Research Article

Commutators of the Fractional Hardy Operator on Weighted Variable Herz-Morrey Spaces

Amjad Hussain,[1] Muhammad Asim,[1] Muhammad Aslam,[2] and Fahd Jarad [3,4]

[1]*Department of Mathematics, Quaid-I-Azam University, Islamabad 44000, Pakistan*
[2]*Department of Mathematics, College of Sciences, King Khalid University, Abha 61413, Saudi Arabia*
[3]*Department of Mathematics, Çankaya University, 06790 Etimesgut, Ankara, Turkey*
[4]*Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan*

Correspondence should be addressed to Fahd Jarad; fahd@cankaya.edu.tr

In the present paper, our aim is to establish the boundedness of commutators of the fractional Hardy operator and its adjoint operator on weighted Herz-Morrey spaces with variable exponents $M\dot{K}_{p,q(\cdot)}^{\alpha(\cdot),\lambda}(w)$.

1. Introduction

Hardy operators and related commutators play an indispensable role in the theory of partial differential equations [1, 2] and the characterization of function spaces [3–5]. Without going into much details, let us first define the fractional Hardy operators [3]

$$Hg(z) = \frac{1}{|z|^{n-\beta}} \int_{|t| \leq |z|} g(t)dt, \quad H^*g(z) = \int_{|t| > |z|} \frac{g(t)}{|t|^{n-\beta}} dt, z \in \mathbb{R}^n/\{0\} \tag{1}$$

and related commutators:

$$[b, H_\beta]g = bHg - H(bg), \quad [b, H_\beta^*]g = bH^*g - H^*(bg). \tag{2}$$

It is important to note that taking $\beta = 0$ in (1), we get multidimensional Hardy operator defined and studied in [6, 7]. Also, (1) reduces to the one dimensional Hardy operator [8] if we choose $\beta = 0$ and $n = 1$. Here, we cite some important literature with regards to the study of Hardy-type operators on different function spaces which include [9–15].

The new development of variable exponent commenced with the work of Kov'aˇcik and R'akosn'ık in [16], where a class of function spaces having variable exponent was defined, and basic properties of variable exponent Lebesgue space were explored. Recently, the theory of variable exponent analysis is modeled in terms of the boundedness of the Hardy Littlewood maximal operator M [17–21]:

$$Mg(z) = \sup_{B:\text{ball}, z \in B} \frac{1}{|B|} \int_{|B|} \int_B |g(t)|dt. \tag{3}$$

Besides, Muckenhoupt A_p theory [22] is generalized in the recent span of time with regard to variable exponent spaces ([23–28]). By taking into account the generalization of function spaces with variable exponents and the same with weights, many results like duality, boundedness of sublinear operators, the wavelet characterization, and commutators of fractional and singular integrals have been studied [29–38].

Recently, authors have studied generalized Herz space in terms of both Muckenhoupt weights and variable exponent [39–41]. Moreover, an idea of combining two function spaces to develop a new one is also an interesting problem in Harmonic analysis. One such problem is considered in [42] in which Herz-Morrey space was defined. Although,

the weighted versions of Herz-Morrey spaces were introduced recently in [43, 44].

In this piece of work, our main focus is on establishing the boundedness of commutators of fractional Hardy operators on a class of function spaces called the weighted Herz-Morrey space with variable exponents. We seek to find the boundedness of these commutators with symbol functions in BMO (bounded mean oscillation) spaces. In establishing such a boundedness, we make use of the boundedness of the fractional integral operator I_β

$$I_\beta(g)(z) = \int_{\mathbb{R}^n} \frac{g(t)}{|z-t|^{n-\beta}} dt \qquad (4)$$

on weighted Lebesgue space which was done in [39].

In the rest of this paper, the symbol C expresses a constant whose value may differ at all of its occurrences. The Greek letter χ_S denotes the characteristics function of a sphere S where S is a measurable subset of R^n and $|S|$ represents its Lebesgue measure. Before turning to our key results, let us first define the relevant variable exponent function spaces.

2. Preliminaries

Let us consider a measurable function $p(\cdot)$ on $\mathbb{R}^n$ having range $[1, \infty)$. The Lebesgue space with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ is the set of all measurable function f such that

$$L^{p(\cdot)}(\mathbb{R}^n) = \left\{ f : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\sigma} \right)^{p(x)} dx < \infty, \text{for some } \sigma > 0 \right\}. \qquad (5)$$

The space $L^{p(\cdot)}(\mathbb{R}^n)$ turns out to be Banach function space under the norm:

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf \left\{ \sigma > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\sigma} \right)^{p(x)} dx \le 1 \right\}. \qquad (6)$$

We denote by $\mathscr{P}(\mathbb{R}^n)$ the set of all measurable functions $p(\cdot): \mathbb{R}^n \longrightarrow (1, \infty)$ such that

$$1 < p_- \le p(x) \le p_+ < \infty, \qquad (7)$$

where

$$p_- := \text{essinf}_{x \in \mathbb{R}^n} p(x), p_+ := \text{esssup}_{x \in \mathbb{R}^n} p(x). \qquad (8)$$

Definition 1. Suppose $p(\cdot)$ is a real valued function on $\mathbb{R}^n$. We say that

(i) $C_{loc}^{\log}(\mathbb{R}^n)$ is the set of all local log-Holder continuous functions $p(\cdot)$ satisfying

$$|p(x) - p(y)| \lesssim \frac{-C}{\log(|x-y|)}, \quad |x-y| < \frac{1}{2}, \quad x, y \in \mathbb{R}^n. \qquad (9)$$

(ii) $\mathscr{C}_0^{\log}(\mathbb{R}^n)$ is the set of all local log-Holder continuous function $p(\cdot)$ satisfying at the origin

$$|p(x) - p(0)| \lesssim \frac{C}{\log(|e + (1/|x|)|)}, \quad |x-y| < \frac{1}{2}, \quad x \in \mathbb{R}^n. \qquad (10)$$

(iii) $\mathscr{C}_\infty^{\log}(\mathbb{R}^n)$ is the set of all log-Holder continuous functions satisfying at infinity

$$|p(x) - p_\infty| \le \frac{C_\infty}{\log(e + |x|)}, \quad x \in \mathbb{R}^n. \qquad (11)$$

(iv) $\mathscr{C}^{\log}(\mathbb{R}^n) = \mathscr{C}_\infty^{\log} \cap \mathscr{C}_{loc}^{\log}$ denotes the set of all global log-Holder continuous functions $p(\cdot)$.

