Al-Omari et al. *Advances in Difference Equations* (2020) 2020:500
https://doi.org/10.1186/s13662-020-02961-x

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

δ-β-Gabor integral operators for a space of locally integrable generalized functions

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Abstract

In this article, we give a definition and discuss several properties of the δ-β-Gabor integral operator in a class of locally integrable Boehmians. We derive delta sequences, convolution products and establish a convolution theorem for the given δ-β-integral. By treating the delta sequences, we derive the necessary axioms to elevate the δ-β-Gabor integrable spaces of Boehmians. The said generalized δ-β-Gabor integral is, therefore, considered as a one-to-one and onto mapping continuous with respect to the usual convergence of the demonstrated spaces. In addition to certain obtained inversion formula, some consistency results are also given.

MSC: Primary 54C40; 14E20; secondary 46E25; 20C20

Keywords: δ-β-Gabor integral; Time-frequency integral; Signal; Gabor integral; Boehmian; Window function

1 Preliminaries

Due to various applications of the Dirac delta distribution and its usual implementation in physics, engineering, and partial differential equations, various distribution spaces have been considered in literature. Distributions or generalized functions are objects that generalize the classical notion of functions. They are widely used in geometry, mathematical physics, stochastic analysis, harmonic analysis, and the theory of linear and nonlinear partial differential equations as it is much easier to establish a distributional solution than a classical one. The space of Boehmians is a space of generalized functions constructed in an algebraic way similar to the construction of the field of quotients. When a multiplication is interpreted as a convolution, the construction of a space of Boehmians, applied to a different function space, yields a different space of Boehmians. Boehmians also allow different identifications of integral operators to be isomorphisms. Therefore, several integral operators have been applied to various spaces of Boehmians in various papers in the recent past. For example, the Stieltjes integral operator was extended to a space of Boehmians of Fox's H-function type (see, e.g., [1]), the Hartley integral operator was extended to a space of strong Boehmians (see, e.g., [2]), the Hilbert integral operator was extended to a space of Boehmians (see, e.g., [3]), the Mellin integral operator was extended to a space of Boehmians of quaternion type (see, e.g., [4]), the short-time Fourier integral operator was extended to a space of H-Boehmians (see, e.g., [5]), the fractional Fourier integral was extended to a space of Boehmians (see, e.g., [6]), the quaternion Fourier integral was extended to a quaternionic set of Boehmians (see, e.g., [7]) and many others, to mention but a few. However, further extensions of integral operators to Boehmian spaces were obtained in [1–10] (see also citations therein).

The Gabor integral transform operator, among other integral transform operators, was proposed as a time-frequency integral operator to perform simultaneous time-frequency analysis of signals. This frequently was used for feature extraction, non-stationary signal processing, radar systems, sonar systems, communications, and space sciences (see, e.g., [11]). For a given window function g and coordinates in the space and the frequency domains q and $\tilde{q}$, the Gabor integral operator of a signal φ is given as follows (see, e.g., [12]):

$$G_g\varphi(q,\tilde{q}) = \int_{\mathbb{R}} \varphi(x)\overline{g(x-\tilde{q})}\exp(i2\pi qx)\,dx, \tag{1}$$

when the integral exists. If g is a given window function satisfying the integral equation

$$\int_{\mathbb{R}} \left|g(x)\right|^2 dx = 1,$$

then the signal function φ can be recovered from the Gabor spectrum integral $G_g\varphi$ as follows:

$$\varphi(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} G_g\varphi(q,\tilde{q})g(x-q)\overline{\exp(i2\pi qx)}\,dq\,d\tilde{q}. \tag{2}$$

The Gabor integral operator of an image has much lower entropy than the pixel representation of the image, and the expansions of the Gabor, for every low bit rates, can provide better signal compression than the discrete cosine integral operator can (see, e.g., [8, 10–18]). If δ and β are real numbers, when β is fixed, then the collection $g_{q,\tilde{q}}^{\beta,\delta}$ of functions is defined by

$$g_{q,\tilde{q}}^{\beta,\delta}(x) = \exp\left(\frac{2iqx - q\tilde{q}}{2}\right)g_\delta^\beta(x-\tilde{q}), \tag{3}$$

where g_δ^β is a family of Gaussian window functions defined by

$$g_\delta^\beta(x) = \delta^{\frac{1}{4}}\exp\left(\frac{i\beta x^2 - 2\delta x^2}{2}\right). \tag{4}$$

With respect to the family g_δ^β of window functions, the δ-β-Gabor integral operator of a signal φ is defined by (see, e.g., [13, Eq. (4)])

$$G_\delta^\beta\varphi(q,\tilde{q}) = \int_{\mathbb{R}} g_{q,\tilde{q}}^{\beta,\delta}(x)\varphi(x)\,dx. \tag{5}$$

