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LETTER

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The stability of the fractional Volterra integro-differential equation by means of Ψ-Hilfer operator revisited

Dumitru Baleanu[1,2] | **Reza Saadati**[3] | **José Sousa**[4]

[1]Department of Mathematics, Cankaya University, Ankara, Turkey
[2]Institute of Space Sciences, Magurele, Romania
[3]Department of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran
[4]Department of Applied Mathematics, Imecc-State University of Campinas, Campinas, Brazil

Correspondence
Reza Saadati, Department of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran.
Email: rsaadati@eml.cc

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In this note, we have as main purpose to investigate the Ulam-Hyers stability of a fractional Volterra integral equation through the Banach fixed point theorem and present an example on Ulam-Hyers stability using operator theory α-resolvent in order to elucidate the investigated result. Our results modify the Theorem 4 of Sousa and et. al. [Sousa JVC, Rodrigues FG, Oliveira EC. Stability of the fractional Volterra integro-differential equation by means of Ψ-Hilfer operator. Math Meth Appl Sci. 2019;42(9):3033-3043.] and present a corrected proof with a modified approximation.

KEYWORDS
fixed point theorem, fractional Volterra integral equation, Ulam-Hyers stability, Ψ-Hilfer fractional derivative

MSC CLASSIFICATION
34K37

1 | INTRODUCTION AND PRELIMINARIES

The first result on stability issues, discussed in 1940 by Ulam,[1,2] imposed the question of the stability of the Cauchy equation, and in 1941, Hyers[3] solved it. After 38 years on the first ideas about the stability of Ulam-Hyers, Rassias[4] provided a generalization of Hyers theorem by proving the existence of unique linear mappings near approximate additive mappings. Numerous important articles with relevant results on the Ulam-Hyers stability of solutions of differential equations have been published.[5–9]

On the other hand, the fractional calculus since its first ideas, and consequently its consolidation in several areas of knowledge, allowed and has belonged and providing results of great impact in the scientific community, in particular, involving applications in biology, medicine, engineering, among others[10–12] and references therein. We want to highlight the fundamental role of the theory of fractional differential equations, since it has been the target of investigation and has been growing during these last decades, in particular, in the investigation of Ulam-Hyers stabilities, existence, and uniqueness.[13–22]

After the introduction of the fractional derivative ψ-Hilfer,[23] which owns a wide class of fractional derivatives, researchers around the world started several researches involving such operator, in particular, on the Ulam-Hyers stability[13,14,18,24] and references therein.

In 2012, Ibrahim[25] discusses the Ulam-Hyers stability of Cauchy fractional differential equations in the unit disk for the linear and non-linear cases. In 2017, Wang and Xu[26] investigate the Hyers-Ulam-Rassias stability of the nonlinear fractional differential equations using the weighted space method and a fixed point theorem. Sousa and Oliveira[13] discussed results on the Ulam-Hyers and Ulam-Hyers-Rassias stabilities of the fractional integro-differential equation using the Banach fixed point theorem.

In 2018, Khan et al[27] discussed conditions to obtain the existence, uniqueness, and Hyers-Ulam stability of solutions for the proposed coupled system of fractional differential equations with the nonlinear p-Laplacian operator and Riemann-Liouville integral boundary conditions via the nonlinear Leray-Schauder alternative and the Banach fixed point theorem. See also the work on stability investigated by Khan et al.[28]

In 2018, Sousa et al[24] investigated the stability of Ulam-Hyers for a fractional integro-differential equation in a Banach space, given by

$$ {}^{H}\mathbb{D}_{0+}^{\alpha,\beta;\psi} x(t) = g(t, x(t)) + \int_0^t K(t, s, x(s))ds \tag{1} $$

where ${}^{H}\mathbb{D}_{0+}^{\alpha,\beta;\psi}(\cdot)$ is the ψ-Hilfer fractional derivative, with $0<\alpha<1$, $0\leq\beta\leq 1$ and $g:[0,T]\times X\to X$, $K:[0,T]\times[0,T]\times X\to X$, $x:[0,T]\to X$ are continuous functions, and X is a Banach space. One of the main objectives of this paper (see Sousa et al.[24]) was to investigate the following result:

