BASIN'S FRACTAL VIA COMPLEX NEWTON'S METHOD

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BASIN'S FRACTAL VIA COMPLEX NEWTON'S METHOD

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ABSTRACT

BASIN'S FRACTAL VIA COMPLEX NEWTON'S METHOD

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In this thesis, we show that when we applied the Newton’s method on the exponential function, $F(z) = P(z)e^{Q(z)}$, where $P(z)$ and $Q(z)$ are polynomials in the complex plane, the attraction basins of roots have finite area when $n \geq 3$. With the help of MATLAB we obtained nice fractals in order to prove the finite basins area when $n \geq 3$ and infinite basins area when $n \leq 2$.

Keywords: Fixed Point, Rational Function, Julia Set, Newton Method, Basins, Exponential Function, Fractal.

ÖZ

NEWTONUN KARMAŞIK METODUYLA HAVZALAR FRAKTALI

MURAD, Zainab
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Tez Yöneticisi: Yrd. Doç. Dr. Dumitru BALEANU

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Bu tezde, $P(z)$ ve $Q(z)$ karmaşık düzlemde polinom olduğu, $F(z)=P(z)e^{Q(z)}$ üstel işlevine Newton metodunu uyguladığımızda, $n\geq 3$ ise, köklerin atraksiyon havzası sonlu alana sahiptir. MATLAB yardımı ile, $n\geq 3$ olduğunda sonlu havza alanlarını kanıtlamak, ve $n\leq 2$ olduğunda sonsuz havza alanlarını kanıtlamak için düzgün fraktaller elde edilmiştir.

Anahtar Kelimeler: Sabit Nokta, Rasyonel Fonksiyon, Julia Set, Newton Metodu, Havzalar, Üstel İşlev, Fraktal.

DEDICATION

Dedicated to My Father, May His Soul Rest in Peace

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Finally, my thanks go to the members of my family for their help and encouragement, and to everyone who helped in one way or another in bringing out this work.

My God bestow health and happiness to all of them.

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CHAPTER 1

INTRODUCTION

The study of functions of a complex variable has many practical use in applied mathematics and in various branches of science and engineering. The impetus to study proper the complex numbers first arose in the 16th century when algebraic solutions for the roots of cubic and quadratic polynomials were discovered by Italian mathematicians. However, it was discovered that these formulas, sometimes implies the manipulation of square roots of negative numbers [1, 2, 3, 4, 5].

The first modern study of iteration was due to Ernst Schroder, a Gymnasium teacher in Germany who published two papers in Mathematische Annalen in 1870–71. Although his treatment is not very rigorous, he was the first to suggest the use of conjugation as a means to studying the dynamical behavior of an analytic function f near a fixed point z_0 [6, 7, 8-12].

Fractals represents a new field of mathematics and art. We recall that the fractal geometry is one of the great advances in mathematics [13]. The researchers have realized that the fractal geometry is an excellent tool for discovering some secrets from a large variety of systems and solving important problems in various branches of science and engineering [13,14, 15, 16,17].

Also almost all studies of fractals, are coming out from iterations of rational functions in the complex domain [18, 19]. Julia set of a rational function is defined as the set of all repelling periodic points and Fatou set is the opposite of Julia set. So, each repelling points belonging to Julia and all attracting fixed point of the rational function belonging to Fatou set [20, 21-30].

The Fatou Flower theorem gives an analytic description of the dynamics around a rationally indifferent fixed point. Thus, the degree of the exponent polynomial Q completely determines the number of petals at infinity [31, 32].

Newton's method is known and introduced in calculation the roots of functions when the analytical methods failed. This method is better, if the initial supposition is close to the real root, iterations will converge very fast to the root. The dynamics of Newton's method in the complex plane, provides exciting of fractals which depend on what kind of functions we used [33, 34, 35].

However, Newton's method for obtaining the solutions of the equations leads to some beautiful images when it is applied to complex functions, that which called a basin of attraction is defined to be the set of all points that converge to the same root. And the connected component of attraction basins which containing the root of the basin is called the immediate basin of attraction [36, 37].

In this thesis we gave a review of Haruta method [31]. The main results of this thesis focus on the areas of the attraction basins. The basin of attraction for a rationally indifferent fixed point is a parabolic basin. The basin is lied in the Fatou set and the parabolic point lied on the border of the basin and in the Julia set. Also we depended on the other authors results [38, 39, 40, 41, 42].

This thesis is organized as follows:

In the second chapter, we recall some fundamental mathematical concept, that will be used in presentation of the results, the complex dynamics and some basic definition of forward and backward orbits, fixed and periodic points. The theorem of Fatou Flower offers an explanation of the local dynamics about a rationally indifferent fixed point around ∞ [31].

In the third chapter, the main idea is based on Haruta's study [31]. We focus on the dynamics of Newton's method on specific complex exponential function, when $n \geq 3$ to prove the finite area of attraction basins.

In the fourth chapter, we simulate our methods with MATLAB, so by our fractal we proved finite basins area when $n \geq 3$ and infinite basins area when $n \leq 2$ by applied Newton's algorithm to complex exponential function $F(z) = P(z)e^{Q(z)}$.

In the fifth chapter we depict our conclusions.

Finally, the MATLAB code is shown in appendices A1.

CHAPTER 2

THE BASIC PRINCIPLES

2.1 Complex Variables

The real numbers have beautiful properties. But we are not able to take the square root of -1. Therefore, we are not able to find a root of the equation $x^2 + 1 = 0$. We know that there is a complex number i that is a root of the equation $x^2 + 1 = 0$, that is, $i^2 = -1$ [1, 2]. The complex numbers that includes *i*, can be an expression by the form $z = x + iy$, where *x* and *y* are real numbers so, $i = \sqrt{-1}$ is a complex number. We denoted *x by* $\operatorname{Re}(z)$ and *y* by $\operatorname{Im}(z)$. The modulus (or absolute value) of z is $|z| = x^2 + y^2$, is a real number which measures distance. For the purposes of describing complex numbers, it is referred to as the complex plane, or the $z-$plane [2]. Using the fact that each point in the plane has an associated vector from the origin to that point, we can establish polar coordinates, r and θ for $z = x + iy$. We have $r = \sqrt{x^2 + y^2}$, where $x = r\cos\theta$ and $y = r\sin\theta$ [3].

Hence, the complex number $z = x + iy$ can be written in the polar form as $z = re^{i\theta}$, and using the Euler's equation we obtain $z = (r\cos\theta + i\sin\theta)$ [2]. A complex-valued function $f(z) = f(x + iy)$ assigns to each z in the domain exactly one complex number $\omega = f(z)$. Just as z decomposes into real and imaginary parts, each complex-valued function can be written by $f(z) = u(x, y) + iv(x, y)$, where u and v are each real valued functions [4]. In essence, $f(z)$ is a pair of real functions of two real variables that maps regions from its domain in the complex plane onto its range in another copy of the complex plane. We call these the z-plane and w-plane respectively. So, the Riemann sphere is a model of the extended complex plane, the complex plane plus a point at infinity. We will refer to Riemann sphere as the symbol $\mathbb{C}_\infty$ [4].

2.2 Basic Definitions

Here we present some important information about the complex sets.

Definition 1 [5]**:** The complex number is a *zero* (or root) of the function $f(z)$ if it is a solution to the equation $f(z) = 0$.

Definition 2 [5]**:** Let $z_0 \in \mathbb{C}$, $r > 0$ the set consisting of all points z satisfying $|z - z_0| < r$ is called the open disc of radius r central at z_0, which denoted by $D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$.

Definition 3 [5]**:** The complement of open disc is the closed disc, we can defined it by $\overline{D}_r(z_0) = \{ z \in \mathbb{C} : | z - z_0 | \leq r \}$.

Definition 4 [5]**:** The boundary of open or close disc is the circle that form $C_r = \{ z \in \mathbb{C} : | z - z_0 | = r \}$.

Definition 5 [5]**:** Let set $\Omega \subset \mathbb{C}$, the point z is said to be limit point of Ω if there exist $z_n \in \Omega$ such that $z \neq z$ and $\lim_{n \to \infty} z_n = z$.

Definition 6 [5]**:** The closures of any set Ω is the union of Ω and its limit points and is often denoted by $\overline{\Omega}$.

Definition 7 [5]**:** The boundary of a set Ω is equal to its closure minus its interior and is often denoted by $\partial\Omega$.

Definition 8 [5]**:** A set Ω is said to be bounded if there exists $M > 0$ such that $|z| < M$, whenever $z \in \Omega$.

Definition 9 [5]**:** A set Ω is said to be compact if it is closed and bounded.

Definition 10 [5]**:** An open set $\Omega \subset \mathbb{C}$ is said to be connected if it is not possible to fined two disjoint non-empty open sets Ω_1 and Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$.

2.3 Complex Derivatives and Analyticity

Suppose that $f : \Omega \to \mathbb{C}$ is a complex valued function, where Ω is an open subset of $\mathbb{C}$ and $z_0 \in \Omega$. Then, the complex derivative of f at z_0 is

$$\frac{df}{dz}(z_0) = f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{2.1}$$

if this limit exists. We say that f is differentiable at z_0 if it has a complex derivative at z_0. When f is differentiable at all points of Ω we say that f is holomorphic in Ω. A complex-valued function $f(z)$ is said to be analytic on an open set Ω if it has a derivative at every point of Ω. If $f(z)$ is analytic on $\mathbb{C}$, then is said to be entire [3].

2.4 Complex Dynamics

In complex dynamics the aim is to know what occurs when an analytic functions repeated in the complex plan or the Riemann sphere [6]. In dynamics the process that is repeated is the application of a function. Also, to iterate a function means to evaluate the function over and over again, using the output of the previous application as the input for the next. The list of iterates around a point is denoted by the orbit of that point [7, 8]. If we have $g(z)$ is any complex function therefore, we could describe the orbit of z by $g^n(z)$ is formed with repeating a function beginning at this point to grow a list of numbers [9, 10]. For example the forward orbit of z of $g^3(z)$ is the set of points in the sequence

$$z, g(z), g(g(z)), g(g(g(z))) \ldots\ldots.$$

Also, the backward orbit of z is the set of points in the sequence

$$z, g^{-1}(z), g^{-1}\left(g^{-1}(z)\right), g^{-1}\left(g^{-1}\left(g^{-1}(z)\right)\right).$$

The point z is a fixed point of the function $g(z)$ if $g(z) = z$ [11].

We can classify the fixed points according to λ, such that $\lambda = g'(z)$ as

1. if $|\lambda| = 0$, z is super attracting,
2. if $|\lambda| < 1$, z is attracting,
3. if $|\lambda| > 1$, z is repelling,
4. if $|\lambda| = 1$, z is neutral.

In the same way the point z is a periodic point for $g(z)$ if $g^n(z) = z$ [11].

Similarly, we can classify a periodic points according to λ, such that $\lambda = g'^n(z)$ as

1- if $|\lambda| = 0$, z is super attracting,

2- if $|\lambda| < 1$, z is attracting,

3- if $|\lambda| > 1$, z is repelling,

4- if $|\lambda| = 1$, z is neutral.

2.5 Complex Polynomial

A complex polynomial is a mathematical expression involving a sum of powers in complex variable multiplied by the complex coefficients. A complex polynomial with constant complex coefficients is given by

$$P(z) = \sum_{m=0}^{k} a_m z^m, \tag{2.2}$$

where the a_m are complex numbers not all zero and z is a complex variable. In particular, a polynomial of degree zero is, by definition, a non-zero constant [12].

The function which is identically zero is often regarded as being a polynomial of degree –infinity. A complex polynomial of degree n has at most n zeros [12].

2.6 A Fractal

The fractal coming from the Latin "fractus" meaning "broken". They are a never-ending pattern. We recall that the fractals represent infinitely complex patterns that are self-similar across different scales. Thus, they are created by repetition of a simple process $z_{n+1} = z_n^2 + c$ over and over. Fractals are images of dynamic systems [13]. Fractal geometry was mainly developed by Benoit Mandelbrot [14] during the sixties and seventies in order to recognize that many phenomena of nature are so irregular and complex therefore, they cannot be described properly by the Euclidean geometry. Many scientists have found that the fractal geometry is a powerful tool for uncovering secrets from a wide variety of systems and solving important problems in applied sciences. Fractal geometry is the better languages to define the nonlinear problems and have fixed over and over notice. Generally the studies of fractals, are coming out from iterations of rational functions in the complex domain [15, 16, 17].

