*fractal and fractional*

Article

# The Marichev-Saigo-Maeda Fractional-Calculus Operators Involving the $(p, q)$-Extended Bessel and Bessel-Wright Functions

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**Abstract:** The goal of this article is to establish several new formulas and new results related to the Marichev-Saigo-Maeda fractional integral and fractional derivative operators which are applied on the $(p, q)$-extended Bessel function. The results are expressed as the Hadamard product of the $(p, q)$-extended Gauss hypergeometric function $F_{p,q}$ and the Fox-Wright function ${}_r\Psi_s(z)$. Some special cases of our main results are considered. Furthermore, the $(p, q)$-extended Bessel-Wright function is introduced. Finally, a variety of formulas for the Marichev-Saigo-Maeda fractional integral and derivative operators involving the $(p, q)$-extended Bessel-Wright function is established.

**Keywords:** operators of fractional calculus; $(p, q)$-extensions of special functions; $(p, q)$-extended Bessel function; $(p, q)$-extended Gauss hypergeometric function; $(p, q)$-extended Bessel-Wright function; Fox-Wright function; Marichev-Saigo-Maeda fractional integral and fractional derivative operators; Euler-Darboux partial differential equation

## 1. Introduction

Many generalizations and extensions of special functions of mathematical physics have witnessed a significant evolution in recent years. This advancement in the theory of special functions serves as an analytic foundation for the majority of problems in mathematical physics and applied sciences, which have been solved exactly and which have found broad practical applications. Further, the importance of Bessel functions appears in many areas of applied mathematics, mathematical physics, astronomy, engineering, *et cetera*. The Bessel function was first introduced by and named after Friedrich Wilhelm Bessel (1784–1846) and it was subsequently developed by (among others) Euler, Lagrange, Bernoulli, and others. The Bessel function is a solution of a homogeneous second-order differential equation which is called the Bessel's differential equation and it is given by (see [1])

$$z^2 \frac{d^2u}{dz^2} + z \frac{du}{dz} + (z^2 - \nu^2)u = 0,$$

where $\nu$ can be a real or complex number. The Bessel function $J_\nu(z)$ of the first kind of order $\nu$ has the following power-series representation (see [1]):

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{2n+\nu}, \tag{1}$$

where $z, \nu \in \mathbb{C}$ and $\Re(\nu) > -1$.

The $(p,q)$-extended Bessel function $J_{\nu,p,q}(z)$ of the first kind of order $\nu$ is defined as follows (see [2]):

$$J_{\nu,p,q}(z) = \frac{\sqrt{\pi}}{\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(-1)^n B\left(n+\frac{1}{2}, \nu+\frac{1}{2}; p, q\right)}{n!\,\Gamma\left(n+\frac{1}{2}\right) B\left(\frac{1}{2}, \nu+\frac{1}{2}\right)} \left(\frac{z}{2}\right)^{2n+\nu}, \tag{2}$$

where $\min\{\Re(p), \Re(q)\} \geqq 0$, and $\Re(\nu) > -1$ when $p = q = 0$ and $B(x,y;p,q)$ is the $(p,q)$-extended Beta function, which is defined as follows (see [3]):

$$B(x,y;p,q) = \int_0^1 t^{x-1}\,(1-t)^{y-1}\,\exp\left(-\frac{p}{t} - \frac{q}{1-t}\right) dt, \tag{3}$$

where

$$\min\{\Re(x), \Re(y)\} > 0 \text{ and } \min\{\Re(p), \Re(q)\} \geqq 0.$$

It should be remarked here that the existing literature on the subject contains much more general extensions of the classical Beta function, especially in the case when $p = q$ (see, for example, [4,5]).

For $p = q = 1$, the $(p,q)$-extended Bessel function of the first kind $J_{\nu,p,q}(z)$ and the $(p,q)$-extended Beta function $B(x,y;p,q)$ reduce to the Bessel function $J_\nu(z)$ of the first kind and the classical Beta function $B(x,y)$, respectively.

The Bessel-Wright function $J_\nu^\mu(z)$ was defined by Edward Maitland Wright (1906–2005) as follows (see [6]):

$$J_\nu^\mu(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\,\Gamma(\mu n+\nu+1)}, \tag{4}$$

where $z, \nu \in \mathbb{C}$ and $\mu > 0$.

Recently, Bessel functions have become widely used in fractional calculus and its applications (see, for example, [7,8]).

The Fox-Wright function ${}_r\Psi_s(z)$ was introduced and studied by Charles Fox (1897–1977) [9] and Wright [10]. It was proposed in the following form:

$${}_r\Psi_s\left[\begin{array}{c} (a_1,A_1), \cdots, (a_r,A_r); \\ \\ (b_1,B_1), \cdots, (b_s,B_s); \end{array} z\right] := \sum_{n=0}^{\infty} \frac{\Gamma(a_1+A_1 n)\cdots\Gamma(a_r+A_r n)}{\Gamma(b_1+B_1 n)\cdots\Gamma(b_s+B_s n)} \frac{z^n}{n!}, \tag{5}$$

where $r,s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z, a_i, b_j \in \mathbb{C}; A_i \in \mathbb{R}^+ (i = 1, \cdots, r); B_j \in \mathbb{R}^+ (j = 1, \cdots, s)$ and

$$1 + \sum_{j=1}^{s} B_j - \sum_{i=1}^{r} A_i > 0.$$

By comparing Equations (4) and (5), it can be easily seen that

$$J_\nu^\mu(z) = {}_0\Psi_1\left[\begin{array}{c} \overline{\phantom{xxxxxx}}; \\ \\ (\nu+1,\mu); \end{array} -z\right],$$

which relates the Bessel-Wright function $J_\nu^\mu(z)$ to the widely- and extensively-investigated Fox-Wright function ${}_r\Psi_s(z)$ defined by Equation (5).

The $(p, q)$-extended Gauss hypergeometric function $F_{p,q}$ is defined as follows (see [3]):

$$F_{p,q}(a, b; c; z) = \sum_{n=0}^{\infty} (a)_n \frac{B(b+n, c-b; p, q)}{B(b, c-b)} \frac{z^n}{n!}, \tag{6}$$

where $|z| < 1$ and $\Re(c) > \Re(b) > 0$.

The *third* Appell function $F_3$ (also known as one of the functions in Horn's list) is defined as follows (see [11]):

$$F_3(\alpha, \alpha', \beta, \beta'; \gamma; x; y) = \sum_{m,n}^{\infty} \frac{(\alpha)_m (\alpha')_n (\beta)_m (\beta')_n}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \qquad (\max\{|x|, |y|\} < 1).$$

Let $f$ and $g$ be two functions having the following power-series representations:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \text{and} \qquad g(z) = \sum_{n=0}^{\infty} b_n z^n.$$

Then the familiar Hadamard product (or convolution) of the functions $f$ and $g$, is given by

$$(f * g)(z) := \sum_{n=0}^{\infty} a_n b_n z^n =: (g * f)(z). \tag{7}$$

The introduction of fractional calculus is a very important development in the field of calculus due to the fact that it has proven to be widely applicable in many fields of mathematical, physical and applied sciences. Initially, fractional calculus is the study of derivative and integral operators with a real or complex order, and thus it is a generalization of the traditional calculus. The fractional derivative was first discussed by l'Hôpital and Leibniz in the 16th century and attracted the attention of many mathematicians such as Euler, Laplace, Fourier, Abel, Liouville, Riemann, Grünwald, Letnikov, Weyl, Lévy, and Riesz. Due to its usefulness in different emerging branches of applied mathematics, physics, engineering, quantum mechanics, electrical engineering, telecommunications, digital image processing, robotics, system identification, chemistry, and biology (see, for example, [12–16]), it has been one of the most significant branches of applied mathematics. The development and study of fractional calculus opens the possibility of generalizations of formulas; furthermore, the generalized Marichev-Saigo-Maeda fractional integral was introduced by Marichev [17] as Mellin-type convolution operators with the Appell function $F_3$ in their kernel.) These operators were rediscovered and studied by Saigo [18] (and, subsequently, by Saigo and Maeda [19]) as generalizations of the Saigo fractional integral operators, which were first studied by Saigo [20] and then applied by Srivastava and Saigo [21] in their systematic investigation of several boundary-value problems involving the Euler-Darboux partial differential equation.

