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The Klein–Fock–Gordon and Tzitzeica dynamical equations with advanced analytical wave solutions

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ABSTRACT

In this manuscript, two mathematical approaches have been functionalized to discover novel wave results of 3rd-order Klein–Gordon and Tzitzeica equations. With the alliance of Mathematica, the competency of these methods for discovering these exact solutions have been more exhibited. As a result, several solitary solutions are constructed and indicated by hyperbolic solutions, diverse combinations of trigonometric and exponential results. Furthermore, employed techniques are more efficient techniques for exploring essential nonlinear waves that enhance a variety of dynamic models that arises in nonlinear fields. All drafting is given out to express the properties of the innovative explicit analytic solutions. Hence our proposed schemes are directed, succinct, and reasonably good for the various nonlinear evaluation equations (NLEEs) related treatment and mathematical physics also.

Introduction

Physical phenomena and processes that occurred naturally have matted with nonlinear features. Nonlinear problems (NLPs) having much attention in engineers, physicists, mathematicians and numerous further scientists. From there nonlinear evaluation equations have been the topic of concern in diverse branches of nonlinear areas such as physics, plasma physics, propagation of shallow water wave, mathematical fluid dynamics, applied mathematics, protein chemistry, geochemistry, chemical kinematics, chemically reactive materials and meteorology, etc. That is why the investigation of solitary wave solutions is fetching a matter of concerning issue gradually [1–6] .

Our planned first model is 3rd-order Klein–Fock–Gordon equation (KFGE) an imperative session of NLEEs, having countless implication for energy particle physics and is applied as a model for various types matter. Sometimes it defined as the equation of relativistic wave related to Schrodinger equation. The KFGE is an importance model used for energy particle in physics and functional several matter, with spread of deviation in crystals and in the basic stuffs of particles. Sometimes it is demarcated as the equation of relativistic wave related to Schrodinger equation. Let KFGE define in [7] has form as;

$$\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial^2 u}{\partial x^2} + \beta u + \partial u^m = g(x,t) \tag{1}$$

In this work, we generally study a special case of nonlinear m order KFG equation when $m = 3$ then Eq. (1) become a 3rd order KFG equation.

Our planned second model is, Tzitzeica equation which has important attention during the last few years, which can written in mathematical form as follow [8].

$$u_{tt} - u_{xx} - e^u + e^{-2u} = 0. \tag{2}$$

The chief goal in this article is to efficiently hire improved form of simple equation and modified F-expansion methods for exact results of KFGE and Tzitzeica equation. These methods are based on a appropriate variables change, which transforms renovate the main problem in ODE, with replacement of a suppositious solution in ODE, and by calculating the unknown coefficients , the central required results of the recommended models are achieved.

Nowadays, exact solutions for nonlinear evaluation equations (NEEs) has been discovered by many authors and they have been used numerous powerful techniques. Several powerful method have been offered such as Hirota's bilinear transformation techniques [9,10],The $\mathrm{Exp}(-\Psi(\xi))$ -expansion techniques [11–13], Exp-function expansion scheme [14], The extended tanh-function technique [15,16], Lie symmetry technique [17–20], The modified simple equation scheme [21, 22], The complex hyperbolic function technique [23], The Bernoulli's

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Sub-ODE scheme [24], The (G'/G)- expansion scheme [25–27], The enhance (G'/G)-expansion technique [28], Jacobi elliptic technique [29], Homogeneous balance scheme [30,31], He's polynomial and Asymptotic methods [32–34], The improved form of Riccati equations expansion scheme and Variational iteration method [35]. Some authors used extended and modified direct algebraic method, extended mapping method and Seadawy techniques to find analytical solutions for some nonlinear partial differential equations [36–41].

The paper is organized as: In section 'Formation of the Proposed Methods', the propose methods have been brief in details. In section 'Applications', applied these schemes to propose nonlinear equations. Finally, conclusion is given in section 'Conclusion'.

Formation of the proposed methods

Let,

$$R_1\left(u, u_{xx}, u_{xz}, u_{xx}, u_{xy}, u_{xtt}, \dots\right) = 0, \tag{3}$$

Consider

$$u = U(\xi), \qquad \xi = kx + ly + mz \pm \omega t, \tag{4}$$

Put (4) in (3),

$$R_2\left(U, U', U'', \dots\right) = 0, \tag{5}$$

Extended simple equation method

Let solution of (5) is,

$$U(\xi) = \sum_{i=-N}^{N} A_i \Psi^i(\xi) \tag{6}$$

Let Ψ satisfies,

$$\Psi' = c_0 + c_1\Psi + c_2\Psi^2 + c_3\Psi^3 \tag{7}$$

The solutions Eq. (7) are;

$$\Psi(\xi) = -\frac{c_1 - \sqrt{4c_0c_2 - c_1^2}\tan\left(\frac{\sqrt{4c_0c_2-c_1^2}}{2}(\xi+\xi_0)\right)}{2c_2}, \quad 4c_0c_2 > c_1^2, \quad c_3 = 0. \tag{8}$$

If $c_0 = 0,\ c_3 = 0$, the Eq. (7) gives solutions as;

$$\Psi = \frac{c_1 e^{c_1(\xi+\xi_0)}}{1 - c_2 e^{c_1(\xi+\xi_0)}}, \qquad c_1 > 0, \tag{9}$$

$$\Psi = \frac{-c_1 e^{c_1(\xi+\xi_0)}}{1 + c_2 e^{c_1(\xi+\xi_0)}}, \qquad c_1 < 0. \tag{10}$$

If $c_1 = 0,\ c_3 = 0$, then Eq. (7) reduces to Riccati equation:

$$\Psi(\xi) = \frac{\sqrt{c_0c_2}}{c_2}\tan(\sqrt{c_0c_2}(\xi+\xi_0)), \qquad c_0c_2 > 0, \tag{11}$$

$$\Psi(\xi) = -\frac{\sqrt{-c_0c_2}}{c_2}\tanh(\sqrt{-c_0c_2}(\xi+\xi_0)), \qquad c_0c_2 < 0. \tag{12}$$

Put (6) with (7) in (5). Solving the obtained systems of equations for the required values parameters. Putting all parameters values and Ψ into Eq. (6), achieved the solution of (3).

Modified F-expansion method

Let Eq. (5) has solution;

$$U = a_0 + \sum_{i=1}^{N} a_i F^i(\xi) + \sum_{i=1}^{N} b_i F^{-i}(\xi) \tag{13}$$

Let $F(\xi)$ gratifies,

$$F' = A + BF + CF^2. \tag{14}$$

Put (13) along (14) in (5) Selecting A, B, C with F from Table 1 and putting a_i, b_i in (13), for destination of Eq. (3).

Table 1
Relation between A, B, C and $F(\xi)$ in Eq. (14).