2.7 Rational Functions Iterations

Let T and S are complex polynomials, then the rational map is defined by

$$R(z) = \frac{T(z)}{S(z)}. \tag{2.2}$$

$R(z)$ is a rational map of a degree d bigger than or equal 2 on the Riemann sphere, the n^{th} iterates of R are defined by R^n which is $R^n = RoRoR....oR$ [18].

The important problem in the dynamics of rational maps is to know the behavior of high iterates $R^n(z) = RoR^{n-1}(z)$ [18].

A point z is a fixed point of $R(z)$ if $R(z) = z$. The derivative $R'(z)$ is defined by the number $\lambda = R'(z)$ and λ is called the multiplier of R at z [19]. We able to classify the fixed point of the rational map according to λ, as follows

1. z is attracting, if $|\lambda| < 1$,
2. z is repelling, if $|\lambda| > 1$,
3. z is neutral, if $|\lambda| = 1$,
4. z is a super attracting fixed point, if $|\lambda| = 0$.

The orbit $\{z_0, z_1, ..., z_n = z_0\}$ is called a cycle. We classify the cycle as (super) attracting, repelling and rationally indifferent or irrationally indifferent according to the type of the fixed point of R^n [18]. For example the cycle is attracting if and only if $\left|R'^n(z)\right| < 1$ [19]. For each rational function $R(z)$ it able to conjugate $R(z)$ with the conversion $z \to 1/z$. Therefore, when $z = \infty$ the behavior of $R(z)$ is the same behavior of $1/S(1/z)$ at 0 [20]. The two polynomials $T(z)$ and $S(z)$ allow us to find the poles and zeros of the rational function, zeroes is the values for z, where $T(z) = 0$ and the pole is the value of z, where $S(z) = 0$. The critical points of a rational function $R(z)$ are those where $R'(z)$ vanishes [21].

Theorem [18]**:** If $d > 1$, a rational function of degree d has precisely $d + 1$ fixed points in $\mathbb{C}_\infty$ counted with multiplicity .

Corollary [25]**:** A rational map of positive degree d has at most $2d - d$ critical points. A polynomial of positive degree d has at most $d - 1$ finite critical points.

Definition 11 [24]**:** A family of complex analytic functions $\{F_n\}$, that is defined on a domain D is called a normal family if every infinite sequence of maps from $\{F_n\}$ contains a subsequence which converges uniformly on every compact subset of D.

Theorem [23]**:** Let R be a rational map of degree at least two. Then, the immediate basin of each super attracting cycle of R contains a critical point of R.

2.8 Julia and Fatou Sets

We recall that there is a Julia set J for every point in the complex plane. Julia set is the set of points z for which the orbit of z under iteration of $z_{n+1} = z_n^2 + c$, remains bounded in the complex plane. Each Julia set has a complex parameter c [20]. Therefore, the enumerating the first few iterations of Julia set by

$$z_0 = z,$$

$$z_1 = z_0^2 + c = z^2 + c,$$

$$z_2 = z_1^2 + c = (z^2 + c)^2 + c.$$

The Julia set is the boundary of the associated filled Julia set. However, Julia set of a rational function is defined as the set of all repelling periodic points and Fatou set is the opposite of Julia set. So, all attracting fixed point of the rational function belonging to Fatou set and each repelling points belonging to Julia set [20].

Definition 12 [20]**:** The Julia set of rational function $R(z)$ is the clouser of the repelling periodic points of $R(z)$ denoted by J_R.

Definition 13 [24]**:** The Fatou set F_R of rational functions denoted to be the set of points $z_0 \in \mathbb{C}_\infty$ so that $\{F^n\}$ is the normal family for about neighborhood of z_0.

Definition 14 [26]**:** If the rational function is a map of a set X into itself, a subset E of X will be:

1. Forward invariant if $R(E) \subset E$,
2. Backward invariant if $R^{-1}(E) \subset E$,
3. Completely invariant if $R(E) = E = R^{-1}(E)$.

Remark [24, 28]**:** It is clear that F_R, is open and completely invariant under f, and J_R is closed and also completely invariant. Julia set of rational function $R(z)$ is not important a bounded set but it certainly is enclosed.

Remark [22]**:** Here we introduce some properties of the Julia set J_R of the rational function R with degree $d \geq 1$.

1. The Julia set is nonempty.
2. The Julia set is completely invariant under R, that is z belongs to J_R, if and only if $R(z)$ belongs to J_R.
3. The Julia set J_R contains no isolated points, that is, J_R is a perfect set.
4. If the Julia set J_R contains an interior point, then J_R must be equal to the entire Riemann sphere.
5. The Julia set J_R is the closure of the repelling periodic points.
6. Every parabolic periodic point belongs to the Julia set.

Theorem [26]**:** The Julia set and the Fatou set are completely invariant.

Theorem [26]**:** The Julia set J_R of a rational function R is nonempty.

Corollary [29]**:** If a rational map has only one fixed point which is repelling or Parabolic with multiplier 1, then its Julia set is connected. In other words, every component of the complement of the Julia set is simply connected. In particular, the Julia set of the Newton's method for a non-constant polynomial is connected.

Corollary [29]**:** If the Julia set of a rational map R is disconnected, then there exist two fixed points of R such that each of them is either repelling or parabolic with multiplier 1, and they belong to different components of the Julia set.

Theorem [18]**:** Let R be a rational map. Then J_R is connected if and only if each component of is F_R simply connected.

Corollary [29]**:** The Julia set of the Newton method of a transcendental entire function is connected.

Remark [19]**:** An analytic complex map continuously divides the plane into tow split subsets, first, is the stable set is called the Fatou set which denoted by F, and the second one is the Julia set which defined by *J*, in which the map is messy. The borders of these sets will create nice figures.

2.9 An Attracting Petal and Repelling Petals

Suppose M is defined in a neighborhood U of the origin. [p] is called an attracting petal for M at fixed point if $M(\overline{p}) \subset [\overline{p}] \cup \{0\}$ and $\bigcap_{n\geq 0} M^n(\overline{p}) = \{0\}$. A repelling petal [p] is an attracting petal for M^{-1} which exists locally since $M^{-1} = 1$. M^{-1}denotes the branch of the inverse of M fixing the origin [30,31].

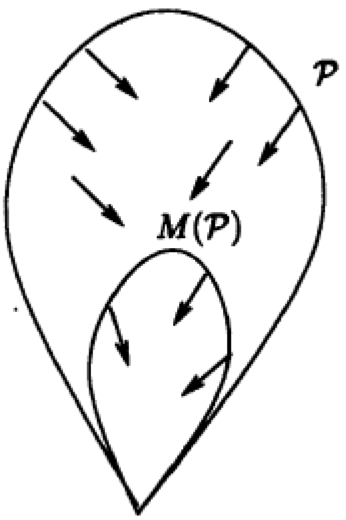

Figure 1 Attracting Petal [41]

Also, if z_0 is a parabolic fixed point of M which multiplier λ=1 and [p] is an attracting petals at 0, we define the parabolic basin (of attraction) of associated to [p] as below

$$A_p = \{z \in C \,|\, f^n(z) \to z, n \to \infty, through\ p\}.$$

2.10 The Leu-Fatuo Flower Theorem

If M is a holomorphic map of the form

$$M(z) = z + Cz^{n+1} + O(z^{n+2}), c \neq 0, n \geq 1. \tag{2.4}$$

Defined in some neighborhood of the origin with $\lambda = 1$. Then zero is a parabolic fixed point and there are n attracting petals and n repelling petals for M at zero. Moreover, these petals alternate with one another [31].

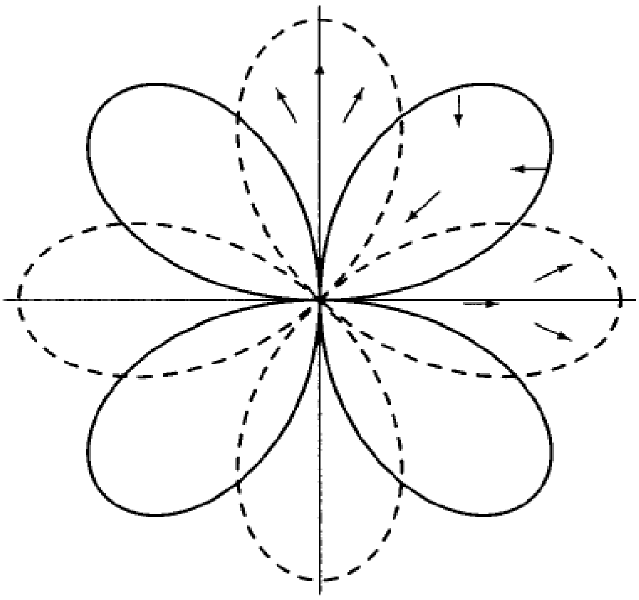

Figure 2 Attracting and Repelling Petal [31]

Remark [31]**:** For simplicity of Leu-Fatuo Flowers theorem [31]. It will assume that (c =1) when λ=1, then there exist respectively attracting and repelling direction rays along which orbits tend to zero and ∞.

Definition 15 [21]: The attraction basin for an attracting fixed point z_0 of some rational function of degree larger than one is defined by

$$A(z_0) = \{z \in C | R^n(z) \to z_0, n \to \infty\} .$$

Definition 16 [21]: Let z_0 is to be the attracting fixed point, then element of $A(z_0)$ containing z_0 is called the immediate basin of attraction of z_0 which defined by $A^*(z_0)$.

2.11 Image and pre-image

Let X and Y be two sets and T a function from x to y[32].Then the image of T is defined as image(T) = {b ∈ Y : there is an a ∈ X with T(a) = b}.

So, let X and Y be two sets and T a function from X to Y. If C is a subset of the range Y then the pre-image, or inverse image, of Z under the function T is the set defined by

$$T^{-1}(C) = \{z \in X: T(x) \in Z\}.$$

Corollary [30]**:** The fixed point zero is the only attracting orbit completely contained in the closure of the union of the attracting petals.

Corollary [30]**:** Accepting pre-images of zero, the orbit of z_0 converges to zero if and only if an image of z_0 lands in one of the attracting petals. It follows that z_0 is in the basin of attraction of zero.

CHAPTER 3

BASIN'S AREA

3.1 Newton's Method

In numerical analysis, the Newton's algorithm is called iteration method to find real or complex roots of differentiable function. Start with an initial value, call z_0, on the complex plane, to get z_1 by the z_0 we have the following equation

$$z_1 = z_0 - \frac{F(z_0)}{F'(z_0)}. \tag{3.1}$$

Therefore, when using this new point z_1 it able to get z_2 by the same way. Repeating these similar steps recursively [33].The pattern is shown as

$$z_{n+1} = z_n - \frac{F(z_n)}{F'(z_n)}, \tag{3.2}$$

such that $F'(z_n) \neq 0$.

We will be call approximation of (n+1) by $z_{n+1} = N(z_n)$, and defining Newton's method $N(z): \mathbb{C} \to \mathbb{C}$ by

$$N(z_n) = z_n - \frac{F(z)}{F'(z)}. \tag{3.3}$$

$N(z)$ is called Newton's transformation for $F(z)$. Therefore, the simple roots of $F(z)$ are fixed points of $N(z)$ satisfying $N(z) = z$. We can determined the nature of the fix points by the derivative of $N(z)$ by

$$N'(z) = \frac{F(z)F''(z)}{(F'(z))^2} = 0. \tag{3.4}$$

Hence, a Newton sequence $\{z_n\}$ given by Newton's method converges to a root of $F(z) = 0$, if z_0 is a proper initial guess.

The Newton's method will be at least quadratically convergent at a simple root and linearly convergent at a multiple root [34, 35].

3.2 Newton's Method Dynamics on $F = Pe^{Q}$

Let P, Q are complexes polynomial, such that $P:\mathbb{C}\to\mathbb{C}$ of degree m, $m \geq 0$, and $Q:\mathbb{C}\to\mathbb{C}$ of degree n $(n \geq 1)$.When we applied the complex Newton's method on the exponential function $F(z) = P(z)e^{Q(z)}$ we obtain

$$N(z) = z - \frac{F(z)}{F'(z)} = z - \frac{P(z)e^{Q(z)}}{P'(z)e^{Q(z)} + Q'(z)e^{Q(z)}P(z)}$$
$$= z - \frac{P(z)}{P'(z) + Q'(z)P(z)}.$$

We will get rational map $N(z)$, as $N(z):\mathbb{C}_\infty\to\mathbb{C}_\infty$. By rational map $N(z)$ we can examine the resulting dynamics of exponential function $F(z)$ Recall that, the fixed point of $N(z)$ coincide with roots of F [31]. We can determined the nature of the fix points by the derivative of $N(z)$ as follows

1- If the fix point $z \neq a$,a is a multiple root, then the derivative of $N(z)$ is

$$N(z) = \frac{F(z)F''(z)}{(F'(z))^2}.$$

Then, the simple root for F is super attracting fixed point for $N(z)$.

