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ABSTRACT

The subdivision scheme is used to illustrate smooth curves and surfaces. It is an algorithmic technique which takes a coarse polygon as an input and produces a refined polygon as an output. In this paper, a 10-point interpolating subdivision scheme is used to develop a numerical algorithm for the solution of fourth order nonlinear singularly perturbed boundary value problems (NSPBVPs). The studies of convergence and approximation order of the numerical algorithm are also presented. The solution of NSPBVPs is presented to see the efficiency of the algorithm.

I. INTRODUCTION

The subdivision scheme is the branch of Computer Aided Geometric Design (CAGD). It is the most popular tool for designing objects. The subdivision scheme is an algorithm to generate curves and surfaces as a sequence of successively refined control polygons. In 1974, de Rham[1] was the first to work on subdivision schemes. In the same year, Chaikin[2] introduced a systematic way for developing a curve from the control polygon. Their schemes are known as corner cutting schemes. Nowadays, subdivision schemes have become a subject of study with many applications in scientific fields and real life. These are used in 2D and 3D computer graphic designs. The shapes of the chromosomes, virus, bacteria, cells, molecules, and inner structures of any organ are designed with the help of subdivision schemes. Subdivision schemes can also be used to design the shapes of cars, aircraft, and engines.

If $Q_i = Q_i^0$ are the 2D points, then by joining them with straight lines, we get a shape called an initial polygon. We get a refined polygon by applying the subdivision rules corresponding to each edge of the polygon. In the 10-point interpolating subdivision scheme (10PISS), we insert a point corresponding to each edge of the polygon by using ten existing points while carrying the previous points. Mathematically, this scheme is defined as[3]

$$\begin{cases} Q_{2i}^{k+1} = Q_i^k, \\ Q_{2i+1}^{k+1} = \left(\frac{1225}{2048} + 14\omega\right)\left(Q_i^k + Q_{i+1}^k\right) - \left(\frac{245}{2048} + 28\omega\right)\left(Q_{i-1}^k + Q_{i+2}^k\right) \\ \qquad + \left(\frac{49}{2048} + 20\omega\right)\left(Q_{i-2}^k + Q_{i+3}^k\right) - \left(\frac{5}{2048} + 7\omega\right)\left(Q_{i-3}^k + Q_{i+4}^k\right) + \omega\left(Q_{i-4}^k + Q_{i+5}^k\right), \end{cases} \tag{1}$$

where $k \geq 0$ represents the iteration of the scheme. This scheme produces a C^4-continuous curve for $0.0005 < \omega < 0.0016$. Its degree of generation and degree of reproduction and approximation order is seven and eight, respectively, as shown in Ref. 4. Its fundamental solution,

$$\Psi(\zeta) = \begin{cases} 1 & \text{for} \quad \zeta = 0 \\ 0 & \text{for} \quad \zeta \neq 0, \end{cases} \tag{2}$$

is fourth time continuously differentiable with support $(-9, 9)$, as shown in Ref. 5.

In physics, nonlinear singularly perturbed boundary value problems (NSPBVPs) arise from the physical phenomenon. These problems also arise in nuclear engineering. The solutions of some of the problems were found by subdivision based algorithms. This technique was first introduced by Qu and Agarwal.[5] They used the interpolatory subdivision algorithm to formulate an algorithm for linear second-order two-point BVPs. Qu and Agarwal[6,7] also solved nonlinear two-point BVPs by interpolating algorithms. In Ref. 8, an algorithm to find approximate solutions of BVPs with deviating arguments is also presented. After a long gap, Mustafa and Ejaz[9] presented an algorithm to handle third order BVPs. They used an 8-point interpolating scheme. Ejaz and Mustafa[11] also formulated an algorithm for non-linear third-order BVPs. After that, Mustafa *et al.*[12,13] solved linear and non-linear fourth-order BVPs by using subdivision schemes. SPBVPs were also solved by using subdivision-based algorithms, as shown in Ref. 14. They constructed the subdivision collocation method for two-point SPBVPs of order three. Fourth order NSPBVPs have not been solved by subdivision based algorithms. Hence, we are interested in solving the following type of fourth order NSPBVPs:

$$-\varepsilon u^{(iv)} = F(t, u, u', u'', u'''), \tag{3}$$

for real numbers a_1, a_2, a_3, b_1, b_2, b_3, $0 < \varepsilon \ll 1$ and subject to boundary conditions (BCs)

$$\begin{cases} u(0) = a_1, u'(0) = a_2, u(1) = b_1, u'(1) = b_2, \\ \text{or} \\ u(0) = a_1, u''(0) = a_3, u(1) = b_1, u''(1) = b_3. \end{cases} \tag{4}$$

The layout of the remaining work is as follows: In Sec. II, a local subdivision matrix of the scheme is presented to find its left and right eigenvectors. These vectors are necessary for the computation of derivatives of basis functions of the scheme. In Sec. III, a subdivision based numerical algorithm for the solution of (3) is derived. The convergence and error estimation of the algorithm are also discussed in this section. Applications and conclusions based on the results obtained are given in Sec. IV.

II. ELEMENTARY CHARACTERISTICS OF THE SCHEMES

The 10PISS can be expressed in the matrix form $Q^{k+1} = SQ^k$, where S is called the subdivision matrix. Here, we first find its eigenvectors corresponding to the eigenvalues; then we find the derivatives of the basis function of the scheme. The finite square portion of the matrix S is defined as

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
f_{4,4} & f_{4,3} & f_{4,2} & f_{4,1} & f_{4,0} & f_{4,0} & f_{4,1} & f_{4,2} & f_{4,3} & f_{4,4} & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & f_{4,4} & f_{4,3} & f_{4,2} & f_{4,1} & f_{4,0} & f_{4,0} & f_{4,1} & f_{4,2} & f_{4,3} & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & f_{4,4} & f_{4,3} & f_{4,2} & f_{4,1} & f_{4,0} & f_{4,0} & f_{4,1} & f_{4,2} & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & f_{4,4} & f_{4,3} & f_{4,2} & f_{4,1} & f_{4,0} & f_{4,0} & f_{4,1} & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & f_{4,4} & f_{4,3} & f_{4,2} & f_{4,1} & f_{4,0} & f_{4,0} & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & f_{4,4} & f_{4,3} & f_{4,2} & f_{4,1} & f_{4,0} & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & f_{4,4} & f_{4,3} & f_{4,2} & f_{4,1} & \cdots & f_{4,4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & f_{4,4} & f_{4,3} & f_{4,2} & \cdots & f_{4,3} & f_{4,4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix},$$

where $f_{4,4} = \omega, f_{4,3} = -\left(\frac{5}{2048} + 7\omega\right), f_{4,2} = \left(\frac{49}{2048} + 20\omega\right), f_{4,1} = -\left(\frac{245}{2048} + 28\omega\right)$, and $f_{4,0} = \left(\frac{1225}{2048} + 14\omega\right)$.