Values of A, B, C	$F(\xi)$
$A=0,\ B=1,\ C=-1$	$\frac{1}{2}+\frac{1}{2}\tanh(\frac{1}{2}\xi)$
$A=0,\ B=-1,\ C=1$	$\frac{1}{2}-\frac{1}{2}\coth(\frac{1}{2}\xi)$
$A=\frac{1}{2},\ B=0,\ C=-\frac{1}{2}$	$\coth(\xi)\pm\csc h(\xi)$
$A=1,\ B=0,\ C=-1$	$\tanh(\xi), \coth(\xi)$
$A=\frac{1}{2},\ B=0,\ C=\frac{1}{2}$	$\sec(\xi)+\tan(\xi)$
$A=-\frac{1}{2},\ B=0,\ C=-\frac{1}{2}$	$\sec(\xi)-\tan(\xi)$
$A=1(-1),\ B=0,\ C=1(-1)$	$\tan(\xi),\ (\cot(\xi))$
$B=A=0,\ C\neq 0$	$-\frac{1}{C\xi+\epsilon}$, (ϵ is arbitrary constant)
$A\neq 0,\ C=B=0$	$A\xi$
$B=A\neq 0,\ C=0$	$\frac{(-A+\exp(B\xi))}{B}$

Applications

The 3rd-order Klein–Fock–Gordon equation

We will successfully constructed solitary solutions of the cubic KFGE. For our expedient, we consider m = 3 to mth order KFG equation [7].

$$u_{tt} + \alpha u_{xx} + \beta u + u^3 = g(x,t) \tag{15}$$

$$U = u(x,t), \quad \xi = mx - bt \tag{16}$$

Put (16) in (15) and after integrating obtained,

$$U''\left(b^2 + \alpha m^2\right) + \delta U^3 + \beta U = 0 \tag{17}$$

Application of extended simple equation method

Let (17) has solution,

$$U = \frac{A_{-1}}{\Psi} + A_1\Psi + A_0 \tag{18}$$

Put (18) with (7) into (17), after solving we have the following solutions cases as following;

CASE 1: $c_3 = 0$,

Family-I

$$A_1 = 0,\ A_0 = \frac{\sqrt{\beta}c_1}{\sqrt{\left(4c_0c_2 - c_1^2\right)\delta}},\ A_{-1} = \frac{2\sqrt{\beta}c_0}{\sqrt{4c_0c_2\delta - c_1^2\delta}}, \quad b = \frac{\sqrt{-2\beta + \alpha c_1^2 m^2 - 4\alpha c_0c_2m^2}}{\sqrt{4c_0c_2 - c_1^2}} \tag{19}$$

Put (19) in (18),

$$U_1 = \frac{\sqrt{\beta}c_1}{\sqrt{\left(4c_0c_2 - c_1^2\right)\delta}} - \frac{4c_0c_2\sqrt{\beta}}{\sqrt{4c_0c_2\delta - c_1^2\delta}\left(c_1 - \sqrt{4c_0c_2 - c_1^2}\tan(\frac{\sqrt{4c_0c_2-c_1^2}}{2}(\xi+\xi_0))\right)}, \quad 4c_0c_2 > c_1^2. \tag{20}$$

Family-II

$$A_1 = \frac{2\sqrt{\beta}c_2}{\sqrt{4c_0c_2\delta - c_1^2\delta}},\ A_0 = \frac{\sqrt{\beta}c_1}{\sqrt{\left(c_1^2 - 4c_0c_2\right)(-\delta)}},\ A_{-1} = 0, \quad b = \frac{\sqrt{-2\beta + \alpha c_1^2 m^2 - 4\alpha c_0c_2m^2}}{\sqrt{4c_0c_2 - c_1^2}} \tag{21}$$

Substitute (21) in (18),

$$U_2 = \frac{\sqrt{\beta}c_1}{\sqrt{(-c_1^2+4c_0c_2)(\delta)}} - \frac{\sqrt{\beta}\left(c_1 - \sqrt{4c_0c_2-c_1^2}\tan(\frac{\sqrt{4c_0c_2-c_1^2}}{2}(\xi+\xi_0))\right)}{\sqrt{4c_0c_2\delta - c_1^2\delta}}, \quad 4c_0c_2 > c_1^2 \tag{22}$$

CASE 2: $c_1 = 0, \quad c_3 = 0,$

Family-I

$$A_1 = 0, \quad A_0 = 0, \quad A_{-1} = -\frac{\sqrt{\beta}\sqrt{c_0}}{\sqrt{c_2}\sqrt{\delta}}, \; b = \frac{\sqrt{-\beta - 2\alpha c_0 c_2 m^2}}{\sqrt{2}\sqrt{c_0}\sqrt{c_2}} \tag{23}$$

Put (23) in (18),

$$U_3 = -\frac{\sqrt{\beta}\sqrt{c_0}}{\sqrt{c_2}\sqrt{\delta}}\left(\frac{c_2}{\sqrt{c_0c_2}\left(\tan(\sqrt{c_0c_2}(\xi+\xi_0))\right)}\right), \quad c_0c_2 > 0, \tag{24}$$

$$U_4 = \frac{\sqrt{\beta}\sqrt{c_0}}{\sqrt{c_2}\sqrt{\delta}}\left(\frac{c_2}{\sqrt{-c_0c_2}\left(\tanh(\sqrt{-c_0c_2}(\xi+\xi_0))\right)}\right), \quad c_0c_2 < 0. \tag{25}$$

Family-II

$$A_1 = -\frac{\sqrt{\beta}\sqrt{c_2}}{\sqrt{c_0}\sqrt{\delta}}, \; A_0 = A_{-1} = 0, \; b = \frac{\sqrt{-\beta - 2\alpha c_0 c_2 m^2}}{\sqrt{2}\sqrt{c_0}\sqrt{c_2}} \tag{26}$$

Put (26) in (18),

$$U_5 = -\frac{\sqrt{\beta}\sqrt{c_2}}{\sqrt{c_0}\sqrt{\delta}}\left(\frac{\sqrt{c_0c_2}\tan(\sqrt{c_0c_2}(\xi+\xi_0))}{c_2}\right), \quad c_0c_2 > 0, \tag{27}$$

$$U_6 = \frac{\sqrt{\beta}\sqrt{c_2}}{\sqrt{c_0}\sqrt{\delta}}\left(\frac{\sqrt{-C_0C_2}\tanh(\sqrt{-C_0C_2}(\xi+\xi_0))}{c_2}\right), \quad c_0c_2 < 0. \tag{28}$$

Family-III

$$A_1 = \frac{\sqrt{\beta}\sqrt{c_2}}{2\sqrt{c_0}\sqrt{\delta}}, \; A_0 = 0, \; A_{-1} = -\frac{\sqrt{\beta}\sqrt{c_0}}{2\sqrt{c_2}\sqrt{\delta}}, \; b = \frac{\sqrt{-\beta - 8\alpha c_0 c_2 m^2}}{2\sqrt{2}\sqrt{c_0}\sqrt{c_2}} \tag{29}$$