The δ-β-Gabor integral operator is the δ-β-extension of the Gabor integral operator when $\delta = 1$ and $\beta = 0$. It is closely related to the Wigner–Ville integral operator

$$W\left(g_\delta^\beta\right)(x,\xi) = C\exp\left(-\frac{2\delta+\beta^2}{2\delta}x^2 - \frac{1}{2\delta}\xi^2 + \frac{\beta}{\delta}x\xi\right),$$

and they are related by (see, e.g., [13])

$$\left|G_\delta^\beta \varphi(q,\tilde{q})\right|^2 = W\left(g_\delta^\beta\right)\sharp W(\varphi),$$

where $\sharp$ is the convolution product of the Fourier convolution type (see, e.g., [19])

$$\varphi\sharp\psi(\xi) = \int_{\mathbb{R}} \psi(t)\varphi(\xi - t)\,dt, \tag{6}$$

and C is some constant. However, the δ-β-Gabor integral of a signal may look much better than the Gabor integral operator since the choice of δ and β, which brings out features best, will depend on the time-frequency content of the signal itself. It, therefore, becomes natural to adapt the choice of parameters δ and β to the phase point $(q,\tilde{q})$. However, this article firstly aims to discuss convolution products and convolution theorems for the δ-β-Gabor integral operator. It then generates two sets of Boehmians and gives some characteristics of the extended δ-β-Gabor integral operator. For the convenience of the reader, we distribute our results into four sections. In Sect. 2, we introduce convolution products and prove a convolution theorem. In Sect. 3, we generate the δ-β-Gabor sets of Boehmians. In Sect. 4, we discuss the δ-β-Gabor integral on the set of Boehmians and obtain an inversion formula as well as radical properties of the generalized integral.

2 Convolutions and convolution theorem

In mathematics, the convolution is a mathematical operation on two functions, producing a third function that is typically viewed as a modified version of one of the original functions, giving the area overlap between the two functions as one of the original functions is translated. Convolutions have applications including probability, statistics, computer vision, image and signal processing, electrical engineering, and differential equations. At any rate, the purpose of this section is to devise the convolution product and derive the convolution theorem for the δ-β-Gabor integral operator for arbitrary real numbers δ and β.

The analysis of the Gabor integral operator successfully brings light to a very beneficial convolution product that works in with the Fourier convolution product. The compatible convolution product is defined as follows.

Definition 1 Let $\varphi \in L^1(\mathbb{R})$ and $F \in L^1(\mathbb{R}^2)$. Then, for φ and F, we define an integral equation $\star_q^{\tilde{q}}$ as follows:

$$\varphi \star_q^{\tilde{q}} F(q,\tilde{q}) = \int_{\mathbb{R}} \varphi(x)\exp\left(\frac{2iqx - qx}{2}\right)F(q,\tilde{q} - x)\,dx, \tag{7}$$

provided the right-hand side integral exists for all coordinates q and $\tilde{q}$.

To establish the δ-β-Gabor convolution theorem, we firstly derive the following preliminary result.

Theorem 2 *Let $g_{q,\tilde{q}}^{\beta,\delta}$ and g_δ^β be defined as in Eq. (3) and Eq. (4), respectively. Then, for real numbers z and x, we have*

$$g_{q,\tilde{q}}^{\beta,\delta}(z + x) = \exp\left(\frac{2iqx - qx}{2}\right)\exp\left(\frac{2iqz - q(\tilde{q} - t)}{2}\right)g_\delta^\beta(z).$$

Proof By using Eq. (3) and employing the change of variables $w = z + x - \tilde{q}$, the above equation routinely becomes

$$g_{q,\tilde{q}}^{\beta,\delta}(z+x) = \exp\left(\frac{2i(q+z) - q\tilde{q}}{2}\right) g_{\delta}^{\beta}(w).$$

Hence, by taking into account the definition of a family of the Gaussian window functions g_{δ}^{β}, we get

$$g_{q,\tilde{q}}^{\beta,\delta}(z+x) = \exp\bigl(iq(z+x)\bigr)\exp\left(-\frac{q\tilde{q}}{2}\right)\delta^{\frac{1}{4}}\exp\left(\frac{i\beta w^2 - 2\delta w^2}{2}\right).$$

This can alternatively be written as

$$g_{q,\tilde{q}}^{\beta,\delta}(z+x) = \exp(ipx)\exp\left(\frac{2ipz - q(\tilde{q}-x) - qx}{2}\right)\delta^{\frac{1}{4}}\exp\left(\frac{i\beta w^2 - 2\delta w^2}{2}\right).$$

Therefore, by using a simple computation and making a rearrangement on the above exponents yield

$$g_{q,\tilde{q}}^{\beta,\delta}(z+x) = \exp\left(\frac{2iq - qx}{2}\right)\exp\left(\frac{2iqz - q(\tilde{q}-x)}{2}\right)\delta^{\frac{1}{4}}\exp\left(\frac{i\beta w^2 - 2\delta w^2}{2}\right).$$

