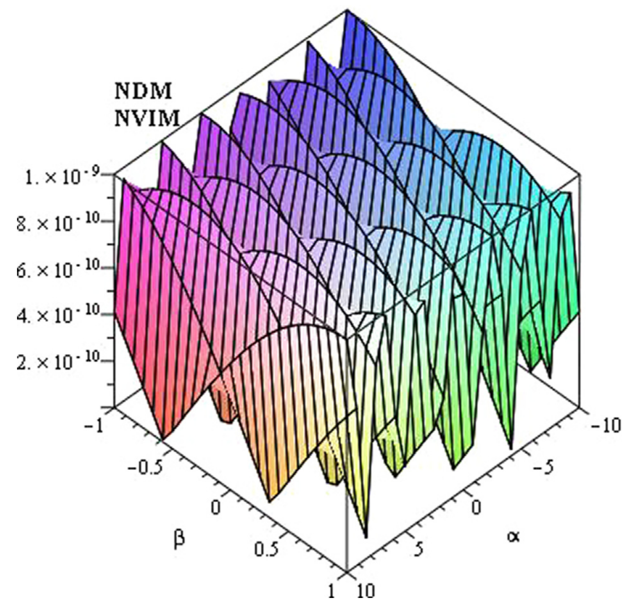

Fig. 11 $\mu(\alpha, \beta, \eta)$ error plot of Example 1, using NDM and NVIM.

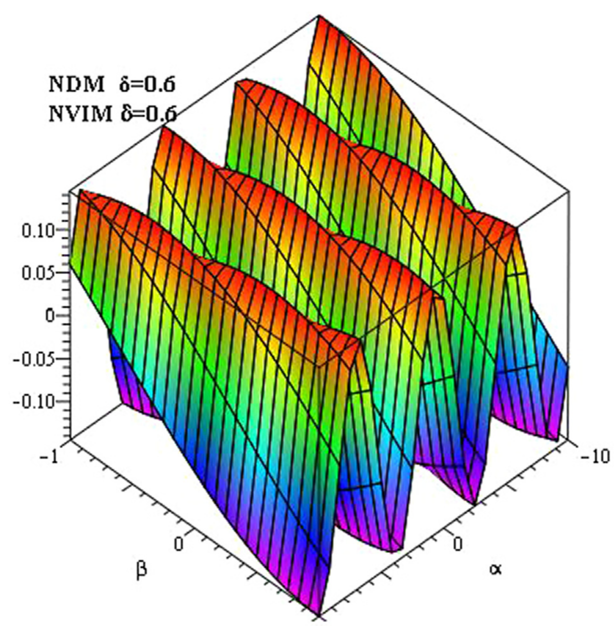

Fig. 10 NDM and NVIM solutions $v(\alpha, \beta, \eta)$ of Example 1 at $\delta = 0.6$.

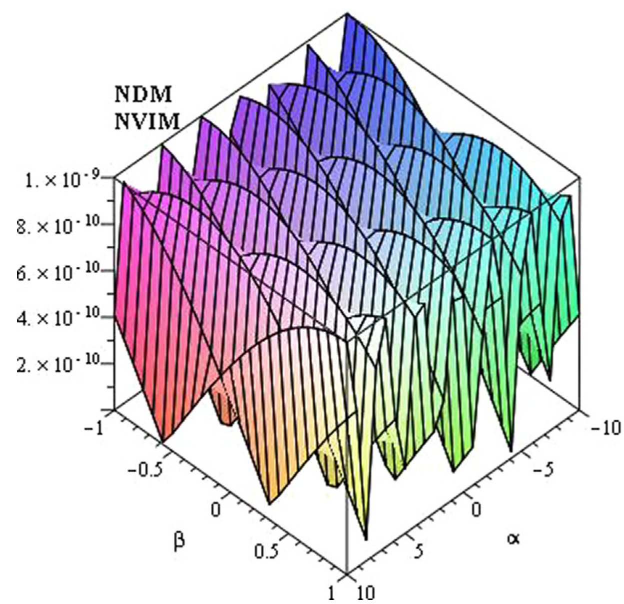

Fig. 12 $v(\alpha, \beta, \eta)$ error plot of Example 1, using NDM and NVIM.