It was proved in [21] that if $p(\cdot) \in \mathscr{P}(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$, then Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Suppose $w(x)$ is a weight function on $\mathbb{R}^n$, which is non-negative and locally integrable on $\mathbb{R}^n$. Let $L^{p(\cdot)}(w)$ be the space of all complex-valued functions f on $\mathbb{R}^n$ such that $fw^{1/p(\cdot)} \in L^{p(\cdot)}(\mathbb{R}^n)$. The space $L^{p(\cdot)}(w)$ is a Banach function space equipped with the norm:

$$\|f\|_{L^{p(\cdot)}(w)} = \left\| fw^{\frac{1}{p(\cdot)}} \right\|_{L^{p(\cdot)}}. \qquad (12)$$

Benjamin Muckenhoupt introduced the theory of $A_p (1 < p < \infty)$ weights on $\mathbb{R}^n$ in [22]. Recently, in [39, 40], Izuki and Noi generalized the Muckenhoupt A_p class by taking p as a variable.

Definition 2. Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$. A weight w is an $A_{p(\cdot)}$ weight if

$$\sup_B \frac{1}{|B|} \left\| w^{1/p(\cdot)} \chi_B \right\|_{L^{p(\cdot)}} \left\| w^{-1/p(\cdot)} \chi_B \right\|_{L^{p'(\cdot)}} < \infty. \qquad (13)$$

In [25], the authors proved that $w \in A_{p(\cdot)}$ if and only if M is bounded on the space $L^{p(\cdot)}$.

Remark 3 (see [39]). Suppose $p(\cdot), q(\cdot) \in \mathscr{P}(\mathbb{R}^n) \cap \mathscr{C}^{\log}(\mathbb{R}^n)$ and $p(\cdot) \le q(\cdot)$, then we have

$$A_1 \subset A_{p(\cdot)} \subset A_{q(\cdot)}. \qquad (14)$$

Definition 4. Suppose $p_1(\cdot), p_2(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and $\beta \in (0, n)$ such that $1/p_2(x) = 1/p_1(x) - \beta/n$. A weight w is said to be $A(p_1(\cdot), p_2(\cdot))$ weight if

$$\|\chi_B\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \|\chi_B\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}' \le C|B|^{1-\frac{\beta}{n}}. \qquad (15)$$

Definition 5 (see [39]). Suppose $p_1(\cdot), p_2(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and $\beta \in (0, n)$ such that $1/p_2(x) = 1/p_1(x) - \beta/n$. Then, $w \in A_{(p_1(\cdot), p_2(\cdot))}$ if and only if $w^{p_2(\cdot)} \in A_{1+p_2(\cdot)/p_1'(\cdot)}$.

Now, we define the variable exponent weighted Morrey-Herz space $M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(w)$. Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$, $A_k = B_k/B_{k-1}$, and $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$.

Definition 6. Let w be a weight on $\mathbb{R}^n$, $\lambda \in [0, \infty)$, $q \in (0, \infty)$, $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and $\alpha(\cdot): \mathbb{R}^n \longrightarrow \mathbb{R}$ with $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. The space $M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(w)$ is the set of all measurable functions which is given by

$$M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(w) = \left\{ f \in L_{loc}^{p(\cdot)}(\mathbb{R}^n/\{0\}, w): \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(w)} < \infty \right\}, \tag{16}$$

where

$$\|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(w)} = \sup_{k_0 \in Z} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(\cdot)q} \|f\chi_k\|_{L^{p(\cdot)}(w)}^q \right)^{1/q}. \tag{17}$$

Obviously, $M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),0}(w) = \dot{K}_{q,p(\cdot)}^{\alpha(\cdot)}(w)$ is the weighted Herz space with variable exponent (see [30]). Here, it is important to refer to some of the pioneering studies of the Herz space with constant exponents made in [45, 46].

3. Some Useful Lemmas

We start this section with some useful lemmas that will be helpful in proving our main results.

Lemma 7 (see [47]). *If X is Banach function space, then*

(i) The associated space X' is also Banach function space

(ii) $\|\cdot\|_{(X')'}$ and $\|\cdot\|_X$ are equivalent

(iii) If $g \in X$ and $f \in X'$, then

$$\int_{\mathbb{R}^n} |f(x)g(x)| \le \|g\|_X \|f\|_{X'} \tag{18}$$

is the generalized Hölder inequality.

Lemma 8 (see [39]). *Suppose X is a Banach function space. Then, we have that for all balls B,*

$$1 \le \frac{1}{|B|} \|\chi_B\|_X \|\chi_B\|_{X'}. \tag{19}$$

Lemma 9 (see [28, 39]). *Let X be a Banach function space. Suppose that the Hardy Littlewood maximal operator M is weakly bounded on X; that is,*

$$\left\| \chi_{\{M_f > \sigma\}} \right\|_X \lesssim \sigma^{-1} \|f\|_X \tag{20}$$

is true for $\sigma > 0$ and for all $f \in X$. Then, we have

$$\sup_{B:ball} \frac{1}{|B|} \|\chi_B\|_X \|\chi_B\|_{X'} < \infty. \tag{21}$$

Lemma 10 (see [39, 48]).

(1) $X(\mathbb{R}^n, W)$ is Banach function space equipped with the norm

$$\|f\|_{X(\mathbb{R}^n, W)} = \|fw\|_X, \tag{22}$$

where

$$X(\mathbb{R}^n, W) = \{f \in M : fW \in X :\}. \tag{23}$$

(2) The associate space $X'(\mathbb{R}^n, W^{-1})$ is also a Banach function space

Lemma 11 (see [39]). *Let X be a Banach function space. Assume that M is bounded on X', then there exists a constant $\delta \in (0, 1)$ for all $B \subset \mathbb{R}^n$ and $E \subset B$,*

$$\frac{\|\chi_E\|_X}{\|\chi_B\|_X} \lesssim \left(\frac{|E|}{|B|} \right)^\delta. \tag{24}$$

The paper [16] shows that $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach function space and the associated space $L^{p'(\cdot)}(R^n)$ with equivalent norm. *Remark 12.* Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, and by comparing the Lebesgue space $L^{p(\cdot)}(w^{p(\cdot)})$ and $L^{p'(\cdot)}(w^{-p'(\cdot)})$ with the definition of $X(\mathbb{R}^n, W)$, we have

(1) If we take $W = w$ and $X = L^{p(\cdot)}(\mathbb{R}^n)$, then we get $L^{p(\cdot)}(\mathbb{R}^n, w) = L^{p(\cdot)}(w^{p(\cdot)})$

(2) If we consider $W = w^{-1}$ and $X = L^{p'(\cdot)}(\mathbb{R}^n)$, then we have $L^{p'(\cdot)}(w^{-p'(\cdot)}) = L^{p'(\cdot)}(\mathbb{R}^n, w^{-1})$

By virtue of Lemma 10, we get $(L^{p(\cdot)}(\mathbb{R}^n, w))' = (L^{p(\cdot)}(w^{p(\cdot)}))' = L^{p'(\cdot)}(w^{-p'(\cdot)}) = L^{p'(\cdot)}(\mathbb{R}^n.w^{-1})$. Next, in view of Lemma 11 and Remark 12, we have the following Lemma.