Indeed, the definition of g_{δ}^{β} gives

$$g_{q,\tilde{q}}^{\beta,\delta}(z+x) = \exp\left(\frac{2iqx - qx}{2}\right)\exp\left(\frac{2iqz - q(\tilde{q}-x)}{2}\right) g_{\delta}^{\beta}(z).$$

This finishes the proof of the theorem. □

Now, as we are implementing the Fourier convolution product $\sharp$ in our next investigation, we have to be very familiar with the convolution properties on the set of integrable functions that we recall (see, e.g., [6]):

$$\varphi\sharp\psi = \psi\sharp\varphi \quad \text{and} \quad \varphi\sharp(\psi\sharp\theta) = (\psi\sharp\varphi)\sharp\theta.$$

On the basis of the above definitions, the convolution theorem of the Gabor integral G_{δ}^{β} can be derived as follows.

Theorem 3 *Let φ and ψ be arbitrarily given in $L^1(\mathbb{R})$, and let q and $\tilde{q}$ be the coordinates in the space and frequency domains, then we have*

$$G_{\delta}^{\beta}(\varphi\sharp\psi)(q,\tilde{q}) = \bigl(G_{\delta}^{\beta}\varphi \star_{q}^{\tilde{q}} \psi\bigr)(q,\tilde{q}).$$

Proof Let φ and ψ be arbitrarily given. Then, by considering the integral relation presented in Eq. (7), we write

$$G_{\delta}^{\beta}(\varphi\sharp\psi)(q,\tilde{q}) = \int_{\mathbb{R}} \psi(t) \int_{\mathbb{R}} g_{q,\tilde{q}}^{\beta,\delta}(x)\varphi(x-t)\,dx\,dt. \tag{8}$$

Hence, the substitution $x = z + t$, consequently, changes Eq. (8) into the integral relation

$$G_\delta^\beta(\varphi \sharp \psi)(q,\tilde{q}) = \int_{\mathbb{R}} \psi(t) \int_{\mathbb{R}} g_{q,\tilde{q}}^{\beta,\delta}(z+t)\varphi(z)\,dz\,dt. \tag{9}$$

Thus, with the aid of Theorem 2, Eq. (9) can be nicely expressed as

$$G_\delta^\beta(\varphi \sharp \psi)(q,\tilde{q}) = \int_{\mathbb{R}} \psi(t) \exp\left(\frac{2iqt - qt}{2}\right)\left(\int_{\mathbb{R}} g_{q,\tilde{q}-t}^{\beta,\delta}(z)\varphi(z)\,dz\right) dt.$$

Therefore, by employing Eq. (5), the Gabor integral of the convolution can be given as follows:

$$G_\delta^\beta(\varphi \sharp \psi)(q,\tilde{q}) = \int_{\mathbb{R}} \psi(t) \exp\left(\frac{2iqt - qt}{2}\right) G_\delta^\beta(\varphi)(q,\tilde{q}-t)\,dt.$$

Hence, by using Eq. (7) we complete the proof of the theorem. □

Indeed, analogous to many magnificent research works related to various integral operators, this convolution theorem still lacks the elegance and simplicity of the Euclidean Fourier integral operator, which states that the Fourier integral operator of a convolution of two functions is the product of their respective Fourier transforms. Although the convolution product and the convolution theorem of the δ-β-Gabor integral operator are not so accessible as the Euclidean Fourier convolution product and convolution theorem, they will serve us nicely to present a simple though rigorous approach to the generalized δ-β-Gabor spaces of Boehmians. It also serves to derive a linear bijection between the Boehmian spaces.

3 δ-β-Gabor spaces of Boehmians

Denote by $D(\mathbb{R})$ the space of smooth functions of compact supports over $\mathbb{R}$ and, at the same time, denote by $L_{loc}^1(\mathbb{R}^2)$ the space of locally integrable functions over $\mathbb{R}^2$. Denote by Δ the set of delta sequences $\{\delta_n\}$ from $(D(\mathbb{R}), \sharp)$ satisfying $\Delta_1 - \Delta_3$, where

$$\Delta_1 : \int_{\mathbb{R}} \delta_n = 1 \text{ for all } n \in \mathbb{N}.$$

$$\Delta_2 : |\delta_n| < M, M \in \mathbb{R}, 0 < M, n \in \mathbb{N}.$$

$$\Delta_3 : supp\delta_n \subseteq (-\gamma_n, \gamma_n), \gamma_n \to 0 \text{ as } n \to \infty.$$

With $L_{loc}^1(\mathbb{R}^2)$, $D(\mathbb{R})$, Δ, the product $\sharp$, and the product $\star_q^{\tilde{q}}$, we generate a space $B(\mathbb{R}^2)$ of Boehmians which works in as a range space of the generalized δ-β-Gabor integral operator. Therefore, we start with the proof of the following fundamental result.