The real eigenvalues of (5) are $\lambda_k = \left(\frac{1}{2}\right)^k$, $k = 0, 1, \ldots, 7$ whereas the remaining unwanted eigenvalues are complex. The normalized right ξ_k and left η_k eigenvectors corresponding to the eigenvalues $\lambda_k = (\frac{1}{2})^k$, $k = 0, 1, 2, 3, 4$ are

$$\xi_k = \Big((-8)^k, (-7)^2, (-6)^k, (-5)^k, (-4)^k, (-3)^k, (-2)^k, (-1)^k, 0, 1, (2)^k, (3)^k, (4)^k, (5)^k, (6)^k, (7)^k, (8)^k\Big)^T,$$

$$\eta_0 = (0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0)^T,$$

$$\eta_1 = \Big(-131\,072\omega^2\alpha_2, -65\,536\omega\alpha_2, \frac{\alpha_3}{32\alpha_1}, -224(147+2048\omega)\alpha_2, -\frac{65\,536\alpha_4}{\alpha_1}, -\frac{327\,680\alpha_5}{\alpha_1}, \frac{7\alpha_6}{32\alpha_1}, -\frac{32\alpha_7}{\alpha_1}, 0, \frac{32\alpha_7}{\alpha_1}, -\frac{7\alpha_6}{32\alpha_1}, \frac{327\,680\alpha_5}{\alpha_1}, \frac{65\,536\alpha_4}{\alpha_1}, 224(147+2048\omega)\alpha_2, -\frac{\alpha_3}{32\alpha_1}, 65\,536\omega\alpha_2, 131\,072\omega^2\alpha_2\Big)^T,$$

$$\eta_2 = \Big(\frac{524\,288\omega^2\alpha_9}{\alpha_8}, \frac{131\,072\omega\alpha_9}{\alpha_8}, -\frac{\alpha_{10}}{8\alpha_8}, \frac{64\alpha_9(517+14\,336\omega)}{\alpha_8}, \frac{\alpha_{11}}{8\alpha_8}, \frac{64\alpha_{12}}{\alpha_8}, \frac{\alpha_{13}}{8\alpha_8}, \frac{64\alpha_{14}}{\alpha_8}, -\frac{\alpha_{15}}{4\alpha_8}, \frac{64\alpha_{14}}{\alpha_8}, \frac{\alpha_{13}}{8\alpha_8}, \frac{64\alpha_{12}}{\alpha_8}, \frac{\alpha_{11}}{8\alpha_8}, \frac{64\alpha_9(517+14\,336\omega)}{\alpha_8}, -\frac{\alpha_{10}}{8\alpha_8}, \frac{131\,072\omega\alpha_9}{\alpha_8}, \frac{524\,288\omega^2\alpha_9}{\alpha_8}\Big)^T,$$

$$\eta_3 = \Big(-\frac{131\,072\omega^2\alpha_{17}}{3\alpha_{16}}, -\frac{16\,384\omega\alpha_{17}}{3\alpha_{16}}, \frac{\alpha_{18}}{96\alpha_{16}}, -\frac{8(261+14\,336\omega)\alpha_{17}}{3\alpha_{16}}, -\frac{\alpha_{19}}{12\alpha_{16}}, -\frac{8\alpha_{20}}{3\alpha_{16}}, \frac{\alpha_{21}}{96\alpha_{16}}, -\frac{256\alpha_{22}}{3\alpha_{16}}, 0, \frac{256\alpha_{22}}{3\alpha_{16}}, -\frac{\alpha_{21}}{96\alpha_{16}}, \frac{8\alpha_{20}}{3\alpha_{16}}, \frac{\alpha_{19}}{12\alpha_{16}}, \frac{8(261+14\,336\omega)\alpha_{17}}{3\alpha_{16}}, -\frac{\alpha_{18}}{96\alpha_{16}}, \frac{16\,384\omega\alpha_{17}}{3\alpha_{16}}, \frac{131\,072\omega^2\alpha_{17}}{3\alpha_{16}}\Big)^T,$$

$$\eta_4 = \Big(\frac{-327\,680\omega^2\alpha_{24}}{\alpha_{23}}, -\frac{20\,480\omega\alpha_{24}}{\alpha_{23}}, \frac{\alpha_{25}}{64\alpha_{23}}, -\frac{70\alpha_{26}\alpha_{24}}{\alpha_{23}}, -\frac{\alpha_{27}}{32\alpha_{23}}, -\frac{2\alpha_{28}}{\alpha_{23}}, -\frac{\alpha_{29}}{64\alpha_{23}}, -\frac{4\alpha_{30}}{\alpha_{23}}, \frac{\alpha_{31}}{16\alpha_{23}}, -\frac{4\alpha_{30}}{\alpha_{23}}, -\frac{\alpha_{29}}{64\alpha_{23}}, -\frac{2\alpha_{28}}{\alpha_{23}}, -\frac{\alpha_{27}}{32\alpha_{23}}, -\frac{70\alpha_{26}\alpha_{24}}{\alpha_{23}}, \frac{\alpha_{25}}{64\alpha_{23}}, -\frac{20\,480\omega\alpha_{24}}{\alpha_{23}}, \frac{-327\,680\omega^2\alpha_{24}}{\alpha_{23}}\Big)^T,$$

where

$$\alpha_1 = 1\,549\,524\,839\,730\,021\,138\,432\omega^4 - 63\,314\,906\,905\,103\,145\,369\omega^3 - 1\,650\,035\,263\,235\,896\,128\omega^2 + 35\,278\,647\,548\,550\omega + 160\,158\,134\,113\,116\,160, \quad (5)$$

$$\alpha_2 = \Big(\frac{1}{\alpha_1}\Big)(12\,582\,912\omega^2 - 28\,672\omega + 5),$$

$$\alpha_3 = 562\,949\,953\,421\,312\omega^4 + 12\,434\,789\,315\,379\omega^3 + 1\,787\,671\,085\,056\omega^2 + 1\,980\,805\,120\omega + 25\,725, \quad (6)$$

$$\alpha_4 = 2\,955\,487\,255\,461\,888\omega^4 - 121\,118\,077\,747\,200\omega^3 - 3\,149\,302\,385\,868\omega^2 + 306\,262\,405\,120\omega + 68\,042\,625, \quad (7)$$

$$\alpha_5 = 94\,489\,280\,512\omega^3 + 560\,778\,444\omega^2 - 209\,420\,288\omega - 200\,655,$$