Lemma 13 (see [41]). *Let $p(\cdot) \in \mathscr{P}(R^n) \cap \mathscr{C} \log(\mathbb{R}^n)$ be a Log Hölder continuous function both at infinity and at origin, if $w^{p2}(\cdot) \in A_{p2(\cdot)}$ implies $w^{-p_2'} \in A_{p_2'(\cdot)}$. Thus, the Hardy Littlewood operator is bounded on $L^{p_2'(\bullet)}(w^{p_2'(\bullet)})$, and there exist constants $\delta_1, \delta_2 \in (0, 1)$ such that*

$$\frac{\|\chi_E\|_{L^{p_2(\bullet)}(w^{p_2(\cdot)})}}{\|\chi_B\|_{L^{p_2(\bullet)}(w^{p_2(\cdot)})}} = \frac{\|\chi_E\|_{\left(L^{p_2'(\bullet)}w^{-p_2'(\bullet)}\right)'}}{\|\chi_B\|_{\left(L^{p_2'(\bullet)}w^{-p_2'(\bullet)}\right)'}} \lesssim \left(\frac{|E|}{|B|} \right)^{\delta_1},$$

$$\frac{\|\chi_E\|_{\left(L^{p_2(\bullet)}w^{p_2(\cdot)}\right)'}}{\|\chi_B\|_{\left(L^{p_2(\bullet)}w^{p_2(\cdot)}\right)'}} \lesssim \left(\frac{|E|}{|B|} \right)^{\delta_1}, \tag{25}$$

for all balls B and all measurable sets $E \subset B$.

Lemma 14 (see [39]). *Let $p_1(\cdot) \in \mathscr{P}(\mathbb{R}^n) \cap \mathscr{C}^{\log}(R^n)$ and $0 < \beta < n/p_{1+}$ and $1/p_2(\cdot) = 1/p_1(\bullet) - \beta/n$. If $\in A(p_1(\cdot), p_2(\cdot))$, then I^β is bounded from $L^{p_1(\cdot)}(w^{P_1(\cdot)})$ to $L^{p_2(\cdot)}(w^{P_2(\cdot)})$.*

4. Main Results and their Proofs

Definition 15. Let $f \in L^1_{loc}(Rn)$ and set

$$\|b\|_{BMO} = \sup_B \int_B |b(x) - b_B|dx, \tag{26}$$

where the supremum is taken all over the balls $B \in \mathbb{R}^n$ and $b_B = |B|^{-1} \int_B b(y)dy$. The function b is a bounded mean oscillation if $\|b\|_{BMO} < \infty$ and $BMO(\mathbb{R}^n)$ consist of all $f \in L^1_{loc}(\mathbb{R}^n)$ with $BMO(\mathbb{R}^n) < \infty$. For a comprehensive review of the BMO space, we suggest the reader to follow the books [49, 50].

Lemma 16. *Let $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and w be an $A_{q(\cdot)}$ weight. Then, for all $b \in BMO$ and all $l, i \in \mathbb{Z}$ with $l > i$, we have*

$$\|b\|_{BMO} \sim \sup_{B:Ball} \frac{1}{\|\chi_B\|_{L^{q(\cdot)}(w^{q(\cdot)})}} \|(b - b_B)\chi_B\|_{L^{q(\cdot)}(w^{q(\cdot)})}, \tag{27}$$

$$\left\|(b - b_{B_i})\chi_{B_i}\right\|_{L^{q(\cdot)}(w^{q(\cdot)})} \le C(l - i)\|b\|_{BMO}\|\chi_{B_i}\|_{L^{q(\cdot)}(w^{q(\cdot)})}. \tag{28}$$

Proof. First part of this lemma is a consequence of [[41], Theorem 18]. Next, we will prove (28), for all $l, i \in \mathbb{Z}$ with $l > i$

$$\left\|(b - b_{B_i})\chi_{B_l}\right\|_{L^{q(\cdot)}(w^{q(\cdot)})}$$
$$\le C\left\|(|b - b_{B_l}| + |b - b_{B_l}|)\chi_{B_l}\right\|_{L^{q(\cdot)}(w^{q(\cdot)})}$$
$$\le C\left\{\left\|(b - b_{B_l})\chi_{B_l}\right\|_{L^{q(\cdot)}(w^{q(\cdot)})} + \left\|(b_{Bl} - b_{B_i})\chi_{B_l}\right\|_{L^{q(\cdot)}(w^{q(\cdot)})}\right\}. \tag{29}$$

In the view of (27), we have

$$\left\|(b - b_{B_i})\chi_{B_l}\right\|_{L^{q(\cdot)}(w^{q(\cdot)})} \le C\|b\|_{BMO}\|\chi_{B_l}\|_{L^{q(\cdot)}(w^{q(\cdot)})}. \tag{30}$$

Also, it is easy to see that

$$\begin{aligned}|b_{Bl} - b_{B_i}| &\le \sum_{n=1}^{l-1}|b_{n+1} - b_n| \\ &\le \sum_{n=1}^{l-1}\frac{1}{|B_n|}\int_{B_n}|b_{n+1} - b(x)|dx \\ &\le C\sum_{n=1}^{l-1}\frac{1}{|B_{n+1}|}\int_{B_n}|b_{n+1} - b(x)|dx \\ &= C(l-i)\|b\|_{BMO(\mathbb{R}^n)}.\end{aligned} \tag{31}$$

Combining (29), (30), and (31), we get (28). □

Proposition 17. *Let $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, $0 < p < \infty$, and $0 \le \lambda < \infty$. If $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathscr{C}^{\log}(\mathbb{R}^n)$, then*

$$\|f\|^p_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(w^{q(\cdot)})} = \sup_{k_0 \in Z} 2^{-k_0\lambda p}\sum_{j=-\infty}^{k_0}2^{j\alpha(\cdot)p}\left\|f\chi_j\right\|^p_{L^{q(\cdot)}(w^{q(\cdot)})}$$

$$\le \max\left\{\substack{\sup \\ k_0 \in Z \\ k_0 < 0}\left(\sum_{j=-\infty}^{k_0}2^{j\alpha(\cdot)p}\left\|f\chi_j\right\|^p_{L^{q(\cdot)}(w^{q(\cdot)})}\right), \substack{\sup \\ k_0 \in Z \\ k_0 < 0}\right.$$

$$\cdot\left(2^{-k_0\lambda p}\left(\sum_{j=-\infty}^{-1}2^{j\alpha(\cdot)p}\left\|f\chi_j\right\|^p_{L^{q(\cdot)}(w^{q(\cdot)})}\right)\right.$$

$$\left.\left.+ 2^{-k_0\lambda p}\left(\sum_{j=0}^{k_0}2^{j\alpha(\infty)p}\left\|f\chi_j\right\|^p_{L^{q(\cdot)}(w^{q(\cdot)})}\right)\right)\right\}. \tag{32}$$