We recall here the generalized Marichev-Saigo-Maeda fractional integral and fractional derivative operators, introduced by Marichev [17], as Mellin type convolution operators with the Appell function $F_3$ in their kernel, which are defined as follows:

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3\left(\alpha, \alpha', \beta, \beta'; \gamma; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)dt, \tag{8}$$

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)} \int_x^{\infty} (t-x)^{\gamma-1} t^{-\alpha} F_3\left(\alpha, \alpha', \beta, \beta'; \gamma; 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t)dt, \tag{9}$$

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f\right)(x) \tag{10}$$

and

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x)=\left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma}f\right)(x), \tag{11}$$

where $\alpha,\alpha',\beta,\beta',\gamma\in\mathbb{C}$. $\Re(\gamma)>0$ and $x>0$.

By first replacing the parameter $\alpha$ by $\alpha+\beta$, and then setting $\alpha'=\beta'=0$, $\beta=-\eta$ and $\gamma=\alpha$ in Equations (8)–(11), the generalized Marichev-Saigo-Maeda fractional integral and fractional derivative operators reduce to the Saigo fractional integral and derivative operators $I_{0+}^{\alpha,\beta,\eta}$, $I_{-}^{\alpha,\beta,\eta}$, $D_{0+}^{\alpha,\beta,\eta}$ and $D_{-}^{\alpha,\beta,\eta}$ involving the hypergeometric function ${}_2F_1$ in their kernel, which are defined by (see [20]; see also [21]):

$$\left(I_{0+}^{\alpha,\beta,\eta}f\right)(x)=\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x}\right)f(t)dt, \tag{12}$$

$$\left(I_{-}^{\alpha,\beta,\eta}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_x^\infty t^{-\alpha-\beta}\,(t-x)^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}\right)f(t)dt, \tag{13}$$

$$\left(D_{0+}^{\alpha,\beta,\eta}f\right)(x)=\left(I_{0+}^{-\alpha,-\beta,\alpha+\eta}f\right)(x) \tag{14}$$

and

$$\left(D_{-}^{\alpha,\beta,\eta}f\right)(x)=\left(I_{-}^{-\alpha,-\beta,\alpha+\eta}f\right)(x), \tag{15}$$

where $\alpha,\beta,\eta\in\mathbb{C}$ and $x>0$.

Moreover, by taking $\beta=-\alpha$ in Equations (12)–(15), the Saigo fractional integral and fractional derivative operators reduce to the Riemann-Liouville integral and fractional derivative operators of the function $f(x)\ \ (x\in\mathbb{R}^+)$ with fractional order $\alpha\in\mathbb{C}\ \ \big(\Re(\alpha)>0\big)$, which are defined as follows (see [22–24]):

$$(I_{0+}^{\alpha}f)(x)=\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}f(t)dt, \tag{16}$$

$$(I_{-}^{\alpha}f)(x)=\frac{1}{\Gamma(\alpha)}\int_x^\infty(t-x)^{\alpha-1}\,f(t)dt, \tag{17}$$

$$\begin{aligned}(D_{0+}^{\alpha}f)(x)&=\left(\frac{d}{dx}\right)^n(I_{0+}^{n-\alpha}f)\\&=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n\int_0^x(x-t)^{n-\alpha-1}\,f(t)dt\\&\qquad\big(n=[\Re(\alpha)]+1\big)\end{aligned} \tag{18}$$

and

$$\begin{aligned}(D_{-}^{\alpha}f)(x)&=\left(-\frac{d}{dx}\right)^n(I_{-}^{n-\alpha}f)(x)(n=[\Re(\alpha)]+1)\\&=\frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n\int_x^\infty(t-x)^{n-\alpha-1}\,f(t)dt\\&\qquad\big(n=[\Re(\alpha)]+1\big).\end{aligned} \tag{19}$$

Further, by taking $\beta=0$ in Equations (12)–(15), the Saigo fractional integral and derivative operators reduce to the Erdélyi-Kober fractional integral and derivative operators of the function $f(x)\ \ (x\in\mathbb{R}^+)$ with fractional order $\alpha\in\mathbb{C}\ \ \big(\Re(\alpha)>0\big)$, which are defined as follows (see [25]):

$$\left(I_{\eta,\alpha}^{+}f\right)(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^x t^{\eta}\ (x-t)^{\alpha-1}\ f(t)dt, \tag{20}$$

$$\left(K_{\eta,\alpha}^{-}f\right)(x) = \frac{x^{\eta}}{\Gamma(\alpha)} \int_x^{\infty} t^{-\alpha-\eta}\ (t-x)^{\alpha-1}\ f(t)dt, \tag{21}$$

$$\left(D_{\eta,\alpha}^{+}f\right)(x) = x^{-\eta}\left(\frac{d}{dx}\right)^{n} \frac{1}{\Gamma(n-\alpha)} \int_0^x t^{\alpha+\eta}\ (x-t)^{n-\alpha-1}\ f(t)dt \tag{22}$$
$$\big(n = [\Re(\alpha)]+1\big)$$

and

$$\left(D_{\eta,\alpha}^{-}f\right)(x) = x^{\alpha+\eta}\left(\frac{d}{dx}\right)^{n} \frac{1}{\Gamma(n-\alpha)} \int_0^x t^{-\eta}\ (x-t)^{n-\alpha-1}\ f(t)dt \tag{23}$$
$$\big(n = [\Re(\alpha)]+1\big).$$

In order to obtain our main results, we shall make use of the following lemma:

**Lemma 1.** *Let $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{C}$ and $x \in \mathbb{R}^+$. Then the Marichev-Saigo-Maeda fractional integral and fractional derivative operators of the power function $t^{\sigma-1}$ are given as follows (see [20]) :*

1. *If $\Re(\gamma) > 0$ and $\Re(\sigma) > \max\{0, \Re(\alpha+\alpha'+\beta-\gamma), \Re(\alpha'-\beta')\}$, then*

$$\begin{aligned}&\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\sigma-1}\right)(x)\\ &\quad= \frac{\Gamma(\sigma)\Gamma(\sigma+\gamma-\alpha-\alpha'-\beta)\Gamma(\sigma+\beta'-\alpha')}{\Gamma(\sigma+\beta')\Gamma(\sigma+\gamma-\alpha-\alpha')\Gamma(\sigma+\gamma-\alpha'-\beta)} x^{\sigma-\alpha-\alpha'+\gamma-1}.\end{aligned} \tag{24}$$