$$\alpha_6 = 1\,688\,849\,860\,263\,936\omega^4 + 19\,574\,742\,948\,249\omega^3 - 134\,880\,428\,032\omega^2 - 254\,803\,210\,240\omega - 839\,145\,825, \quad (8)$$

$$\alpha_7 = 420\,906\,795\,008\omega^3 + 3\,400\,741\,683\omega^2 - 710\,922\,240\omega - 23\,692\,725,$$

$$\alpha_8 = 12\,666\,373\,951\,979\,520\omega^4 + 1\,298\,303\,383\,764\,992\omega^3 + 15\,775\,183\,352\,629\omega^2 - 1\,961\,550\,099\,552\omega - 2\,232\,416\,340, \quad (9)$$

$$\alpha_9 = 71\,303\,168\omega^2 - 19\,660\omega + 55,$$

$$\alpha_{10} = 2\,251\,799\,813\,685\,248\omega^4 + 306\,213\,988\,335\,616\omega^3 + 233\,944\,855\,347\omega^2 + 529\,038\,540\omega + 142\,175,$$

$$\alpha_{11} = 6\,473\,924\,464\,345\,088\omega^4 - 874\,661\,499\,895\,808\omega^3 - 42\,011\,457\,486\,848\omega^2 + 407\,469\,989\,888\omega + 94\,410\,921, \quad (10)$$

$$\alpha_{12} = 3\,135\,326\,126\,080\omega^3 + 95\,852\,429\,312\omega^2 - 1\,757\,814\,784\omega - 2\,633\,081,$$

$$\alpha_{13} = 2\,251\,799\,813\,685\,248\omega^4 - 960\,423\,406\,862\,336\omega^3 - 44\,316\,781\,182\,976\omega^2 + 55\,061\,806\,284\omega + 6\,231\,737\,567,$$

$$\alpha_{14} = 5\,592\,047\,419\,392\omega^3 + 23\,500\,685\,312\omega^2 - 12\,401\,812\,480\omega - 46\,079\,693, \quad (11)$$

$$\alpha_{15} = 6\,772\,991\,627\,100\,160\omega^4 + 2\,924\,426\,051\,977\,216\omega^3 + 9\,795\,746\,240\,921\omega^2 - 1\,228\,513\,042\,432\omega - 18\,600\,375\,255,$$

$$\alpha_{16} = 8\,444\,249\,301\,319\,680\omega^4 + 1\,315\,089\,780\,257\,589\omega^3 - 2\,279\,379\,409\,408\omega^2 + 33\,613\,562\,880\omega - 224\,286\,609,$$

$$\alpha_{17} = 117\,440\,512\omega^2 + 4096\omega + 45, \quad (12)$$

$$\alpha_{18} = 5\,066\,549\,580\,791\,808\omega^4 + 29\,009\,927\,104\,102\omega^3 + 1\,276\,771\,303\,424\omega^2 + 1\,168\,257\,024\omega + 58\,725,$$

$$\alpha_{19} = 2\,973\,079\,441\,506\,304\omega^4 - 41\,682\,657\,607\,680\omega^3 - 2\,262\,396\,043\,264\omega^2 + 4\,182\,429\,696\omega - 357\,309,$$

$$\alpha_{20} = 4\,501\,125\,726\,208\omega^3 + 71\,512\,883\,200\omega^2 - 53\,849\,088\omega - 486\,243, \quad (13)$$

$$\alpha_{21} = 34\,339\,947\,158\,700\,032\omega^4 + 2\,069\,246\,523\,736\,064\omega^3 + 19\,453\,106\,454\,528\omega^2 - 20\,893\,673\,472\omega + 795\,524\,607,$$

$$\alpha_{22} = 142\,807\,662\,592\omega^3 + 301\,701\,529\omega^2 - 16\,296\,384\omega - 149\,031,$$

$$\alpha_{23} = 42\,559\,016\,478\,651\,200\omega^4 - 2\,949\,137\,575\,811\,514\omega^3 - 10\,958\,744\,951\,226\omega^2 + 3\,535\,283\,405\omega + 2\,346\,120, \quad (14)$$

$$\alpha_{24} = 16\,777\,216\omega^2 + 11,$$

$$\alpha_{25} = 2\,251\,799\,813\,685\,248\omega^4 + 8\,851\,068\,603\,596\omega^3 + 27\,947\,486\,412\omega^2 + 339\,992\,576\omega + 36\,575,$$

$$\alpha_{26} = 2048\omega + 19, \quad (15)$$

$$\alpha_{27} = 154\,811\,237\,190\,860\omega^4 - 144\,585\,779\,052\,544\omega^3 - 1\,468\,367\,110\,144\omega^2 + 2\,593\,153\,024\omega - 802\,389,$$

$$\alpha_{28} = 3\,882\,650\,435\,584\omega^3 + 33\,705\,426\,944\omega^2 - 105\,957\,376\omega - 115\,577,$$

$$\alpha_{29} = 2\,251\,799\,813\,685\,248\omega^4 - 650\,361\,127\,829\,504\omega^3 - 7\,229\,201\,711\,104\omega^2 + 1\,975\,621\,222\omega + 71\,626\,751, \quad (16)$$

$$\alpha_{30} = 3\,968\,549\,781\,504\omega^3 + 44\,610\,617\,344\omega^2 - 120\,830\,976\omega - 585\,067,$$

$$\alpha_{31} = 1\,724\,034\,232\,352\,768\omega^4 + 33\,040\,324\,414\,668\omega^3 + 3\,358\,754\,603\,008\omega^2 - 9\,888\,702\,464\omega - 46\,824\,645. \quad (17)$$

Lemma II.1 If $\Psi(\zeta)$ is the fundamental solution of 10PISS which is integrally supported on $[-8, 8]$, then its derivatives at integers are defined as

$$\Psi^{(d)}(\zeta) = 2^d \operatorname{sgn}(t) v_{|\zeta|}^T \eta_d, \quad d = 1, 2, 3, 4, \quad (18)$$

whereas for $t \in [0, 8]$,

$$\operatorname{sgn}(\zeta) = \begin{cases} -1, & \zeta < 0, \\ 0, & \zeta = 0, \\ 1, & \zeta > 0, \end{cases} \quad (19)$$

and

$$v_\zeta = \left(v_{8\zeta}, v_{7\zeta}, v_{6\zeta}, v_{5\zeta}, v_{4\zeta}, v_{3\zeta}, v_{2\zeta}, v_{1\zeta}, v_{0\zeta}, v_{-1\zeta}, v_{-2\zeta}, v_{-3\zeta}, v_{-4\zeta}, v_{-5\zeta}, v_{-6\zeta}, v_{-7\zeta}, v_{-8\zeta}\right)^T, \quad v_{\gamma\zeta} = \begin{cases} 1, & \gamma = \zeta, \\ 0, & \gamma \neq \zeta. \end{cases} \quad (20)$$