Proof. The proof is similar to the proof of Proposition 17 in [44]. So, we omit the details. □

Theorem 18. *Let $0 < p_1 \le p_2 < \infty$, $q_2(\cdot) \in \mathscr{P}(\mathbb{R}^n) \cap \mathscr{C}^{\log}(\mathbb{R}^n)$, and $q_1(\cdot)$ be such that $1/q_1(\cdot) = 1/q_2(\bullet) - \beta/n$.*

Also, let $w^{q2}(\cdot) \in A_1$, $b \in BMO(\mathbb{R}^n)$, $\lambda > 0$, and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathscr{C}^{\log}(\mathbb{R}^n)$ be log Hölder continuous at the origin, with $\alpha(0) \le \alpha(\infty) < \lambda + n\delta_2 - \beta$, where $0 < \delta_2 < 1$, then

$$\left\|[b, H_\beta]f\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_2,q_2(\cdot)}(w^{q2(\cdot)})} \le C\|b\|_{BMO}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(w^{q2(\cdot)})}. \tag{33}$$

Proof. For any $f \in M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(w^{q_1(\cdot)})$, if we denote $f_l = f \cdot \chi_l = f \cdot \chi_{A_l}$, and for each $l \in \mathbb{Z}$,

$$f(x) = \sum_{1=-\infty}^{\infty}f(x)\cdot\chi_l(x) = \sum_{1=-\infty}^{\infty}f_l(x), \tag{34}$$

then it is not difficult to see that

$$\begin{aligned}\left|[b, H_\beta]f(x)\chi_j(x)\right| &\le \frac{1}{|x|^{n-\beta}}\int_{B_j}|(b(x) - b(y))f(y)|dy \cdot \chi_j(x) \\ &\le 2^{-j(n-\beta)}\sum_{l=-\infty}^{j}\int_{B_j}|(b(x) - b(y))f(y)|dy \cdot \chi_j(x) \\ &\le 2^{-j(n-\beta)}\sum_{l=-\infty}^{j}\int_{B_j}\left|(b(x) - b_{B_l})f(y)\right|dy \cdot \chi_j(x) \\ &\quad + 2^{-j(n-\beta)}\sum_{l=-\infty}^{j}\int_{B_j}\left|(b(x) - b_{B_l})f(y)\right|dy \cdot \chi_j(x) \\ &= E_1 + E_2.\end{aligned} \tag{35}$$

The generalized Hölder inequality (Lemma 7) yields the following inequality for E_1:

$$
\begin{aligned}
E_1 &= 2^{-j(n-\beta)} \sum_{l=-\infty}^{j} \int_{B_l} \left| (b(x) - b_{B_l}) f(y) \right| dy \cdot \chi_j(x) \\
&\leq 2^{-j(n-\beta)} \sum_{l=-\infty}^{j} \left| (b(x) - b_{B_l}) \right| \cdot \chi_j(x) \|f_l\|_{L^{q_1(\bullet)}(w^{q_1(\cdot)})} \\
&\quad \times \left\| \chi_{B_l} \right\|_{(L^{q_1(\bullet)}(w^{q_1(\cdot)}))'}.
\end{aligned}
\tag{36}
$$

Applying the norm on both sides and using Lemma 16, we get

$$
\begin{aligned}
\|E_1\|_{(L^{q_2(\bullet)}(w^{q_2(\cdot)}))} &\leq 2^{-j(n-\beta)} \sum_{l=-\infty}^{j} \left\| (b(x) - b_{B_l} \cdot \chi_{B_j}) \right\|_{(L^{q_2(\bullet)}(w^{q_2(\cdot)}))} \\
&\quad \cdot \|f_l\|_{L^{q_1(\bullet)}(w^{q_1(\cdot)})} \left\| \chi_{B_l} \right\|_{(L^{q_1(\bullet)}(w^{q_1(\cdot)}))'} \\
&\leq 2^{-j(n-\beta)} \sum_{l=-\infty}^{j} (j-l) \|b\|_{BMO} \left\| \chi_{B_l} \right\|_{(L^{q_2(\bullet)}(w^{q_2(\cdot)}))'} \\
&\quad \cdot \|f_l\|_{L^{q_1(\bullet)}(w^{q_1(\cdot)})} \left\| \chi_{B_l} \right\|_{(L^{q_1(\bullet)}(w^{q_1(\cdot)}))'}.
\end{aligned}
\tag{37}
$$

Now, we turn to estimate E_2. For this, we have

$$
\begin{aligned}
E_2 &\leq 2^{-j(n-\beta)} \sum_{l=-\infty}^{j} \left\| (b(y) - b_{B_l} \cdot \chi_l) \right\|_{(L^{q_1(\bullet)}(w^{q_1(\cdot)}))'} \\
&\quad \cdot \|f_l\|_{L^{q_1(\bullet)}(w^{q_1(\cdot)})} \cdot \chi_j(x) \\
&\leq 2^{-j(n-\beta)} \sum_{l=-\infty}^{j} \left\| (b(y) - b_{B_l} \cdot \chi_{B_l}) \right\|_{(L^{q_1(\bullet)}(w^{q_1(\cdot)}))'} \\
&\quad \cdot \|f_l\|_{L^{q_1(\bullet)}(w^{q_1(\cdot)})} \cdot \chi_j(x).
\end{aligned}
\tag{38}
$$

Similar to the estimation for E_1, we take the norm on both sides of above inequality and use Lemma 16 to obtain

$$
\begin{aligned}
\|E_2\|_{L^{q_2(\bullet)}(w^{q_2(\cdot)})} &\leq 2^{-j(n-\beta)} \sum_{l=-\infty}^{j} \left\| (b(y) - b_{B_l}) \cdot \chi_l \right\|_{(L^{q_1(\bullet)}(w^{q_1(\cdot)}))'} \\
&\quad \cdot \|f_l\|_{L^{q_1(\bullet)}(w^{q_1(\cdot)})} \cdot \left\| \chi_j \right\|_{L^{q_2(\bullet)}(w^{q_2(\cdot)})} \\
&\leq 2^{-j(n-\beta)} \sum_{l=-\infty}^{j} \|b\|_{BMO} \left\| \chi_{B_l} \right\|_{(L^{q_1(\bullet)}(w^{q_1(\cdot)}))'} \\
&\quad \cdot \|f_l\|_{L^{q_1(\bullet)}(w^{q_1(\cdot)})} \left\| \chi_{B_j} \right\|_{(L^{q_2(\bullet)}(w^{q_2(\cdot)}))}.
\end{aligned}
\tag{39}
$$