Lemma 4 *Let $F \in L_{loc}^1(\mathbb{R}^2)$ and $\varphi, \psi \in D(\mathbb{R})$. Then we have*

$$F \star_q^{\tilde{q}} (\varphi \sharp \psi)(q,\tilde{q}) = \left(F \star_q^{\tilde{q}} \varphi\right) \star_q^{\tilde{q}} \psi(q,\tilde{q}) \quad \text{in } L_{loc}^1\left(\mathbb{R}^2\right),$$

where $(q,\tilde{q}) \in \mathbb{R}^2$.

Proof By employing the definitions of the convolution products $\star_q^{\tilde{q}}$ and $\sharp$, we respectively obtain

$$\begin{aligned} F \star_q^{\tilde{q}} (\varphi \sharp \psi)(q,\tilde{q}) &= \int_{\mathbb{R}} F(q,\tilde{q}-x) \exp\left(\frac{2iqx-qx}{2}\right)(\varphi\sharp\psi)(x)\,dx \\ &= \int_{\mathbb{R}} F(q,\tilde{q}-x) \exp\left(\frac{2iqx-qx}{2}\right) \int_{\mathbb{R}} \varphi(x-y)\psi(y)\,dy\,dx. \end{aligned}$$

Hence, with the aid of Fubini's theorem, we simplify the above equation to write

$$F \star_q^{\tilde{q}} (\varphi \sharp \psi)(q,\tilde{q}) = \int_{\mathbb{R}} \psi(y) \int_{\mathbb{R}} F(q,\tilde{q}-x) \exp\left(\frac{2iqx-qx}{2}\right)\varphi(x-y)\,dx\,dy. \quad (10)$$

Therefore, a proper change in the variables changes Eq. (10) into the standard form

$$\begin{aligned} &F \star_q^{\tilde{q}} (\varphi \sharp \psi)(q,\tilde{q}) \\ &\quad= \int_{\mathbb{R}} \psi(y) \exp\left(\frac{2iqy-qy}{2}\right) \int_{\mathbb{R}} F\bigl(q(\tilde{q}-y)-z\bigr) \exp\left(\frac{2iqx-qx}{2}\right)\varphi(z)\,dz\,dy \\ &\quad= \int_{\mathbb{R}} \psi(y) \exp\left(\frac{2iqy-qy}{2}\right)\bigl(F \star_q^{\tilde{q}} \varphi\bigr)(q,\tilde{q}-y)\,dy. \end{aligned}$$

Hence, we have reached the conclusion that $F \star_q^{\tilde{q}} (\varphi \sharp \psi) = (F \star_q^{\tilde{q}} \varphi) \star_q^{\tilde{q}} \psi$. To complete the proof of this lemma, we have to show

$$F \star_q^{\tilde{q}} \varphi \in L^1_{loc}\bigl(\mathbb{R}^2\bigr) \quad (11)$$

for every $F \in L^1_{loc}(\mathbb{R}^2)$ and $\phi \in D(\mathbb{R})$. Let K be a compact subset of R^2, then we have

$$\begin{aligned} \int_K \bigl|\bigl(F \star_q^{\tilde{q}} \varphi\bigr)(q,\tilde{q})\bigr|\,d(q,\tilde{q}) &= \int_K \left|\int_{\mathbb{R}} F(q,\tilde{q}-x)\varphi(x) \exp\left(\frac{2iqx-qx}{2}\right)dx\right| d(q,\tilde{q}) \\ &\le \int_{\mathbb{R}} \bigl|\varphi(x)\bigr| \int_K \bigl|F(q,\tilde{q}-x)\bigr|\,d(q,\tilde{q})\,dx \\ &\le A \int_{\mathbb{R}} \bigl|\varphi(x)\bigr|\,dx, \end{aligned}$$

where A is a certain positive constant such that

$$\int_K \bigl|F(q,\tilde{q}-x)\bigr|\,d(q,\tilde{q}) \le A \text{ as } F \in L^1_{loc}\bigl(\mathbb{R}^2\bigr). \quad (12)$$

Also, as $\varphi \in D(\mathbb{R})$ and φ is a smooth function of compact support, we have

$$\int_{\mathbb{R}} \bigl|\varphi(x)\bigr|\,dx \le B$$

for some $B \in \mathbb{R}$. Hence, we have obtained $F \star_q^{\tilde{q}} \varphi \in L^1_{loc}(\mathbb{R}^2)$ for all $F \in L^1_{loc}(\mathbb{R}^2)$ and $\varphi \in D(\mathbb{R})$.

The proof of the theorem is, therefore, finished. □

Each of the identities of the following lemma has a routine verification and, hence, the proof is left to the reader. Details are, therefore, omitted.