$$
\begin{aligned}
&\mu_3(\alpha,\beta,\gamma,\eta) = \mu_2(\alpha,\beta,\gamma,\eta)\\
&-N^{-}\left[\tfrac{u^\delta}{s^\delta} N^{+}\left\{\tfrac{s^\delta}{u^\delta}\tfrac{\partial \mu_2}{\partial \eta} + \mu_2 \times \tfrac{\partial \mu_2}{\partial \alpha} + v_2 \tfrac{\partial \mu_2}{\partial \beta} + \omega_2 \tfrac{\partial \mu_2}{\partial \gamma} + \rho\left(\tfrac{\partial^2 \mu_2}{\partial \alpha^2} + \tfrac{\partial^2 \mu_2}{\partial \beta^2} + \tfrac{\partial^2 \mu_2}{\partial \gamma^2}\right) + q_1\right\}\right],\\
&v_3(\alpha,\beta,\gamma,\eta) = v_2(\alpha,\beta,\gamma,\eta)\\
&-N^{-}\left[\tfrac{u^\delta}{s^\delta} N^{+}\left\{\tfrac{s^\delta}{u^\delta}\tfrac{\partial v_2}{\partial \eta} + \mu_2 \times \tfrac{\partial v_2}{\partial \alpha} + v_2 \tfrac{\partial v_2}{\partial \beta} + \omega_2 \tfrac{\partial v_2}{\partial \gamma} + \rho\left(\tfrac{\partial^2 v_2}{\partial \alpha^2} + \tfrac{\partial^2 v_2}{\partial \beta^2} + \tfrac{\partial^2 v_2}{\partial \gamma^2}\right) + q_2\right\}\right],\\
&\omega_3(\alpha,\beta,\gamma,\eta) = \omega_2(\alpha,\beta,\gamma,\eta)\\
&-N^{-}\left[\tfrac{u^\delta}{s^\delta} N^{+}\left\{\tfrac{s^\delta}{u^\delta}\tfrac{\partial v_2}{\partial \eta} + \mu_2 \tfrac{\partial \omega_2}{\partial \alpha} + v_2 \tfrac{\partial \omega_2}{\partial \beta} + \omega_2 \tfrac{\partial \omega_2}{\partial \gamma} + \rho\left(\tfrac{\partial^2 \omega_2}{\partial \alpha^2} + \tfrac{\partial^2 \omega_2}{\partial \beta^2} + \tfrac{\partial^2 \omega_2}{\partial \gamma^2}\right) + q_3\right\}\right],
\end{aligned}
$$

$$
\begin{aligned}
&\mu_3(\alpha,\beta,\gamma,\eta) = -0.5\alpha + \beta + \gamma - \tfrac{2.25\alpha\eta^\delta}{\Gamma(\delta+1)} + \tfrac{2(2.25)\alpha\eta^{2\delta}}{\Gamma(2\delta+1)}(-0.5\alpha + \beta + \gamma)\\
&\qquad - \tfrac{(2.25)^2\alpha\left(4(\Gamma(\delta+1))^2 + \Gamma(2\delta+1)\right)\eta^{3\delta}}{\Gamma(2\delta+1)(\Gamma(\delta+1))^2},\\
&v_3(\alpha,\beta,\gamma,\eta) = \alpha - 0.5\beta + \gamma - \tfrac{2.25\beta\eta^\delta}{\Gamma(\delta+1)} + \tfrac{2(2.25)\beta\eta^{2\delta}}{\Gamma(2\delta+1)}\\
&(\alpha - 0.5\beta + \gamma) - \tfrac{(2.25)^2\beta\left(4(\Gamma(\delta+1))^2 + \Gamma(2\delta+1)\right)\eta^{3\delta}}{\Gamma(2\delta+1)(\Gamma(\delta+1))^2},\\
&\omega_3(\alpha,\beta,\gamma,\eta) = \alpha + \beta - 0.5\gamma - \tfrac{2.25\gamma\eta^\delta}{\Gamma(\delta+1)} + \tfrac{2(2.25)\gamma\eta^{2\delta}}{\Gamma(2\delta+1)}\\
&(\alpha + \beta - 0.5\gamma) - \tfrac{(2.25)^2\gamma\left(4(\Gamma(\delta+1))^2 + \Gamma(2\delta+1)\right)\eta^{3\delta}}{\Gamma(2\delta+1)(\Gamma(\delta+1))^2}.
\end{aligned}
$$