2- If the fix point $z = a,$ then a is a multiple root of F with multiplicity. The derivative of $N(z)$ is given by

$$N'(z) = \frac{m-1}{m},\ m \text{ is multiplicity of } a.$$

3- If $z = a$ is a critical point of F, then $N(z)$ has a pole at a if and only if a is not a root of F. If a is a critical point and not a root of F, then a will be send to infinity under single iteration of $N(z)$ [31].

Remark [31]**:** The speed of the convergence to a root depends on the multiplicity, therefore, there is an inverse relationship between them, and the higher multiplicity gives slower converge.

Proposition [31]**:** Infinity is a parabolic fixed point, with multiplier equal to 1, for Newton's method applied to $F(z) = P(z)e^{Q(z)}$, where P and Q are complex polynomials, P is not identically zero and Q is not constant.

Proof [31]**:**

Let P is a complex polynomial of order m, ($m \geq 0$) and Q is a complex polynomial of order n, $(n \geq 1)$**.**

Therefore, the Newton's method for exponential function, $F(z) = P(z)e^{Q(z)}$ is

$$N(z) = z - \frac{F(z)}{F'(z)} = z - \frac{P(z)e^{Q(z)}}{(P(z)e^{Q(z)})'}.$$

Since, the degree of numerator of Newton's method$= m + n$, and the order for its denominator$= m + n - 1$. Then, we have $\lim_{z \to \infty} N(z) = \infty$. Therefore, ∞ is a fixed point of Newton's method.

To prove that ∞ is a parabolic fix point of $N(z)$ we must determine the nature of ∞**.** So, we will map ∞ to zero via $g(z) = \frac{1}{z}$. The conjugate function M given by

$$M(v) = g(N(\frac{1}{v})),$$

$$\frac{1}{N(\frac{1}{v})} = \frac{1}{\frac{1}{v} - \frac{P(\frac{1}{v})}{P'(\frac{1}{v}) + Q'(\frac{1}{v})P(\frac{1}{v})}}$$

$$= \frac{1}{\frac{P'(\frac{1}{v}) + Q'(\frac{1}{v})P(\frac{1}{v})}{v(P'(\frac{1}{v}) + Q'(\frac{1}{v})P(\frac{1}{v}))} - \frac{vP(\frac{1}{v})}{v(P'(\frac{1}{v}) + Q'(\frac{1}{v})P(\frac{1}{v}))}}$$

$$= \frac{1}{\dfrac{P'(\frac{1}{v}) + Q'(\frac{1}{v})P(\frac{1}{v}) - vP(\frac{1}{v})}{v(P'(\frac{1}{v}) + Q'(\frac{1}{v})P(\frac{1}{v}))}}$$

$$= \frac{vP'(\frac{1}{v}) + vQ'(\frac{1}{v})P(\frac{1}{v})}{P'(\frac{1}{v}) + Q'(\frac{1}{v})P(\frac{1}{v}) - vP(\frac{1}{v})}.$$

Let us consider

$$H(v) = \frac{vP(\frac{1}{v})}{P'(\frac{1}{v}) + Q'(\frac{1}{v})P(\frac{1}{v}) - vP(\frac{1}{v})}.$$

Then, we have $M(v) = v + v\,H(v)$. Since, $v = 0$, then, $H(v) = 0$.
So since, $M'(v) = 1 + H(v) + v\,H'(v)$, then $M'(0) = 1$. Therefore, infinity is the parabolic fixed point of $N(z)$.

Leau–Fatou Flower application gives the information of local dynamics for $N(z)$ near infinity through the study of M near the origin.The series expansion of M is

$$M(v) = v + cv^{n+1} + Ov^{n+2},$$

where $c = \frac{1}{n}$.

Therefore, the degree of the exponent polynomial Q completely determines the number of petals at ∞. By the Fatou Flower theorem, the following propositions is proved [31].

Proposition [31]**:** There is just n repelling petals, n attracting for the neutral fixed point infinity if Q has degree of n.

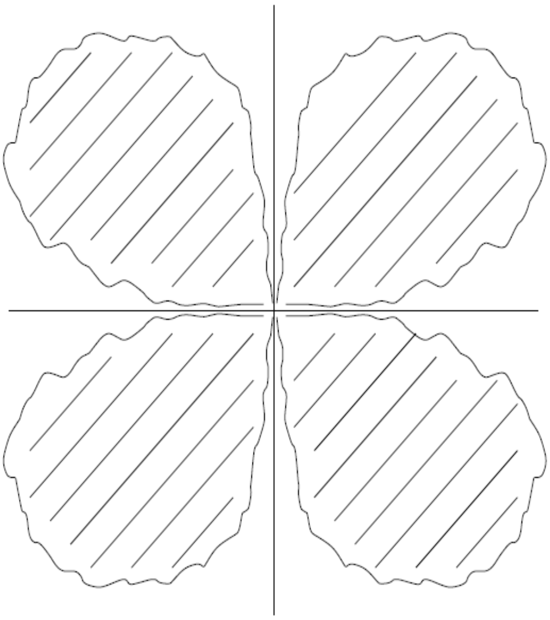

Figure 3 The v- Plane [31]

3.3 Basin's

Newton's approach to obtain the solutions of the equations leads to some nice images when it's applied to complex functions. That which called a basin of attraction is defined to be the set of all points that converge to the same root (ξ^*) when $(n \to \infty)$ and we denoted by $A(\xi^*)$, here (ξ^*) is the root of $F(z) = 0$. Since (ξ^*) is the super-attracting fixed point of the Newton's method then the $A(\xi^*)$ is an open region including (ξ^*). By coloring each basin of attraction a different color, the boundaries between the basins are defined Julia set, in other words we can define Julia set by

$$J_F = \partial A(\xi_1^*) = \ldots = \partial A(\xi_n^*).$$

These boundary points are points that do not converge in any of Newton's method and form fractal image [36, 37]. The connected component of attraction basins $A(\xi^*)$ which containing (ξ^*) of the basin is called the immediate basin of (ξ^*) [35, 36].

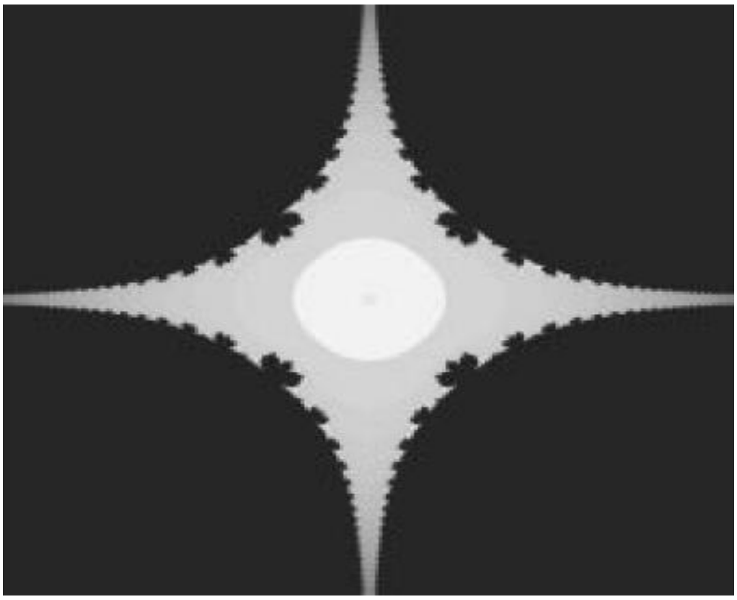

Figure 4 Basin Fractal of Zero for n=4 [31]

3.4 Basin's Area

Theorem [31]**:** If Newton's method is applied to $F = Pe^{Q}$, where P and Q are complex valued polynomials, P is not identically zero and Q is non-constant with degree of Q ≥ 3, then the area of the attractive basin of a root of F has finite area.

Proof [31]**:**

There are more steps for prove the theorem, we will explain it step by step.

Step 1. Basin Tail at ∞ Has Finite Area

Because the infinity is a parabolic fixed point, therefore we have *n* attracting petals and *n* repelling petals at infinity when n bigger than or equal 3, and that petals being consistently spaced. The attracting petals define parabolic basins of attraction for infinity. Also, the union for a basin of infinitywith Julia set of each roots basins has finite area. The tails of roots of the basin continue until infinity so these should lies between two attracting petals for infinity [31].

Then, these tails have finite area, by constructing attracting petals. Note that the petals are pairwise disjoint, and that each of them sub tend an angle $2\pi / n$ at ∞. So, the total angle subtend at ∞ by all petals is 2π. The line of symmetry of a petal is the ray $\theta_k = k\pi / n, k = 1,2,..., n.$ and is called the axis of the petal [31].

If *k*even, then the axis is an attracting ray and *k* odd determines a repelling ray. Every repelling ray of petals is a repelling direction for infinity, also every repelling axis ray to infinity lies within a basin tail. Since axis rays differ only by rotation, we may consider a tail of the which continue until infinityalong of a positive x- axis. We need to find a curve γ that lies inside the attractive basin of infinity, and hence bounds the basin tail from above for values of t near infinity [31].

In Figure 4 the lines closer to the real axis represent the boundary of the basin tail, which is bounded above by γ and below by $-\gamma$. Because, an attracting petal must located in the immediate basin for infinity. Since, if γ located in, or denotes the boundary of a petal near infinity therefore the orbits of points on γ it coming until infinity under iterations. The symmetric argument illustrates this $-\gamma$ bounds a basin tail from below. Hence the tail of the basin is locate in the bounded area, it has finite area [31].

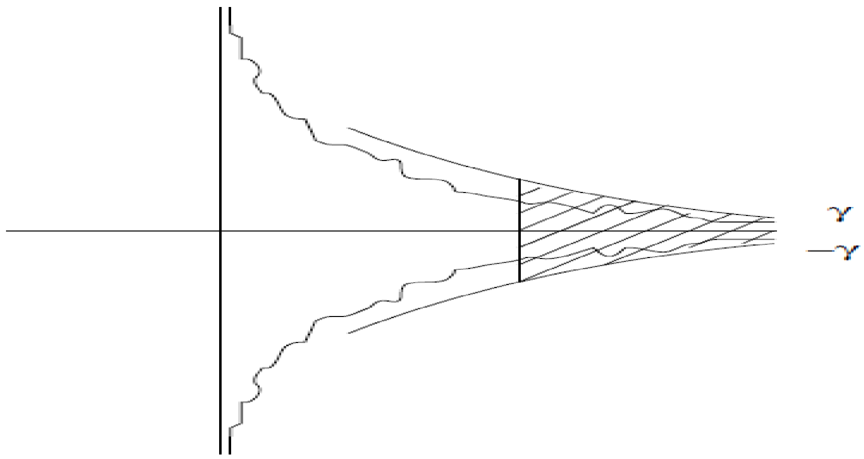

Figure 5 Basin Tail Bounded Above by γ and Below by $-\gamma$ [31]

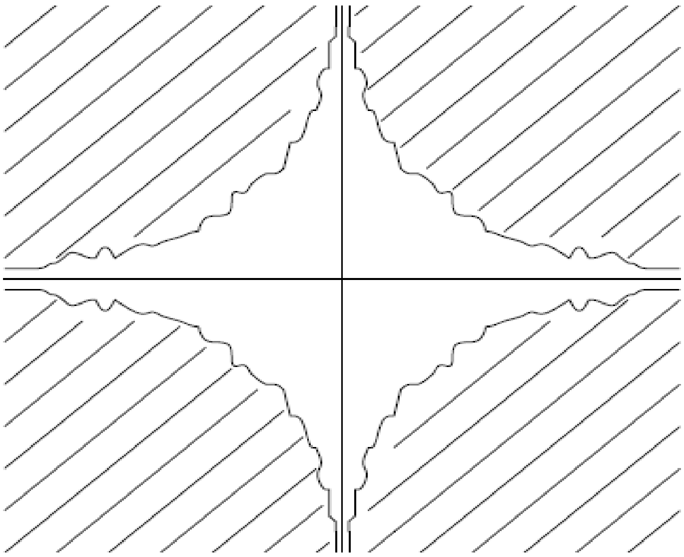

Figure 6 The z -Plane [31]

Step 2. Construction of an Attracting Petal at ∞

The function of Newton's algorithm $N(z)$ acts on the z-plane. When send infinity to zero by $g(z) = 1/z$ in order to study the nature of the fixed point at ∞, we will be work with M and the attracting and repelling petals from the Fatou Flower Theorem on the v-plane [31].