Theorem 1. *(Theorem 4)[24] Assume that Ω is a Banach space and $\mathcal{L}, \mathcal{L}_1, \mathcal{L}_2$, and P are positive constants for which $0 < \mathcal{L}_1 + \mathcal{L}_2\mathcal{L} < 1$. Let $\zeta : [0,P]\times\Omega\to\Omega$, $\mathcal{K} : [0,P]^2\times\Omega\to\Omega$ and continuous map $\psi : [0,P]\to(0,\infty)$ satisfying*

$$ \|\zeta(\tau,\omega) - \zeta(\tau,\nu)\|_{C_{1-\xi;\Psi}} \leq \mathcal{L}_1\|\omega-\nu\|_{C_{1-\xi;\Psi}[0,P]}, \tag{2} $$

$$ \|\mathcal{K}(\tau,\sigma,\omega(\sigma)) - \mathcal{K}(\tau,\sigma,\nu(\sigma))\|_{C_{1-\xi;\Psi}} \leq \mathcal{L}_2\|\omega-\nu\|_{C_{1-\xi;\Psi}[0,P]}, \tag{3} $$

and

$$ \frac{1}{\Gamma(\varrho)}\int_0^\tau \mathcal{Q}_\Psi^\varrho(\tau,\sigma)\psi(\sigma)d\sigma \leq \mathcal{L}\psi(\tau), \tag{4} $$

for all $\tau,\sigma\in[0,P]$, and $\omega,\nu\in\Omega$. If the continuous map $\eta : [0,P]\to\Omega$ satisfying

$$ \left\|\eta(\tau) - \zeta(\tau,\eta(\tau)) - \frac{1}{\Gamma(\varrho)}\int_0^\tau \mathcal{Q}_\Psi^\varrho(\tau,\sigma)\mathcal{K}(\tau,\sigma,\eta(\sigma))d\sigma\right\|_{C_{1-\xi;\Psi}[0,P]} \leq \psi(\tau), \tag{5} $$

$0\leq\tau\leq P$; thus, we can find a unique continuous map $\eta_0 : [0,P]\to\Omega$ such that for every $\tau\in[0,P]$,

$$ \eta_0(\tau) = \zeta(\tau,\eta_0(\tau)) + \frac{1}{\Gamma(\varrho)}\int_0^\tau \mathcal{Q}_\Psi^\varrho(\tau,\sigma)\mathcal{K}(\tau,\sigma,\eta_0(\sigma))d\sigma, \tag{6} $$

and

$$ \|\eta(\tau) - \eta_0(\tau)\|_{C_{1-\xi;\Psi}} \leq \frac{1}{1-[\mathcal{L}_1+\mathcal{L}_2\mathcal{L}]}\psi(\tau). \tag{7} $$

The next theorem was introduced by Diaz and Margolis[29] and used by Cădariu and Radu for studying the concept of stability of functional equations.[30–32]

Theorem 2 (Alternative FPT). *Let (Γ,δ) be a complete $[0,\infty]$-valued metric space. Assume that $\Lambda : \Gamma\times\Gamma$ is a strictly contractive operator with the Lipschitz constant $\mathcal{L} < 1$. If there exists a nonnegative integer k such that $\delta(\Lambda^{k+1}\omega, \Lambda^k\omega) < 1$, for some $\omega\in\Gamma$. Thus,*

1. *$\{\Lambda^n\omega\}$ converges to a FP ω^* of Λ.*
2. *$\Lambda\omega^* = \omega^*$, also ω^* is the unique in*

$$ \Gamma^* = \{\nu\in\Gamma; \delta(\Lambda^k\omega,\nu) < \infty\}. $$

3. *If $\nu\in\Gamma^*$, $\delta(\nu,\omega^*) \leq \frac{1}{1-\mathcal{L}}d(\Lambda\nu,\nu)$.*