Lemma 5 *Let $F,\{F_n\}\in L^1_{loc}(\mathbb{R}^2)$ and $\varphi,\psi\in D(\mathbb{R})$ and $\alpha\in\mathbb{C}$. Then we have $F\star_q^{\tilde{q}}(\varphi+\psi)=F\star_q^{\tilde{q}}\varphi+F\star_q^{\tilde{q}}\psi$, $(\alpha F)\star_q^{\tilde{q}}\varphi=\alpha(F\star_q^{\tilde{q}}\varphi)$ and*

$$F_n\star_q^{\tilde{q}}\varphi\to F\star_q^{\tilde{q}}\varphi\quad \text{as } n\to\infty \text{ as } F_n\to F \text{ as } n\to\infty.$$

Lemma 6 *For $\{\delta_n\}\in\Delta$ and $F\in L^1_{loc}(\mathbb{R}^2)$, we have*

$$F\star_q^{\tilde{q}}\delta_n\to F\quad \text{in } L^1_{loc}\bigl(\mathbb{R}^2\bigr) \text{ as } n\to\infty. \tag{13}$$

Proof Let $K\subseteq\mathbb{R}^2$ be compact. Then, by using the concept of Δ_1, we have

$$\begin{aligned}
&\int_K\bigl|\bigl(F\star_q^{\tilde{q}}\delta_n-F\bigr)(q,\tilde{q})\bigr|\,d(q,\tilde{q})\\
&\quad=\int_K\left|\bigl(F\star_q^{\tilde{q}}\delta_n\bigr)(q,\tilde{q})-F(q,\tilde{q})\int_{\mathbb{R}}\delta_n(x)\,dx\right|\,d(q,\tilde{q})\\
&\quad\le\int_K\left(\int_{\mathbb{R}}\left|F(q,\tilde{q}-x)\exp\left(\frac{2iqx-qx}{2}\right)-F(q,\tilde{q})\right|\bigl|\delta_n(x)\bigr|\,dx\right)d(q,\tilde{q})\\
&\quad\le\int_K\left(\int_{-\gamma_n}^{\gamma_n}\bigl|F(q,\tilde{q}-x)-F(q,\tilde{q})\bigr|\bigl|\delta_n(x)\bigr|\,dx\right)d(q,\tilde{q})\\
&\quad\le M\int_K\int_{-\gamma_n}^{\gamma_n}\bigl|F(q,\tilde{q}-x)-F(q,\tilde{q})\bigr|\,dx\,d(q,\tilde{q}).
\end{aligned}$$

The last inequality follows from Δ_2 and the fact that $\{\delta_n\}\subseteq D(\mathbb{R})$. Hence, as $F\in L^1_{loc}(\mathbb{R}^2)$, by pursuing simple computations, we write

$$\int_K\bigl|\bigl(F\star_q^{\tilde{q}}\delta_n\to F\bigr)(q,\tilde{q})\bigr|\,d(q,\tilde{q})\le MA\mu(K)(2\gamma_n),$$

where $\mu(K)$ is the Lebesgue measure of K and A is some positive constant. Hence, by Δ_3, we have

$$\bigl\|F\star_q^{\tilde{q}}\delta_n-F\bigr\|\le MA\mu(K)(2\gamma_n)\to 0$$

as $n\to\infty$. This finishes the proof of the theorem. □

The Boehmian space $B(\mathbb{R}^2)$ with the sets $(L^1_{loc}(\mathbb{R}^2),\star_q^{\tilde{q}})$, $(D(\mathbb{R}),\sharp)$, $\Delta(\mathbb{R})$ is defined.
The sum of the Boehmians φ_n/δ_n and g_n/ε_n in $B(\mathbb{R}^2)$ is given as

$$\varphi_n/\delta_n+g_n/\varepsilon_n=\bigl(\varphi_n\star_q^{\tilde{q}}\delta_n+g_n\star_q^{\tilde{q}}\delta_n\bigr)/(\delta_n\sharp\varepsilon_n),$$

whereas a multiplication of a Boehmian φ_n/δ_nin $B(\mathbb{R}^2)$ by a complex number $\gamma\in\mathbb{C}$ is defined as $\gamma(\varphi_n/\delta_n)=(\gamma\varphi_n/\delta_n)$. On the other hand, the extension of $\star_q^{\tilde{q}}$ and D^α to $B(\mathbb{R}^2)$ is introduced as follows:

$$(\varphi_n/\delta_n)\star_q^{\tilde{q}}(g_n/\varepsilon_n)=\bigl(\varphi_n\star_q^{\tilde{q}}g_n\bigr)/(\delta_n\sharp\varepsilon_n)\quad\text{and}\quad D^\alpha(\varphi_n/\delta_n)=\bigl(D^\alpha\varphi_n/\delta_n\bigr),\alpha\in\mathbb{R}.$$