$M(v)$ is moving to the w-plane where $G(\omega)$ by semi-conjugacy $w = \pi(v) = \frac{1}{v^n}$ which send zero back to ∞. Note that the corresponding transformation in the w- plane is

$$\omega \to G(\omega) = \pi o M o\, \pi^{-1}(\omega),$$

where $\pi^{-1}(\omega) = \frac{1}{\omega^{\frac{1}{n}}}$.

Conjugate $N(z)$ by $g(z) = \frac{1}{z}$ to M from ∞ to near zero as before.

Then we have

$$M(v) = v + cv^{n+1} + c_1 v^{n+2} + O\left(v^{n+3}\right),$$

where $c = \dfrac{1}{n}$ and $c_1 = -\dfrac{n-1}{n^2}$.

$$
\begin{tikzcd}
z \arrow[r, "N"] \arrow[d, "g"'] & z \arrow[d, "g"] \\
v \arrow[r, "M"] \arrow[d, "\pi"'] & v \arrow[d, "\pi"] \\
w \arrow[r, "G"] & w
\end{tikzcd}
$$

Figure 7 Transformation Diagram [31]

With the semi-conjugation and by choosing the branch of the inverse associated with the $G(\omega)$, attracting direction $\dfrac{\pi}{n}$, we obtain [31]:

$$G(\omega) = \frac{1}{\left[\dfrac{1}{\omega^{\frac{1}{n}}} + nc\dfrac{1}{\omega}\cdot\dfrac{1}{\omega^{\frac{1}{n}}} + nc_1\dfrac{1}{\omega^2}\cdot\dfrac{1}{\omega^{\frac{1}{n}}} + O\left(\dfrac{1}{\left(\omega^{\frac{1}{n}}\right)^{n+3}}\right)\right]^n},$$

$$G(\omega) = \frac{1}{\left[\dfrac{1}{\omega} + nc\cdot\dfrac{1}{\omega^2} + nc_1\dfrac{1}{\omega^2}\cdot\dfrac{1}{\omega^{\frac{1}{n}}} + O\left(\dfrac{1}{\left(\omega^{\frac{n+3}{n}}\right)}\right)\right]},$$

$$G(\omega)=\frac{1}{\left[\frac{1}{\omega}+nc\cdot\frac{1}{\omega^2}+nc_1\frac{1}{\omega^2}\cdot\frac{1}{\omega^{\frac{1}{n}}}+O\left(\frac{1}{\left(\omega^{\frac{3}{n}}\right)}\right)\right]},$$

$$G(\omega)=\omega\left[\left[\frac{1}{\omega}+nc\cdot\frac{1}{\omega}+nc_1\frac{1}{\omega}\cdot\frac{1}{\omega^{\frac{1}{n}}}+O\left(\omega^{\frac{-3}{n}}\right)\right]\right]^{-1},$$

$$G(\omega)=\omega\left[1-nc\cdot\omega^{-1}-nc_1\omega^{-1}\cdot\frac{1}{\omega^{\frac{1}{n}}}-O\left(\omega^{\frac{-3}{n}}\right)\right],$$

$$G(\omega)=\omega-nc\cdot\omega\omega^{-1}-nc_1\omega\omega^{-1}\cdot\frac{1}{\omega^{\frac{1}{n}}}-O\left(\omega^{\frac{-3}{n}}\right),$$

$$G(\omega)=\omega-nc-nc_1\cdot\frac{1}{\omega^{\frac{1}{n}}}-O\left(\omega^{\frac{-3}{n}}\right).$$

Since $c=\frac{1}{n}$ and $c_1=-\frac{n-1}{n^2}$ we have :

$$G(\omega)=\omega-n\frac{1}{n}+n\frac{n-1}{n^2}\cdot\frac{1}{\omega^{\frac{1}{n}}}-O\left(\omega^{\frac{-3}{n}}\right),$$

$$G(\omega)=\omega-1+\frac{n-1}{n}\cdot\frac{1}{\omega^{\frac{1}{n}}}-lower\ term,$$

$$G(\omega)=\omega-1+\frac{\alpha}{\omega^{\frac{1}{n}}}+lower\ term,$$

where $\alpha=\frac{n-1}{n}$.

Step 3. An Open Wedge-Shaped Region Ω Construct in the w - plane

In fact, near ∞ iteration of *G* is effectively the translation via one to the left given any small number $\varepsilon \in (0, \pi)$, which it will be convenient to write as
$\sin \varepsilon > 0,$ we can choose a radius $r_0 > 0$ for $|\omega| > r_0$
such that

$$|\omega - 1 - G(\omega)| < \sin \varepsilon$$

and

$$\Omega = \{\omega \in \mathbb{C} | \mathcal{E} < \arg(\omega + \mathrm{A})\},$$

$$G(\overline{\Omega}^{+}) \subset \Omega^{+}, 0 < \varepsilon < \pi .$$

Taking an appropriate branch of the inverse map, $\pi^{-1}(\Omega^{+})$ gets a lone attracting petal for 0. The lasting *n* – 1 branches get the rest of the attracting petals. The repelling petals are constructed in the similar way but by the change of variable $z^n = \dfrac{-1}{nc\omega}$.

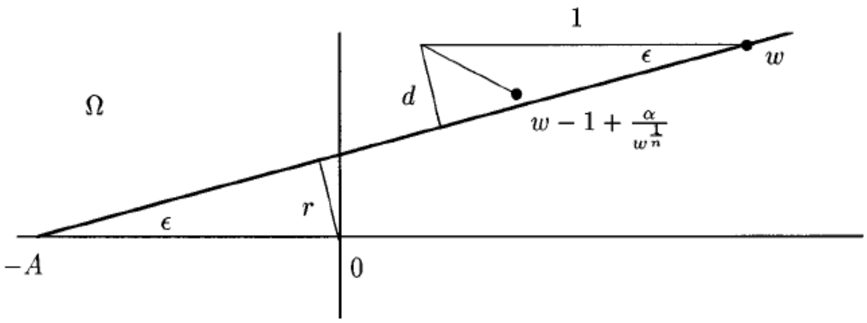

Figure 8 *G*(*w*) [31]

It able to select $A \in R^{+}$ big sufficient for satisfy $|\omega| > r_0$. Also, the closure of Ω^{+} maps in itself under *G*. To increase the wedge size by decreasing the angle ε, we have to shift Ω to the left since $A = \mu / \tan \varepsilon$ *for* $\mu > 0.$ It can be give the estimates for the minimum value of *A*, with $0 < \varepsilon$.

The closure of Ω maps in itself. Therefore, symmetry of Ω with the real line allows us, dose not loss of generality, for another arguments in the upper half plane.

Not that d = sin ε, and $G(\omega)$ lies in Ω^{+} if

$$\left|\frac{\alpha}{\omega^{\frac{1}{n}}}\right| < \frac{\sin\varepsilon}{2}, where \alpha = \frac{n-1}{n}.$$

Note that $\left|\frac{\alpha}{\omega^{\frac{1}{n}}}\right|$ is largest when $|\omega|$ is smallest. Writing $\omega = re^{i\theta}$, the minimum value of $\left|re^{i\theta}\right| = r$ occurs for $r = A\sin\varepsilon$. Thus, we have $\left|\frac{\alpha}{\omega^{\frac{1}{n}}}\right| \leq \frac{\alpha}{(\sin\varepsilon A)^{\frac{1}{n}}}$.

Set

$$\sin\varepsilon > \frac{\alpha}{(\sin\varepsilon A)^{\frac{1}{n}}},$$

then, we have

$$A > \frac{\alpha^{n}}{(\sin\varepsilon)^{n+1}}.$$

From, $A > \max\left\{\frac{\alpha^{n}}{\sin^{n+1}\varepsilon}, \frac{r_0}{\sin\varepsilon}\right\}$, then $|\omega| > r_0$ produces a wedge Ω such that images of points in Ω under G remain in Ω. Let $\rho = \pi_0 g$. Then ρ semi-conjugates $N(z)$ and G. Bringing Ω for z-plane with the appropriate branch of ρ^{-1} producing a single attracting petal at infinity with attracting direction π / n.

Step 4. Modified Bigger Petal

To find a petal at ∞ which in the end limits the area of an attracting basin tail. In so far, it have construct the region Ω that is image under below a branch for ρ^{-1}, got a single attracting petal at infinity on z-plane.

It able to try use horizontal lines of denoting a boundary of Ω in the right half plane [31]. Under a branch of ρ^{-1}, a parabolic curve is sent a curve of degree $t^{1-\frac{n}{2}}$ near infinity on *z*-plane. This approach yields finite area for $n \geq 6$, but says nothing about the cases where n <6. Hence, it can be construct a modification of the wedge Ω denoted $\hat{\Omega}$, that is boundary close infinity is denoted with some curve Γ [31].

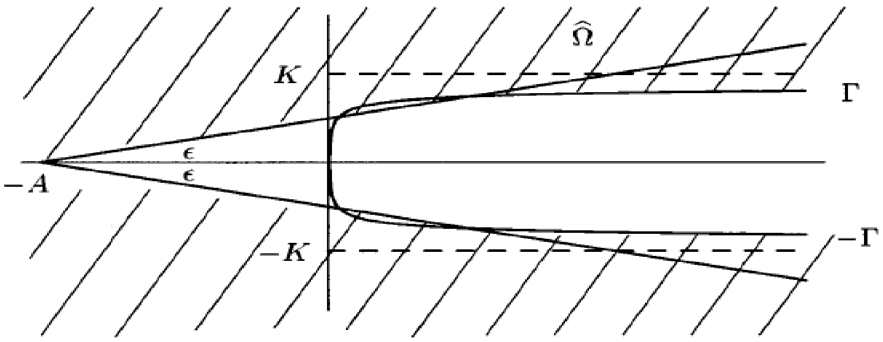

Figure 9 The $\hat{\Omega}$ Modified [31]

We will require Γ to be the image of another curve γ that lie at *z*-plane where

$$\int_{t_0}^{\infty} \gamma dt \text{ for some } t_0 > 0 .$$

In the end, the necessary is that $\hat{\Omega}$ satisfy $G\left(\overline{\hat{\Omega}}\right) \subset \overline{\hat{\Omega}}$ standard calculus shows that the area under the curve

$$\gamma(t) = t + i\left(\frac{a}{t^{n-1}} - \frac{b}{t^n}\right)$$

is bounded for $n \geq 3$ and $a, b \in (0, +\infty)$ when t is big enough and positive.

Now define, $\hat{\Omega} = \Omega \cup \{\omega | \text{Im } \omega > \text{Im}\,\Gamma(\omega), \text{ Re } \omega > \omega_0\}$, where $\omega_0 = \max_{\Gamma \cap \partial\Omega} \{\text{Re } \omega\}$ and Γ is the curve at $w-plane$ satisfy the following four argument :

i. Γ is the image of γ under ρ.
ii. Points on Γ map above the curve Γ under G.
iii. Γ Interests a boundary for Ω.
iv. Points map above both Γ and $\partial\Omega$ under G.

To limit a precise expression for the curve Γ, apply $P(z) = z^n$ to γ, where $n \geq 3$ and $a, b \in R^+$, to obtain

$$[\gamma(t)]^n \approx t^n + in\left(a - \frac{b}{t}\right).$$

Re-parameterizing by set $x = t^n$ results in $y \approx na - \frac{nb}{t^{\frac{1}{n}}}$, therefore, we will let

$$\Gamma(t) = t + i\left(K - \frac{k}{t^{\frac{1}{n}}}\right)$$

with $K = na$ and $k = nb$ to satisfy condition (i). Note that Γ is asymptotic to $y = K$.

The region $\hat{\Omega}$ will map inside of itself under G as long as Γ satisfies conditions (ii), (iii), (iv) above.