Table 1 $\mu(\alpha, \beta, \eta)$ Comparison of NDM, NVIM and FRDTM [50] of Example 1 at $\rho = 0.5$.

$\eta = 1$		AE of FRDTM	AE of NVIM	AE of NDM
α	β	$\delta = 1$	$\delta = 1$	$\delta = 1$
1	1	1.0911579134E−09	1.0911469120E−09	1.0911567125E−09
2	2	9.0816287990E−10	9.0816299440E−10	9.0816299660E−10
3	3	3.3529869788E−10	3.3529859780E−10	3.3529849783E−10
4	4	1.1872298960E−09	1.1872298960E−09	1.1872298960E−09
5	5	6.5282533310E−10	6.5282533310E−10	6.5282533310E−10
6	6	6.4388750160E−10	6.4388750160E−10	6.4388750160E−10
7	7	1.1887288270E−09	1.1887288270E−09	1.1887288270E−09
8	8	3.4548398000E−10	3.4548398000E−10	3.4548398000E−10
9	9	9.0118469620E−10	9.0118469620E−10	9.0118469620E−10
10	10	1.0955343010E−09	1.0955343010E−09	1.0955343010E−09

Table 2 $v(\alpha, \beta, \eta)$ Comparison of NDM, NVIM and FRDTM [50] of Example 1 at $\rho = 0.5$.

$\eta = 2$		AE of FRDTM	AE of NVIM	AE of NDM
α	β	$\delta = 1$	$\delta = 1$	$\delta = 1$
1	1	2.90975176E−09	2.90975176E−09	2.90975176E−09
2	2	2.42176798E−09	2.42176798E−09	2.42176798E−09
3	3	8.94129594E−10	8.94129594E−10	8.94129594E−10
4	4	3.16594638E−09	3.16594638E−09	3.16594638E−09
5	5	1.74086755E−09	1.74086755E−09	1.74086755E−09
6	6	1.71703333E−09	1.71703333E−09	1.71703333E−09
7	7	3.16994353E−09	3.16994353E−09	3.16994353E−09
8	8	9.21290613E−10	9.21290613E−10	9.21290613E−10
9	9	2.40315919E−09	2.40315919E−09	2.40315919E−09
10	10	2.92142484E−09	2.92142483E−09	2.92142480E−09

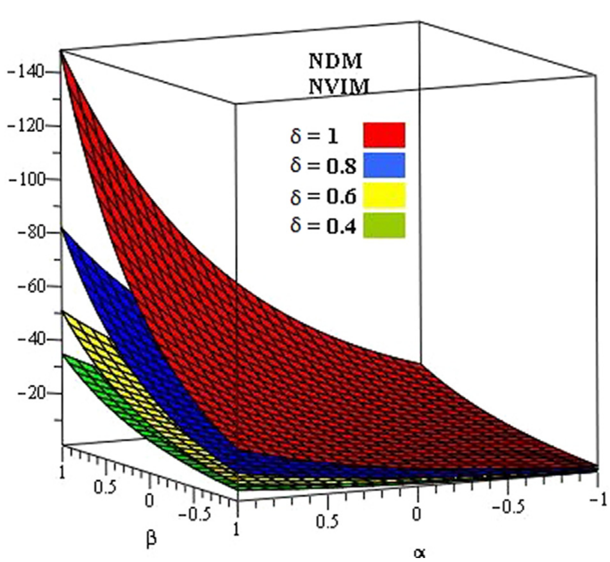

Fig. 13 NDM and NVIM solutions $\mu(\alpha, \beta, \eta)$ of Example 2 at different fractional-order δ.