2. *If $\Re(\gamma) > 0$ and $\Re(\sigma) < 1 + \min\{\Re(-\beta), \Re(\alpha+\alpha'-\gamma), \Re(\alpha+\beta'-\gamma)\}$, then*

$$\begin{aligned}&\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\sigma-1}\right)(x)\\ &\quad= \frac{\Gamma(1-\sigma-\beta)\Gamma(1-\sigma-\gamma+\alpha+\alpha')\Gamma(1-\sigma+\alpha+\beta'-\gamma)}{\Gamma(1-\sigma)\Gamma(1-\sigma+\alpha+\alpha'+\beta'-\gamma)\Gamma(1-\sigma+\alpha-\beta)} x^{\sigma-\alpha-\alpha'+\gamma-1}.\end{aligned} \tag{25}$$

3. *If $\Re(\gamma) > 0$ and $\Re(\sigma) > \max\{0, \Re(\gamma-\alpha-\alpha'-\beta'), \Re(\beta-\alpha)\}$, then*

$$\begin{aligned}&\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\sigma-1}\right)(x)\\ &\quad= \frac{\Gamma(\sigma)\Gamma(\sigma-\gamma+\alpha+\alpha'+\beta')\Gamma(\sigma-\beta+\alpha)}{\Gamma(\sigma-\beta)\Gamma(\sigma-\gamma+\alpha+\alpha')\Gamma(\sigma-\gamma+\alpha+\beta')} x^{\sigma+\alpha+\alpha'-\gamma-1}.\end{aligned} \tag{26}$$

4. *If $\Re(\gamma) > 0$ and $\Re(\sigma) < 1 + \min\{\Re(\beta'), \Re(\gamma-\alpha-\alpha'), \Re(\gamma-\alpha'-\beta)\}$, then*

$$\begin{aligned}&\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\sigma-1}\right)(x)\\ &\quad= \frac{\Gamma(1-\sigma+\beta')\Gamma(1-\sigma+\gamma-\alpha-\alpha')\Gamma(1-\sigma-\alpha'-\beta+\gamma)}{\Gamma(1-\sigma)\Gamma(1-\sigma-\alpha-\alpha'-\beta+\gamma)\Gamma(1-\sigma-\alpha'+\beta')} x^{\sigma+\alpha+\alpha'-\gamma-1}.\end{aligned} \tag{27}$$

The study of fractional calculus provides many important tools for dealing with derivative and integral equations involving certain special functions and provides the generalized integrals and derivatives of arbitrary fractional order (see, for example, [21,26]). For several general results associated with the Marichev-Saigo-Maeda fractional integrals and derivatives, see the recent work by Srivastava et al. [27].

Motivated by these applications, in this work, we establish various formulas for the Marichev-Saigo-Maeda fractional derivative and integral operators involving the $(p,q)$-

extended Bessel function of the first kind of order $\nu$ in terms of the Hadamard product of the Fox-Wright function and the $(p,q)$-extended Gauss hypergeometric function.

In the next section, we establish formulas for the Marichev-Saigo-Maeda fractional integrals and derivatives involving the $(p,q)$-extended Bessel function $J_{\nu,p,q}(z)$ in terms of the Fox-Wright function ${}_r\Psi_s$ and the $(p,q)$-extended Gauss hypergeometric function.

## 2. Marichev-Saigo-Maeda Fractional Integral of the Function $J_{\nu,p,q}(z)$

Now, we establish the Marichev-Saigo-Maeda fractional integral formulas involving the $(p,q)$-extended Bessel function of the first kind of order $\nu$; the results are expressed as the Hadamard product of the Fox-Wright function and the $(p,q)$-extended Gauss hypergeometric function.

**Theorem 1.** *The Marichev-Saigo-Maeda fractional integral $I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$ $(\alpha,\alpha',\beta,\beta',\gamma,\sigma,\nu \in \mathbb{C})$ of the $(p,q)$-extended Bessel function of the first kind $J_{\nu,p,q}(t)$ is given by*

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\sigma-1} J_{\nu,p,q}(t)\right)(x) = \frac{\sqrt{\pi}x^{\sigma+\nu-\alpha-\alpha'+\gamma-1}}{2^{\nu}\Gamma(\nu+1)} F_{p,q}\left(1,\frac{1}{2};\nu+1;-\frac{x^2}{4}\right)$$
$$* \, {}_3\Psi_4\left[\begin{array}{c} (\sigma+\nu,2),\ (\sigma+\nu+\gamma-\alpha-\alpha'-\beta,2),\ (\sigma+\nu+\beta'-\alpha',2); \\ \left(\frac{1}{2},1\right),\ (\sigma+\nu+\beta',2),\ (\sigma+\nu+\gamma-\alpha-\alpha',2),\ (\sigma+\nu+\gamma-\alpha'-\beta,2); \end{array} -\frac{x^2}{4}\right], \tag{28}$$

*where* $\Re(\sigma+\nu) > \max\{0, \Re(\alpha+\alpha'+\beta-\gamma), \Re(\alpha'-\beta')\}$, $\min\{\Re(p),\Re(q)\} \geqq 0$, $\Re(\nu) > -1$ *when* $p=q=1$ *and* $\Re(\gamma) > 0$.

**Proof.** Applying the Marichev-Saigo-Maeda fractional integral operator $I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$, which is given by Equation (8) on the function $J_{\nu,p,q}(t)$, which is given by Equation (2), we have

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\sigma-1} J_{\nu,p,q}(t)\right)(x)$$
$$* \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{\pi} B\left(n+\frac{1}{2},\nu+\frac{1}{2};p,q\right)}{n! \cdot 2^{2n+\nu} B\left(\frac{1}{2},\nu+\frac{1}{2}\right)\Gamma(\nu+1)\Gamma\left(n+\frac{1}{2}\right)} \left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\sigma+\nu+2n-1}\right)(x). \tag{29}$$

Now, taking advantage of Equation (24) in Lemma 1 and Equation (29), which is satisfied under the conditions of Theorem 1, we get

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\sigma-1} J_{\nu,p,q}(t)\right)(x) = \frac{\sqrt{\pi}x^{\sigma+\nu+\gamma-\alpha-\alpha'-1}}{2^{\nu}\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{B\left(n+\frac{1}{2},\nu+\frac{1}{2};p,q\right)}{B\left(\frac{1}{2},\nu+\frac{1}{2}\right)}$$
$$* \sum_{n=0}^{\infty} \left[\frac{\Gamma(\sigma+\nu+2n)\Gamma(\sigma+\nu+\gamma-\alpha-\alpha'-\beta+2n)\Gamma(\sigma+\nu+\beta'-\alpha'+2n)}{\Gamma\left(n+\frac{1}{2}\right)\Gamma(\sigma+\nu+\beta'+2n)\Gamma(\sigma+\nu+\gamma-\alpha-\alpha'+2n)}\right.$$
$$\left. \cdot \frac{1}{n!\,\Gamma(\sigma+\nu+\gamma-\alpha'-\beta+2n)}\left(-\frac{x^2}{4}\right)^n\right]. \tag{30}$$

Therefore, by expressing the above Equation (30) as the Hadamard product of the Fox-Wright function ${}_r\Psi_s$, which is given by Equation (5) and the $(p,q)$-extended Gauss hypergeometric function $F_{p,q}$, which is given by Equation (6), we obtain the right-hand side of Equation (28). □

**Theorem 2.** *The Marichev-Saigo-Maeda fractional integral $I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}$ $(\alpha,\alpha',\beta,\beta',\gamma,\sigma,\nu \in \mathbb{C})$ of the $(p,q)$-extended Bessel function of the first kind is given by*

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\sigma-1}\, J_{\nu,p,q}\left(\frac{1}{t}\right)\right)(x) = \frac{\sqrt{\pi}x^{\sigma-\nu-\alpha-\alpha'+\gamma-1}}{2^{\nu}\Gamma(\nu+1)} F_{p,q}\left(1,\frac{1}{2};\nu+1;-\frac{1}{4x^2}\right)$$
$$*\ {}_3\Psi_4\left[\begin{array}{c} (1-\sigma+\nu-\beta,2),\ (1-\sigma+\nu-\gamma+\alpha+\alpha',2),\ (1-\sigma+\nu+\alpha+\beta'-\gamma,2); \\ \left(\frac{1}{2},1\right),\ (1-\sigma+\nu,2),\ (1-\sigma+\nu+\alpha+\alpha'+\beta'-\gamma,2),\ (1-\sigma+\nu+\alpha-\beta,2); \end{array} -\frac{1}{4x^2}\right], \tag{31}$$

*where* $\Re(\sigma-\nu) < 1+\min\{\Re(-\beta),\Re(\alpha+\alpha'-\gamma),\Re(\alpha+\beta'-\gamma)\}$, $\min\{\Re(p),\Re(q)\} \geqq 0$, $\Re(\nu) > -1$ *when* $p=q=1$ *and* $\Re(\gamma) > 0$.