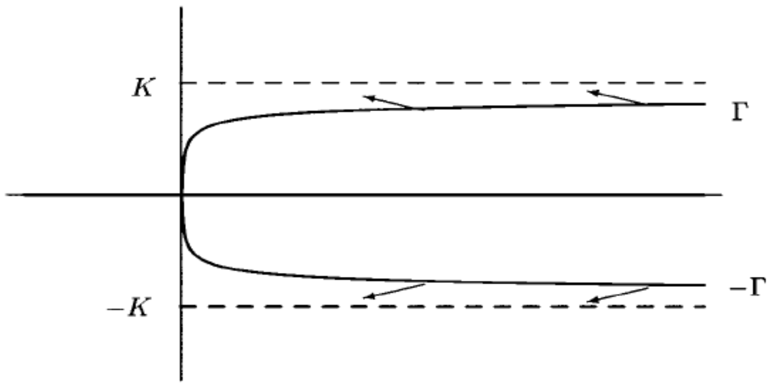

Figure 10 Points on Γ Map Above Γ [31]

First we will show, as in the Figure 10, the points at Γ are sent to points above Γ with

$$G(\omega) = \omega - 1 + \frac{\alpha}{\omega^{\frac{1}{n}}} + \ldots \ .$$

Denote the image of Γ under G by $\tilde{\Gamma}$. It follows that [31]

$$\tilde{\Gamma}(t) = t + i\left(K - \frac{k}{t^{\frac{1}{n}}}\right) - 1 + \alpha\left(\frac{1}{t^2 + (K - \frac{k}{t^{\frac{1}{n}}})^2}\right)^{\frac{1}{n}} \cdot \left(t^{\frac{1}{n}} - \frac{i}{n} t^{\frac{1}{n}-1}\left(K - \frac{k}{t^{\frac{1}{n}}}\right) - \ldots\right).$$

We require that

$$\operatorname{Im}\left(\hat{\Gamma}(t)\right) \; > \; \operatorname{Im}\left[\Gamma\left(\operatorname{Re}(\hat{\Gamma}(t))\right)\right]. \tag{3.5}$$

Substituting in the real and imaginary parts of $\tilde{\Gamma}$ from above it gives [31]

$$\frac{k}{t^{\frac{1}{n}}} + \frac{\frac{\alpha}{n} t^{\frac{1}{n}-1}\left(K - \frac{k}{t^{\frac{1}{n}}}\right)}{(t^2 + (K - \frac{k}{t^{\frac{1}{n}}})^2)^{\frac{1}{n}}} + \cdots < \frac{k}{(t - 1 + \frac{\alpha t^{\frac{1}{n}}}{(t^2 + (K - \frac{k}{t^{\frac{1}{n}}})^2)^{\frac{1}{n}}} + \ldots)^{\frac{1}{n}}}. \tag{3.6}$$

Denote the left side of the inequality by L. Since $\lim_{t\to\infty} t^{\frac{1}{n}} \cdot L = k$ and [31]

$$\lim_{t\to\infty} t^{1+\frac{1}{n}} \cdot \left(L - \frac{k}{t^{\frac{1}{n}}}\right) = \frac{\alpha K}{n}$$

we have

$$L = \frac{k}{t^{\frac{1}{n}}} + \frac{\alpha K}{n t^{1+\frac{1}{n}}} + \ldots. \quad .$$

Define the right side of the inequality by R and let $R = kB^{\frac{1}{n}}$, where

$$B = \frac{1}{t - 1 + \frac{\alpha t^{\frac{1}{n}}}{\left(t^2 + \left(K - \frac{k}{t^{\frac{1}{n}}}\right)^2\right)^{\frac{1}{n}}} + \ldots}.$$

Since

$$\lim_{t\to\infty} t\cdot B = 1,$$

and

$$\lim_{t\to\infty} t^2 \cdot \left(B - \frac{1}{t}\right) = 1,$$

we have

$$B = \frac{1}{t} + \frac{1}{t^2} + \cdots \quad .$$

Thus, we conclude [31]

$$R = \frac{k}{t^{\frac{1}{n}}} + \frac{k}{nt^{\frac{1}{n}+1}} + O\left(\frac{1}{t^{\frac{1}{n}+2}}\right).$$

Inequality (3.6) can be written by as

$$\frac{k}{t^{\frac{1}{n}}} + \frac{\alpha k}{nt^{\frac{1}{n}+1}} + \cdots + < \frac{k}{t^{\frac{1}{n}}} + \frac{k}{nt^{\frac{1}{n}+1}} + \cdots ,$$

where $\alpha = \dfrac{n-1}{n}$, $K = na$ and $k = nb$.

It's clear that this expression holds if $k > \alpha K$, or equivalently if $\dfrac{b}{a} > \dfrac{n-1}{n}$. Therefore, Γ provides a suitable boundary for $\hat{\Omega}$ in that, for sufficiently large t and appropriate choice of *a* and *b*, condition (ii) satisfied [31].

More precisely, we may choose a number $\delta > 1$ and assume that $k = (\alpha\delta)K$. Then there exists the number T_0dependingonly on P and Q like that (3.6) and subsequently (ii) are satisfied. For the remainder of the proof, we should work with these fixed values for *k* and T_0.

We are now prepared to show condition (iii), that Γ and the boundary of Ω intersect for t > 0. Seethe Figure 9. The approach is to find the point $t_0 > 0$ at which $\Gamma'(t_0)$ and the slope of the line defining δΩ are equal [31].

If we determine that the curve lies above Ω at t_0 i.e..,

$$\tan\varepsilon(t_0 + A) < \mathrm{Im}[\Gamma(t_0)] \ . \qquad (3.7)$$

Hence, $\Gamma(t)$ located below Ω for big values of t.

Therefore, fix $\varepsilon > 0$ and subsequently A. Not $k > 0$, $K > 0$ where Γ locate inside the upper half plane. Whenever, the slope of a line is $\tan\varepsilon$ so the derivative of the curve will be [31]

$$\Gamma'(t_0) = \frac{k}{n} t_0^{\frac{-1}{n}-1} \ .$$

By set these equal and solving for t_0 yields

$$t_0 = \left(\frac{k}{n\tan\varepsilon}\right)^{\frac{n}{n+1}} .$$

It remain to demonstrate that

$$\tan\varepsilon\left(\left(\frac{k}{n\tan\varepsilon}\right)^{\frac{n}{n+1}} + A\right) < K - \frac{k}{\left(\frac{k}{n\tan\varepsilon}\right)^{\frac{1}{n+1}}}$$

or equivalently

$$A\tan\varepsilon + k^{\frac{n}{n+1}}\left[\left(\frac{1}{n\tan\varepsilon}\right)^{\frac{n}{n+1}} + (n\tan\varepsilon)^{\frac{1}{1+n}}\right] < K \ . \qquad (3.8)$$

Remember that $A > \frac{\alpha}{(\sin\varepsilon)^{n+1}}$ and $\alpha = \frac{n-1}{n}$ are fixed via the polynomial Q and our choice of ε. Therefore, for $K = na$ and $k = nb$ the value for n is predetermined, but the choice of a and b still open. Therefore, in equation (3.8) it replace all fixed expressions denotes by constant v_1 and v_2 and k by $(\alpha\delta)K$ with resulting in [31]

$$v_1 + ((\alpha\delta)K)^{\frac{n}{n+1}} v_2 < K \qquad (3.9)$$

or equivalently

$$v_1 < K\left(1 - \frac{(\alpha\delta)^{\frac{n}{n+1}} v_2}{(K)^{\frac{1}{n+1}}}\right)$$

which holds for K sufficiently large.

Hence, (3.9) and in turn (3.7), holds, consequently satisfying condition (iii).

To show condition (iv), suppose t_1 and also t_2 be the values of which Γ intersects the original wedge Ω with

$$T_0 \le t_1 \le t_2$$

also

$$t_2 - t_1 \ge 2$$

denote the boundary of $\hat{\Omega}$ to be

$$\hat{\Omega} = \begin{cases} \partial\Omega, & t < t_2, \\ \Gamma, & t \ge t_2, \end{cases}$$

since

$$\left|\mathrm{Re}[\Gamma(t)] - \mathrm{Re}[G(\Gamma(t_0))]\right| < 2 \ .$$

After that $t \ge t_2$ implies $t \ge T_0$. Therefore, for large enough values for K, the condition (iv) satisfy [31].

CHAPTER 4

BASINS FRACTAL

In this chapter, we will use the functions that shape $F(z) = P(z)e^{Q(z)}$ (which Haruta [31] used it in his studies) to generate fractal on z-plane with different value of (n = 1, 2,..., 14) in some examples to check up the basins area and showing of our results. We simulate our methods with MATLAB. So, we used $|N^k(z)| < 0.001$ and $|k| \le 50$, which k is the number of iteration of the Newton's method [31, 38-42].

1.4 Our Results

Part I:

When $n = 1, 2$ which means $n < 3$ we will get infinite area of basins (see Example1 and Example 2).

Example 1.

When $n = 1$, the exponential function be $F(z) = ze^z$. Therefore, the complex Newton's method of $F(z) = ze^z$ is

$$N(z) = z - \frac{ze^z}{(ze^z)'} = z - \frac{z}{1+z}.$$

When $z = 0$, we have $N(0) = 0$. Since, $N(0) = 0$, then zero is a simple root of F .
To determine the nature of zero by the derivative of $N(z)$ as follows

$$N^{'}(z) = 1 - \frac{1}{(1+z)^2}$$

For $z = 0$, $N^{'}(0) = 0$. Since, $N^{'}(0) = 0$, then, zero is super attracting fixed point of $N(z)$.

Also, since, $\lim_{z\to\infty} N(z) = \infty$, therefore, ∞ is a fixed point of $N(z)$.

For checking the nature of ∞, it is needed to conjugate $N(z)$ by $g(z) = \frac{1}{z}$ to M from infinity to zero as

$$M(v) = g(N(\frac{1}{v})) = \frac{v(v+1)}{1+v}.$$

Let $H(v) = \frac{v}{1+v}$, then, $M(v) = v + vH(v)$. The series expansion of M is given by

$$M(v) = v + v^{2},$$

$$M'(v) = 1 + 2v .$$

When v = 0, $M'(0) = 1$, therefore, we have ∞ is a parabolic fixed point of $N(z)$. $z \in (0,\infty)$. $[0,\infty)$ is a repelling direction for ∞ and $(-\infty,0]$ is an attraction direction for ∞. Since $\deg(Q) = 1$, there is one attracting and one repelling petal of $N(z)$ at ∞. The basin of attraction has infinite area, and the attracting basin lies in the region of the origin of Julia set. The attracting petals are not symmetrical about x-axis and y-axis since n is odd.

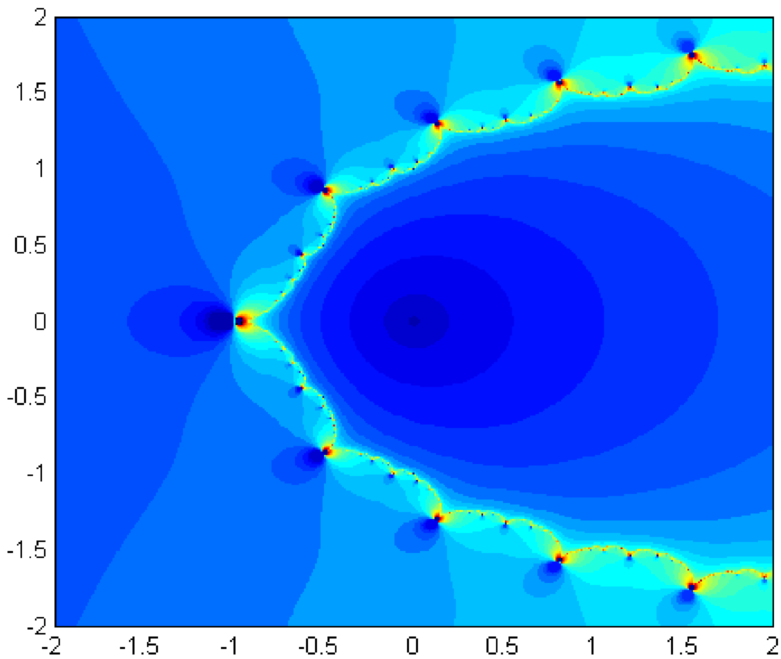

Figure 11 Basins Fractal of $F(z) = ze^{z}$ for $n = 1$

We can change $P(z)$ such as to (z^2-1) and (z^4-1)