Particularly, the first four derivatives at integer values −8 to 8 are defined as

$$\begin{cases} \Psi'(0) = 0, & \Psi'(\pm 1) = \mp \frac{64\alpha_7}{\alpha_1}, \quad \Psi'(\pm 2) = \pm \frac{7\alpha_6}{16\alpha_1}, \quad \Psi'(\pm 3) = \mp \frac{655\,360\alpha_5}{\alpha_1}, \\ \Psi'(\pm 4) = \pm \frac{131\,072\alpha_4}{\alpha_1}, & \Psi'(\pm 5) = \pm 448(147 + 2048\omega)\alpha_2, \quad \Psi'(\pm 6) = \mp \frac{\alpha_3}{16\alpha_1}, \\ \Psi'(\pm 7) = \pm 131\,072\omega\alpha_2, & \Psi'(\pm 8) = \pm 262\,144\omega^2\alpha_2, \end{cases} \quad (21)$$

$$\begin{cases} \Psi''(0) = -\frac{\alpha_{15}}{\alpha_8}, & \Psi''(\pm 1) = \frac{256\alpha_{14}}{\alpha_8}, & \Psi''(\pm 2) = -\frac{\alpha_{13}}{2\alpha_8}, \\ \Psi''(\pm 3) = \frac{256\alpha_{12}}{\alpha_8}, & \Psi''(\pm 4) = \frac{\alpha_{11}}{2\alpha_8}, & \Psi''(\pm 5) = \frac{256\alpha_9(517+14\,336\omega)}{\alpha_8}, \\ \Psi''(\pm 6) = -\frac{\alpha_{10}}{2\alpha_8}, & \Psi''(\pm 7) = \frac{524\,288\omega\alpha_9}{\alpha_8}, & \Psi''(\pm 8) = \frac{2\,097\,152\omega^2\alpha_9}{\alpha_8}, \end{cases} \quad (22)$$

$$\begin{cases} \Psi'''(0) = 0, & \Psi'''(\pm 1) = \pm \frac{2048\alpha_{22}}{3\alpha_{16}}, & \Psi'''(\pm 2) = \mp \frac{\alpha_{21}}{12\alpha_{16}}, \\ \Psi'''(\pm 3) = \pm \frac{64\alpha_{20}}{3\alpha_{16}}, & \Psi'''(\pm 4) = \pm \frac{2\alpha_{19}}{3\alpha_{16}}, & \Psi'''(\pm 5) = \pm \frac{64(261+14\,336\omega)\alpha_{17}}{3\alpha_{16}}, \\ \Psi'''(\pm 6) = \mp \frac{\alpha_{18}}{12\alpha_{16}}, & \Psi'''(\pm 7) = \pm \frac{131\,072\omega\alpha_{17}}{3\alpha_{16}}, & \Psi'''(\pm 8) = \pm \frac{1\,048\,576\omega^2\alpha_{17}}{3\alpha_{16}}, \end{cases} \quad (23)$$

$$\begin{cases} \Psi^{(iv)}(0) = \frac{\alpha_{31}}{16\alpha_{23}}, & \Psi^{(iv)}(\pm 1) = -\frac{4\alpha_{30}}{\alpha_{23}}, & \Psi^{(iv)}(\pm 2) = -\frac{\alpha_{29}}{64\alpha_{23}}, \\ \Psi^{(iv)}(\pm 3) = -\frac{2\alpha_{28}}{\alpha_{23}}, & \Psi^{(iv)}(\pm 4) = -\frac{\alpha_{27}}{32\alpha_{23}}, & \Psi^{(iv)}(\pm 5) = -\frac{70\alpha_{26}\alpha_{24}}{\alpha_{23}}, \\ \Psi^{(iv)}(\pm 6) = \frac{\alpha_{25}}{64\alpha_{23}} & \Psi^{(iv)}(\pm 7) = -\frac{20\,480\omega\alpha_{24}}{\alpha_{23}}, & \Psi^{(iv)}(\pm 8) = \frac{-327\,680\omega^2\alpha_{24}}{\alpha_{23}}, \end{cases} \quad (24)$$

where α_i's are defined in (5)–(15).

III. FORMULATION OF THE NUMERICAL ALGORITHM

This section includes the numerical algorithm which is formulated for the numerical solution of NSPBVPs defined by (3). The numerical algorithm is divided into two main steps.

A. First step: Construction of a nonlinear system of equations

In this step, we construct a nonlinear system of equations. The main goal is to seek an approximate solution of (3). Let N be an integer such that $N \geq 8$; then the approximate solution can be defined as

$$G(t) = \sum_{i=-8}^{N+8} g_i \Psi\left(\frac{t-t_i}{h}\right), \quad 0 \leq t \leq 1, \tag{25}$$

where $\{g_i\}$ are the unknowns to be determined. Here, we define some parameters for further use. Let $h = \frac{1}{N} = ih$, $t_i = ih$, $i = -8, -7, \ldots, N + 8$, and $G(t_i) = g_i$. Now substituting (25) into (3) leads to

$$-\varepsilon G^{(iv)}(t) = F(t, G(t), G'(t), G''(t), G'''(t)), \tag{26}$$

where

$$G^{(d)}(t) = \frac{1}{h^d}\sum_{i=-8}^{N+8} g_i \Psi^{(d)}\left(\frac{t_j-t_i}{h}\right), \quad d = 1,2,3,4. \tag{27}$$

Here, d represents the order of derivatives. Using Eqs. (25) and (27) in (26), we have

$$-\varepsilon\frac{1}{h^4}\sum_{i=-8}^{N+8} g_i \Psi^{(iv)}\left(\frac{t_j-t_i}{h}\right) = F\left(t_i, \frac{1}{h}\sum_{i=-8}^{N+8} g_i \Psi'\left(\frac{t_j-t_i}{h}\right), \frac{1}{h^2}\sum_{i=-8}^{N+8} g_i \Psi''\left(\frac{t_j-t_i}{h}\right), \frac{1}{h^3}\sum_{i=-8}^{N+8} g_i \Psi'''\left(\frac{t_j-t_i}{h}\right)\right). \tag{28}$$

Here, we introduce some notations,

$$\Psi(j-i) = \Psi_{j-i}, \quad \Psi_{-i}^{(iv)} = \Psi_i^{(iv)}, \quad \Psi'''_{-i} = -\Psi'''_i, \quad \Psi''_{-i} = \Psi''_i, \quad \Psi'_{-i} = -\Psi'_i. \tag{29}$$