Put (29) in (18), (see Fig. 1.)

$$U_7 = -\frac{\sqrt{\beta}\sqrt{c_0}}{2\sqrt{c_2}\sqrt{\delta}}\left(\frac{c_2}{(\sqrt{c_0c_2}\tan(\sqrt{c_0c_2}(\xi+\xi_0)))}\right) - \frac{\sqrt{\beta}\sqrt{c_2}}{2\sqrt{c_0}\sqrt{\delta}}\left(\frac{\sqrt{c_0c_2}\tan(\sqrt{c_0c_2}(\xi+\xi_0))}{c_2}\right), \quad c_0c_2 > 0. \tag{30}$$

$$U_8 = \frac{\sqrt{\beta}\sqrt{c_0}}{2\sqrt{c_2}\sqrt{\delta}}\left(\frac{c_2}{(\sqrt{-c_0c_2}\tanh(\sqrt{-c_0c_2}(\xi+\xi_0)))}\right) - \frac{\sqrt{\beta}\sqrt{c_2}}{2\sqrt{c_0}\sqrt{\delta}}\left(\frac{2b_1k\sqrt{-c_0c_2}\tanh(\sqrt{-c_0c_2}(\xi+\xi_0))}{a_1}\right), \quad c_0c_2 < 0. \tag{31}$$

Applications of modified F-expansion method

Let solution of (17) is;

$$U = a_0 + a_1F(\xi) + \frac{b_1}{F(\xi)} \tag{32}$$

Let F satisfies,

$$F' = A + BF(\xi) + CF^2(\xi) \tag{33}$$

Substitute (32) in (17) with (33), obtained following possible solutions cases as;

For $A = 1, \; B = 0, \; C = -1,$

$$a_1 = \frac{\sqrt{\beta}}{\sqrt{2}\sqrt{\delta}}, \quad a_0 = 0, \quad b_1 = -\frac{\sqrt{\beta}}{\sqrt{2}\sqrt{\delta}}, \quad b = \frac{1}{2}\sqrt{-\beta - 4\alpha m^2} \tag{34}$$

Put (34) in (32),

$$U_{11} = \frac{\sqrt{\beta}}{\sqrt{2}\sqrt{\delta}}\left(\tanh(\xi) - \frac{1}{\tanh(\xi)}\right), \tag{35}$$

For $A = \frac{1}{2}, \; B = 0, \; C = \frac{1}{2},$

Family-I

$$a_1 = \frac{\sqrt{\beta}}{\sqrt{\delta}}, \quad a_0 = 0, \quad b_1 = 0, \quad b = -\sqrt{\alpha\left(-m^2\right) - 2\beta} \tag{36}$$

Put (36) in (32), (see Fig. 2.)

$$U_{12} = \frac{\sqrt{\beta}}{\sqrt{\delta}}(\sec(\xi) + \tan(\xi)) \tag{37}$$

Family-II

$$a_1 = 0, \; a_0 = 0, \quad b_1 = \frac{\sqrt{\beta}}{\sqrt{\delta}}, \; b = -\sqrt{\alpha\left(-m^2\right) - 2\beta} \tag{38}$$

Put (38) in (5),

$$U_{13} = \frac{\sqrt{\beta}}{\sqrt{\delta}}\left(\frac{1}{\sec(\xi) + \tan(\xi)}\right) \tag{39}$$

Family-III

$$a_1 = \frac{\sqrt{\beta}}{2\sqrt{\delta}}, \quad a_0 = 0, \quad b_1 = -\frac{\sqrt{\beta}}{2\sqrt{\delta}}, \quad b = -\frac{\sqrt{-\beta - 2\alpha m^2}}{\sqrt{2}} \tag{40}$$

By putting Eq. (40) in (32),

$$U_{14} = \frac{\sqrt{\beta}}{2\sqrt{\delta}}\left((\sec(\xi) + \tan(\xi)) - \left(\frac{1}{\sec(\xi) + \tan(\xi)}\right)\right) \tag{41}$$

For $A = -\frac{1}{2}, \; B = 0, \; C = -\frac{1}{2},$

Family-I

$$a_1 = \frac{\sqrt{\beta}}{\sqrt{\delta}}, \quad a_0 = 0, \quad b_1 = 0, \quad b = \sqrt{\alpha\left(-m^2\right) - 2\beta} \tag{42}$$

Put (54) in (32),

$$U_{15} = \frac{\sqrt{\beta}}{\sqrt{\delta}}(\sec(\xi) - \tan(\xi)) \tag{43}$$

Family-II

$$a_1 = 0, \quad a_0 = 0, \quad b_1 = \frac{\sqrt{\beta}}{\sqrt{\delta}}, \quad b = \sqrt{\alpha\left(-m^2\right) - 2\beta} \tag{44}$$

Put (56) in (32), (see Fig. 3.)

$$U_{16} = \frac{\sqrt{\beta}}{\sqrt{\delta}}\left(\frac{1}{\tan(\xi) - \sec(\xi)}\right) \tag{45}$$

Family-III

$$a_1 = \frac{\sqrt{\beta}}{2\sqrt{\delta}}, \quad a_0 = 0, \quad b_1 = -\frac{\sqrt{\beta}}{2\sqrt{\delta}}, \quad b = \frac{\sqrt{-\beta - 2\alpha m^2}}{\sqrt{2}} \tag{46}$$

Put (58) in (32),

$$U_{17} = \frac{\sqrt{\beta}}{2\sqrt{\delta}}\left((\sec(\xi) - \tan(\xi)) - \frac{1}{\tan(\xi) - \sec(\xi)}\right) \tag{47}$$

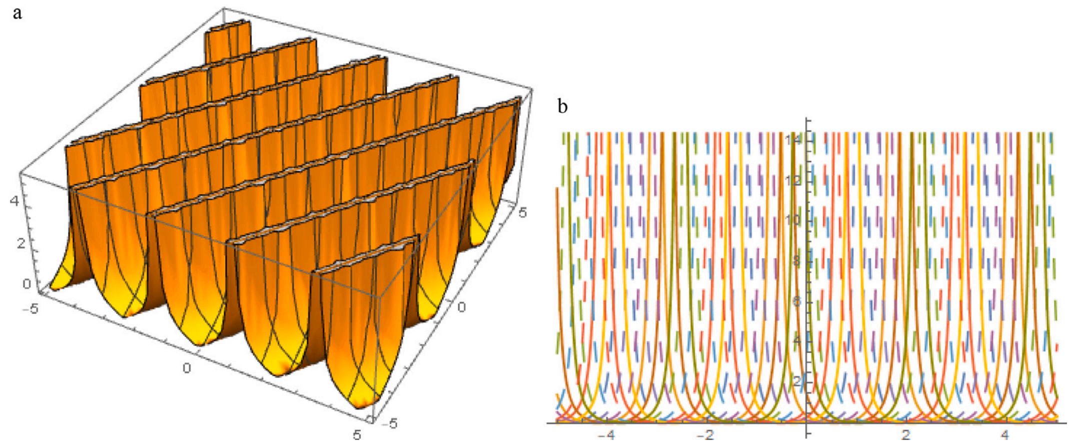

Fig. 1. Soliton shape solution of U_1 for $c_1=-1,\ \alpha=-2,\ \beta=5.5,\ c_2=1,\ c_0=2,\ \delta=12,\ m=1,\ \epsilon=2.5$.