Hence, from inequalities (35), (37), and (39), one has $\left\| [b, H_\beta] f \chi_j \right\|_{L^{q_2(\bullet)}(w^{q_2(\cdot)})} \leq 2^{-j(n-\beta)} \|b\|_{BMO} \sum_{l=-\infty}^{j} (j-1)$

$\|f_l\|_{L^{q_1(\bullet)}(w^{q_1(\cdot)})} \left\| \chi_{B_j} \right\|_{(L^{q_2(\bullet)}(w^{q_2(\cdot)}))} \left\| \chi_{B_l} \right\|_{(L^{q_1(\bullet)}(w^{q_1(\cdot)}))'}$, which by virtue of Lemma 9 reduces to

$$
\begin{aligned}
\left\| [b, H_\beta] f \chi_j \right\|_{L^{q_2(\bullet)}(w^{q_2(\cdot)})} &\leq 2^{j\beta} \|b\|_{BMO} \sum_{l=-\infty}^{j} (j-l) \|f_l\|_{L^{q_1(\bullet)}(w^{q_1(\cdot)})} \\
&\quad \cdot \left\| \chi_{B_l} \right\|_{(L^{q_1(\bullet)}(w^{q_1(\cdot)}))'} \left\| \chi_{B_j} \right\|_{(L^{q_2(\bullet)}(w^{q_2(\cdot)}))'}^{-1}.
\end{aligned}
\tag{40}
$$

Now using Lemma 13, we learn

$$
\begin{aligned}
&\left\| [b, H_\beta] f \chi_j \right\|_{L^{q_2(\bullet)}(w^{q_2(\cdot)})} \\
&\leq 2^{j\beta} \|b\|_{BMO} \sum_{l=-\infty}^{j} (j-l) \|f_l\|_{L^{q_1(\bullet)}(w^{q_1(\cdot)})} \\
&\quad \times \frac{\left\| \chi_{B_l} \right\|_{(L^{q_1(\bullet)}(w^{q_1(\cdot)}))'}}{\left\| \chi_{B_l} \right\|_{(L^{q_2(\bullet)}(w^{q_2(\cdot)}))'}} \frac{\left\| \chi_{B_l} \right\|_{(L^{q_2(\bullet)}(w^{q_2(\cdot)}))'}}{\left\| \chi_{B_j} \right\|_{(L^{q_2(\bullet)}(w^{q_2(\cdot)}))'}} \\
&\leq 2^{j\beta} \|b\|_{BMO} \sum_{l=-\infty}^{j} (j-l) 2^{(l-j)n\delta_2} \|b\|_{BMO} \|f_l\|_{L^{q_1(\bullet)}(w^{q_1(\cdot)})} \\
&\quad \times \frac{\left\| \chi_{B_l} \right\|_{(L^{q_1(\bullet)}(w^{q_1(\cdot)}))'}}{\left\| \chi_{B_l} \right\|_{(L^{q_2(\bullet)}(w^{q_2(\cdot)}))'}}.
\end{aligned}
\tag{41}
$$

In the definition of the fraction integral I_β, we replace f by χ_{Bl} to obtain

$$
I_\beta (\chi_{B_l})(x) \geq C 2^{l\beta} \chi_{B_l}(x),
\tag{42}
$$

from which we infer that

$$
\chi_{B_l}(x) \leq C 2^{-l\beta} I_\beta (\chi_{B_l})(x).
\tag{43}
$$

Taking the norm on both sides and using Lemmas 14 and 9, respectively, we get

$$
\begin{aligned}
\left\| \chi_{B_l} \right\|_{L^{q_2(\bullet)}(w^{q_2(\cdot)})} &\leq C 2^{l\beta} \left\| I_\beta (\chi_{B_l}) \right\|_{L^{q_2(\bullet)}(w^{q_2(\cdot)})} \\
&\leq C 2^{l\beta} \left\| \chi_{B_l} \right\|_{L^{q_1(\bullet)}(w^{q_1(\cdot)})} \\
&\leq C 2^{l(n-\beta)} \left\| \chi_{B_l} \right\|_{(L^{q_1(\bullet)}(w^{q_1(\cdot)}))'}^{-1}.
\end{aligned}
\tag{44}
$$

In view of Lemmas 8 and 9, the use of (44) into (41) results in the following inequality:

$$
\begin{aligned}
&\left\| [b, H_\beta] f \chi_j \right\|_{L^{q_2(\bullet)}\left(w^{q_2(\cdot)}\right)} \\
&\leq C \|b\|_{BMO} \sum_{l=-\infty}^{j} 2^{l(n-\beta)} 2^{j\beta} (j-l) 2^{(l-j)n\delta_2} \|f_l\|_{L^{q_1(\bullet)}\left(w^{q_1(\cdot)}\right)} \\
&\quad \times \left(\|\chi_l\|_{\left(L^{q_2(\bullet)}\left(w^{q_2(\cdot)}\right)\right)} \|\chi_l\|_{\left(L^{q_2(\bullet)}\left(w^{q_2(\cdot)}\right)\right)'} \right)^{-1} \\
&\leq C \|b\|_{BMO} \sum_{l=-\infty}^{j} 2^{(j-l)(\beta-n\delta_2)} (j-l) \|f_l\|_{L^{q_1(\bullet)}\left(w^{q_1(\cdot)}\right)} \\
&\quad \times \left(2^{-\ln} \|\chi_l\|_{\left(L^{q_2(\bullet)}\left(w^{q_2(\cdot)}\right)\right)} \|\chi_l\|_{\left(L^{q_2(\bullet)}\left(w^{q_2(\cdot)}\right)\right)'} \right)^{-1} \\
&\leq C \|b\|_{BMO} \sum_{l=-\infty}^{j} 2^{(j-l)(\beta-n\delta_2)} (j-l) \|f_l\|_{L^{q_1(\bullet)}\left(w^{q_1(\cdot)}\right)}.
\end{aligned}
\tag{45}
$$