**Proof.** Performing the Marichev-Saigo-Maeda fractional integral operator $I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}$, which is given by Equation (9) on the function $J_{\nu,p,q}(.)$, which is given by Equation (2), we have

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\sigma-1} J_{\nu,p,q}\left(\frac{1}{t}\right)\right)(x)$$
$$*\sum_{n=0}^{\infty} \frac{(-1)^n\sqrt{\pi}B\left(n+\frac{1}{2},\nu+\frac{1}{2};p,q\right)}{n!\cdot 2^{2n+\nu}\Gamma\left(n+\frac{1}{2}\right)\Gamma(\nu+1)B\left(\frac{1}{2},\nu+\frac{1}{2}\right)} \left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\sigma-\nu-2n-1}\right)(x). \tag{32}$$

Because of Lemma 1, and by using Equation (25) and Equation (32), which are satisfied under the conditions stated in Theorem 2, we get

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\sigma-1}\, J_{\nu,p,q}\left(\frac{1}{t}\right)\right)(x) = \frac{\sqrt{\pi}x^{\sigma+\gamma-\nu-\alpha-\alpha'-1}}{2^{\nu}\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{B\left(n+\frac{1}{2},\nu+\frac{1}{2};p,q\right)}{B\left(\frac{1}{2},\nu+\frac{1}{2}\right)}$$
$$*\sum_{n=0}^{\infty}\left[\frac{\Gamma(1-\sigma+\nu-\beta+2n)\Gamma(1-\sigma+\nu-\gamma+\alpha+\alpha'+2n)}{\Gamma\left(n+\frac{1}{2}\right)\Gamma(1-\sigma+\nu+2n)\Gamma(1-\sigma+\nu+\alpha+\alpha'+\beta'-\gamma+2n)}\right.$$
$$\left.\cdot\frac{\Gamma(1-\sigma+\nu+\alpha+\beta'-\gamma+2n)}{n!\,\Gamma(1-\sigma+\nu+\alpha-\beta+2n)}\left(-\frac{1}{4x^2}\right)^n\right]. \tag{33}$$

Therefore, by expressing Equation (33) as the Hadamard product of the Fox-Wright function ${}_r\Psi_s$, which is given by Equation (5) and the $(p,q)$-extended Gauss hypergeometric function $F_{p,q}$, which is given by Equation (6), we obtain the right-hand side of Equation (31). □

If we take $\alpha=\alpha+\beta$, $\alpha'=\beta'=0$, $\beta=-\eta$ and $\gamma=\alpha$ in Theorems 1 and 2, we get the image formula of the Saigo hypergeometric fractional integrals involving the $(p,q)$-extended Bessel function of the first kind $J_{\nu,p,q}(t)$, respectively, as follows.

**Corollary 1.** *Let* $\alpha,\beta,\eta,\sigma,\nu\in\mathbb{C}$ *be such that* $\min\{\Re(p),\Re(q)\}\geqq 0$, $\Re(\nu) > -1$ *when* $p=q=1$ *and* $\Re(\sigma+\nu) > \max\{0,\Re(\beta-\eta)\}$. *Then the following Saigo hypergeometric fractional integral of the function* $J_{\nu,p,q}(t)$ *holds true*:

$$\left(I_{0+}^{\alpha,\beta,\eta} t^{\sigma-1} J_{\nu,p,q}(t)\right)(x) = \frac{\sqrt{\pi}x^{\sigma+\nu-\beta-1}}{2^{\nu}\Gamma(\nu+1)} F_{p,q}\left(1,\frac{1}{2};\nu+1;-\frac{x^2}{4}\right)$$
$$*\ {}_2\Psi_3\left[\begin{array}{c} (\sigma+\nu+\eta-\beta,2),\ (\sigma+\nu,2); \\ \left(\frac{1}{2},1\right),\ (\sigma+\nu-\beta,2),\ (\sigma+\nu+\alpha+\eta,2); \end{array} -\frac{x^2}{4}\right]. \tag{34}$$

**Corollary 2.** *Let $\alpha, \beta, \eta, \sigma, \nu \in \mathbb{C}$ be such that $\min\{\Re(p), \Re(q)\} \geqq 0$, $\Re(\nu) > -1$ when $p = q = 1$ and $\Re(\sigma - \nu) > 1 + \min\{\Re(\beta), \Re(\eta)\}$. Then the following Saigo hypergeometric fractional integral $I_{-}^{\alpha,\beta,\eta}$ of the function $J_{\nu,p,q}\left(\frac{1}{t}\right)$ holds true*:

$$\left(I_{-}^{\alpha,\beta,\eta} t^{\sigma-1} J_{\nu,p,q}\left(\frac{1}{t}\right)\right)(x) = \frac{\sqrt{\pi}\, x^{\sigma-\nu-\beta-1}}{2^{\nu}\Gamma(\nu+1)} F_{p,q}\left(1, \frac{1}{2}; \nu+1; -\frac{1}{4x^2}\right)$$
$$* \, {}_2\Psi_3\left[\begin{array}{c} (1-\sigma+\beta+\nu, 2),\ (1+\nu-\sigma+\eta, 2); \\ \left(\frac{1}{2}, 1\right),\ (1-\sigma+\nu, 2),\ (1-\sigma+\nu+\alpha+\beta+\eta, 2); \end{array} \ -\frac{1}{4x^2}\right]. \tag{35}$$

If we take $\beta = -\alpha$ in Corollary 1 and Corollary 2, we obtain the Riemann-Liouville fractional integrals of the function $J_{\nu,p,q}$ as follows.