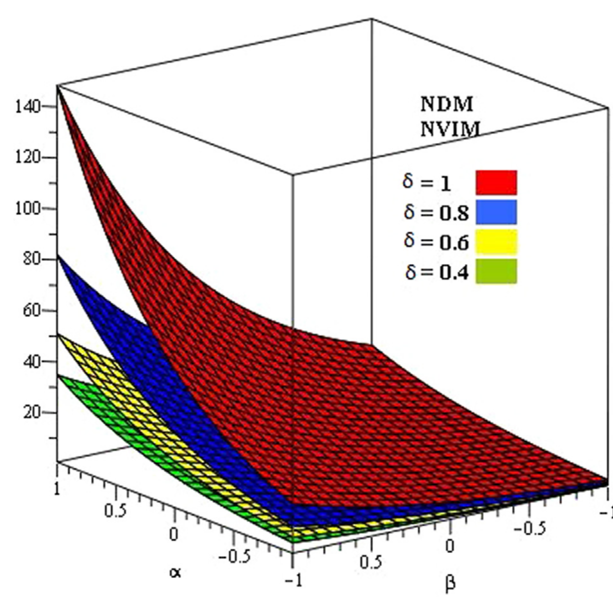

Fig. 14 NDM and NVIM solutions $v(\alpha, \beta, \eta)$ of Example 2 at different fractional-order δ.

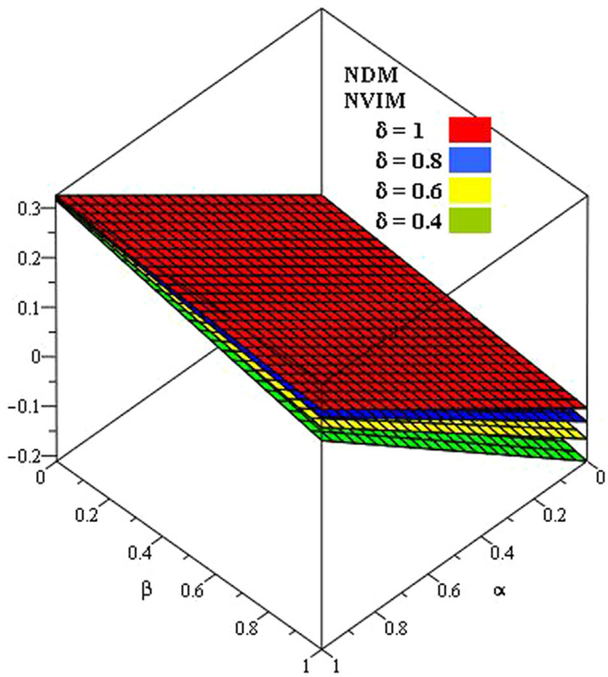

Fig. 15 NDM and NVIM solutions $\mu(\alpha,\beta,\gamma,\eta)$ of Example 3 at different fractional-order δ.