Moreover, an extension of $\star_q^{\tilde{q}}$ to $B(\mathbb{R}^2) \star_q^{\tilde{q}} L^1_{loc}(\mathbb{R}^2)$, where (φ_n/δ_n) is in $B(\mathbb{R}^2)$ and ω in $L^1_{loc}(\mathbb{R}^2)$, is given as

$$(\varphi_n/\delta_n) \star_q^{\tilde{q}} \omega = \left(\varphi_n \star_q^{\tilde{q}} \omega\right)/\delta_n.$$

Definition 7 Let $\beta_n, \beta \in B(\mathbb{R}^2)$ for $n = 1, 2, 3, \ldots$. Then the sequence $\{\beta_n\}$ is δ-convergent to β, denoted by $\delta - \lim_{n\to\infty} \beta_n = \beta (\beta_n \xrightarrow{\delta} \beta)$, provided there can be found a delta sequence $\{\delta_n\}$ such that

(a) $(\beta_n \star_q^{\tilde{q}} \delta_k)$ and $(\beta \star_q^{\tilde{q}} \delta_k) \in L^1_{loc}(\mathbb{R}^2)$ for all $n, k \in \mathbb{N}$,

(b) $\lim_{n\to\infty} \beta_n \star_q^{\tilde{q}} \delta_k = \beta \star_q^{\tilde{q}} \delta_k$ in $L^1_{loc}(\mathbb{R}^2)$ for every $k \in \mathbb{N}$.

Or, equivalently, $\delta - \lim_{n\to\infty} \beta_n = \beta$ if and only if there are $\varphi_{n,k}, \varphi_k \in L^1_{loc}(\mathbb{R}^2)$ and $\{\delta_k\} \in \Delta$ such that (i) $\beta_n = \varphi_{n,k}/\delta_k$, $\beta = \varphi_k/\delta_k$ (ii) $\lim_{n\to\infty} \varphi_{n,k} = \varphi_k \in L^1_{loc}(\mathbb{R}^2)$ to every $k \in \mathbb{N}$.

Definition 8 Let $\beta_n, \beta \in B(\mathbb{R}^2)$ for $n = 1, 2, 3, \ldots$. Then the sequence $\{\beta_n\}$ is Δ-convergent to β, denoted by $\Delta\text{-}\lim_{n\to\infty} \beta_n = \beta (\beta_n \xrightarrow{\Delta} \beta)$, provided there can be found a delta sequence $\{\delta_n\}$ such that

(i) $(\beta_n - \beta) \star_q^{\tilde{q}} \delta_n \in L^1_{loc}(\mathbb{R}^2)$ $(\forall n \in \mathbb{N})$ (ii) $\lim_{n\to\infty} (\beta_n - \beta) \star_q^{\tilde{q}} \delta_n = 0$ in $L^1_{loc}(\mathbb{R}^2)$.

Defining the space H with the sets $(L^1_{loc}(\mathbb{R}), \sharp)$, $(D(\mathbb{R}), \sharp)$, and $\Delta(\mathbb{R})$ is quite similar to the construction of the space of L^p- Boehmians; for details, we refer to [9]. In H, addition of φ_n/δ_n and g_n/ε_n in H and $\sharp$ is, respectively, defined as

$$\varphi_n/\delta_n + g_n/\varepsilon_n = (\varphi_n \sharp \delta_n + g_n \sharp \delta_n)/(\delta_n \sharp \varepsilon_n)$$

and

$$(\varphi_n/\delta_n) \sharp (g_n/\varepsilon_n) = (\varphi_n \sharp g_n)/(\delta_n \sharp \varepsilon_n).$$

Multiplication of φ_n/δ_n in H by a complex number $\gamma \in \mathbb{C}$ is defined as

$$\gamma(\varphi_n/\delta_n) = (\gamma \varphi_n/\delta_n).$$

D^α of φ_n/δ_n in H is introduced as $D^\alpha(\varphi_n/\delta_n) = D^\alpha \varphi_n/\delta_n$, $\alpha \in \mathbb{R}$. For every φ_n/δ_n in H and κ in $L^1_{loc}(\mathbb{R})$, $\sharp$ can be extended to $H \sharp L^1_{loc}(\mathbb{R})$ by $(\varphi_n/\delta_n) \sharp \kappa = (\varphi_n \sharp \kappa)/\delta_n$.

Definition 9 Let $\beta_n, \beta \in H$ for $n = 1, 2, 3, \ldots$. Then the sequence $\{\beta_n\}$ is δ-convergent to β, denoted by $\delta - \lim_{n\to\infty} \beta_n = \beta (\beta_n \xrightarrow{\delta} \beta)$, provided there can be found a delta sequence $\{\delta_n\}$ such that

(i) $\beta_n \sharp \delta_k$ and $\beta \sharp \delta_k \in L^1_{loc}(\mathbb{R})$ for all $n, k \in \mathbb{N}$,

(ii) $\lim_{n\to\infty} \beta_n \sharp \delta_k = \beta \sharp \delta_k$ in $L^1_{loc}(\mathbb{R})$ for every $k \in \mathbb{N}$.