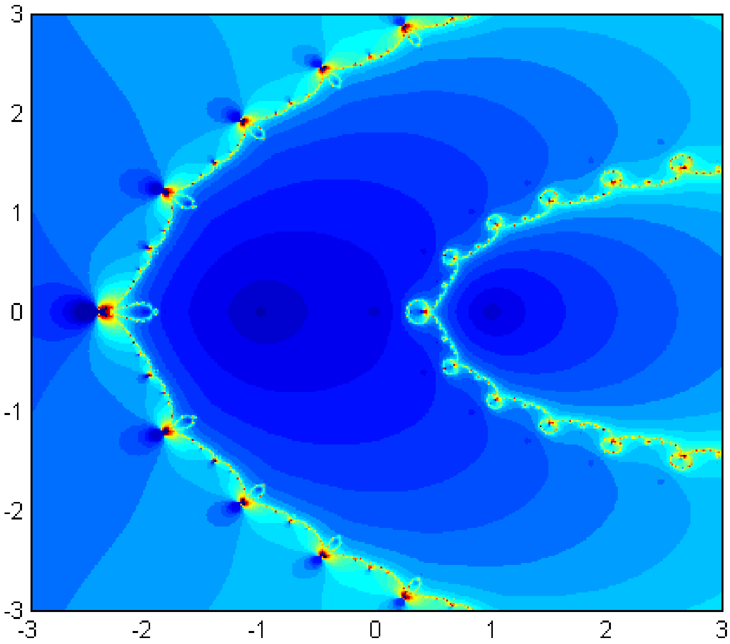

Figure 12 Basins Fractal of $F(z)=(z^2-1)e^Z$ for $n=1$

We Have Two Roots ∓ 1

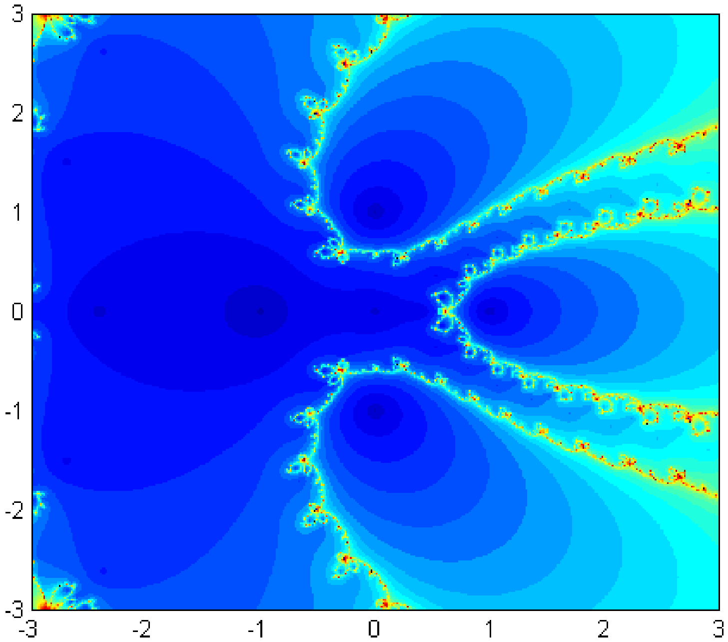

Figure 13 Basins Fractal of $F(z)=(z^4-1)e^Z$ for $n=1$

We Have Four Roots ∓ 1 , $\mp i$

Example 2.

Since, $\deg(Q) = 2$ then basins area is infinite, so, when $n = 2$, the exponential function it will be by this shape $F(z) = ze^{z^2}$. Therefore, the complex Newton's method of $F(z) = ze^{z^2}$ is $N(z) = z - \dfrac{ze^{z^2}}{(ze^{z^2})'} = z - \dfrac{z}{1+2z^2}$.

When $z = 0$, $N(0) = 0$. Then, zero is a simple root of F. To determine the nature of zero by the derivative of $N(z)$ as follows $N'(z) = 1 - \dfrac{1-2z^2}{(1+2z^2)^2}$, when $z = 0$, $N'(0) = 0$. Since, $N'(0) = 0$. Then, zero is super attracting fixed point of $N(z)$. Also, since, $\lim_{z\to\infty} N(z) = \infty$, therefore, ∞ is a fixed point of $N(z)$. For checking the nature of ∞, it is needed to conjugate $N(z)$ by $g(z) = \dfrac{1}{z}$ to M from infinity to zero as before. Then, we have $M(v) = v + \dfrac{1}{2}v^3$. When v = 0, $M'(v) = 1$, then ∞ is a parabolic fixed point of $N(z)$.

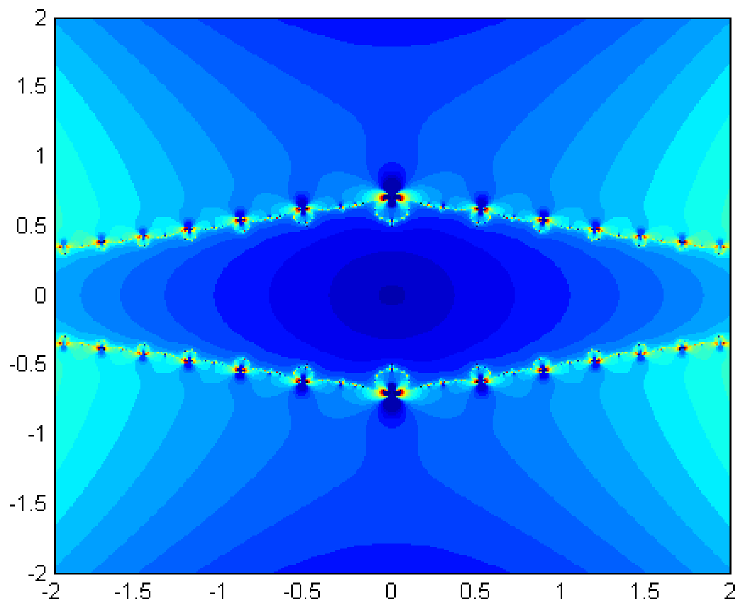

Figure 14 Basins Fractal of $F(z) = ze^{z^2}$ for $n = 2$

There are two attracting and two repelling petals of $N(z)$ at ∞. Different color denote to different iterative number. The attracting petals are symmetrical about y-axis.

Also we can change $P(z)$ such as to (z^2-1) and (z^4+16)

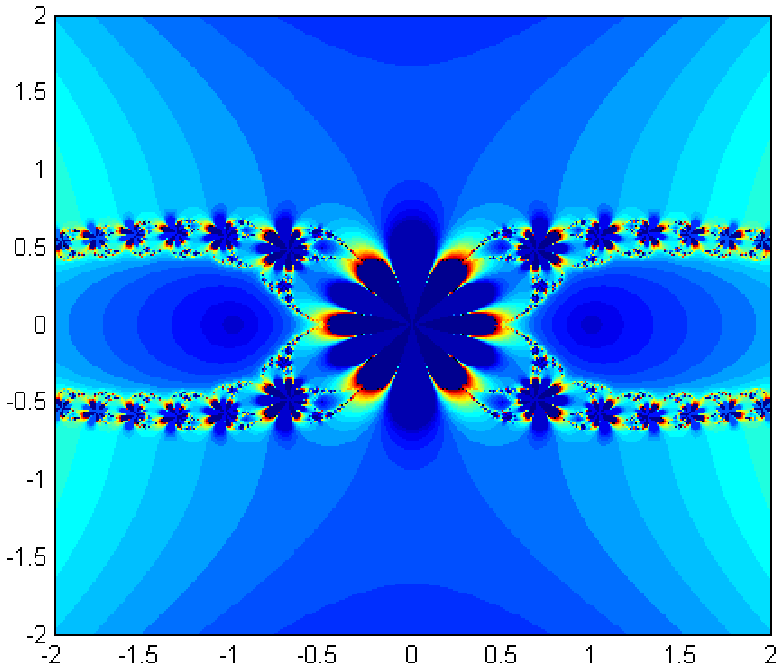

Figure 15 Basins Fractal of $F(z)=(z^2-1)e^{z^{z^2}}$ for $n=2$

We Have Two Roots of ∓ 1

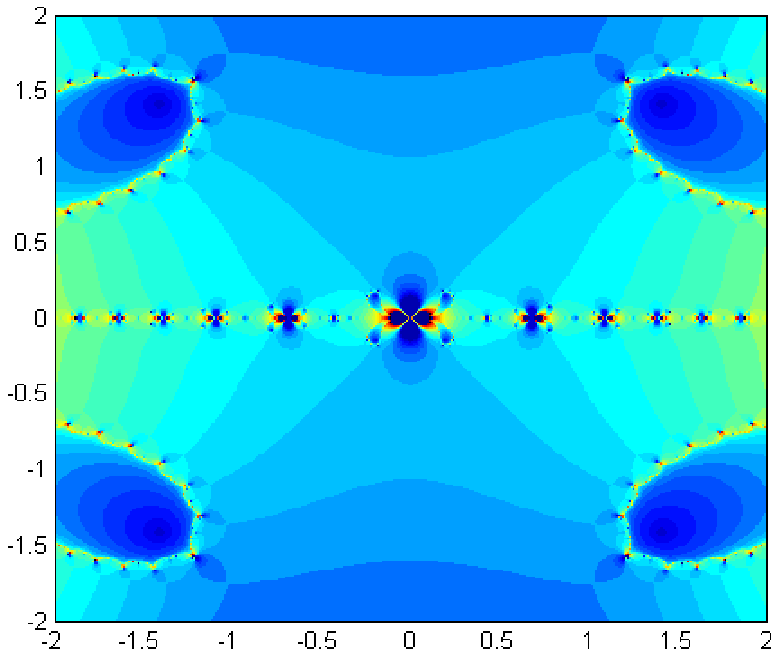

Figure 16 Basins Fractal of $F(z)=(z^4+16)e^{z^{z^2}}$ for $n=2$

We Have Four Roots $\mp(1+i)$, $\mp(1-i)$

Iterative number	Value of RGB			color
	R	G	B	
1	0	0	0	
2	0.5625	0	0	
3	0.6250	0	0	
4	0.6340	0	0	
5	0.7540	0	0	
.	.	.	.	.
.	.	.	.	.
10	1.000	0.1143	0	
11	1.0000	0.1250	0	
12	1.0000	0.1875	0	
13	1.0000	0.2123	0	
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
16	1.0000	0.4375	0	
17	1.0000	0.5210	0	
18	1.0000	0.5839	0	
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
40	1.0000	0.2410	1.000	
.	.	.	.	.
.	.	.	.	.
62	0	0	0.6875	
63	0	0	0.6250	
64	0	0	0.5625	

Table 1 Color Palette's Descriptions Infinite Basins Area

Part II:

When $n \geq 3$ we will get finite area of basins (see Example 3, Example 4, Example 5, Example 6 and Example 7).

Example 3.

When $n = 3$, the exponential function it is $F(z) = ze^{z^3}$. Therefore, the complex Newton's method of $F(z) = ze^{z^3}$ gives $N(z) = z - \dfrac{ze^{z^3}}{(ze^{z^3})'} = z - \dfrac{z}{1+3z^3}$.

When $z = 0$, we have $N(0) = 0$. Since, $N(0) = 0$, then zero is a simple root of F. To determine the nature of zero by the derivative of $N(z)$ as follows

$$N'(z) = 1 - \frac{1-6z^3}{(1+3z^3)^2}.$$

When $z = 0$, we have $N'(0) = 0$. Since, $N'(0) = 0$, then, zero is super attracting fixed point of $N(z)$. Since, $\lim\limits_{z\to\infty} N(z) = \infty$, then, ∞ is a fixed point of $N(z)$. Conjugate $N(z)$, by $g(z) = \dfrac{1}{z}$ to M from infinity *to* zero as $M(v) = v + \dfrac{1}{3}v^4$.

When v = 0, $M'(v) = 1$, since, $M'(v) = 1$, then, ∞ is a parabolic fixed point of $N(z)$.