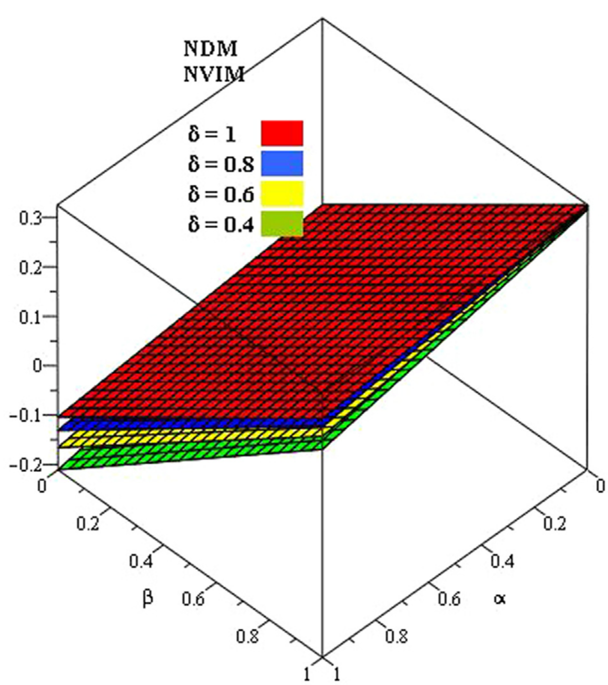

Fig. 16 NDM and NVIM solutions $v(\alpha,\beta,\gamma,\eta)$ of Example 3 at different fractional-order δ.

In the same procedure, the remaining μ_m, v_m and ω_m $(m \geqslant 3)$ components of the NDM solution can be obtained smoothly.

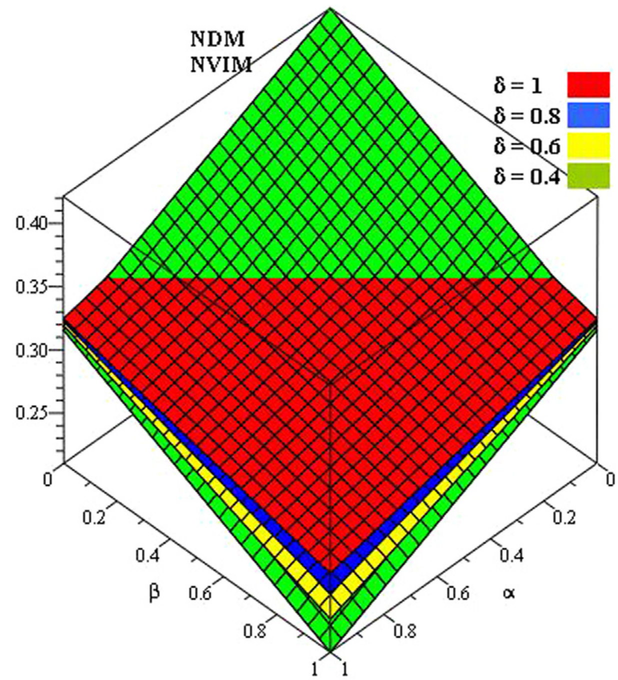

Fig. 17 NDM and NVIM solutions $\omega(\alpha,\beta,\gamma,\eta)$ of Example 3 at different fractional-order δ.

$$\mu_3(\alpha,\beta,\gamma,\eta) = -0.5\alpha+\beta+\gamma-\frac{2.25\alpha\eta^{\delta}}{\Gamma(\delta+1)}+\frac{2(2.25)\alpha\eta^{2\delta}}{\Gamma(2\delta+1)}$$
$$(-0.5\alpha+\beta+\gamma)-\frac{(2.25)^2\alpha\left(4(\Gamma(\delta+1))^2+\Gamma(2\delta+1)\right)\eta^{3\delta}}{\Gamma(2\delta+1)(\Gamma(\delta+1))^2}+\cdots,$$
$$v_3(\alpha,\beta,\gamma,\eta) = \alpha-0.5\beta+\gamma-\frac{2.25\beta\eta^{\delta}}{\Gamma(\delta+1)}+\frac{2(2.25)\beta\eta^{2\delta}}{\Gamma(2\delta+1)}$$
$$(\alpha-0.5\beta+\gamma)-\frac{(2.25)^2\beta\left(4(\Gamma(\delta+1))^2+\Gamma(2\delta+1)\right)\eta^{3\delta}}{\Gamma(2\delta+1)(\Gamma(\delta+1))^2}+\cdots,$$
$$\omega_3(\alpha,\beta,\gamma,\eta) = \alpha+\beta-0.5\gamma-\frac{2.25\gamma\eta^{\delta}}{\Gamma(\delta+1)}+\frac{2(2.25)\gamma\eta^{2\delta}}{\Gamma(2\delta+1)}$$
$$(\alpha+\beta-0.5\gamma)-\frac{(2.25)^2\gamma\left(4(\Gamma(\delta+1))^2+\Gamma(2\delta+1)\right)\eta^{3\delta}}{\Gamma(2\delta+1)(\Gamma(\delta+1))^2}+\cdots,$$