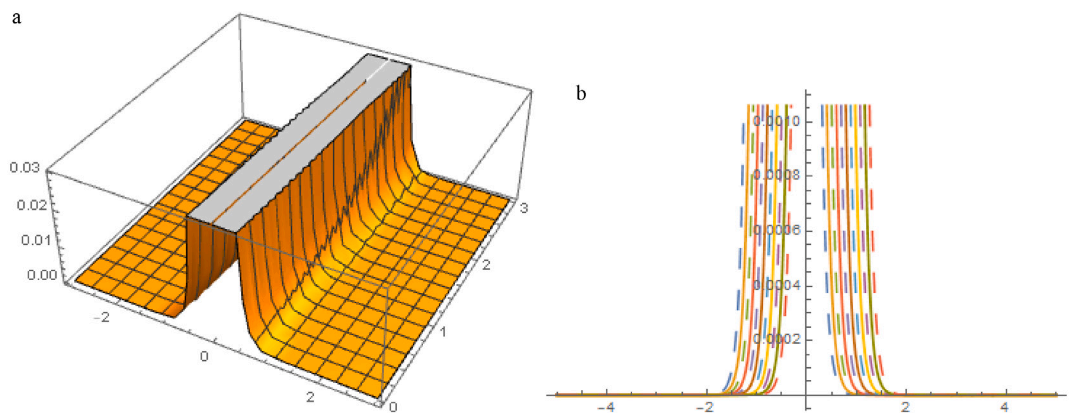

Fig. 2. Soliton shape solution of U_{11} for $m=5,\ \alpha=-0.01,\ \beta=0.1,\ \delta=0.112$.

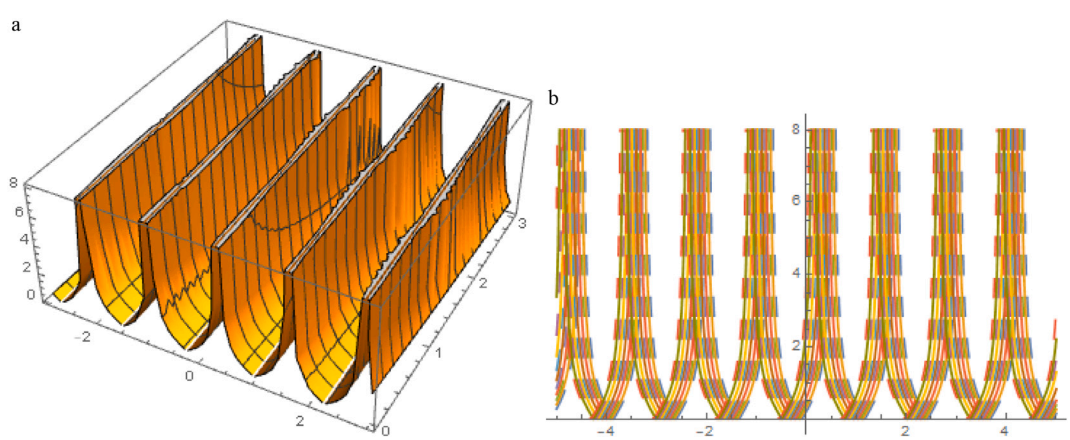

Fig. 3. Soliton shape solution of U_{12} for $m=5,\ \alpha=-0.01,\ \beta=0.1,\ \delta=0.112$.

$A=-1,\ B=0,\ C=-1,$

Family-I

$$a_1=\frac{\sqrt{\beta}}{\sqrt{\delta}},\quad a_0=0,\quad b_1=0,\quad b=-\frac{\sqrt{-\beta-2\alpha m^2}}{\sqrt{2}} \tag{48}$$

Put (60) in (5),

$$U_{18}=\frac{\sqrt{\beta}}{\sqrt{\delta}}(\tan(\xi)) \tag{49}$$

Family-II

$$a_1=0,\quad a_0=0,\quad b_1=-\frac{\sqrt{\beta}}{\sqrt{\delta}},\quad b=\frac{\sqrt{-\beta-2\alpha m^2}}{\sqrt{2}} \tag{50}$$

Put (62) in (32),

$$U_{19}=-\frac{\sqrt{\beta}}{\sqrt{\delta}}\left(\frac{1}{(\tan(\xi))}\right) \tag{51}$$

Family-III

$$a_1 = -\frac{\sqrt{\beta}}{2\sqrt{\delta}},\; a_0 = 0,\; b_1 = \frac{\sqrt{\beta}}{2\sqrt{\delta}},\; b = \frac{\sqrt{-\beta - 8\alpha m^2}}{2\sqrt{2}} \tag{52}$$

Put (64) in (32), (see Fig. 4.)

$$U_{20} = -\frac{\sqrt{\beta}}{2\sqrt{\delta}}\left(\tan(\xi) - \frac{1}{(\tan(\xi))}\right) \tag{53}$$

The Tzitzeica equation

Consider Painleve transformation $u = Ln(V)$ in (2),

$$VV_{tt} - V_t^2 - VV_{xx} + V_x^2 - V^3 + 1 = 0. \tag{54}$$

Consider the travel transformation;

$$V = U(x,t), \quad \xi = kx + \lambda t \tag{55}$$

Put Eq. (55) in Eq. (54),

$$\left(\lambda^2 - k^2\right)\left(UU'' - (U')^2\right) - U^3 + 1 = 0, \tag{56}$$

Application of extended simple equation method

Let Eq. (2) has solution,

$$U = A_2\Psi^2 + A_1\Psi + \frac{A_{-2}}{\Psi^2} + \frac{A_{-1}}{\Psi} + A_0 \tag{57}$$

Put (57) with (7) in (56), after solving we have the following solutions cases as following;

CASE 1: $c_3 = 0$,

Family-I

$$A_{-2} = -\frac{6c_0^2}{4c_0c_2 - c_1^2},\quad A_{-1} = -\frac{6c_0c_1}{4c_0c_2 - c_1^2},\quad A_2 = 0,\quad A_1 = 0,$$
$$A_0 = \frac{c_1^2 + 2c_0c_2}{c_1^2 - 4c_0c_2},\quad \lambda = -\frac{\sqrt{c_1^2k^2 - 4c_0c_2k^2 + 3}}{\sqrt{c_1^2 - 4c_0c_2}} \tag{58}$$