**Corollary 3.** *Let $\alpha, \nu, \sigma \in \mathbb{C}$ be such that $\min\{\Re(p), \Re(q)\} \geqq 0$, $\Re(\nu) > -1$ when $p = q = 1$ and $\Re(\sigma + \nu) > 0$. Then the following Riemann-Liouville fractional integral $I_{0+}^{\alpha}$ of $J_{\nu,p,q}(t)$ holds true*:

$$\left(I_{0+}^{\alpha} t^{\sigma-1} J_{\nu,p,q}(t)\right)(x) = \frac{\sqrt{\pi} x^{\sigma+\nu+\alpha-1}}{2^{\nu}\Gamma(\nu+1)} F_{p,q}\left(1, \frac{1}{2}; \nu+1; -\frac{x^2}{4}\right)$$
$$* \, {}_1\Psi_2\left[\begin{array}{c} (\sigma+\nu, 2); \\ \left(\frac{1}{2}, 1\right),\ (\sigma+\nu+\alpha, 2); \end{array} \ -\frac{x^2}{4}\right]. \tag{36}$$

**Corollary 4.** *Let $\alpha, \sigma, \nu \in \mathbb{C}$ be such that $\min\{\Re(p), \Re(q)\} \geqq 0$, $\Re(\nu) > -1$ when $p = q = 1$ and $\Re(\sigma - \nu) > 1 + \min\{\Re(\alpha)\}$. Then the following Riemann-Liouville fractional integral $I_{-}^{\alpha}$ of the function $J_{\nu,p,q}\left(\frac{1}{t}\right)$ holds true*:

$$\left(I_{-}^{\alpha} t^{\sigma-1} J_{\nu,p,q}\left(\frac{1}{t}\right)\right)(x) = \frac{\sqrt{\pi} x^{\sigma-\nu+\alpha-1}}{2^{\nu}\Gamma(\nu+\frac{1}{2})} F_{p,q}\left(1, \frac{1}{2}; \nu+1; -\frac{1}{4x^2}\right)$$
$$* \, {}_1\Psi_2\left[\begin{array}{c} (1-\sigma-\alpha+\nu, 2); \\ \left(\frac{1}{2}, 1\right),\ (1-\sigma+\nu, 2); \end{array} \ -\frac{1}{4x^2}\right]. \tag{37}$$

Upon setting $\beta = 0$ in Corollary 1 and Corollary 2, we obtain the Erdélyi-Kober fractional integrals of the function $J_{\nu,p,q}$ as follows.

**Corollary 5.** *Let $\alpha, \sigma, \eta, \nu \in \mathbb{C}$ be such that $\min\{\Re(p), \Re(q)\} \geqq 0$, $\Re(\nu) > -1$ when $p = q = 1$ and $\Re(\sigma + \nu) > -\Re(\eta)$. Then, the following Erdélyi-Kober fractional integral $I_{\alpha,\eta}^{+}$, which is given by Equation (21) and involves the function $J_{\nu,p,q}(t)$, holds true*:

$$\left(I_{\alpha,\eta}^{+} t^{\sigma-1} J_{\nu,p,q}(t)\right)(x) = \frac{\sqrt{\pi} x^{\sigma+\nu-1}}{2^{\nu}\Gamma(\nu+1)} F_{p,q}\left(1, \frac{1}{2}; \nu+1; -\frac{x^2}{4}\right)$$
$$* \, {}_1\Psi_2\left[\begin{array}{c} (\sigma+\nu+\eta, 2); \\ \left(\frac{1}{2}, 1\right),\ (\sigma+\nu+\alpha+\eta, 2); \end{array} \ -\frac{x^2}{4}\right]. \tag{38}$$

**Corollary 6.** *Let $\alpha, \sigma, \eta, \nu \in \mathbb{C}$ be such that $\min\{\Re(p), \Re(q)\} \geqq 0$, $\Re(\nu) > -1$ when $p = q = 1$ and $\Re(\sigma - \nu) < 1 + \Re(\eta)$. Then the following Erdélyi-Kober fractional integral $K_{\alpha,\eta}^{-}$, given by Equation (22) and involving the function $J_{\nu,p,q}\left(\frac{1}{t}\right)$, holds true*:

$$\left(K_{\alpha,\eta}^{-}t^{\sigma-1}J_{\nu,p,q}\left(\frac{1}{t}\right)\right)(x)=\frac{\sqrt{\pi}x^{\sigma-\nu-1}}{2^{\nu}\Gamma(\nu+1)}F_{p,q}\left(1,\frac{1}{2};\nu+1;-\frac{1}{4x^2}\right)$$
$$*\,{}_1\Psi_2\left[\begin{array}{c}(1+\nu-\sigma+\eta,2);\\ \left(\frac{1}{2},1\right),\ (1-\sigma+\nu+\alpha+\eta,2);\end{array}\ -\frac{1}{4x^2}\right]. \tag{39}$$

**Remark 1.** *If we take $p=q=1$ in Theorems 1 and 2, we get the results established by Purohit et al. [8].*

## 3. Marichev-Saigo-Maeda Fractional Derivative of the Function $J_{\nu,p,q}(z)$

In this section, we establish Marichev-Saigo-Maeda fractional derivatives of the $(p,q)$-extended Bessel function of the first kind of order $\nu$.

**Theorem 3.** *The Marichev-Saigo-Maeda fractional derivative $D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$ $(\alpha,\alpha',\beta,\beta',\gamma,\sigma,\nu\in\mathbb{C})$ of the function $J_{\nu,p,q}(t)$ is given by*

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\sigma-1}J_{\nu,p,q}(t)\right)(x)=\frac{\sqrt{\pi}x^{\sigma+\nu+\alpha+\alpha'-\gamma-1}}{2^{\nu}\Gamma(\nu+1)}F_{p,q}\left(1,\frac{1}{2};\nu+1;-\frac{x^2}{4}\right)$$
$$*\,{}_3\Psi_4\left[\begin{array}{c}(\sigma+\nu,2),\ (\sigma+\nu-\gamma+\alpha+\alpha'+\beta',2),\ (\sigma+\nu-\beta+\alpha,2);\\ \left(\frac{1}{2},1\right),\ (\sigma+\nu-\beta,2),\ (\sigma+\nu-\gamma+\alpha+\alpha',2),\ (\sigma+\nu-\gamma+\alpha+\beta',2);\end{array}\ -\frac{x^2}{4}\right], \tag{40}$$

*where $\Re(\sigma+\nu)>\max\{0,\Re(\gamma-\alpha-\alpha'-\beta'),\Re(\beta-\alpha)\}$, $\min\{\Re(p),\Re(q)\}\geqq 0$, $\Re(\nu)>-1$ when $p=q=1$ and $\Re(\gamma)>0$.*

**Proof.** Benefiting from Equations (10) and (2) in the left-hand side of Equation (40), we have

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\sigma-1}\ J_{\nu,p,q}(t)\right)(x)$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n\sqrt{\pi}\ B\left(n+\frac{1}{2},\nu+\frac{1}{2};p,q\right)}{n!\cdot 2^{2n+\nu}B\left(\frac{1}{2},\nu+\frac{1}{2}\right)\Gamma(\nu+1)\Gamma\left(n+\frac{1}{2}\right)}\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\nu+\sigma+2n-1}\right)(x), \tag{41}$$

Thus, by using Equations (26) and (41), which are satisfied under the conditions stated with Theorem 3, we get

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\sigma-1}\ J_{\nu,p,q}(t)\right)(x)=\frac{\sqrt{\pi}\ x^{\sigma+\nu-\gamma+\alpha+\alpha'-1}}{2^{\nu}\ \Gamma(\nu+1)}\sum_{n=0}^{\infty}\left[\frac{B\left(n+\frac{1}{2},\nu+\frac{1}{2};p,q\right)}{B\left(\frac{1}{2},\nu+\frac{1}{2}\right)}\right.$$
$$\cdot\frac{\Gamma(\sigma+\nu+2n)\Gamma(\sigma+\nu-\gamma+\alpha+\alpha'+\beta'+2n)}{\Gamma\left(n+\frac{1}{2}\right)\Gamma(\sigma+\nu-\beta+2n)\Gamma(\sigma+\nu-\gamma+\alpha+\alpha'+2n)}$$
$$\left.\cdot\frac{\Gamma(\sigma+\nu-\beta+\alpha+2n)}{n!\ \Gamma(\sigma+\nu-\gamma+\alpha+\beta'+2n)}\left(-\frac{x^2}{4}\right)^n\right]. \tag{42}$$