Now, by virtue of the condition $p_1 \leq p_2$ and Proposition 17, we have

$$
\begin{aligned}
&\left\| [b, H_\beta] f \chi_j \right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_2,q_2(\cdot)}\left(w^{q_2(\cdot)}\right)}^{p_1} \\
&\leq \max \left\{ \sup_{\substack{k_0 \in Z \\ k_0 < 0}} 2^{-k_0\lambda p_1} \left(\sum_{j=-\infty}^{k_0} 2^{j\alpha(0)p_1} \left\| [b, H_\beta] f \chi_j \right\|_{L^{q_2(\bullet)}\left(w^{q_2(\cdot)}\right)}^{p_1} \right), \sup_{\substack{k_0 \in Z \\ k_0 \geq 0}} \right. \\
&\quad \times \left(2^{-k_0\lambda p_1} \left(\sum_{j=-\infty}^{k_0} 2^{j\alpha(0)p_1} \left\| [b, H_\beta] f \chi_j \right\|_{L^{q_2(\bullet)}\left(w^{q_2(\cdot)}\right)}^{p_1} \right) \right. \\
&\quad \left. \left. + 2^{-k_0\lambda p_1} \left(\sum_{j=0}^{k_0} 2^{j\alpha(\infty)p_1} \left\| [b, H_\beta] f \chi_j \right\|_{L^{q_2(\bullet)}\left(w^{q_2(\cdot)}\right)}^{p_1} \right) \right) \right\} \\
&= \max \{X_1, X_2, X_3\},
\end{aligned}
\tag{46}
$$

where

$$
\begin{aligned}
X_1 &= \sup_{\substack{k_0 \in Z \\ k_0 < 0}} 2^{-k_0\lambda p_1} \left(\sum_{j=-\infty}^{k_0} 2^{j\alpha(0)p_1} \left\| [b, H_\beta] f \chi_j \right\|_{L^{q_2(\bullet)}\left(w^{q_2(\cdot)}\right)}^{p_1} \right), \\
X_2 &= \sup_{\substack{k_0 \in Z \\ k_0 \geq 0}} 2^{-k_0\lambda p_1} \left(\sum_{j=-\infty}^{-1} 2^{j\alpha(0)p_1} \left\| [b, H_\beta] f \chi_j \right\|_{L^{q_2(\bullet)}\left(w^{q_2(\cdot)}\right)}^{p_1} \right), \\
X_3 &= \sup_{\substack{k_0 \in Z \\ k_0 \geq 0}} 2^{-k_0\lambda p_1} \left(\sum_{j=0}^{k_0} 2^{j\alpha(0)p_1} \left\| [b, H_\beta] f \chi_j \right\|_{L^{q_2(\bullet)}\left(w^{q_2(\cdot)}\right)}^{p_1} \right).
\end{aligned}
\tag{47}
$$

To estimate X_1, X_2, and X_3, we make use of the conditions on $\alpha(\cdot)$, such that for $l < 0$, we have

$$
\begin{aligned}
\|f_l\|_{L^{q_1(\bullet)}\left(w^{q_1(\cdot)}\right)} &= 2^{-l\alpha(0)} \left(2^{j\alpha(0)p_1} \|f_l\|_{L^{q_1(\bullet)}\left(w^{q_1(\cdot)}\right)}^{p_1} \right)^{\frac{1}{p_1}} \\
&\leq 2^{-l\alpha(0)} \left(\sum_{1=-\infty}^{l} 2^{i\alpha(0)p_1} \|f_i\|_{L^{q_1(\bullet)}\left(w^{q_1(\cdot)}\right)}^{p_1} \right)^{\frac{1}{p_1}} \\
&\leq 2^{l(\lambda-\alpha(0))} 2^{-l\lambda} \left(\sum_{1=-\infty}^{l} 2^{i\alpha(\cdot)p_1} \|f_i\|_{L^{q_1(\bullet)}\left(w^{q_1(\cdot)}\right)}^{p_1} \right)^{\frac{1}{p_1}} \\
&\leq C 2^{l(\lambda-\alpha(0))} \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}\left(w^{q_1(\cdot)}\right)},
\end{aligned}
\tag{48}
$$

and for $l \geq 0$, we obtain

$$
\begin{aligned}
\|f_l\|_{L^{q_1(\bullet)}\left(w^{q_1(\cdot)}\right)} &= 2^{-l\alpha(\infty)} \left(2^{l\alpha(\infty)p_1} \|f_l\|_{L^{q_1(\bullet)}\left(w^{q_1(\cdot)}\right)}^{p_1} \right)^{\frac{1}{p_1}} \\
&\leq 2^{-l\alpha(\infty)} \left(\sum_{1=-\infty}^{l} 2^{i\alpha(\infty)p_1} \|f_i\|_{L^{q_1(\bullet)}\left(w^{q_1(\cdot)}\right)}^{p_1} \right)^{\frac{1}{p_1}} \\
&\leq 2^{l(\lambda-\alpha(\infty))} 2^{-l\lambda} \left(\sum_{1=-\infty}^{l} 2^{i\alpha(\cdot)p_1} \|f_i\|_{L^{q_1(\bullet)}\left(w^{q_1(\cdot)}\right)}^{p_1} \right)^{\frac{1}{p_1}} \\
&\leq C 2^{l(\lambda-\alpha(\infty))} \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}\left(w^{q_1(\cdot)}\right)}.
\end{aligned}
\tag{49}
$$

In order to estimate X_1, we need to use $\alpha(0) \leq \alpha(\infty) < n\delta_2 + \lambda - \beta$.

$$
\begin{aligned}
X_1 &\leq \sup_{\substack{k_0 \in Z \\ k_0 < 0}} 2^{-k_0\lambda p_1} \sum_{j=-\infty}^{k_0} 2^{j\alpha(0)p_1} \\
&\quad \cdot \left(\sum_{l=-\infty}^{j} (j-l) 2^{(j-l)(\beta-n\delta_2)} \|b\|_{BMO} \|f_l\|_{L^{q_1(\bullet)}\left(w^{q_1(\cdot)}\right)} \right)^{p_1} \\
&\leq C \sup_{\substack{k_0 \in Z \\ k_0 < 0}} 2^{-k_0\lambda p_1} \sum_{j=-\infty}^{k_0} 2^{j\alpha(0)p_1} \\
&\quad \cdot \left(\sum_{l=-\infty}^{j} (j-l) 2^{(j-l)(\beta-n\delta_2)} 2^{l(\lambda-\alpha(0))} \|b\|_{BMO} \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}\left(w^{q_1(\cdot)}\right)} \right)^{p_1} \\
&\leq C \sup_{\substack{k_0 \in Z \\ k_0 < 0}} 2^{-k_0\lambda p_1} \sum_{j=-\infty}^{k_0} 2^{j\alpha(0)p_1} \left(\sum_{l=-\infty}^{j} (j-l) 2^{(j-l)(\beta-n\delta_2)} 2^{l(\lambda-\alpha(0))} \right)^{p_1} \\
&\quad \cdot \|b\|_{BMO}^{p_1} \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}\left(w^{q_1(\cdot)}\right)}^{p_1}
\end{aligned}
$$