Put Eq. (58) in Eq. (57),

$$u_1 = Ln\left(\frac{12c_0c_2\left(\sqrt{4c_0c_2 - c_1^2}\,c_1\tan\left(\frac{1}{2}\sqrt{4c_0c_2 - c_1^2}(\xi + \epsilon)\right) - c_1^2 + 2c_0c_2\right)}{\left(c_1^2 - 4c_0c_2\right)\left(c_1 - \sqrt{4c_0c_2 - c_1^2}\tan\left(\frac{1}{2}\sqrt{4c_0c_2 - c_1^2}(\xi + \epsilon)\right)\right)^2} + \frac{c_1^2 + 2c_0c_2}{c_1^2 - 4c_0c_2}\right),$$
$$4c_0c_2 > c_1^2. \tag{59}$$

Family-II

$$A_{-2} = 0,\quad A_{-1} = 0,\quad A_2 = -\frac{6c_2^2}{4c_0c_2 - c_1^2},\quad A_1 = \frac{6c_1c_2}{c_1^2 - 4c_0c_2},$$
$$A_0 = \frac{c_1^2 + 2c_0c_2}{c_1^2 - 4c_0c_2},\quad \lambda = -\frac{\sqrt{c_1^2k^2 - 4c_0c_2k^2 + 3}}{\sqrt{c_1^2 - 4c_0c_2}} \tag{60}$$

Substitute Eq. (60) in Eq. (57),

$$u_2 = Ln\left(\frac{3}{\cos\left(\sqrt{4c_0c_2 - c_1^2}(\xi + \epsilon)\right) + 1} - \frac{6c_1\tan\left(\frac{1}{2}\sqrt{4c_0c_2 - c_1^2}(\xi + \epsilon)\right)}{\sqrt{4c_0c_2 - c_1^2}} - \frac{12c_0c_2}{c_1^2 - 4c_0c_2} - 5\right),$$
$$4c_0c_2 > c_1^2 \tag{61}$$

CASE 2: $c_1 = 0, \quad c_3 = 0$,

Family-I

$$A_0 = -\frac{1}{2},\; A_{-2} = -\frac{3c_0}{2c_2},\; A_{-1} = 0,\; A_2 = 0,\; A_1 = 0,\; \lambda = \frac{\sqrt{4c_0c_2k^2 - 3}}{2\sqrt{c_0}\sqrt{c_2}} \tag{62}$$

Put Eq. (62) in Eq. (57),

$$u_3 = Ln\left(-\frac{3c_0}{(2c_2)\left(\sqrt{\frac{c_0c_2}{c_2}}\tan\left(\sqrt{c_0c_2}\,(\xi + \xi_0)\right)\right)^2} - \frac{1}{2}\right),\quad c_0c_2 > 0, \tag{63}$$

$$u_4 = \mathrm{Ln}\left(-\frac{3c_0}{(2c_2)\left(\sqrt{-\frac{c_0c_2}{c_2}}\tanh\left(\sqrt{-c_0c_2}\,(\xi + \xi_0)\right)\right)^2} - \frac{1}{2}\right),\quad c_0c_2 < 0. \tag{64}$$

Family-II

$$A_0 = -\frac{1}{2},\; A_{-2} = 0,\; A_{-1} = 0,\; A_2 = -\frac{3c_2}{2c_0},\; A_1 = 0,\; \lambda = -\frac{\sqrt{4c_0c_2k^2 - 3}}{2\sqrt{c_0}\sqrt{c_2}} \tag{65}$$

Put (65) in (57),

$$u_5 = \mathrm{Ln}\left(-\frac{(3c_2)\left(\sqrt{\frac{c_0c_2}{c_2}}\tan\left(\sqrt{c_0c_2}\,(\xi + \xi_0)\right)\right)^2}{2c_0} - \frac{1}{2}\right),\quad c_0c_2 > 0, \tag{66}$$

$$u_6 = \mathrm{Ln}\left(-\frac{(3c_2)\left(\sqrt{-\frac{c_0c_2}{c_2}}\tanh\left(\sqrt{-c_0c_2}\,(\xi + \xi_0)\right)\right)^2}{2c_0} - \frac{1}{2}\right),\quad c_0c_2 < 0. \tag{67}$$

Family-III

$$A_0 = \frac{1}{4},\; A_{-2} = -\frac{3c_0}{8c_2},\; A_{-1} = 0,\; A_2 = -\frac{3c_2}{8c_0},\; A_1 = 0,\; \lambda = \frac{\sqrt{16c_0c_2k^2 - 3}}{4\sqrt{c_0}\sqrt{c_2}} \tag{68}$$

Put (68) in (57),

$$u_7 = \frac{1}{8}\mathrm{Ln}\left(-3c_2\tan^2\left(\sqrt{c_0c_2}\,(\xi + \xi_0)\right) - \frac{3\cot^2\left(\sqrt{c_0c_2}\,(\xi + \xi_0)\right)}{c_2} + 2\right)$$
$$c_0c_2 > 0. \tag{69}$$

$$u_8 = \frac{1}{8}\mathrm{Ln}\left(-3c_2\tanh^2\left(\sqrt{-c_0c_2}\,(\xi + \xi_0)\right) - \frac{3\coth^2\left(\sqrt{-c_0c_2}\,(\xi + \xi_0)\right)}{c_2} + 2\right)\quad c_0c_2 < 0. \tag{70}$$

CASE 3: $c_0 = 0, \quad c_3 = 0$,

$$A_0 = 1,\; A_{-2} = 0,\; A_{-1} = 0,\; A_2 = \frac{6c_2^2}{c_1^2},\; A_1 = \frac{6c_2}{c_1},\; \lambda = \frac{\sqrt{c_1^2k^2 + 3}}{c_1} \tag{71}$$

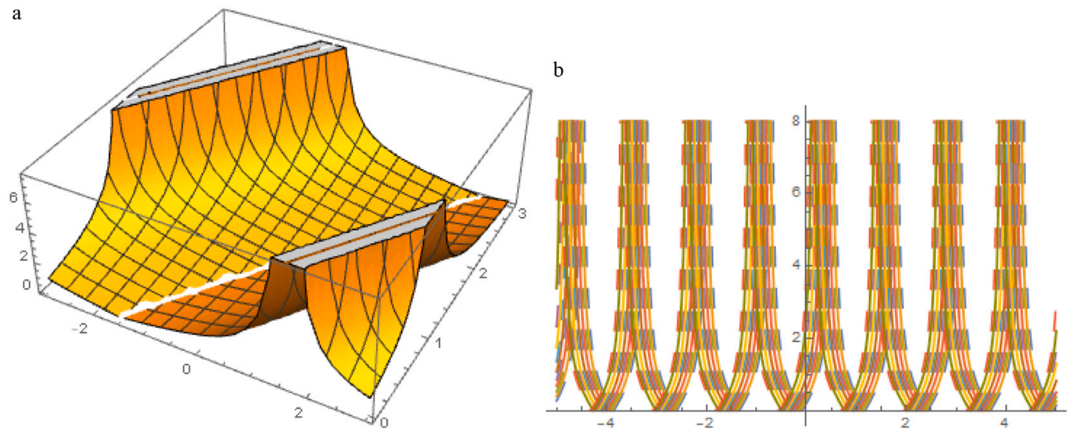

Fig. 4. Soliton shape solution of U_{16} for $m=1,\ \alpha=-2,\ \beta=0.1,\ \delta=0.112$.