Therefore, by expressing the above equation (42) as the Hadamard product of the Fox-Wright function ${}_r\Psi_s$ given by Equation (5) and the $(p,q)$-extended Gauss hypergeometric function $F_{p,q}$ given by Equation (6), we obtain the right-hand side of Equation (40). □

**Theorem 4.** *The Marichev-Saigo-Maeda fractional derivative* $D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}$ $(\alpha,\alpha',\beta,\beta',\gamma,\sigma,\nu \in \mathbb{C})$ *of the function* $J_{\nu,p,q}\left(\frac{1}{t}\right)$ *is given by*

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\sigma-1}J_{\nu,p,q}\left(\frac{1}{t}\right)\right)(x) = \frac{\sqrt{\pi}x^{\sigma-\nu+\alpha+\alpha'-\gamma-1}}{2^{\nu}\Gamma(\nu+1)}F_{p,q}\left(1,\frac{1}{2};\nu+1;-\frac{1}{4x^2}\right)$$
$$*\,{}_3\Psi_4\left[\begin{matrix}(1-\sigma+\nu+\beta',2),\ (1-\sigma+\nu+\gamma-\alpha-\alpha',2),\ (1-\sigma+\nu-\alpha'-\beta+\gamma,2);\\ \left(\frac{1}{2},1\right),\ (1-\sigma+\nu,2),(1-\sigma+\nu-\alpha-\alpha'-\beta+\gamma,2),\ (1-\sigma+\nu-\alpha'+\beta',2);\end{matrix}\ -\frac{1}{4x^2}\right], \quad (43)$$

*where* $\Re(\sigma-\nu) < 1+\min\{\Re(\beta'),\Re(\gamma-\alpha-\alpha'),\Re(\gamma-\alpha'-\beta)\}$, $\min\{\Re(p),\Re(q)\} \geqq 0$, $\Re(\nu) > -1$ *when* $p=q=1$ *and* $\Re(\gamma) > 0$.

**Proof.** In view of Equations (11), (2) and the left-hand side of Equation (43), we have

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\sigma-1}J_{\nu,p,q}\left(\frac{1}{t}\right)\right)(x)$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n\sqrt{\pi}B\left(n+\frac{1}{2},\nu+\frac{1}{2};p,q\right)}{n!\cdot 2^{2n+\nu}\Gamma(\nu+1)\Gamma\left(n+\frac{1}{2}\right)B\left(\frac{1}{2},\nu+\frac{1}{2}\right)}\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\sigma-\nu-2n-1}\right)(x), \quad (44)$$

Thus, making use of Equation (27) in Lemma 1 and Equation (44), which is satisfied under the conditions stated with Theorem 4, we get

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\sigma-1}J_{\nu,p,q}\left(\frac{1}{t}\right)\right)(x) = \frac{\sqrt{\pi}x^{\sigma-\gamma-\nu+\alpha+\alpha'-1}}{2^{\nu}\Gamma(\nu+1)}\sum_{n=0}^{\infty}\frac{B\left(n+\frac{1}{2},\nu+\frac{1}{2};p,q\right)}{B\left(\frac{1}{2},\nu+\frac{1}{2}\right)}$$
$$\cdot\sum_{n=0}^{\infty}\left[\frac{\Gamma(1-\sigma+\beta'+\nu+2n)\Gamma(1-\sigma+\nu+\gamma-\alpha-\alpha'+2n)}{\Gamma\left(n+\frac{1}{2}\right)\Gamma(1-\sigma+\nu+2n)\Gamma(1-\sigma+\nu-\alpha-\alpha'-\beta+\gamma+2n)}\right.$$
$$\left.\cdot\frac{\Gamma(1-\sigma+\nu-\alpha'-\beta+\gamma+2n)}{n!\,\Gamma(1-\sigma+\nu-\alpha'+\beta'+2n)}\left(-\frac{1}{4x^2}\right)^n\right]. \quad (45)$$

Therefore, by expressing the above Equation (45) as the Hadamard product of the Fox-Wright function ${}_r\Psi_s$, given by Equation (5) and the $(p,q)$-extended Gauss hypergeometric function $F_{p,q}$ given by Equation (6), we obtain the right-hand side of Equation (43). □

If we take $\alpha=\alpha+\beta,\alpha'=\beta'=0,\beta=-\eta$ and $\gamma=\alpha$ in Theorem 3 and Theorem 4, we get the Saigo hypergeometric fractional derivatives of the $(p,q)$-extended Bessel function of the first kind $J_{\nu,p,q}$, respectively, as follows.

**Corollary 7.** *Let* $\alpha,\beta,\eta,\nu,\sigma \in \mathbb{C}$ *be such that* $\min\{\Re(p),\Re(q)\} \geqq 0$, $\Re(\nu) > -1$ *when* $p=q=1$ *and* $\Re(\sigma+\nu) > -\max\{0,\Re(\beta),\Re(\alpha+\beta+\eta)\}$. *Then the following Saigo hypergeometric fractional derivative of the function* $J_{\nu,p,q}(t)$ *holds true*:

$$\left(D_{0+}^{\alpha,\beta,\eta}t^{\sigma-1}J_{\nu,p,q}(t)\right)(x) = \frac{\sqrt{\pi}x^{\sigma+\nu+\beta-1}}{2^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)}F_{p,q}\left(1,\frac{1}{2};\nu+1;-\frac{x^2}{4}\right)$$
$$*\,{}_2\Psi_3\left[\begin{matrix}(\sigma+\nu,2),\ (\sigma+\nu+\alpha+\beta+\eta,2);\\ \left(\frac{1}{2},1\right),\ (\sigma+\nu+\eta,2),\ (\sigma+\nu+\beta,2);\end{matrix}\ -\frac{x^2}{4}\right]. \quad (46)$$

**Corollary 8.** *Let $\alpha, \beta, \eta, \sigma, \nu \in \mathbb{C}$ be such that $\min\{\Re(p), \Re(q)\} \geqq 0$, $\Re(\nu) > -1$ when $p = q = 1$ and $\Re(\sigma - \nu) > 1 + \min\{\Re(-\beta), \Re(\alpha + \eta)\}$. Then, the following Saigo hypergeometric fractional derivative of the function $J_{\nu,p,q}\left(\frac{1}{t}\right)$ holds true:*

$$\left(D_{-}^{\alpha,\beta,\eta} t^{\sigma-1} J_{\nu,p,q}\left(\frac{1}{t}\right)\right)(x) = \frac{\sqrt{\pi} x^{\sigma-\nu+\beta-1}}{2^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)} F_{p,q}\left(1, \frac{1}{2}; \nu+1; -\frac{1}{4x^2}\right)$$
$$* {}_2\Psi_3\left[\begin{array}{c} (1-\sigma-\beta+\nu, 2),\ (1+\nu-\sigma+\alpha+\eta, 2); \\ \left(\frac{1}{2}, 1\right),\ (1-\sigma+\nu, 2),\ (1-\sigma+\nu-\beta+\eta, 2); \end{array} -\frac{1}{4x^2}\right]. \tag{47}$$

If we take $\beta = -\alpha$ in Corollaries 7 and 8, we obtain the Riemann-Liouville fractional derivatives of the function $J_{\nu,p,q}$ as follows.