Or, equivalently,

$\delta - \lim_{n\to\infty} \beta_n = \beta$ if and only if there are $\varphi_{n,k}, \varphi_k \in L^1_{loc}(\mathbb{R})$and $\{\delta_k\} \in \Delta$ such that

(i) $\beta_n = \varphi_{n,k}/\delta_k$, $\beta = \varphi_k/\delta_k$,

(ii) to every $k \in \mathbb{N}$, we have $\lim_{n\to\infty} \varphi_{n,k} = \varphi_k$ in $L^1_{loc}(\mathbb{R})$.

Definition 10 Let $\beta_n, \beta \in H$ for $n = 1, 2, 3, \ldots$. Then the sequence $\{\beta_n\}$ is Δ-convergent to β, denoted by $\Delta\text{-}\lim_{n\to\infty} \beta_n = \beta (\beta_n \xrightarrow{\Delta} \beta)$, provided there can be found a delta sequence $\{\delta_n\}$ such that

(i) $(\beta_n - \beta)\sharp\delta_n \in L^1_{loc}(\mathbb{R})$ $(\forall n \in \mathbb{N})$,
(ii) $\lim_{n\to\infty}(\beta_n - \beta)\sharp\delta_n = 0$ in $L^1_{loc}(\mathbb{R})$.

4 The X^β_δ and S^β_δ integrals

With the aid of the previous investigation, we estimate the Gabor X^β_δ integral and its inversion S^β_δ on the respective spaces $H(\mathbb{R})$ and $B(\mathbb{R}^2)$ as follows.

Definition 11 Let $\varphi_n/\delta_n \in H(\mathbb{R})$, then the estimated δ-β-Gabor integral of φ_n/δ_n is defined as

$$X^\beta_\delta(\varphi_n/\delta_n) = G^\beta_\delta \varphi_n/\delta_n, \tag{14}$$

which lies in the space $B(\mathbb{R}^2)$.

We recite some properties of the extension X^β_δ of G^β_δ with the help of the following theorems.

Theorem 12 *(i) $X^\beta_\delta : H(\mathbb{R}) \to B(\mathbb{R}^2)$ is well defined.*
(ii) $X^\beta_\delta : H(\mathbb{R}) \to B(\mathbb{R}^2)$ is linear, one-to-one, and onto.
(iii) $X^\beta_\delta : H(\mathbb{R}) \to B(\mathbb{R}^2)$ is continuous with respect to δ and Δ-convergence.
(iv) $X^\beta_\delta : H(\mathbb{R}) \to B(\mathbb{R}^2)$ is consistent with the operator G^β_δ.
(v) $X^\beta_\delta((\varphi_n/\delta_n)\sharp(g_n/\varepsilon_n)) = X^\beta_\delta(\varphi_n/\delta_n) \star^{\tilde{q}}_q X^\beta_\delta(g_n/\varepsilon_n)$.

Proof (i) Let $\varphi_n/\delta_n \in H(\mathbb{R})$. Then $\varphi_n\sharp\delta_m = g_m\sharp\delta_n$ for all $m, n \in \mathbb{N}$. But then, by Theorem 3, we have $G^\beta_\delta(\varphi_n\sharp\delta_m) = G^\beta_\delta(g_m\sharp\delta_n)$. Therefore, $G^\beta_\delta\varphi_n \star^{\tilde{q}}_q \delta_m = G^\beta_\delta g_m \star^{\tilde{q}}_q \delta_n$.

Hence, we have

$$G^\beta_\delta \varphi_n/\delta_n \in B\bigl(\mathbb{R}^2\bigr).$$

Now, we show that X^β_δ is independent of the representative. Let $\varphi_n/\delta_n = g_n/\varepsilon_n$ in $H(\mathbb{R})$, then $\varphi_n\sharp\varepsilon_m = g_m\sharp\delta_n$ for all $m, n \in \mathbb{N}$. Applying Theorem 3 gives $G^\beta_\delta\varphi_n \star^{\tilde{q}}_q \varepsilon_m = G^\beta_\delta g_m \star^{\tilde{q}}_q \delta_n$. Hence, $G^\beta_\delta\varphi_n/\delta_n = G^\beta_\delta g_n/\varepsilon_n$. Thus, $G^\beta_\delta(\varphi_n/\delta_n) = G^\beta_\delta(g_n/\varepsilon_n)$.