$$\leq C \sup_{\substack{k_0 \in Z \\ k_0 < 0}} 2^{-k_0 \lambda p_1} \sum_{j=-\infty}^{k_0} 2^{j\lambda p_1} \left(\sum_{l=-\infty}^{j} (j-l) 2^{(l-j)(-\beta + n\delta_2 - \alpha(0) + \lambda)} \right)^{p_1}$$

$$\cdot \|b\|_{BMO}^{p_1} \|f\|_{M\dot{K}_{p_1 q_1(\cdot)}^{\alpha(\cdot),\lambda}(w^{q_1(\cdot)})}^{p_1}$$

$$\leq C \|b\|_{BMO}^{p_1} \|f\|_{M\dot{K}_{p_1 q_1(\cdot)}^{\alpha(\cdot),\lambda}(w^{q_1(\cdot)})}^{p_1}.$$

$$(50)$$

The result of X_2 is similar to that of X_1. Next, we will estimate X_3 below

$$X_3 \leq \sup_{\substack{k_0 \in Z \\ k_0 < 0}} 2^{-k_0 \lambda p_1} \sum_{j=-\infty}^{k_0} 2^{j\alpha(\infty)p_1}$$

$$\cdot \left(\sum_{l=-\infty}^{j} (j-l) 2^{(j-l)(\beta - n\delta_2)} \|b\|_{BMO} \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \right)^{p_1}$$

$$\leq C \sup_{\substack{k_0 \in Z \\ k_0 < 0}} 2^{-k_0 \lambda p_1} \sum_{j=-\infty}^{k_0} 2^{j\alpha(\infty)p_1}$$

$$\cdot \left(\sum_{l=-\infty}^{j} (j-l) 2^{(j-l)(\beta - n\delta_2)} 2^{l(\lambda - \alpha(\infty))} \|b\|_{BMO} \|f\|_{M\dot{K}_{p_1 q_1(\cdot)}^{\alpha(\cdot),\lambda}(w^{q_2(\cdot)})} \right)^{p_1}$$

$$\leq C \sup_{\substack{k_0 \in Z \\ k_0 < 0}} 2^{-k_0 \lambda p_1} \sum_{j=-\infty}^{k_0} 2^{j\alpha(\infty)p_1} \left(\sum_{l=-\infty}^{j} (j-l) 2^{(j-l)(\beta - n\delta_2)} 2^{l(\lambda - \alpha(\infty))} \right)^{p_1}$$

$$\cdot \|b\|_{BMO}^{p_1} \|f\|_{M\dot{K}_{p_1 q_1(\cdot)}^{\alpha(\cdot),\lambda}(w^{q_2(\cdot)})}^{p_1}$$

$$\leq C \sup_{\substack{k_0 \in Z \\ k_0 < 0}} 2^{-k_0 \lambda p_1} \sum_{j=-\infty}^{k_0} 2^{j\lambda p_1} \left(\sum_{l=-\infty}^{j} (j-l) 2^{(l-j)(-\beta + n\delta_2 - \alpha(\infty) + \lambda)} \right)^{p_1}$$

$$\cdot \|b\|_{BMO}^{p_1} \|f\|_{M\dot{K}_{p_1 q_1(\cdot)}^{\alpha(\cdot),\lambda}(w^{q_2(\cdot)})}^{p_1}$$

$$\leq C \|b\|_{BMO}^{p_1} \|f\|_{M\dot{K}_{p_1 q_1(\cdot)}^{\alpha(\cdot),\lambda}(w^{q_2(\cdot)})}^{p_1}.$$

$$(51)$$

Finally, we combine the estimates for $X_i (i = 1, 2, 3)$, to have the desired result. □

Theorem 19. Let $p_1, p_2, q_1(\cdot), q_2(\cdot), \beta, \alpha(\cdot)$ and w be as in Theorem 18. In addition, if $\lambda - n\delta_1 < \alpha(0) \leq \alpha(\infty)$, where $1 < \delta_1 < 0$, then

$$\left\| \left[b, H_\beta^* \right] f \right\|_{M\dot{K}_{p_2 q_2(\cdot)}^{\alpha(\cdot),\lambda}(w^{q_2(\cdot)})} \leq C \|b\|_{BMO} \|f\|_{M\dot{K}_{p_1 q_1(\cdot)}^{\alpha(\cdot),\lambda}(w^{q_2(\cdot)})}.$$

$$(52)$$

Proof. We write

$$\left[b, H_\beta^* \right] f(x) \chi_j(x) \leq \int_{\mathbb{R}^n \setminus B_j} |y|^{\beta-n} |(b(x) - b(y)) f(y)| dy \cdot \chi_j(x)$$

$$\leq \sum_{l=j+1}^{\infty} \int_{B_l} |y|^{\beta-n} |(b(x) - b(y)) f(y)| dy \cdot \chi_j(x)$$

$$\leq \sum_{l=j+1}^{\infty} \int_{B_l} |y|^{\beta-n} |b(x) - b_{B_l}| f(y) dy \cdot \chi_j(x) +$$

$$\leq \sum_{l=j+1}^{\infty} \int_{B_l} |y|^{\beta-n} |(b(y) - b_{B_l}) f(y)| dy \cdot \chi_j(x)$$

$$= F_1 + F_2.$$

$$(53)$$

We estimate F_1 and F_2 separately. A use of generalized inequality results in the following:

$$F_1 \leq C \sum_{l=j+1}^{\infty} 2^{-l(n-\beta)} \int_{B_l} |(b(x) - b_{B_l}) f(y)| dy \cdot \chi_j(x)$$

$$\leq C \sum_{l=j+1}^{\infty} 2^{-l(n-\beta)} \|f_i\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \|\chi_{B_l}\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})'}$$

$$\cdot |b(x) - b_{B_l}| \cdot \chi_j.$$

$$(54)$$

Applying the weighted Lebesgue space norm on both sides and using Lemma 16, we obtain

$$\|F_1\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \leq C \sum_{l=j+1}^{\infty} 2^{-l(n-\beta)} \left\| \left(b(x) - b_{B_j} \right) \chi_j \right\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}$$

$$\cdot \|f_i\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})'} \|\chi_{B_l}\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})'}$$

$$\leq C \sum_{l=j+1}^{\infty} 2^{-l(n-\beta)} \|b\|_{BMO} \left\| \left(b(x) - b_{B_j} \right) \chi_j \right\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}$$

$$\cdot \|f_i\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \|\chi_{B_l}\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})'}.$$