For $j = 0, 1, 2, \ldots, N$, we get

$$-\varepsilon\sum_{i=-8}^{N+8} g_i \Psi^{(iv)}\left(\frac{t_j-t_i}{h}\right) = h^4 F\left(t_i, \frac{1}{h}\sum_{i=-8}^{N+8} g_i \Psi'\left(\frac{t_j-t_i}{h}\right), \frac{1}{h^2}\sum_{i=-8}^{N+8} g_i \Psi''\left(\frac{t_j-t_i}{h}\right)\frac{1}{h^3}\sum_{i=-8}^{N+8} g_i \Psi'''\left(\frac{t_j-t_i}{h}\right)\right). \tag{30}$$

In general, we can write (30) as

$$-\varepsilon f_4^{(j)} = F(t_i, f_1^{(j)}, f_2^{(j)}, f_3^{(j)}), \quad j = 0,1,2,\ldots,N, \tag{31}$$

and with $d = 1, 2, 3, 4$,

$$f_d^{(j)} = \frac{1}{h^d}\sum_{i=-8}^{N+8} C_i^{(j)} g_i, \text{ and } C_i^{(j)} = \Psi^{(d)}\left(\frac{t_j-t_i}{h}\right). \tag{32}$$

Theorem III.1 *The system (31) can be expressed in the matrix form,*

$$\mathbb{C}\mathbb{G} = \mathbb{D}. \tag{33}$$

For $r = 1, 2, \ldots, N + 1$, $s = 1, 2, \ldots, N + 17$,

$$\mathbb{C} = [\tau_{s-r-8}^{(rs)}]_{(N+1)\times(N+17)}, \tag{34}$$

with

$$\tau_l = \begin{cases} \Psi_l^{(iv)} & \text{for} \quad l \in [-8,8], \\ 0 & \text{for} \quad l \notin [-8,8], \end{cases}$$

$$\mathbb{D}(g) = [R^{(j)}(g)], \; R^{(j)}(g) = F(t_i, f_1^{(j)}, f_2^{(j)}, f_3^{(j)}), \tag{35}$$

$$\mathbb{G} = [g_i]_{(N+17)\times 1}. \tag{36}$$

$f_d^{(j)}$ is defined in (32).

Proof System (31) for $j = 0$ is

$$-\varepsilon f_4^{(0)} = F(t_i, f_1^{(0)}, f_2^{(0)}, f_3^{(0)}), \tag{37}$$

where for $d = 1, 2, 3, 4$,

$$f_d^{(0)} = \frac{1}{h^d}\sum_{-8}^{N+8} C_i^{(0)} g_i, \quad \text{with} \quad C_i^{(0)} = \Psi^{(iv)}\left(\frac{t_0-t_i}{h}\right).$$

Now consider the left hand side of (37),

$$-\varepsilon f_4^{(0)} = -\frac{\varepsilon}{h^4}\sum_{-8}^{N+8} C_i^{(0)} g_i.$$

This implies

$$-\varepsilon f_4^{(0)} = \frac{\varepsilon}{h^4}\Big[C_{-8}^{(0)} g_{-8} + C_{-7}^{(0)} g_{-7} + C_{-6}^{(0)} g_{-6} + \cdots + C_{N+7}^{(0)} g_{N+7} + C_{N+8}^{(0)} g_{N+8}\Big]. \tag{38}$$

As $C_i^{(0)} = \Psi_{0-i}^{(iv)} = \Psi_{-i}^{(iv)} = \Psi_i^{(iv)}$ and using (38) in (37), we get

$$-\varepsilon f_4^{(0)} = -\frac{\varepsilon}{h^4}\Big[\Psi_8^{(iv)} g_{-8} + \Psi_7^{(iv)} g_{-7} + \cdots + \Psi_{N+7}^{(iv)} g_{N+7} + \Psi_{N+8}^{(iv)} g_{N+8}\Big] = F(t_0, f_1^{(0)}, f_2^{(0)}, f_3^{(0)}).$$

Similarly for $1 \leq j \leq N$,

$$-\varepsilon f_4^{(1)} = -\frac{\varepsilon}{h^4}\Big[\Psi_9^{(iv)} g_{-8} + \Psi_8^{(iv)} g_{-7} + \cdots + \Psi_{-N-6}^{(iv)} g_{N+7} + \Psi_{-N-7}^{(iv)} g_{N+8}\Big] = F(t_1, f_1^{(1)}, f_2^{(1)}, f_3^{(1)}),$$

$$-\varepsilon f_4^{(2)} = -\frac{\varepsilon}{h^4}\Big[\Psi_{10}^{(iv)} g_{-8} + \Psi_9^{(iv)} g_{-7} + \cdots + \Psi_{-N-5}^{(iv)} g_{N+7} + \Psi_{-N-6}^{(iv)} g_{N+8}\Big] = F(t_0, f_1^{(2)}, f_2^{(2)}, f_3^{(2)}),$$

$$\vdots$$

$$-\varepsilon f_4^{(N-1)} = -\frac{\varepsilon}{h^4}\Big[\Psi_{N+7}^{(iv)} g_{-8} + \Psi_{N+6}^{(iv)} g_{-7} + \cdots + \Psi_{-8}^{(iv)} g_{N+7} + \Psi_{-9}^{(iv)} g_{N+8}\Big] = F(t_{N-1}, f_1^{(N-1)}, f_2^{(N-1)}, f_3^{(N-1)}),$$

$$-\varepsilon f_4^{(N)} = -\frac{\varepsilon}{h^4}\Big[\Psi_{N+8}^{(iv)} g_{-8} + \Psi_{N+7}^{(iv)} g_{-7} + \cdots + \Psi_{-7}^{(iv)} g_{N+7} + \Psi_{-8}^{(iv)} g_{N+8}\Big] = F\left(t_N, f_1^{(N)}, f_2^{(N)}, f_3^{(N)}\right).$$