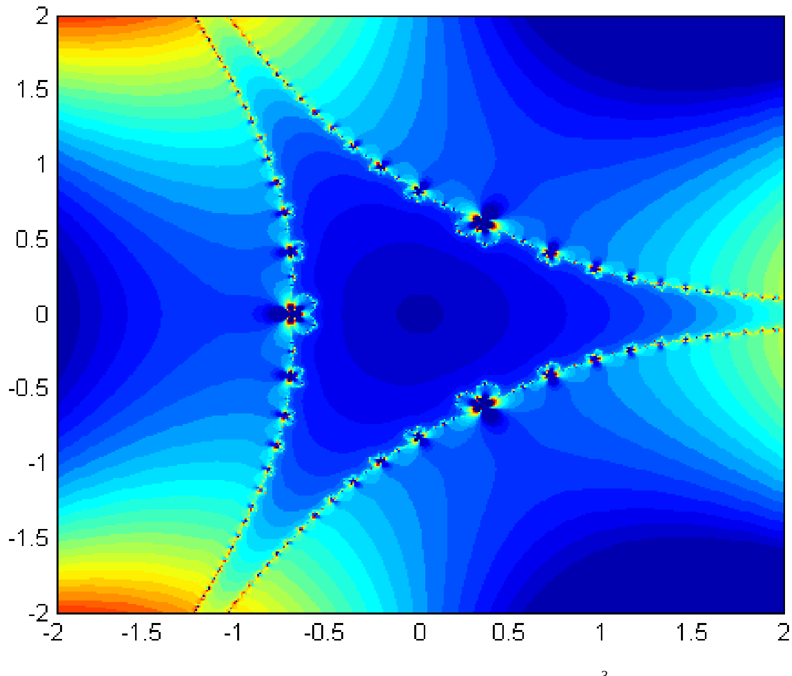

Figure 17 Basins Fractal of $F(z) = ze^{z^3}$ for $n = 3$

Also we can change $P(z)$ such as to $(z^2+4z+1)(z^2+4)(z-2)$ and (z^3-1)

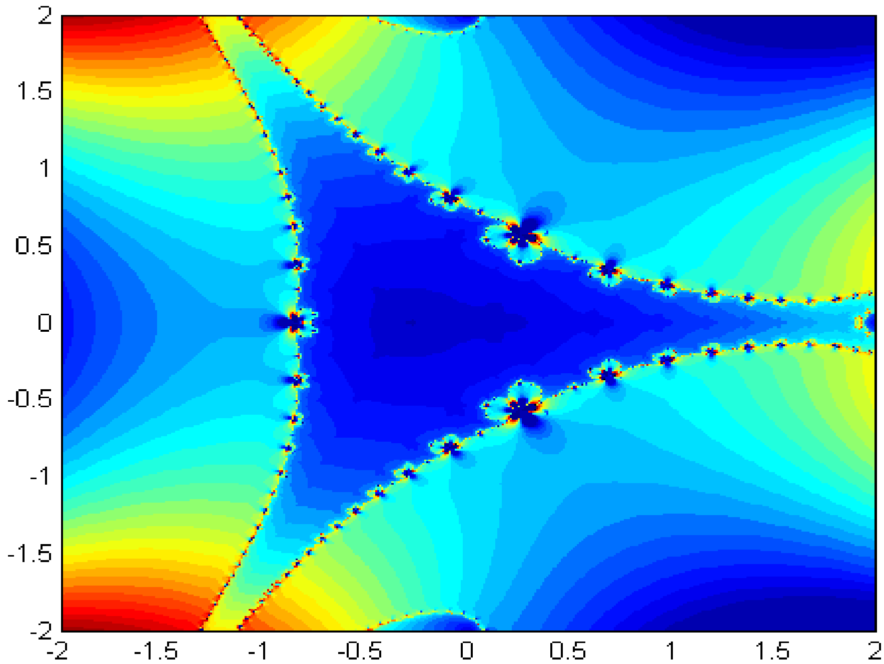

Figure 18 Basins Fractal of $F(z)=(z^2+4z+1)(z^2+4)(z-2)e^{z^3}$

for $n=3$ We Have Five Roots, 2+i, 2-i, 2i, -2i , +2

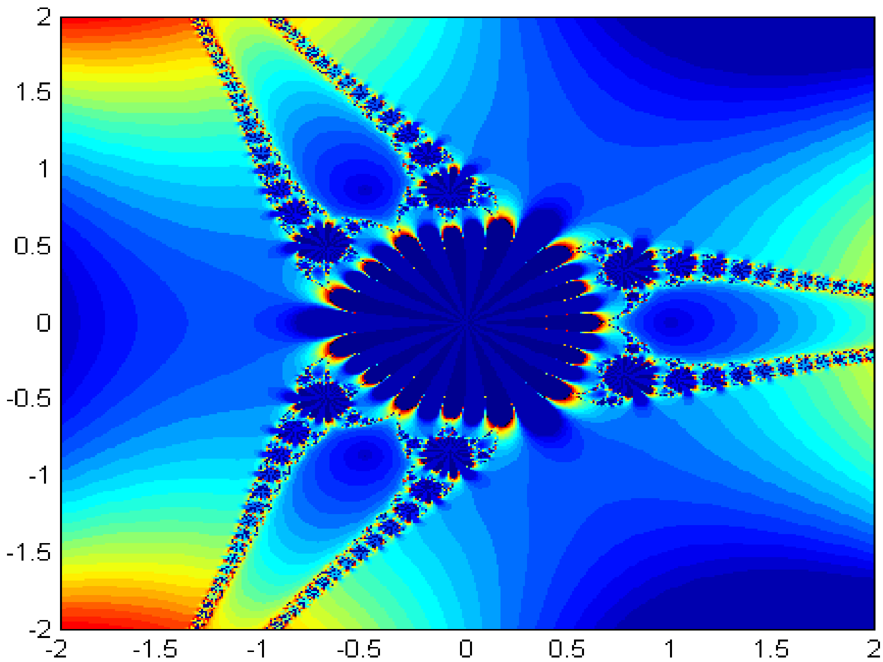

Figure 19 Basins Fractal of $F(z)=(z^3-1)e^{z^3}$ for $n=3$

We Have Three Roots 1, $-\frac{1}{2}+\frac{\sqrt{3}}{2}i,\quad -\frac{1}{2}-\frac{\sqrt{3}}{2}i$

Example 4.

When $n=4$, the exponential function is $F(z)=ze^{z^4}$. Therefore, the complex Newton's method of $F(z)=ze^{z^4}$ is

$$N(z)=z-\frac{ze^{z^4}}{(ze^{z^4})'}=z-\frac{z}{1+4z^4}.$$

When $z=0$, we have $N(0)=0$. Then, zero is a simple root of F. To determine the nature of zero by the derivative of $N(z)$ as follows

$$N^{'}(z)=1-\frac{1-12z^4}{(1+4z^4)^2}.$$

When $z=0$, we have $N'(0)=0$.

Then, zero is super attracting fixed point of $N(z)$.

Since, $\lim\limits_{z\to\infty} N(z)=\infty$, the series expansion of M is $M(v)=v+\frac{1}{4}v^5$.

When v = 0, we have $M'(v)=1$. Therefore, ∞ is a parabolic fixed point of $N(z)$.

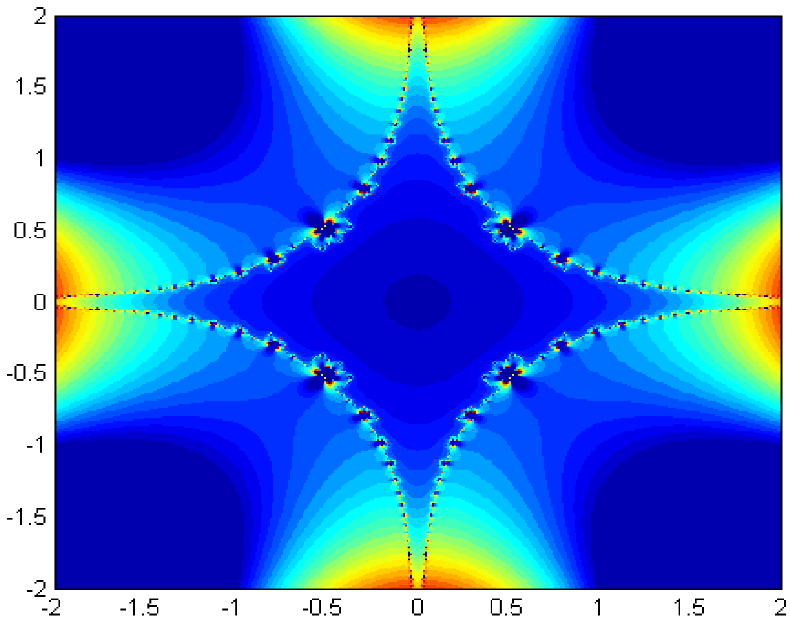

Figure 20 Basins Fractal of $F(z)=ze^{z^4}$ for $n=4$

There are four attracting and four repelling petals for $N(z)$ at ∞. Since, n is even the attracting petals are symmetrical about x-axis and y-axis. The basin lies in the Fatou set and the parabolic point lies on the boundary of the basin and in the Julia set.

Also we can change $P(z)$ such as to (z^4-1) and $(z^4-1)(z^4-16)$

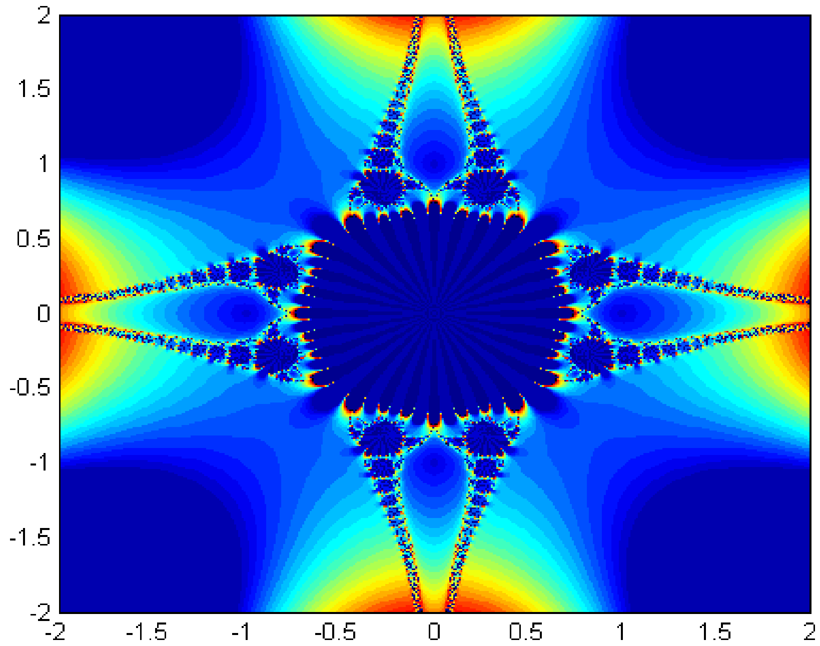

Figure 21 Basins Fractal of $F(z)=(z^4-1)e^{z^4}$ for $n=4$

We Have Four Roots, ∓ 1 , $\mp i$

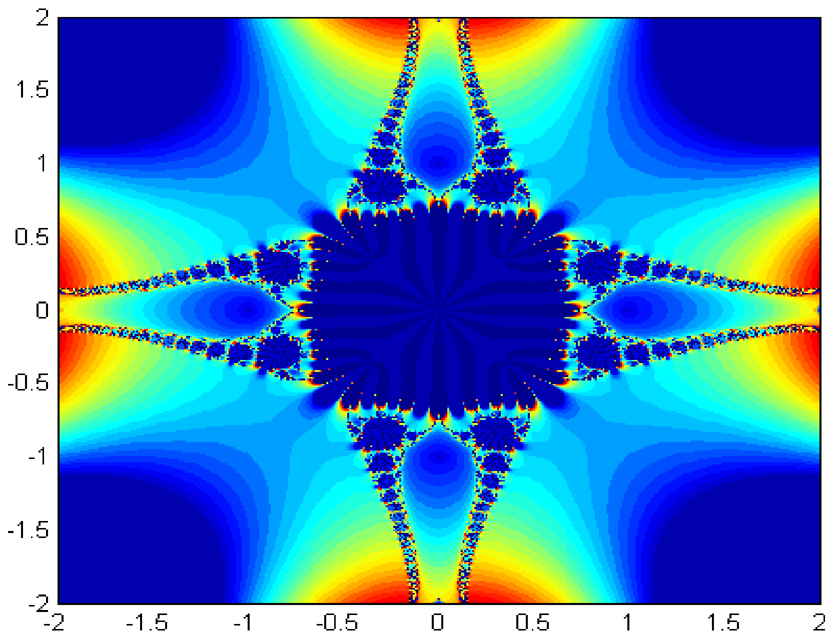

Figure 22 Basins Fractal of $F(z)=(z^4-1)(z^4-16)e^{z^4}$

for $n=4$,We Have Eight Roots $\mp 1, \mp i$, $\mp 2i$, ∓ 2

Example 5.