**Corollary 9.** *Let $\alpha, \eta, \nu, \sigma \in \mathbb{C}$ be such that $\min\{\Re(p), \Re(q)\} \geqq 0$, $\Re(\nu) > -1$ when $p = q = 1$ and $\Re(\sigma + \nu) > \max\{0, \Re(\alpha), \Re(-\eta)\}$. Then the following Riemann-Liouville fractional derivative of the function $J_{\nu,p,q}(t)$ holds true:*

$$\left(D_{0+}^{\alpha} t^{\sigma-1} J_{\nu,p,q}(t)\right)(x) = \frac{\sqrt{\pi} x^{\sigma+\nu-\alpha-1}}{2^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)} F_{p,q}\left(1, \frac{1}{2}; \nu+1; -\frac{x^2}{4}\right)$$
$$* {}_1\Psi_2\left[\begin{array}{c} (\sigma+\nu, 2); \\ \left(\frac{1}{2}, 1\right),\ (\sigma+\nu-\alpha, 2); \end{array} -\frac{x^2}{4}\right]. \tag{48}$$

**Corollary 10.** *Let $\alpha, \eta, \sigma, \nu \in \mathbb{C}$ be such that $\min\{\Re(p), \Re(q)\} \geqq 0$, $\Re(\nu) > -1$ when $p = q = 1$ and $\Re(\sigma - \nu) > 1 + \min\{\Re(\alpha), \Re(\alpha + \eta)\}$. Then the following Riemann-Liouville fractional derivative of the function $J_{\nu,p,q}\left(\frac{1}{t}\right)$ holds true:*

$$\left(D_{-}^{\alpha} t^{\sigma-1} J_{\nu,p,q}\left(\frac{1}{t}\right)\right)(x) = \frac{\sqrt{\pi} x^{\sigma-\nu-\alpha-1}}{2^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)} F_{p,q}\left(1, \frac{1}{2}; \nu+1; -\frac{1}{4x^2}\right)$$
$$* {}_1\Psi_2\left[\begin{array}{c} (1-\sigma+\alpha+\nu, 2); \\ \left(\frac{1}{2}, 1\right),\ (1-\sigma+\nu, 2); \end{array} -\frac{1}{4x^2}\right]. \tag{49}$$

Furthermore, for $\beta = 0$, Corollaries 7 and 8 yield the following results for the Erdélyi-Kober fractional derivatives, respectively.

**Corollary 11.** *Let $\alpha, \eta, \nu, \sigma \in \mathbb{C}$ be such that $\min\{\Re(p), \Re(q)\} \geqq 0$, $\Re(\nu) > -1$ when $p = q = 1$ and $\Re(\sigma + \nu) > -\max\{0, \Re(\alpha + \eta)\}$. Then the following Erdélyi-Kober fractional derivative $D_{0+}^{\alpha,\eta}$ of the function $J_{\nu,p,q}(t)$ holds true:*

$$\left(D_{0+}^{\alpha,\eta} t^{\sigma-1} J_{\nu,p,q}(t)\right)(x) = \frac{\sqrt{\pi} x^{\sigma+\nu-1}}{2^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)} F_{p,q}\left(1, \frac{1}{2}; \nu+1; -\frac{x^2}{4}\right)$$
$$* {}_1\Psi_2\left[\begin{array}{c} (\sigma+\nu+\alpha+\eta, 2); \\ \left(\frac{1}{2}, 1\right),\ (\sigma+\nu+\eta, 2); \end{array} -\frac{x^2}{4}\right]. \tag{50}$$

**Corollary 12.** *Let $\alpha, \eta, \sigma, \nu \in \mathbb{C}$ be such that $\min\{\Re(p), \Re(q)\} \geqq 0$, $\Re(\nu) > -1$ when $p = q = 1$ and $\Re(\sigma - \nu) > 1 + \Re(\alpha + \eta)$. Then the following Erdélyi-Kober fractional derivative $D_{-}^{\alpha,\eta}$ of the function $J_{\nu,p,q}\left(\frac{1}{t}\right)$ holds true:*

$$\left(D_{-}^{\alpha,\eta}t^{\sigma-1}J_{\nu,p,q}\left(\frac{1}{t}\right)\right)(x)=\frac{\sqrt{\pi}x^{\sigma-\nu-1}}{2^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)}F_{p,q}\left(1,\frac{1}{2};\nu+1;-\frac{1}{4x^{2}}\right)$$
$$*\,{}_1\Psi_2\left[\begin{array}{c}(1+\nu-\sigma+\alpha+\eta,2);\\ \left(\frac{1}{2},1\right),\ (1-\sigma+\nu+\eta,2);\end{array}\ -\frac{1}{4x^{2}}\right]. \tag{51}$$

## 4. Marichev-Saigo-Maeda Fractional Calculus of the $(p,q)$-Extended Bessel-Wright Function

In this section, we first introduce the $(p,q)$-extended Bessel-Wright function. We then establish the results for the Marichev-Saigo-Maeda fractional integrals and fractional derivatives involving the $(p,q)$-extended Bessel-Wright function.

The $(p,q)$-extended Bessel-Wright function $J_{\nu,p,q}^{\mu}(z)$ is defined, in terms of the $(p,q)$-extended Beta function, by

$$J_{\nu,p,q}^{\mu}(z)=\frac{1}{\Gamma\left(\nu+\frac{1}{2}\right)}\sum_{n=0}^{\infty}\frac{B\left(\mu n+\frac{1}{2},\nu+\frac{1}{2};p,q\right)}{\Gamma\left(\mu n+\frac{1}{2}\right)}\ \frac{(-z)^{n}}{n!}, \tag{52}$$

or, equivalently, by

$$J_{\nu,p,q}^{\mu}(z)=\frac{\sqrt{\pi}}{\Gamma(\nu+1)}\sum_{n=0}^{\infty}\frac{B\left(\mu n+\frac{1}{2},\nu+\frac{1}{2};p,q\right)}{B\left(\frac{1}{2},\nu+\frac{1}{2}\right)\Gamma\left(\mu n+\frac{1}{2}\right)}\ \frac{(-z)^{n}}{n!}, \tag{53}$$

where $\nu,z\in\mathbb{C}$, $\mu>0$, $\min\{\Re(p),\Re(q)\}\geqq 0$, $\Re(\nu)>-1$ when $p=q=1$ and $B(x,y;p,q)$ is given by the Equation (3).

We now establish the following formulas of the Marichev-Saigo-Maeda fractional integrals involving the $(p,q)$-extended Bessel-Wright function $J_{\nu,p,q}^{\mu}(z)$ in terms of the Hadamard product of the Fox-Wright function ${}_r\Psi_s$ and the $(p,q)$-extended Gauss hypergeometric function.