Since the support of the function $\Psi_i^{(iv)}(t)$ is $[-8, 8,]$ $\Psi_i^{(iv)}(t) = 0$ for $t \notin [-8, 8]$. This leads to the system of $(N + 1)$ equations with $(N + 17)$ unknowns $\{g_i\}$,

$$\begin{cases} -\frac{\varepsilon}{h^4}\Big[\Psi_8^{(iv)}g_{-8}+\Psi_7^{(iv)}g_{-7}+\cdots+\Psi_7^{(iv)}g_7+\Psi_8^{(iv)}g_8\Big] & = F(t_0,f_1^{(0)},f_2^{(0)},f_3^{(0)}),\\ -\frac{\varepsilon}{h^4}\Big[\Psi_8^{(iv)}g_{-7}+\Psi_7^{(iv)}g_{-6}+\cdots+\Psi_7^{(iv)}g_8+\Psi_8^{(iv)}g_9\Big] & = F(t_1,f_1^{(1)},f_2^{(1)},f_3^{(1)}),\\ -\frac{\varepsilon}{h^4}\Big[\Psi_8^{(iv)}g_{-6}+\Psi_7^{(iv)}g_{-5}+\cdots+\Psi_7^{(iv)}g_9+\Psi_8^{(iv)}g_{10}\Big] & = F(t_2,f_1^{(2)},f_2^{(2)},f_3^{(2)}),\\ \quad\vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots & = \quad\vdots\\ \quad\vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots & = \quad\vdots\\ -\frac{\varepsilon}{h^4}\Big[\Psi_8^{(iv)}g_{N-9}+\Psi_7^{(iv)}g_{N-8}+\cdots+\Psi_{-7}^{(iv)}g_{N+6}+\Psi_{-8}^{(iv)}g_{N+7}\Big] & = F(t_{N-1},f_1^{(N-1)},f_2^{(N-1)},f_3^{(N-1)}),\\ -\frac{\varepsilon}{h^4}\Big[\Psi_8^{(iv)}g_{N-8}+\Psi_7^{(iv)}g_{N-7}+\cdots+\Psi_{-7}^{(iv)}g_{N+7}+\Psi_{-8}^{(iv)}g_{N+8}\Big] & = F(t_N,f_1^{(N)},f_2^{(N)},f_3^{(N)}). \end{cases} \tag{39}$$

This system is close to system (33) with the coefficient matrix $\mathbb{C}$, unknown vector $\mathbb{G}$, and column vector $\mathbb{D}$, where

$$\mathbb{C} = -\varepsilon\begin{pmatrix} \Psi_8^{(iv)} & \Psi_7^{(iv)} & \Psi_6^{(iv)} & \Psi_5^{(iv)} & \Psi_4^{(iv)} & \Psi_3^{(iv)} & \Psi_2^{(iv)} & \Psi_1^{(iv)} & \Psi_0^{(iv)} & \Psi_{-1}^{(iv)} & \Psi_{-2}^{(iv)} & \cdots\cdots & 0 & 0 & 0\\ 0 & \Psi_8^{(iv)} & \Psi_7^{(iv)} & \Psi_6^{(iv)} & \Psi_5^{(iv)} & \Psi_4^{(iv)} & \Psi_3^{(iv)} & \Psi_2^{(iv)} & \Psi_1^{(iv)} & \Psi_0^{(iv)} & \Psi_{-1}^{(iv)} & \cdots\cdots & 0 & 0 & 0\\ 0 & 0 & \Psi_8^{(iv)} & \Psi_7^{(iv)} & \Psi_6^{(iv)} & \Psi_5^{(iv)} & \Psi_4^{(iv)} & \Psi_3^{(iv)} & \Psi_2^{(iv)} & \Psi_1^{(iv)} & \Psi_0^{(iv)} & \cdots\cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots\cdots & \Psi_{-8}^{(iv)} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots\cdots & \Psi_{-7}^{(iv)} & \Psi_{-8}^{(iv)} & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots\cdots & \Psi_{-6}^{(iv)} & \Psi_{-7}^{(iv)} & \Psi_{-8}^{(iv)} \end{pmatrix},$$

$$\mathbb{G} = (g_{-8}, g_{-7}, g_{-6}, g_{-5}, g_{-4}, \ldots, g_{N+8})^T, \tag{40}$$

and

$$\mathbb{D} = \Big(h^4F\big(t_0,f_1^{(0)},f_2^{(0)},f_3^{(0)}\big), h^4F\big(t_1,f_1^{(1)},f_2^{(1)},f_3^{(1)}\big),\ldots,\\ h^4F\big(t_{N-1},f_1^{(N-1)},f_2^{(N-1)},f_3^{(N-1)}\big),\\ h^4F\big(t_N,f_1^{(N)},f_2^{(N)},f_3^{(N)}\big)\Big)^T. \tag{41}$$

This completes the proof.

Since system (33) has $N + 1$ equations and $N + 17$ unknowns, we need 16 more equations.

1. Approximations of BC and its derivatives

Six boundary conditions can be approximated at the left end points and six more conditions from the right end points,

$$\sum_{j=0}^{10}\binom{10}{j}(-1)^j g(t_{j-i}) = 0,\ i = 7, 6, 5, 4, 3, 2, \tag{42}$$

$$\sum_{j=0}^{10}\binom{4}{j}(-1)^j g(t_{i-j}) = 0, \tag{43}$$

where $i = N + 2, N + 3, N + 4, N + 5, N + 6, N + 7$. The approximations of first and second derivatives of boundary conditions (BC) at end points are defined as

$$G'(0) = \frac{1}{840h}(-2283g_0 + 6720g_1 - 11\,760g_2 + 15\,680g_3 \\ - 14\,700g_4 + 9408g_5 - 3920g_6 + 960g_7 - 105g_8) \\ + O(h^8), \tag{44}$$

$$G'(N) = \frac{1}{840h}(2283g_N - 6720g_{N+1} + 11\,760g_{N+2} - 15\,680g_{N+3} \\ + 14\,700g_{N+4} - 9408g_{N+5} + 3920g_{N+6} \\ - 960g_{N+7} + 105g_{N+8}) + O(h^8), \tag{45}$$

$$G''(0) = \frac{1}{10\,080h}(32\,575g_0 - 165\,924g_1 + 422\,568g_2 \\ - 704\,368g_3 + 818\,874g_4 - 667\,800g_5 + 375\,704g_6 \\ - 139\,248g_7 + 30\,663g_8 - 3044g_9) + O(h^8), \tag{46}$$

$$G''(N) = \frac{1}{10\,080h}(-32\,575g_N + 165\,924g_{N+1} - 422\,568g_{N+2} \\ + 704\,368g_{N+3} - 818\,874g_{N+4} + 667\,800g_{N+5} \\ - 375\,704g_{N+6} + 139\,248g_{N+7} - 30\,663g_{N+8} \\ + 3044g_{N+9}) + O(h^8). \tag{47}$$

2. Consistent nonlinear systems of equations

Finally, we get the following new nonlinear system of $(N + 17)$ equations and $(N + 17)$ unknowns $\{g_i\}$, in which $N + 1$ equations are obtained from (33), 12 from (42) and (43), and four from (44)–(47),

$$\mathbb{S}\mathbb{G} = \mathbb{N}, \tag{48}$$

where the coefficient matrix $\mathbb{S} = (\mathbb{S}_0^T, \mathbb{C}^T, \mathbb{S}_1^T)^T$, $\mathbb{C}$ is defined in (40), the first six rows of the matrix $\mathbb{S}_0$ are obtained from (42), and the last two rows are obtained from (4) and (44) or (46), respectively, by writing the highest order derivative to the lowest order derivative conditions.