The exact solution of Eq. (39) at $\delta = 1$ and $q_1 = q_2 = q_3 = 0$,

$$\begin{aligned}\mu(\alpha,\beta,\gamma,\eta) &= \frac{-0.5\alpha+\beta+\gamma-2.25\alpha\eta}{1-2.25\eta^2},\\ v(\alpha,\beta,\gamma,\eta) &= \frac{\alpha-0.5\beta+\gamma-2.25\beta\eta}{1-2.25\eta^2},\\ \omega(\alpha,\beta,\gamma,\eta) &= \frac{\alpha+\beta-0.5\gamma-2.25\gamma\eta}{1-2.25\eta^2},\end{aligned} \tag{46}$$

6. Results and discussion

The aim of the present work is to find an analytical solution of fractional-order Navier–Stokes equations, using an efficient analytical techniques. The Natural decomposition method and Natural variational iteration method are used to solve the targeted problems. The Caputo definition of fractional derivative is used to express fractional-derivative. To check the validity of the suggested techniques, the solution of some illustrative examples are presented. The solutions graphs are plotted for both fractional and integer-order problems. In Fig. 1, the exact solution $\mu(\alpha,\beta,\eta)$ of Example 1 is shown. Similarly in Fig. 2, the graph of NDM solution is discussed in Fig. 3 at $\delta = 1$. It is observed that the exact, NDM and NVIM solution are in closed agreement with the exact solutions of the problems. Also in Figs. 4–6 the NDM and NVIM solutions of Example 1 are calculated at different fractional-order $\delta = 0.8, 0.6, 0.4$. It is confirmed that NDM and NVIM solutions are in strong agreement with each other. The similar graphical investigation and discussion can be made for the solutions $v(\alpha, \beta, \eta)$ of Example 1 in Figs. 7–10. In Figs. 11 and 12 the error graph have been plotted for NDM and NVIM respectively. In these graphs it is analyzed that both the techniques have the sufficient degree of accuracy. In Tables 1 and 2 the NDM, NVIM and FRDTM solutions are compared in terms of absolute errors for $\mu(\alpha, \beta, \eta)$ and $v(\alpha, \beta, \eta)$ respectively. It has been shown that the proposed method have the identical accuracy. Figs. 11 and 12 represents the NDM and NVIM solutions $\mu(\alpha, \beta, \eta)$ and $v(\alpha, \beta, \eta)$ of Example 2 at different fractional-orders $\delta = 1, 0.8, 0.6$ and 0.4. It is analyzed that solution of fractional-order problems are convergent to an integer-order solution as fractional-order approaches to integer-order. At the end Figs. 13–15 are plotted to show the solutions $\mu(\alpha, \beta, \gamma, \eta), v(\alpha, \beta, \gamma, \eta)$ and $\omega(\alpha, \beta, \gamma, \eta)$ at different fractional-order $\delta = 1, 0.8, 0.6, 0.4$ respectively for Example 3. The same convergence phenomena of the fractional-order solutions towards integer-order solutions is observed (see Figs. 16 and 17).

7. Conclusion

In the current article, different hybrid technique are used to solve fractional-order multi-dimensional Navier–Stokes equations. The analytical solution of some examples are calculated to confirmed the reliability and effectiveness of the current techniques. The solutions graphs are plotted to show the closed contact between the exact and obtained solutions. It is also investigated that natural decomposition method simple and straight forward as compared natural variational iteration method. Furthermore, the proposed methods provide the series form solutions with easily computable components. It is analyzed that the obtained series form solutions have the higher rate of convergence towards the exact solutions of the problems. The suggested techniques have a small number of calculations to obtain the analytical solutions. On the basis of sufficient degree of accuracy the current techniques are preferred to solve other complicated non-linear fractional-order partial differential equations.

Declaration of Competing Interest

None.

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