Proof (ii) Linearity of X^β_δ follows from linearity of G^β_δ. Let $\varphi_n/\delta_n, g_n/\varepsilon_n \in H(\mathbb{R})$ be such that $X^\beta_\delta(\varphi_n/\delta_n) = X^\beta_\delta(g_n/\varepsilon_n) \in B(\mathbb{R}^2)$. Then by Eq. (14) we get $G^\beta_\delta\varphi_n/\delta_n = G^\beta_\delta g_n/\varepsilon_n$. This means that $G^\beta_\delta\varphi_n \star^{\tilde{q}}_q \varepsilon_m = G^\beta_\delta g_m \star^{\tilde{q}}_q \delta_n$ for all $m, n \in \mathbb{N}$. Hence Theorem 3 gives $G^\beta_\delta(\varphi_n\sharp\varepsilon_m) = G^\beta_\delta(g_m\sharp\delta_n)$. As $G^\beta_\delta : L^1(\mathbb{R}) \to L^1(\mathbb{R}^2)$ is one-to-one, we have $\varphi_n\sharp\varepsilon_m = g_m\sharp\delta_n$ for all $m, n \in \mathbb{N}$. This in turn yields $\varphi_n/\delta_n = g_n/\varepsilon_n \in H(\mathbb{R})$. The onto condition is clear.

We prove Part (iv) as similar proofs for Part (iii) and the convolution theorem in Part (v) may be followed in [9, 12]. Let $\rho \in L^1_{loc}(\mathbb{R}^2)$, then $(\rho\sharp\delta_n)/\delta_n$ is the representative of ρ in $H(\mathbb{R})$, $\{\delta_n\} \in \Delta$ $(\forall n \in \mathbb{N})$. Clearly, for all $n \in \mathbb{N}$, $\{\delta_n\}$ is independent of the representative. Hence, by the convolution theorem, we get

$$X^\beta_\delta\bigl((\rho\sharp\delta_n)/\delta_n\bigr) = G^\beta_\delta(\rho\sharp\delta_n)/\delta_n = \bigl(G^\beta_\delta\rho \star^{\tilde{q}}_q \delta_n\bigr)/\delta_n = G^\beta_\delta\rho \star^{\tilde{q}}_q (\delta_n/\delta_n).$$

Thus, $(G^\beta_\delta\rho \star^{\tilde{q}}_q \delta_n)/\delta_n$ is the representative of $G^\beta_\delta\rho$ in the space $L^1_{loc}(\mathbb{R}^2)$.

The proof is therefore finished. □

We introduce the inverse operator of X^β_δ as follows.

Definition 13 Let $G_\delta^\beta \varphi_n/\delta_n \in B(\mathbb{R}^2)$. We define the inverse X_δ^β integral of a Boehmian $G_\delta^\beta \varphi_n/\delta_n$ in $B(\mathbb{R}^2)$ as follows:

$$S_\delta^\beta \left(G_\delta^\beta \varphi_n/\delta_n\right) = \varphi_n/\delta_n$$

for each $\{\delta_n\} \in \Delta$.

Theorem 14 *Let $G_\delta^\beta \varphi_n/\delta_n \in B(\mathbb{R}^2)$ and $\varphi \in L^1_{loc}(\mathbb{R}^2)$ be given. We have*

$$S_\delta^\beta \left(\left(G_\delta^\beta \varphi_n/\delta_n\right) \star_q^{\tilde{q}} \varphi\right) = (\varphi_n/\delta_n)\sharp\varphi \quad \textit{and} \quad X_\delta^\beta \left((\varphi_n/\delta_n)\sharp\varphi\right) = \left(G_\delta^\beta \varphi_n/\delta_n\right) \star_q^{\tilde{q}} \varphi.$$

Proof Assume $G_\delta^\beta \varphi_n/\delta_n \in B(\mathbb{R}^2)$. For every $\varphi \in L^1_{loc}(\mathbb{R}^2)$, by using the convolution theorem and Definition 11, we have

$$\begin{aligned} S_\delta^\beta \left(\left(G_\delta^\beta \varphi_n/\delta_n\right) \star_q^{\tilde{q}} \varphi\right) &= S_\delta^\beta \left(\left(G_\delta^\beta \varphi_n \star_q^{\tilde{q}} \varphi\right)/\delta_n\right) \\ &= S_\delta^\beta \left(G_\delta^\beta (\varphi_n \sharp\varphi)\right)/\delta_n \\ &= (\varphi_n\sharp\varphi)/\delta_n \\ &= (\varphi_n/\delta_n)\sharp\varphi. \end{aligned}$$

As the proof of the second part is similar, we omit the details. This completely finishes the proof of the theorem. □

Acknowledgements
The authors would like to thank their friends and Springer Nature for their support.

Funding
No funding sources to be declared.

Availability of data and materials
Please contact the author for data requests.

Competing interests
Authors declare that they have no competing interests regarding the publication of the article.

Authors' contributions
The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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Publisher's Note

Received: 11 May 2020 Accepted: 9 September 2020 Published online: 17 September 2020

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