Put Eq. (71) in Eq. (57), (see Fig. 5.)

$$u_9=\mathrm{Ln}\left(\frac{(6c_2)\left(c_1\exp\left(c_1\left(\xi+\xi_0\right)\right)\right)}{c_1\left(1-c_2\exp\left(c_1\left(\xi+\xi_0\right)\right)\right)}+\frac{(6c_2^2)\left(\frac{c_1\exp(c_1(\xi+\xi_0))}{1-c_2\exp(c_1(\xi+\xi_0))}\right)^2}{c_1^2}+1\right),\quad c_1>0, \tag{72}$$

$$u_{10}=\mathrm{Ln}\left(-\frac{(6c_2)\left(c_1\exp\left(c_1\left(\xi+\xi_0\right)\right)\right)}{c_1\left(c_2\exp\left(c_1\left(\xi+\xi_0\right)\right)+1\right)}+\frac{(6c_2^2)\left(-\frac{c_1\exp(c_1(\xi+\xi_0))}{c_2\exp(c_1(\xi+\xi_0))+1}\right)^2}{c_1^2}+1\right),\quad c_1<0. \tag{73}$$

Applications of modified F-expansion method

Let solution of (56),

$$U=a_2F^2+a_1F+a_0+\frac{b_2}{F^2}+\frac{b_1}{F} \tag{74}$$

Put (74) with (14) in (56), solve the achieved equation of system. We have the following possible solutions cases as follow;;

Case-1

For $A=\frac{1}{2},\ B=0,\ C=-\frac{1}{2}$,

Family-I

$$a_0=-\frac{1}{2},a_2=0,\ a_1=0,\ b_1=0,\ b_2=\frac{3}{2},\ k=-\sqrt{\lambda^2-3} \tag{75}$$

Put (75) in (74),

$$u_{11}=\mathrm{Ln}\left(\frac{3}{2(\coth\xi\pm\mathrm{csch}\xi)^2}-\frac{1}{2}\right) \tag{76}$$

Family-II

$$a_0=-\frac{1}{2},\ a_2=\frac{3}{2},\ a_1=0,\ b_1=0,\ b_2=0,\ k=-\sqrt{\lambda^2-3} \tag{77}$$

Put (77) in (74),

$$u_{12}=\mathrm{Ln}\left(\frac{3}{2}(\coth\xi\pm\mathrm{csch}\xi)^2-\frac{1}{2}\right) \tag{78}$$

Case-2

$A=1,\ B=0,\ C=-1$

Family-I

$$a_0=-\frac{1}{2},\ a_2=0,\ a_1=0,\ b_1=0,\ b_2=\frac{3}{2},\ k=-\frac{1}{2}\sqrt{4\lambda^2-3} \tag{79}$$

Put (79) in (74),

$$u_{13}=\mathrm{Ln}\left(\frac{3}{2\tanh^2(\xi)}-\frac{1}{2}\right) \tag{80}$$

Family-II

$$a_0=-\frac{1}{2},\ a_2=\frac{3}{2},\ a_1=0,\ b_1=0,\ b_2=0,\ k=-\frac{1}{2}\sqrt{4\lambda^2-3} \tag{81}$$

Put (81) in (74), (see Fig. 6.)

$$u_{14}=\mathrm{Ln}\left(\frac{3\tanh^2(\xi)}{2}-\frac{1}{2}\right) \tag{82}$$

Case-3

For $A=\frac{1}{2},\ B=0,\ C=\frac{1}{2}$,

Family-I

$$a_0=-\frac{1}{2},\ a_2=-\frac{3}{2},\ a_1=0,\ b_1=0,\ b_2=0,\ k=-\sqrt{\lambda^2+3} \tag{83}$$

Put (83) in (74),

$$u_{15}=\mathrm{Ln}\left(-\frac{1}{2}3(\tan(\xi)+\sec(\xi))^2-\frac{1}{2}\right) \tag{84}$$

Family-II

$$a_0=-\frac{1}{2},\ a_2=0,\ a_1=0,\ b_1=0,\ b_2=-\frac{3}{2},\ k=-\sqrt{\lambda^2+3} \tag{85}$$

Put (85) in (74),

$$u_{16}=\mathrm{Ln}\left(-\frac{3}{2(\tan(\xi)+\sec(\xi))^2}-\frac{1}{2}\right) \tag{86}$$

Case-4

For $A=-\frac{1}{2},\ B=0,\ C=-\frac{1}{2}$,

Family-I

$$a_0=-\frac{1}{2},\ a_2=-\frac{3}{2},\ a_1=0,\ b_1=0,\ b_2=0,\ k=-\sqrt{\lambda^2+3} \tag{87}$$

Put (87) in (74),

$$u_{17}=\mathrm{Ln}\left(\frac{1}{2}(-3)(\sec(\xi)-\tan(\xi))^2-\frac{1}{2}\right) \tag{88}$$

Family-II

$$a_0=-\frac{1}{2},\ a_2=0,\ a_1=0,\ b_1=0,\ b_2=-\frac{3}{2},\ k=-\sqrt{\lambda^2+3} \tag{89}$$

Put (89) in (74),

$$u_{18}=\mathrm{Ln}\left(-\frac{3}{2(\sec(\xi)-\tan(\xi))^2}-\frac{1}{2}\right) \tag{90}$$

Case-5

$A=-1,\ B=0,\ C=-1$

Family-I

$$a_0=-\frac{1}{2},\ a_2=-\frac{3}{2},\ a_1=0,\ b_1=0,\ b_2=0,\ k=-\frac{1}{2}\sqrt{4\lambda^2+3} \tag{91}$$