The matrix $\mathbb{S}_1$ can be written as follows: the first two rows are obtained from (4) and (45) or (47), respectively, by writing the lowest order derivative to the highest order derivative conditions, and the last six rows are obtained from (43); the vector $\mathbb{G}$ is defined in (40), $\mathbb{D}$ is defined in (41), and

$$\mathbb{N} = (0,0,0,0,0,0,g^{(d)}(0),g(0),\mathbb{D}^T,g(1),g^{(d)}(1),0,0,0,0,0,0)^T, \tag{49}$$

for $d = 1$ or 2.

B. Second step: Algorithm its convergence and error estimation

Here, we suggest an iterative algorithm with three steps for system (48).

(*i*) *Initial approximation*

First of all, the initial approximation $\mathbb{N}^0$ is chosen for the system $\mathbb{S}\mathbb{G}^0 = \mathbb{N}^0$, where for $d = 1$ or 2,

$$\begin{cases} \mathbb{N}^0 = (0,0,0,0,0,0,0,u^d(0),u(0),g_1,\ldots,g_N,u(1),\\ u^d(0),0,0,0,0,0,0,0)^T,\\ g_i = h^4 f(t_i,\mathbb{L}_i,\rho_0,\rho_1,\rho_2), i = 1,2,\ldots,N-1,\\ \mathbb{L}_i = u_l + ih(u_r - u_l), i = 1,2,\ldots,N-1,\\ \rho_0 = u_r - u_l,\\ \rho_1 = u_r' - u_l',\\ \rho_2 = u_r'' - u_l''. \end{cases} \tag{50}$$

(*ii*) *Iterative algorithm*

Use the following iterative scheme:

$$\mathbb{S}\mathbb{G}^{(k+1)} = \mathbb{N}(g^{(k)}), \quad k = 0,1,2,\ldots. \tag{51}$$

(*iii*) *Breaking off iteration*

If for a given error tolerance ε

$$\|\mathbb{G}^{(k)} - \mathbb{G}^{(k-1)}\|_\infty \le \varepsilon, \tag{52}$$

then the iterative process will break off. The following result guarantees the convergence of the iterative algorithm. We refer to Ref. 13 for its proof.

Theorem III.2 *The sequence $\mathbb{G}^{(k)}$ generated by the iterative scheme (51) converges linearly to the solution of fourth order NSP-BVPs defined by (3) for a small mesh size h. In other words,*

$$\|\mathbb{S}^{-1}\|_\infty(L_0h^4 + L_1h^3\|D_1\|_\infty + L_2h^2\|D_2\|_\infty + L_3h\|D_3\|_\infty) \approx (\|\mathbb{S}^{-1}\|_\infty L_3h\|D_3\|_\infty) < 1,$$

where L_0, L_1, L_2, and L_3 are Lipschitz constants,

$$h < \|D_3\|_\infty^{-1} L_3^{-1}\|\mathbb{S}^{-1}\|_\infty^{-1},$$

$$\|D_1\|_\infty = \frac{\beta_1}{\beta_2}, \quad \|D_2\|_\infty = \frac{\beta_3}{\beta_4}, \quad \|D_3\|_\infty = \frac{\beta_5}{\beta_6},$$

with

$$\beta_1 = 131\,072(2\,955\,487\,255\,461\,888\omega^4 - 121\,118\,077\,747\,200\omega^3 - 3\,149\,302\,385\,868\omega^2 + 306\,262\,405\,120\omega + 68\,042\,625),$$

$$\beta_2 = 1\,549\,524\,839\,730\,021\,138\,432\omega^4 - 63\,314\,906\,905\,103\,145\,369\omega^3 - 1\,650\,035\,263\,235\,896\,128\omega^2 + 160\,158\,134\,113\,116\,160\omega + 35\,278\,647\,548\,550,$$

$$\beta_3 = 6\,772\,991\,627\,100\,160\omega^4 + 2\,924\,426\,051\,977\,216\omega^3 + 9\,795\,746\,240\,921\omega^2 - 1\,228\,513\,042\,432\omega - 18\,600\,375\,255,$$

$$\beta_4 = +12\,666\,373\,951\,979\,520\omega^4 + 1\,298\,303\,383\,764\,992\omega^3 + 15\,775\,183\,352\,629\omega^2 1\,961\,550\,099\,552\omega - 2\,232\,416\,340,$$

$$\beta_5 = 2048(142\,807\,662\,592\omega^3 + 301\,701\,529\omega^2 - 16\,296\,384\omega - 149\,031),$$

$$\beta_6 = 3(8\,444\,249\,301\,319\,680\omega^4 + 1\,315\,089\,780\,257\,589\omega^3 - 2\,279\,379\,409\,408\omega^2 + 33\,613\,562\,880\omega - 224\,286\,609).$$

The proof of the following result is similar to the proof of proposition.[5,9]

Theorem III.3 *Let $u(t) \in C^4[0,1]$ and $g(t)$ be the accurate and approximate solutions of fourth order NSPBVPs, respectively; then,*

$$\|g(t_i) - u(t_i)\|_\infty \le O(h^4).$$

IV. NUMERICAL EXAMPLES AND COMPARISON

In this section, we find the solutions of fourth order NSPBVP. We also compute the absolute errors (AE) between the exact and approximate solutions. The conclusion is also presented.

A. Numerical examples

The following examples have been chosen for their solutions.

Example IV.1 Consider the fourth order NSPBVP,[10]

$$-\varepsilon u^{(iv)} - u''' + u'' - u^2(t) = l(t), \ t \in [0,1]$$
$$u(0) = 1, \ u(1) = 1, \ u''(0) = 1, \ u''(1) = \sin(1),$$

TABLE I. The exact u_i and approximate g_i solutions of Example IV.1 for $\varepsilon = (2)^{-2}$.