$$(55)$$

Similarly,

$$F_2 \leq C \sum_{l=j+1}^{\infty} 2^{-l(n-\beta)} \int_{B_l} |(b(y) - b_{B_l}) f(y)| dy \cdot \chi_j(x)$$

$$\leq C \sum_{l=j+1}^{\infty} 2^{-l(n-\beta)} \|b(y) - b_{B_l} \cdot \chi_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}$$

$$\cdot \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})'} \cdot \chi_j(x)(x).$$

$$(56)$$

In view of the weighted Lebesgue norm and Lemma 16, we get

$$
\begin{aligned}
\left\| F_2 \right\|_{L^{q_2(\cdot)}\left(w^{q_2(\cdot)}\right)} &\leq C \sum_{l=j+1}^{\infty} 2^{-l(n-\beta)} \left\| (b(y) - b_{B_l}) \cdot \chi_j \right\|_{\left(L^{q_1(\cdot)}\left(w^{q_1(\cdot)}\right)\right)'} \\
&\quad \cdot \left\| f_l \right\|_{L^{q_1(\cdot)}\left(w^{q_1(\cdot)}\right)} \cdot \left\| \chi_j \right\|_{L^{q_2(\cdot)}\left(w^{q_2(\cdot)}\right)} \\
&\leq C \sum_{l=j+1}^{\infty} 2^{-l(n-\beta)}(l-j) \|b\|_{BMO} \|\chi_l\|_{\left(L^{q_1(\cdot)}\left(w^{q_1(\cdot)}\right)\right)'} \\
&\quad \cdot \left\| f_l \right\|_{L^{q_1(\cdot)}\left(w^{q_1(\cdot)}\right)} \cdot \left\| \chi_j \right\|_{L^{q_2(\cdot)}\left(w^{q_2(\cdot)}\right)}.
\end{aligned}
\tag{57}
$$

Hence, from (53), (55), and (57), we obtain

$$
\begin{aligned}
&\left\| \left[b, H_\beta^* \right] f \chi_j \right\|_{L^{q_2(\cdot)}\left(w^{q_2(\cdot)}\right)} \\
&\leq \sum_{l=j+1}^{\infty} 2^{-l(n-\beta)}(l-j) \|b\|_{BMO} \left\| \chi_j \right\|_{L^{q_2(\cdot)}\left(w^{q_2(\cdot)}\right)} \\
&\quad \times \|f_l\|_{L^{q_1(\cdot)}\left(w^{q_1(\cdot)}\right)} \cdot \left\| \chi_{B_l} \right\|_{\left(L^{q_1(\cdot)}\left(w^{q_1(\cdot)}\right)\right)'} \\
&\leq \sum_{l=j+1}^{\infty} 2^{-l(n-\beta)}(l-j) \|b\|_{BMO} \|f_l\|_{L^{q_1(\cdot)}\left(w^{q_1(\cdot)}\right)} \\
&\quad \times \left\| \chi_{B_l} \right\|_{\left(L^{q_1(\cdot)}\left(w^{q_1(\cdot)}\right)\right)'} \|\chi_l\|_{L^{q_2(\cdot)}\left(w^{q_2(\cdot)}\right)} \frac{\left\| \chi_j \right\|_{L^{q_2(\cdot)}\left(w^{q_2(\cdot)}\right)}}{\|\chi_l\|_{L^{q_2(\cdot)}\left(w^{q_2(\cdot)}\right)}} \\
&\leq \sum_{l=j+1}^{\infty} 2^{n\delta(j-1)} 2^{-l(n-\beta)}(l-j) \|b\|_{BMO} \|f_1\|_{L^{q_1(\cdot)}\left(w^{q_1(\cdot)}\right)} \\
&\quad \times \left\| \chi_{B_l} \right\|_{\left(L^{q_1(\cdot)}\left(w^{q_1(\cdot)}\right)\right)'} \|\chi_l\|_{L^{q_2(\cdot)}\left(w^{q_2(\cdot)}\right)}.
\end{aligned}
\tag{58}
$$

Using the condition of $A(q_1(\cdot), q_2(\cdot))$ weights given in the Definition 4, the above inequality reduces to

$$
\begin{aligned}
&\left\| \left[b, H_\beta^* \right] f \chi_j \right\|_{L^{q_2(\cdot)}\left(w^{q_2(\cdot)}\right)} \\
&\leq \sum_{l=j+1}^{\infty} 2^{n\delta_1(j-l)}(l-j) \|b\|_{BMO} \|f_l\|_{L^{q_1(\cdot)}\left(w^{q_1(\cdot)}\right)}.
\end{aligned}
\tag{59}
$$

Next, the condition $p_1 < p_2$ and Proposition 17 help us to write

$$
\left\| \left[b, H_\beta^* \right] f \chi_j \right\|_{M\dot{K}_{p_2, q_2(\cdot)}^{\alpha(\cdot), \lambda}\left(w^{q_2(\cdot)}\right)}^{p_1} = \max \left\{ Y_1, Y_2, Y_3 \right\},
\tag{60}
$$

where

$$
Y_1 = \sup_{\substack{k_0 \in Z \\ k_0 < 0}} 2^{-k_0 \lambda p_1} \left(\sum_{j=-\infty}^{k_0} 2^{j\alpha(0)p_1} \left\| \left[b, H_\beta^* \right] f \cdot \chi_j \right\|_{L^{q_2(\cdot)}\left(w^{q_2(\cdot)}\right)}^{p_1} \right),
$$

$$
Y_2 = \sup_{\substack{k_0 \in Z \\ k_0 \geq 0}} 2^{-k_0 \lambda p_1} \left(\sum_{j=-\infty}^{-1} 2^{j\alpha(0)p_1} \left\| \left[b, H_\beta^* \right] f \cdot \chi_j \right\|_{L^{q_2(\cdot)}\left(w^{q_2(\cdot)}\right)}^{p_1} \right),
$$

$$
Y_3 = \sup_{\substack{k_0 \in Z \\ k_0 \geq 0}} 2^{-k_0 \lambda p_1} \left(\sum_{j=0}^{k_0} 2^{j\alpha(\infty)p_1} \left\| \left[b, H_\beta^* \right] f \cdot \chi_j \right\|_{L^{q_2(\cdot)}\left(w^{q_2(\cdot)}\right)}^{p_1} \right).
$$

$$
\tag{61}
$$

Lastly, in view of the condition $-n\delta_1 + \lambda < \alpha(0) \leq \alpha(\infty)$, we estimate $Y_i, i = 1, 2, 3$, as we estimated $X_i, i = 1, 2, 3$, in Theorem 18. Hence, we finish the proof. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Acknowledgments

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