In the proof of Theorem 1, the authors defined the mapping, $\Lambda : \mathcal{M} \to \mathcal{M}$, by

$$\Lambda(\omega(\tau)) = \mathcal{I}_{0^+}^{\varrho;\Psi}\left[\zeta(\tau,\omega(\tau)) + \frac{1}{\Gamma(\varrho)}\int_0^t \mathcal{Q}_\Psi^\varrho(\tau,\sigma)\mathcal{K}(\varsigma,\sigma,\omega(\sigma))d\sigma\right]$$
$$= \mathcal{I}_{0^+}^{\varrho;\Psi}[\zeta(\tau,\omega(\tau))] + \mathcal{I}_{0^+}^{\varrho;\Psi}\left[\frac{1}{\Gamma(\varrho)}\int_0^t \mathcal{Q}_\Psi^\varrho(\tau,\sigma)\mathcal{K}(\varsigma,\sigma,\omega(\sigma))d\sigma\right], \quad (8)$$

which is not suitable for fractional integral equations. More exactly, since by Theorem 2 the fixed point of Λ is the unique solution of (9) and the operator Λ helps to find a solution for integral Equation (9), so could not be an integral operator as above, then Sousa et al[24] could not find an accurate approximation.

Consider continuous map $\omega : [0,P] \to \Omega$ which Ω is a Banach space. Motivate by Equation(1) and by Equation(8), in this paper, we apply a fixed point theorem (FPT) to study Ulam-Hyers stability (UH-stability) of the solution of the fractional Volterra integral equation

$$\omega(\tau) = \zeta(\tau,\omega(\tau)) + \mathcal{I}_{0^+}^{\varrho;\Psi}\mathcal{K}(\tau,\sigma,\omega(\sigma)), \quad (9)$$

where $\mathcal{I}_{0^+}^{\varrho;\Psi}$ is the Ψ-Riemann-Liouville fractional integral defined by

$$\mathcal{I}_{0^+}^{\varrho;\Psi}\omega(\tau) = \frac{1}{\Gamma(\varrho)}\int_0^\tau \mathcal{Q}_\Psi^\varrho(\tau,\sigma)\omega(\sigma)d\sigma, \quad (10)$$

where $\mathcal{Q}_\Psi^\varrho(\tau,\sigma) := \Psi'(\sigma)(\Psi(\tau)-\Psi(\sigma))^{\varrho-1}$,[10,23] with $0<\varrho<1$, and $\zeta : [0,P]\times\Omega \to \Omega$, $\mathcal{K} : [0,P]^2\times\Omega \to \Omega$. In this sense, in order to elucidate the investigated result, we present an example using the α-resolvent operator theory.

We work on the weighted space $C_{1-\xi;\Psi}[0,P]$ of continuous mappings ω which defined in previous studies[23,24] and with norm $\|.\|_{C_{1-\xi;\Psi}[0,P]}$.

For a continuous map $\omega(\tau)$, satisfying

$$\left\|\zeta(\tau,\omega(\tau)) + \mathcal{I}_{0^+}^{\varrho;\Psi}\mathcal{K}(\tau,\sigma,\omega(\sigma)) - \omega(\tau)\right\|_{C_{1-\xi;\Psi}} \le \psi(\tau),$$

with $\psi(\tau)>0$ and $0\le\tau\le P$, we can find a solution $\nu(\tau)$ of (9) such that

$$\|\omega(\tau)-\nu(\tau)\|_{C_{1-\xi;\Psi}[0,P]} \le K\psi(\tau),$$

for some $K>0$, so (9) has the UH-stability. Note that (9) can be considered as integral equation

$$\omega(\tau) = \zeta(\tau,\omega(\tau)) + \frac{1}{\Gamma(\varrho)}\int_0^t \mathcal{Q}_\Psi^\varrho(\tau,\sigma)\mathcal{K}(\tau,\sigma,\omega(\sigma))d\sigma. \quad (11)$$

In Proposition 1 of Sousa et al,[24] the authors proved that if $(\Omega, \|.\|_{C_{1-\xi;\Psi}})$ is a normed space, then the function $\delta : \Omega\times\Omega \to \mathbb{R}$, $\delta(\omega,\nu) = \|\omega-\nu\|_{C_{1-\xi;\Psi}}$ is a metric on Ω. By Kreyszig,[33] every normed space induce a metric by equality $\delta(\omega,\nu) = \|\omega-\nu\|$; then Proposition 1 of Sousa et al[24] is obvious and no need any proof.