Put (91) in (74),

$$u_{19}=\mathrm{Ln}\left(-\frac{1}{2}3\tan^2(\xi)-\frac{1}{2}\right) \tag{92}$$

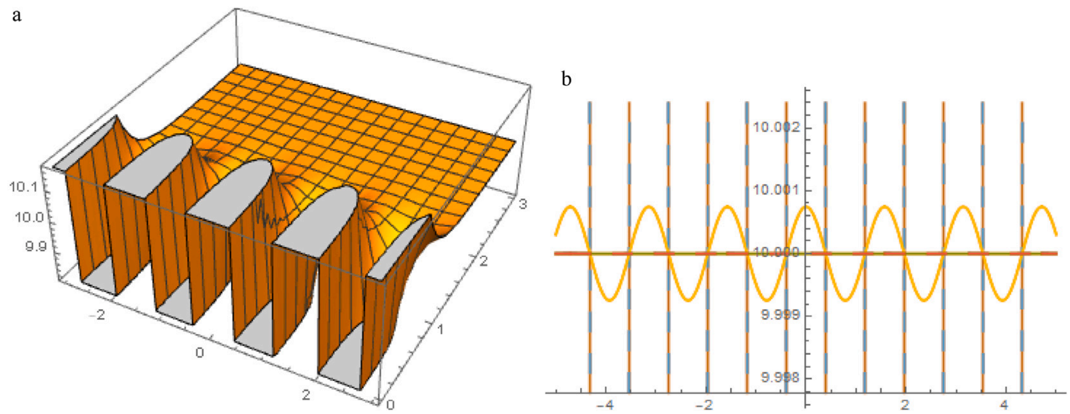

Fig. 5. Soliton shape solution of U_{20} for $m = 1,\ \alpha = 1.5\ \beta = 1,\ \delta = -0.01$.

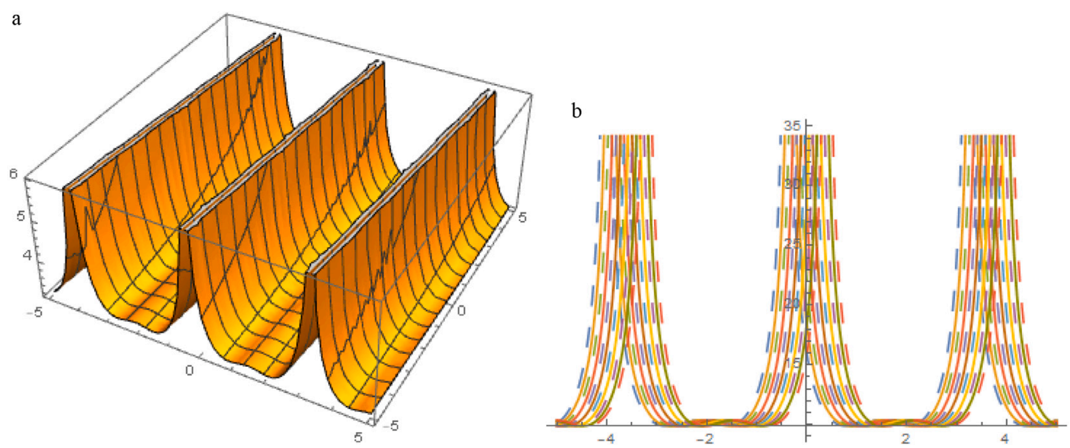

Fig. 6. Soliton shape solution of u_{20} for $\lambda = 0.07,\ k = -\frac{1}{2}\sqrt{4\lambda^2+3}$.

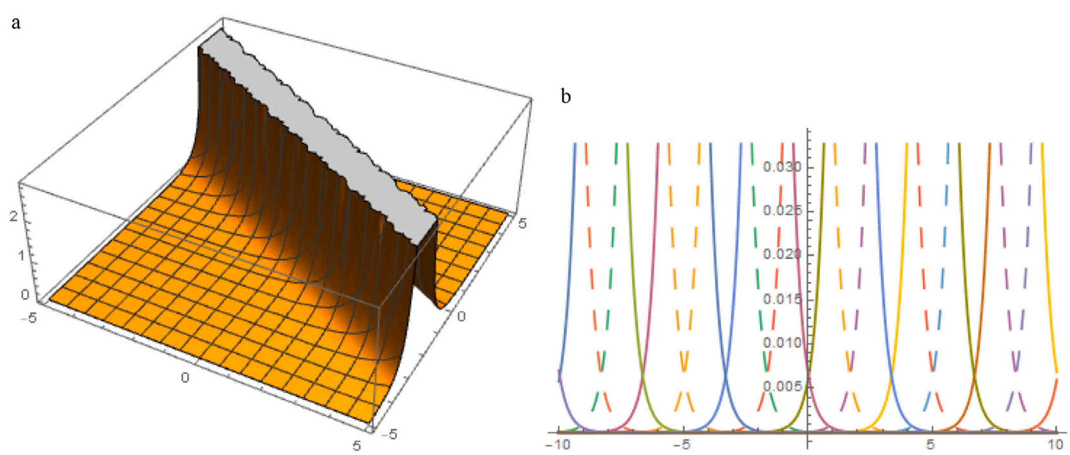

Fig. 7. Soliton shape solution of u_{21} for $A = -0.11,\ B = 0.13,\ k = -10$.

Family-II

$$a_0 = -\frac{1}{2},\ a_2 = 0,\ a_1 = 0,\ b_1 = 0,\ b_2 = -\frac{3}{2},\ k = -\frac{1}{2}\sqrt{4\lambda^2+3} \tag{93}$$

Put (93) in (74),

$$u_{20} = \mathrm{Ln}\left(-\frac{3}{2\tan^2(\xi)} - \frac{1}{2}\right) \tag{94}$$

Case-6

$$a_0 = 1,\ a_2 = 0,\ a_1 = 0,\ b_1 = \frac{6A}{B},\ b_2 = \frac{6A^2}{B^2},\ \lambda = -\frac{\sqrt{B^2k^2+3}}{B} \tag{95}$$

Put (95) in (74), (see Fig. 7.)

$$u_{21} = \mathrm{Ln}\left(1 + \frac{6A}{B}\left(\frac{1}{\frac{(\mathrm{Exp}([B\xi])-A)}{B}}\right) + \frac{6A^2}{B^2}\frac{1}{\left(\frac{(\mathrm{Exp}([B\xi])-A)}{B}\right)^2}\right) \tag{96}$$

Conclusion

In this study, two mathematical methods have been successfully applied to discover new solitary wave solutions for nonlinear wave equation of 3rd-Order KFGE and the Tzitzeica equation. We obtained various novel traveling wave solutions including hyperbolic function solutions, trigonometric function solutions and exponential solutions. The results are clear to us that our proposed schemes are reliable, effective and reasonable good for nonlinear evolution equations.