**Theorem 5.** *The Marichev-Saigo-Maeda fractional integral $I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$ $(\alpha,\alpha',\beta,\beta',\gamma,\sigma,\nu\in\mathbb{C})$ of the $(p,q)$-extended Bessel-Wright function $J_{\nu,p,q}^{\mu}(t)$ is given by*

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\sigma-1}\ J_{\nu,p,q}^{\mu}(t)\right)(x)=\frac{\sqrt{\pi}x^{\sigma-\alpha-\alpha'+\gamma-1}}{\Gamma(\nu+1)}F_{p,q}\left(1,\frac{1}{2};\nu+1;-x\right)$$
$$*\,{}_3\Psi_4\left[\begin{array}{c}(\sigma,1),\ (\sigma+\gamma-\alpha-\alpha'-\beta,1),\ (\sigma+\beta'-\alpha',1);\\ \left(\frac{1}{2},\mu\right),\ (\sigma+\beta',1),\ (\sigma+\gamma-\alpha-\alpha',1),\ (\sigma+\gamma-\alpha'-\beta,1);\end{array}\ -x\right], \tag{54}$$

*where $\Re(\sigma)>\max\{0,\Re(\alpha+\alpha'+\beta-\gamma),\Re(\alpha'-\beta')\}$, $\min\{\Re(p),\Re(q)\}\geqq 0$, $\Re(\nu)>-1$ when $p=q=1$, $\Re(\gamma)>0$ and $\mu>0$.*

**Theorem 6.** *The Marichev-Saigo-Maeda fractional integral $I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}$ $(\alpha,\alpha',\beta,\beta',\gamma,\sigma,\nu\in\mathbb{C})$ of the $(p,q)$-extended Bessel-Wright function $J_{\nu,p,q}^{\mu}(t)$ is given by*

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\sigma-1}J_{\nu,p,q}^{\mu}\left(\frac{1}{t}\right)\right)(x)=\frac{\sqrt{\pi}x^{\sigma-\alpha-\alpha'+\gamma-1}}{\Gamma(\nu+1)}F_{p,q}\left(1,\frac{1}{2};\nu+1;-\frac{1}{x}\right)$$
$$*\,{}_3\Psi_4\left[\begin{array}{c}(1-\sigma-\beta,1),\ (1-\sigma-\gamma+\alpha+\alpha',1),\ (1-\sigma+\alpha+\beta'-\gamma,1);\\ \left(\frac{1}{2},\mu\right),\ (1-\sigma,1),\ (1-\sigma+\alpha+\alpha'+\beta'-\gamma,1),\ (1-\sigma+\alpha-\beta,1);\end{array}\ -\frac{1}{x}\right], \tag{55}$$

*where $\Re(\sigma)<1+\min\{\Re(-\beta),\Re(\alpha+\alpha'-\gamma),\Re(\alpha+\beta'-\gamma)\}$, $\min\{\Re(p),\Re(q)\}\geqq 0$, $\Re(\nu)>-1$ when $p=q=1$, $\Re(\gamma)>0$ and $\mu>0$.*

We also establish the following results of the Marichev-Saigo-Maeda fractional derivatives involving the $(p,q)$-extended Bessel-Wright function $J^{\mu}_{\nu,p,q}(z)$ in terms of the Hadamard product of the Fox-Wright function ${}_r\Psi_s$ and the $(p,q)$-extended Gauss hypergeometric function.

**Theorem 7.** *The Marichev-Saigo-Maeda fractional derivative $D^{\alpha,\alpha',\beta,\beta',\gamma}_{0+}$ $(\alpha,\alpha',\beta,\beta',\gamma,\sigma,\nu \in \mathbb{C})$ involving the $(p,q)$-extended Bessel-Wright function $J^{\mu}_{\nu,p,q}(t)$ of the first kind is given by*

$$\left(D^{\alpha,\alpha',\beta,\beta',\gamma}_{0+} t^{\sigma-1} J^{\mu}_{\nu,p,q}(t)\right)(x) = \frac{\sqrt{\pi}x^{\sigma+\alpha+\alpha'-\gamma-1}}{2^{\nu}\Gamma(\nu+1)} F_{p,q}\left(1,\frac{1}{2};\nu+1;-x\right)$$
$$* {}_3\Psi_4\left[\begin{matrix} (\sigma,1),\ (\sigma-\gamma+\alpha+\alpha'+\beta',1),\ (\sigma-\beta+\alpha,1); \\ \left(\frac{1}{2},\mu\right),\ (\sigma-\beta,1),\ (\sigma-\gamma+\alpha+\alpha',1),\ (\sigma-\gamma+\alpha+\beta',1); \end{matrix} \ -x\right], \quad (56)$$

*where* $\Re(\sigma) > \max\{0, \Re(\gamma-\alpha-\alpha'-\beta'), \Re(\beta-\alpha)\}$, $\min\{\Re(p),\Re(q)\} \geqq 0$, $\Re(\nu) > -1$ *when* $p=q=1$, $\Re(\gamma) > 0$ *and* $\mu > 0$.

**Theorem 8.** *The Marichev-Saigo-Maeda fractional derivative $D^{\alpha,\alpha',\beta,\beta',\gamma}_{-}$ $(\alpha,\alpha',\beta,\beta',\gamma,\sigma,\nu \in \mathbb{C})$ of the $(p,q)$-extended Bessel-Wright function $J^{\mu}_{\nu,p,q}\left(\frac{1}{t}\right)$ is given by*

$$\left(D^{\alpha,\alpha',\beta,\beta',\gamma}_{-} t^{\sigma-1} J^{\mu}_{\nu,p,q}\left(\frac{1}{t}\right)\right)(x) = \frac{\sqrt{\pi}x^{\sigma+\alpha+\alpha'-\gamma-1}}{\Gamma(\nu+1)} F_{p,q}\left(1,\frac{1}{2};\nu+1;-\frac{1}{x}\right)$$
$$* {}_3\Psi_4\left[\begin{matrix} (1-\sigma+\beta',1),\ (1-\sigma+\gamma-\alpha-\alpha',1),\ (1-\sigma-\alpha'-\beta+\gamma,1); \\ \left(\frac{1}{2},\mu\right),\ (1-\sigma,1),\ (1-\sigma-\alpha-\alpha'-\beta+\gamma,1),\ (1-\sigma-\alpha'+\beta',1); \end{matrix} \ -\frac{1}{x}\right], \quad (57)$$

*where* $\Re(\sigma) < 1+\min\{\Re(\beta'), \Re(\gamma-\alpha-\alpha'), \Re(\gamma-\alpha'-\beta)\}$, $\min\{\Re(p),\Re(q)\} \geqq 0$, $\Re(\nu) > -1$ *when* $p=q=1$, $\Re(\gamma) > 0$ *and* $\mu > 0$.

The proofs of Theorems 5, 6, 7 and 8 are similar to those that we have already fully described for Theorems 1, 2, 3 and 4, respectively. We, therefore, choose to omit the details involved.

## 5. Conclusions

Motivated by the demonstrated usages and the potential for applications of the various operators of fractional calculus (that is, fractional integral and fractional derivative) and also of the considerably large spectrum of special functions and higher transcendental functions in mathematical, physical, engineering, biological and statistical sciences, we have established here several new formulas and new results for the Marichev-Saigo-Maeda fractional integral and fractional derivative operators, which are applied on the $(p,q)$-extended Bessel function $J_{\nu,p,q}(z)$. Our results have been expressed as the Hadamard product of the $(p,q)$-extended Gauss hypergeometric function $F_{p,q}(a,b;c;z)$ and the Fox-Wright function ${}_r\Psi_s(z)$. Some special cases of our main results have also been considered. Furthermore, we have introduced and investigated the $(p,q)$-extended Bessel-Wright function $J^{\mu}_{\nu,p,q}(z)$. Finally, we have proved several new formulas for the Marichev-Saigo-Maeda fractional integral and fractional derivative operators involving the $(p,q)$-extended Bessel-Wright function $J^{\mu}_{\nu,p,q}(z)$.

In concluding this investigation, we choose to indicate the possibility of further researches involving basic or quantum (or $\mathfrak{q}$-) extensions of the results which we have presented in this paper. At the same time, in order not to encourage the current trend of some amateurish-type publications, the authors should refer the interested reader to the well-demonstrated observations in [24] (pp. 1511–1512) that this trend of trivially and inconsequentially translating known $\mathfrak{q}$-results into the corresponding $(\mathfrak{p},\mathfrak{q})$-results leads to no more than a straightforward and shallow variation of the known $\mathfrak{q}$-results by means of a forced-in redundant (or superfluous) parameter $\mathfrak{p}$.

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