2 | ULAM-HYERS STABILITY (PROOF TO THEOREM 1)

In this section, we present a correct proof to Theorem 1 by defining a suitable function $\Lambda : \mathcal{M} \to \mathcal{M}$.

Proof. Set $\mathcal{M} = \{\omega : [0,P] \to \Omega, \omega \text{ is continuous}\}$ and define a mapping $\delta : \mathcal{M}^2 \to [0,\infty]$ by

$$\delta(\omega,\nu) = \bigwedge\{C \ge 0 : \|\omega(\sigma)-\nu(\sigma)\|_{C_{1-\xi;\Psi}} \le C\psi(\sigma), 0 \le \sigma \le P\}.$$

In previous studies,[24,30–32] the authors proved that $(\mathcal{M}, \delta)$ is a complete $[0, \infty]$-valued metric space. Consider $\Lambda : \mathcal{M} \to \mathcal{M}$, such that

$$\Lambda(\omega(\tau)) = \zeta(\tau, \omega(\tau)) + \frac{1}{\Gamma(\varrho)} \int_0^\tau \mathcal{Q}_\Psi^\varrho(\tau, \sigma)\mathcal{K}(\tau, \sigma, \omega(\sigma))d\sigma.$$

Suppose $\omega, \nu \in \Omega$, $C_{\omega\nu} \in [0, \infty]$, and $\delta(\omega, \nu) \le C_{\omega\nu}$. So, for every $\tau \in [0, P]$,

$$\|\omega(\tau) - \nu(\tau)\|_{C_{1-\xi;\Psi}} < C_{\omega\nu}\phi(t).$$

Using (2) to (4), we get

$$\begin{aligned}
\|\Lambda\omega(\tau) - \Lambda\nu(\tau)\|_{C_{1-\xi;\Psi}} &\le \|\zeta(\tau, \omega) - \zeta(\tau, \nu)\|_{C_{1-\xi;\Psi}} + \left\|\mathcal{I}_{0^+}^{\varrho;\Psi}(\mathcal{K}(\varsigma, \sigma, \omega(\sigma)) - \mathcal{K}(\varsigma, \sigma, \nu(\sigma))\right\|_{C_{1-\xi;\Psi}} \\
&\le \mathcal{L}_1\|\omega(\tau) - \nu(\tau)\|_{C_{1-\xi;\Psi}} + \mathcal{I}_{0^+}^{\varrho;\Psi}\mathcal{L}_2\|\omega(\tau) - \nu(\tau)\|_{C_{1-\xi;\Psi}} \\
&\le \mathcal{L}_1 C_{\omega\nu}\psi(\tau) + \mathcal{L}_2\mathcal{I}_{0^+}^{\varrho;\Psi}[C_{\omega\nu}\psi(\sigma)] \\
&\le [\mathcal{L}_1 + \mathcal{L}_2\mathcal{L}]C_{\omega\nu}\psi(\tau).
\end{aligned}$$

In this sense, we have

$$\delta(\Lambda\omega, \Lambda\nu) \le [\mathcal{L}_1 + \mathcal{L}_2\mathcal{L}]\delta(\omega, \nu), \tag{12}$$

which implies that the contraction property of Λ, because $0 < \mathcal{L}_1 + \mathcal{L}_2\mathcal{L} < 1$. Since $\eta \in \mathcal{M}$ and using (5), we obtain

$$\|\Lambda\eta(\tau) - \eta(\tau)\|_{C_{1-\xi;\Psi}} = \left\|\zeta(\tau, \omega(\tau)) + \frac{1}{\Gamma(\varrho)} \int_0^\tau \mathcal{Q}_\Psi^\varrho(\tau, \sigma)\mathcal{K}(\tau, \sigma, \omega(\sigma))d\sigma - \eta(\tau)\right\|_{C_{1-\xi;\Psi}} \le \psi(\tau),$$

and so,

$$\delta(\Lambda\eta, \eta) \le 1 < \infty. \tag{13}$$

Now, using Theorem 2, let us find a unique map $\eta_0 \in \mathcal{M}^* = \{\nu \in \mathcal{M} : \delta(\Lambda\eta, \nu) < \infty\}$ such that $\Lambda\eta_0 = \eta_0$ and so

$$\eta_0(\tau) = \zeta(\tau, \eta_0(\tau)) + \mathcal{I}_{0^+}^{\varrho;\Psi}[\mathcal{K}(\varsigma, \sigma, \eta_0(\sigma))]. \tag{14}$$