t_i	u_i	g_i for		
		$\omega = 0.006$	$\omega = 0.0089$	$\omega = 0.0012$
0.0	1	0.999 789 810 8	0.999 790 655 1	0.999 791 324 4
0.1	0.970 413 831 1	0.970 294 936 9	0.970 294 585 5	0.970 294 305 2
0.2	0.948 558 386 0	0.948 426 238 6	0.948 426 361 4	0.948 426 459 2
0.3	0.933 139 882 8	0.933 008 285 9	0.933 007 400 3	0.933 005 615 7
0.4	0.923 596 810 7	0.923 370 648 8	0.923 370 643 2	0.923 370 639 6
0.5	0.919 842 306 4	0.919 725 816 8	0.919 726 692 5	0.919 728 467 1
0.6	0.922 088 572 6	0.921 982 720 4	0.921 982 749 7	0.921 982 710 9
0.7	0.930 726 478 6	0.930 634 307 6	0.930 635 175 6	0.930 636 943 9
0.8	0.946 242 370 7	0.946 161 951 0	0.946 162 083 5	0.946 162 190 9
0.9	0.969 160 067 6	0.969 098 326 8	0.969 097 993 0	0.969 097 731 5
1.0	1	0.999 841 914 7	0.999 842 630 9	0.999 843 199 4

TABLE II. AE of Example IV.1 for $\varepsilon = (2)^{-2}$.

t_i	AE = $\|u_i - g_i\|$		
	For $\omega = 0.006$	For $\omega = 0.0089$	For $\omega = 0.0012$
0.0	0.000 210 189 2	0.000 209 344 9	0.000 208 675 6
0.1	0.000 118 894 2	0.000 119 245 6	0.000 119 525 9
0.2	0.000 132 147 4	0.000 132 024 6	0.000 131 926 8
0.3	0.000 131 596 9	0.000 132 482 5	0.000 134 267 1
0.4	0.000 226 161 9	0.000 226 167 5	0.000 226 171 1
0.5	0.000 116 489 6	0.000 115 613 9	0.000 113 839 3
0.6	0.000 105 852 2	0.000 105 822 9	0.000 105 861 7
0.7	0.000 092 171 0	0.000 091 303 0	0.000 089 534 7
0.8	0.000 080 419 7	0.000 080 287 2	0.000 080 179 8
0.9	0.000 061 740 8	0.000 062 074 6	0.000 062 336 1
1.0	0.000 158 085 3	0.000 157 369 1	0.000 156 800 6

where

$$l(t) = \cos(t) - (\varepsilon + 1)\sin(t) - \frac{e^{-\frac{1}{\varepsilon}} - e^{-\frac{t}{\varepsilon}}}{e^{-\frac{1}{\varepsilon}} - 1} + \Bigg(\sin(1)t + \varepsilon^2 - \sin(t) + \frac{2\varepsilon^2 + (t^2 - t)e^{-\frac{1}{\varepsilon}} - s\varepsilon^2 e^{\frac{-t}{\varepsilon}}}{2(e^{\frac{-1}{\varepsilon}} - 1)}\Bigg)^2.$$

Its exact solution is

$$u(t) = \sin(1)t + \varepsilon^2 t - \sin(t) + 1 + \frac{2\varepsilon^2 + (t^2 - t)e^{-\frac{1}{\varepsilon}} - 2\varepsilon^2 e^{-\frac{t}{\varepsilon}}}{2(e^{-\frac{1}{\varepsilon}} - 1)}.$$

The approximate solutions are given in Tables I and II.

Example IV.2 Consider the fourth order NSPBVP,[10]

$$-\varepsilon u^{(iv)} - u''' = -\cos(t) + \sin(t), \quad t \in [0, 1]$$
$$u(0) = 1, \; u(1) = 1, \; u''(0) = 1, \; u''(1) = \sin(1).$$

TABLE III. The exact u_i and approximate g_i solutions of Example IV.2 for $\varepsilon = (2)^{-4}$.

t_i	u_i	g_i for		
		$\omega = 0.006$	$\omega = 0.0089$	$\omega = 0.0012$
0.0	1	1.000 099 968	1.000 099 966	1.000 115 853
0.1	0.981 586 719 9	0.981 676 179 0	0.981 676 174 9	0.981 692 089 9
0.2	0.966 659 102 1	0.966 737 210 1	0.966 737 208 6	0.966 753 094 7
0.3	0.954 218 872 5	0.954 284 246 6	0.954 283 370 1	0.954 300 707 0
0.4	0.944 832 805 1	0.944 885 969 8	0.944 885 969 9	0.944 901 822 3
0.5	0.939 358 152 8	0.939 398 571 1	0.939 399 447 8	0.939 413 821 4
0.6	0.938 677 895 1	0.938 703 963 9	0.938 703 963 8	0.938 719 817 3
0.7	0.943 640 192 0	0.943 652 862 2	0.943 653 738 9	0.943 668 107 4
0.8	0.955 039 466 2	0.955 037 433 4	0.955 037 435 0	0.955 053 254 2
0.9	0.973 606 358 5	0.973 590 239 7	0.973 590 243 7	0.973 606 039 7
1.0	1	0.999 969 883 1	0.999 969 883 0	0.999 985 736 0

TABLE IV. AE of Example IV.2 for $\varepsilon = (2)^{-4}$.

	AE = $\|u_i - g_i\|$		
t_i	For $\omega = 0.006$	For $\omega = 0.0089$	For $\omega = 0.0012$
0.0	0.000 099 968	0.000 099 966	0.000 115 853
0.1	0.000 089 459 1	0.000 089 455 0	0.000 105 370 0
0.2	0.000 078 108 0	0.000 078 106 5	0.000 093 992 6
0.3	0.000 065 374 1	0.000 064 497 6	0.000 081 834 5
0.4	0.000 053 164 7	0.000 053 164 8	0.000 069 017 2
0.5	0.000 040 418 3	0.000 041 295 0	0.000 055 668 6
0.6	0.000 026 068 8	0.000 026 068 7	0.000 041 922 2
0.7	0.000 012 670 2	0.000 013 546 9	0.000 027 915 4
0.8	0.000 002 032 8	0.000 002 031 2	0.000 013 788 0
0.9	0.000 016 118 8	0.000 016 114 8	3.188×10^{-7}
1.0	0.000 030 116 9	0.000 030 117 0	0.000 014 264 0

Its exact solution is

$$u(t) = \sin(1)t + \varepsilon^2 t - \sin(t) + 1 + \frac{2\varepsilon^2 + (t^2 - t)e^{-\frac{1}{\varepsilon}} - 2\varepsilon^2 e^{-\frac{t}{\varepsilon}}}{2(e^{-\frac{1}{\varepsilon}} - 1)}.$$

The approximate solutions are presented in Tables III and IV.

B. Conclusion

We offered a numerical algorithm for the approximate solutions of fourth order NSPBVPs. 10PISS has been used to develop an algorithm. The numerical results at different values of the parameter used in 10PISS are presented. It has been observed that for a small value of the parameter ω, the numerical algorithm gives a better approximation.

AUTHORS' CONTRIBUTIONS

Authors contributed equally to this work.

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DATA AVAILABILITY

The data that support the findings of this study are available within the article.

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