CRediT authorship contribution statement

Aly R. Seadawy: Conceptualization, Methodology, Software, Supervision. **Asghar Ali:** Data curation, Writing - original draft. **Hanadi Zahed:** review & editing. **Dumitru Baleanu:** Visualization, Investigation, Software, Validation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] Ablowitz MJ, Clarkson PA. Solitons, nonlinear evolution equations and inverse scattering. Cambridge: Cambridge University Press; 1991.
[2] Seadawy AR. Stability analysis for Zakharov-Kuznetsov equation of weakly nonlinear ion-acoustic waves in a plasma. Comput Math Appl 2014;67:172–80.
[3] Seadawy AR. Stability analysis for two-dimensional ion-acoustic waves in quantum plasmas. Phys Plasmas 2014;21:052107.
[4] Dong SH, Zhong QM. Exact solutions to the Schrodinger equation for the potential $V(r) = ar^2 + br^{-4} + cr^{-6}$ in two dimensions. J Phys A: Math Gen 1998;31(49):9855.
[5] Dong SH. Wave equations in higher dimensions. Springer; 2011.
[6] Qiang WC, Dong SH. Analytical approximations to the solutions of the manning rosen potential with centrifugal term. Phys Lett A 2007;368:13–7.
[7] Hafez MG, Alam MN, Akbar MA. Exact traveling wave solutions to the Klein-Gordon equation using the novel (G'/G)-expansion method. Results Phys 2014;4:177–84.
[8] Tzitzeica G. Geometric infinitesimate-sur une nouvelle class de surface. C R Math Acad Sci Paris 1910;150:227–50.
[9] Akira N. Surface impurity localized diode vibration of the toda lattice: Perturbation theory based on Hirota's bilinear transformation method. Progr Theoret Phys 1979;61:427–42.
[10] Jarmo H. A search for bilinear equations passing Hirota's three-soliton condition. I. KdV-type bilinear equations. J Math Phys 1987;28:1732–42.
[11] Akbar K. Application of the $\exp(-\psi(\xi))$-expansion method to find the exact solutions of modified Benjamin-Bona-Mahony equation. World Appl Sci J 2013;24(10):1373–7.
[12] Zhao M, Li C. The $\exp(-\psi(\xi))$-expansion method applied to nonlinear evolution equations. 2008.
[13] Seadawy AR. Nonlinear wave solutions of the three-dimensionalZakharov-Kuznetsov-Burgers equation in dusty plasma. Physica A 2015;439:124–31.
[14] Yinghui H, Li Shaolinand, Yao L. Exact solutions of the Klein-Gordon equation by modified Exp-function method. Int Math Forum 2012;7(4).
[15] Seadawy AR, Rashidy KE. Rayleigh–Taylor instability of the cylindrical flow with mass and heat transfer. Pramana J Phys 2016;87:20.
[16] Engui F. Extended tanh-function method and its applications to nonlinear equations. Phys Lett A 2000;277:212–8.
[17] Wang GW, Zhou T. Group analysis and new explicit solutions of simplified modified Kawahara equation with variable coefficients. Abstr Appl Anal 2013.
[18] Seadawy AR. Three-dimensional nonlinear modified Zakharov-Kuznetsov equation of ion-acoustic waves in a magnetized plasma. Comput Math Appl 2016;71:201–12.
[19] Seadawy AR. New exact solutions for the KdV equation with higher order nonlinearity by using the variational method. Comput Math Appl 2011;62:3741–55.
[20] Wang GW, Qiang Liu L, Zhang Y. New explicit solutions of the fifth-order KdV equation with variable coefficients. Bull Malays Math Sci Soc (2) 2014;37:769–78.
[21] Jaafar AJ, Petković MD, Biswas A. Modified simple equation method for nonlinear evolution equations. Appl Math Comput 2010;217(2):869–77.
[22] Helal MA, Seadawy AR. Benjamin-Feir-instability in nonlinear dispersive waves. Comput Math Appl 2012;64:3557–68.
[23] Cheng BL, Zhao H. Complex hyperbolic-function method and its applications to nonlinear equations. Phys Lett A 2006;355:32–8.
[24] Wang M, Li X, Zhang J. Sub-ODE method and solitary wave solutions for higher order nonlinear Schrödinger equation. Phys Lett A 2007;363:96–101.
[25] Wang M, Li X, Zhang J. The $(G'G)$-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. Phys Lett A 2008;372(4):417–23.
[26] Seadawy AR. Solitary wave solutions of tow-dimensional nonlinear Kadomtsev–Petviashvili dynamic equation in a dust acoustic plasmas. Pramana J Phys 2017;89:1–11.
[27] Seadawy AR, Rashidy KE. Rayleigh–Taylor instability of the cylindrical flow with mass and heat transfer. Pramana J Phys 2016;87.
[28] Islam MH, Akbar K, Salam MA. Exact traveling wave solutions of modified KdV–Zakharov–Kuznetsov equation and viscous Burgers equation. Springer Plus 2014;3105.
[29] Ahmad A. New generalized Jacobi elliptic function rational expansion method. J Comput Appl Math 2011;14:4117–27.
[30] Engui F, Zhang H. A note on the homogeneous balance method. Phys Lett A 1998;246:403–6.
[31] Wang M. Exact solutions for a compound KdV-Burgers equation. Phys Lett A 1996;213(5–6):279–87.
[32] Sharma D, Singh P, Chauhan S. Homotopy perturbation transform method with He's polynomial for solution of coupled nonlinear partial differential equations. Nonlinear Engineering 2016;5(1):17–23.
[33] Bayat M, Iman P, Domairry G. Recent developments of some asymptotic methods and their applications for nonlinear vibration equations in engineering problems: A review. Lat Am J Solids Struct 2012;9(2):1–93.
[34] Seadawy AR. Two-dimensional interaction of a shear flow with a free surface in a stratified fluid and its a solitary wave solutions via mathematical methods. Eur Phys J Plus 2017;132:518.
[35] Huan J. Variational iteration method for autonomous ordinary differential systems. Appl Math Comput 2000;114(2–3):115–23.
[36] Seadawy Aly R. Three-dimensional weakly nonlinear shallow water waves regime and its travelling wave solutions. Int J Comput Math 2018;15:1850017.
[37] Farah N, Seadawy AR, Ahmad S, Rizvi ST, Younis M. Interaction properties of soliton molecules and Painleve analysis for nano bioelectronics transmission model. Opt Quantum Electron 2020;52:1–15, ID: 329.
[38] Ahmad H, Seadawy AR, Tufail A. Analytic approximate solutions analytic approximate solutions for some nonlinear parabolic dynamical wave equations. Taibah Univ J Sci 2020;14:346–58.
[39] Yesim Yesim G, Yasar E, Seadawy AR. A third-order nonlinear Schrodinger equation: the exact solutions, group-invariant solutions and conservation laws. J Taibah Univ Sci 2020;14:585–97.
[40] Helal MA, Seadawy AR, Zekry MH. Stability analysis of solitary wave solutions for the fourth-order nonlinear Boussinesq water wave equation. Appl Math Comput 2014;232:1094–103.
[41] Seadawy AR, Rashidy KE. Dispersive solitary wave solutions of Kadomtsev-Petviashivili and modified Kadomtsev-Petviashivili dynamical equations in unmagnetized dust plasma. Results Phys 2018;8:1216–22.