Using Theorem 2 and (13), we have

$$\delta(\eta, \eta_0) \le \frac{1}{1 - [\mathcal{L}_1 + \mathcal{L}_2\mathcal{L}]}d(\Lambda f, f) \le \frac{1}{1 - [\mathcal{L}_1 + \mathcal{L}_2\mathcal{L}]},$$

which implies that (7). Set

$$\beta = \frac{1}{1 - [\mathcal{L}_1 + \mathcal{L}_2\mathcal{L}]}.$$

Consider another continuous map θ satisfying (6) and (7). Thus, $\eta \in \mathcal{M}$, $\delta(\eta, \theta) < \beta$, and

$$\theta(\tau) = \zeta(\tau, \theta(\tau)) + \frac{1}{\Gamma(\varrho)} \int_0^\tau \mathcal{Q}_\Psi^\varrho(\tau, \sigma)\mathcal{K}(\tau, \sigma, \theta(\sigma))d\sigma. \tag{15}$$

We prove that, $\theta \in \mathcal{M}^*$ is a FP of Λ.

By (15), we have $\Lambda\theta = \theta$. Now, we are ready to prove that $\delta(\Lambda\eta, \theta) < \infty$. Using (15) and $\delta(\eta, \theta) < \beta$, imply that

$$\begin{aligned}\|\Lambda\eta(\tau) - \theta(\tau)\|_{C_{1-\xi;\Psi}} &= \left\| \zeta(\tau, \eta(\tau)) + \frac{1}{\Gamma(\varrho)} \int_0^\tau \mathcal{Q}_\Psi^\varrho(\tau, \sigma)\mathcal{K}(\tau, \sigma, \eta(\sigma))d\sigma - \eta(\tau) + \eta(\tau) - \theta(\tau) \right\|_{C_{1-\xi;\Psi}} \\ &\leq \left\| \zeta(\tau, \eta(\tau)) + \frac{1}{\Gamma(\varrho)} \int_0^\tau \mathcal{Q}_\Psi^\varrho(\tau, \sigma)\mathcal{K}(\tau, \sigma, \eta(\sigma))d\sigma - \eta(\tau) - \eta(\tau) \right\|_{C_{1-\xi;\Psi}} \\ &\quad + \|\eta(\tau) - \theta(\tau)\|_{C_{1-\xi;\Psi}} \\ &\leq (1 + \beta)\psi(\tau),\end{aligned}$$

which implies that $\delta(\Lambda\eta, \theta) \leq 1 + \beta < \infty$. □

3 | EXAMPLE

To conclude this paper, in this section, we present an example, in order to elucidate the result investigated in Section 2.

We recall that for a Banach space Ω, a one parameter family $\{\mathbb{S}_\alpha(t)\}_{t\geq 0}$, $\alpha > 0$ in $\mathcal{B}(\Omega)$, the space of all bounded linear operators, is called an α-resolvent operator function if the following conditions are satisfied:

1. $\mathbb{S}_\alpha(t)$ is strongly continuous on $\mathbb{R}^+$ and $\mathbb{S}_\alpha(0) = I$;
2. $\mathbb{S}_\alpha(t)\mathbb{S}_\alpha(s) = \mathbb{S}_\alpha(s)\mathbb{S}_\alpha(t)$, for all $t, s \geq 0$;
3. The functional equation
$$\mathbb{S}_\alpha(s)I_{0+}^{\alpha;\psi}\mathbb{S}_\alpha(t) - I_{0+}^{\alpha;\psi}\mathbb{S}_\alpha(s)\mathbb{S}_\alpha(t) = I_{0+}^{\alpha;\psi}\mathbb{S}_\alpha(s) - I_{0+}^{\alpha;\psi}\mathbb{S}_\alpha(t)$$
holds for all $t, s \geq 0$. The generator $\mathcal{A}$ of $\mathbb{S}_\alpha$ is defined by
$$\mathcal{D}(\mathcal{A}) = \left\{ x \in \Omega, \lim_{t\to 0} \frac{\mathbb{S}_\alpha(t)x - x}{g_{\alpha+1}(t)} \text{ exists} \right\}$$
where $g_{\alpha+1}(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}$ and $\mathcal{A}x = \lim\limits_{t\to 0} \frac{\mathbb{S}_\alpha(t)x - x}{g_{\alpha+1}(t)}$, $x \in \mathcal{D}(\mathcal{A})$. Note that, $\lim\limits_{t\to 0}\mathbb{S}_\alpha(t)x = x$ for $x \in \Omega$.

Consider the following conditions:

C1. Let $g : [0, P] \times \Omega \to \Omega$, $K : [0, P] \times [0, P] \times \Omega \to \Omega$ and $\phi : [0, P] \to (0, \infty)$ be continuous functions satisfying

$$\|g(t, x) - g(t, y)\| \leq L_1 \|x - y\|, \tag{16}$$

$$\|K(t, s, x) - K(t, s, y)\| \leq L_2 \|x - y\|, \tag{17}$$

$$\int_0^t \phi(s)ds \leq L\phi(t) \tag{18}$$

for all $s, t \in [0, P]$ and $x, y \in \Omega$.

C2. Assume that $\mathcal{A}$ generates an α-resolvent family $\{\mathbb{S}_\alpha(t)\}_{t\geq 0}$ such that $\|\mathbb{S}_\alpha(t)\| \leq e^{-\delta t}M$ $(M > 0, \alpha > 0)$ for all $t \geq 0$.

Example 1. Suppose Ω is a Banach space, $P \in (0, \infty)$ and $\mathbb{S}_\alpha(t)$ is a α-resolvent operator with generator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$. Let $B \in C([0, P], B(\Omega))$, the space of all continuous function from $[0, P]$ into $B(\Omega)$. For $x_0 \in \mathcal{D}(\mathcal{A})$, consider the integral equation

$$u(t) = \mathbb{S}_\alpha(t)x_0 + \int_0^t \mathbb{S}_\alpha(t - s) B(s)u(s)ds, \tag{19}$$

$t \in [0, P]$. This equation has a solution.

Define $g:[0,P]\times\Omega\to\Omega$ and $K:[0,P]\times[0,P]\times\Omega\to\Omega$ by

$$g(t,x)=\mathbb{S}_\alpha(t)x_0,$$

$$K(t,s,x)=\mathbb{S}_\alpha(t-s)B(s)x.$$

Note that, for any $x,y\in\Omega$, $\|g(t,x)-g(t,y)\|=0$. Also, from condition **C2**, we know that there exist $m,\omega>0$, such that for all $t\geq 0$, $\|\mathbb{S}_\alpha(t)\|\leq M_0$, for some $M_0>0$ and all $s\in[0,P]$. Thus,

$$\begin{aligned}\|K(t,s,x)-K(t,s,y)\| &= \|\mathbb{S}_\alpha(t-s)B(s)(x-y)\|\\ &\leq MM_0e^{P\omega}\|x-y\|.\end{aligned}$$

Now suppose $0<L\leq\frac{1}{MM_0e^{P\omega}}$. For fixed $\alpha\geq\frac{1}{L}$ and $\rho>0$, if $\phi(t)=\rho e^{\alpha t}$, then the conditions (16), (17), and (18) hold. Thus, if

$$\left\|u(t)-\mathbb{S}_\alpha(t)x_0-\int_0^t\mathbb{S}_\alpha(t-s)B(s)u(s)\,ds\right\|\leq\phi(t)$$

$t\in[0,P]$, by Theorem 1, there exists a unique solution $u_0(t)$ of (19) such that

$$\|u(t)-u_0(t)\|\leq\frac{1}{1-MM_0e^{P\omega}L}\phi(t).$$

Hence, it conclude the Hyers-Ulam stability of (19).

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CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

AUTHOR CONTRIBUTIONS

All authors' contributions to this manuscript are the same. All authors read and approved the final manuscript.

ORCID

Dumitru Baleanu https://orcid.org/0000-0002-0286-7244
Reza Saadati https://orcid.org/0000-0002-6770-6951
José Sousa https://orcid.org/0000-0002-6986-948X

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