When $n=5$, the exponential function is $F(z)=ze^{z^5}$ **.** Therefore, the complex Newton's method of $F(z)=ze^{z^5}$ is

$$N(z)=z-\frac{ze^{z^5}}{(ze^{z^5})'}=z-\frac{z}{1+5z^5}.$$

Zero is a simple root of F . So, the derivative of $N(z)$ has the form

$$N^{'}(z)=1-\frac{1-20z^5}{(1+5z^5)^2}.$$

Then zero is super attracting fixed point of $N(z)$. Since $\lim_{z\to\infty} N(z)=\infty$, so, ∞ is a fixed point of $N(z)$. The series expansion of M is $M(v)=v+\frac{1}{5}v^6$.

When v = 0, $M'(0)=1$.Then, ∞ is a parabolic fixed point of $N(z)$.

$z\in(0,\infty)$. $[0,\infty)$ is a repelling direction for ∞ . Also, $(-\infty,0]$ is an attraction direction for ∞.

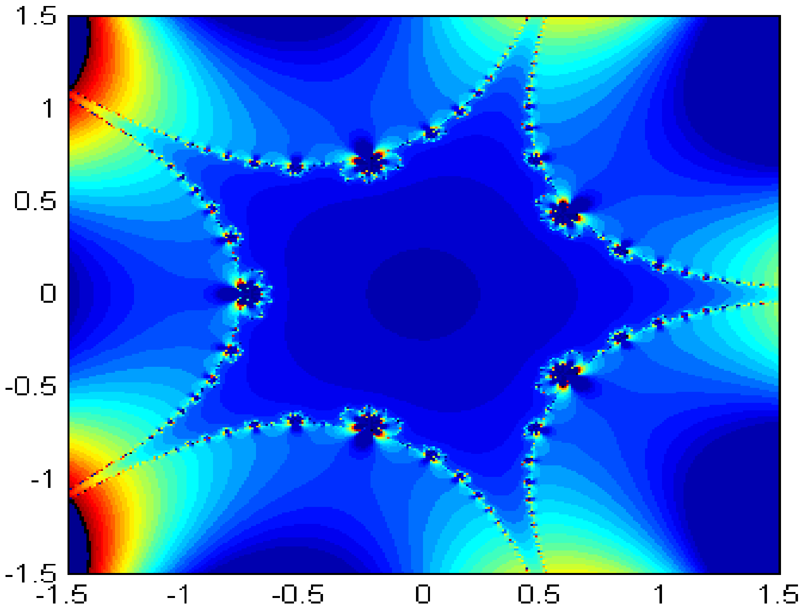

Figure 23 Basins Fractal of $F(z)=ze^{z^5}$ for $n=5$

Since, n =5, then, there are 5 attracting and 5 repelling petal. Julia set is boundary of the basin. Since n is odd then, the attracting petals not symmetrical on any axis. Basin tails of roots extend to ∞ and it most lies between pairs of attracting petals for ∞.

Also we can change $P(z)$ such as to $(z-5)$ and $(z-2)$

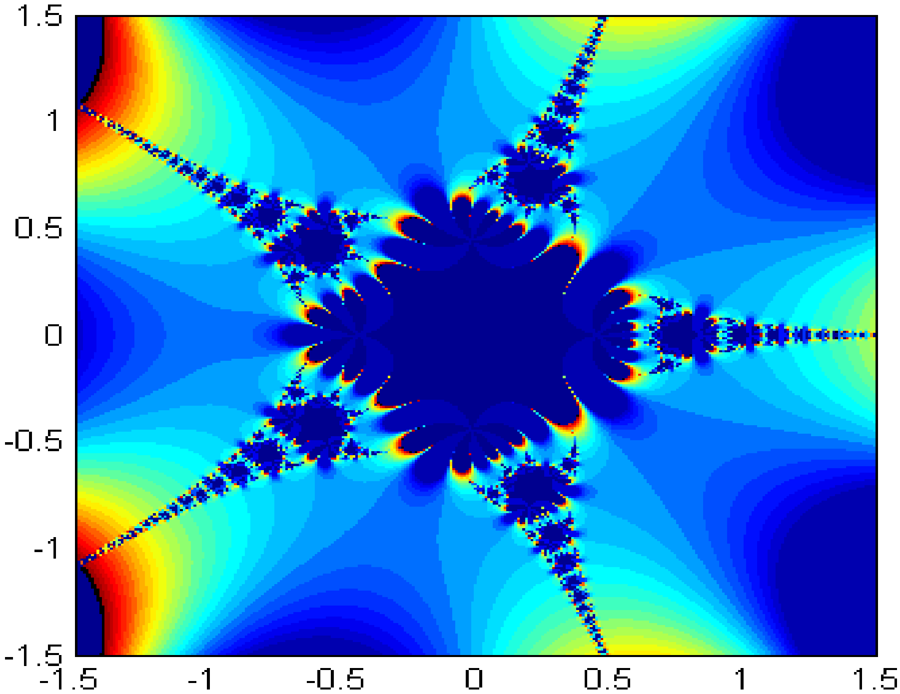

Figure 24 Basins Fractal of $F(z)=(z-5)e^{z^5}$ for $n=5$

We Have One Root 5

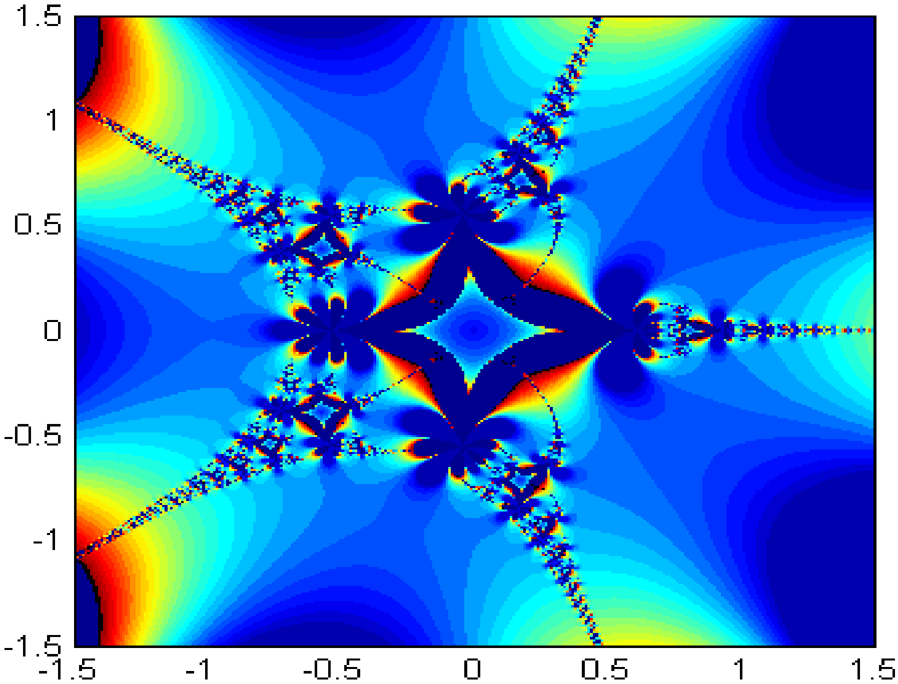

Figure 25 Basins Fractal of $F(z)=(z-2)e^{z^5}$ for $n=5$

We Have One Root

Example 6.

When $n = 6$, the exponential function is $F(z) = ze^{z^6}$. Therefore, the complex Newton's method of $F(z) = ze^{z^6}$ is

$$N(z) = z - \frac{ze^{z^6}}{(ze^{z^6})'} = z - \frac{z}{1+6z^6}.$$

Zero is a simple root of F. So, the derivative of $N_F(z)$ is

$$N^{'}(z) = 1 - \frac{1-30z^6}{(1+6z^6)^2}.$$

Then, zero is super attracting fixed point of $N(z)$. Since $\lim_{z\to\infty} N(z) = \infty,$ then ∞ is a parabolic fixed point of $N(z)$. The series expansion of M is $M(v) = v + \frac{1}{6}v^7$.

So, the derivative of $M(v)$ is $M'(v) = 1 + \frac{7}{6}v^6$.

When $v = 0, M'(0) = 1$, then, ∞ is a parabolic fixed point of $N(z)$.

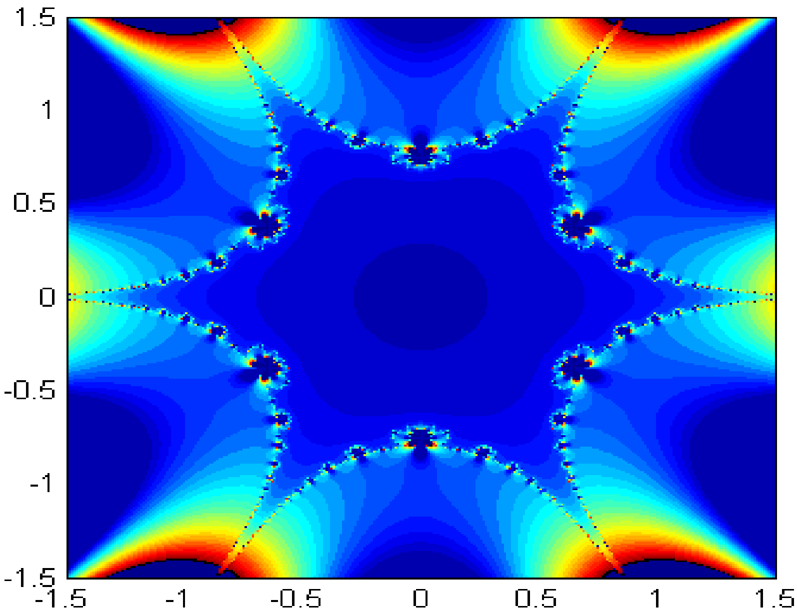

Figure 26 Basins Fractal of $F(z) = ze^{z^6}$ for $n = 6$

Since deg(Q)= 6, then there are six attracting and six repelling petals for $N(z)$ at ∞. The basin is lied in Fatu set and the parabolic point lies on the boundary of the basin and in the Julia set. The basin has finite area.

Also we can change $P(z)$ such as to (z^3-1) and $(z^3-1)(z^3+9)$

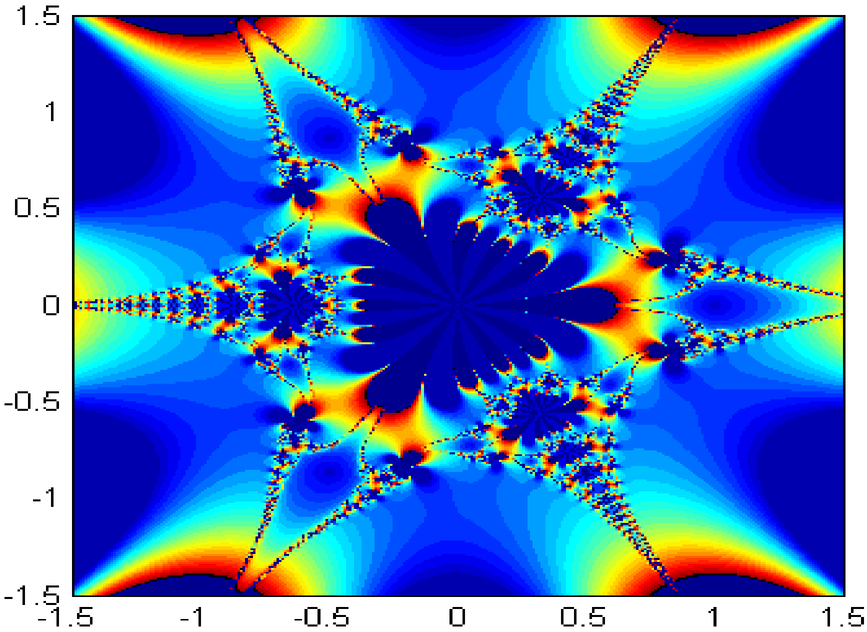

Figure 27 Basins Fractal of $F(z)=(z^3-1)e^{z^6}$ for $n=6$

We Have Three Roots $1, \quad -\frac{1}{2}+\frac{\sqrt{3}}{2}i, -\frac{1}{2}-\frac{\sqrt{3}}{2}i$

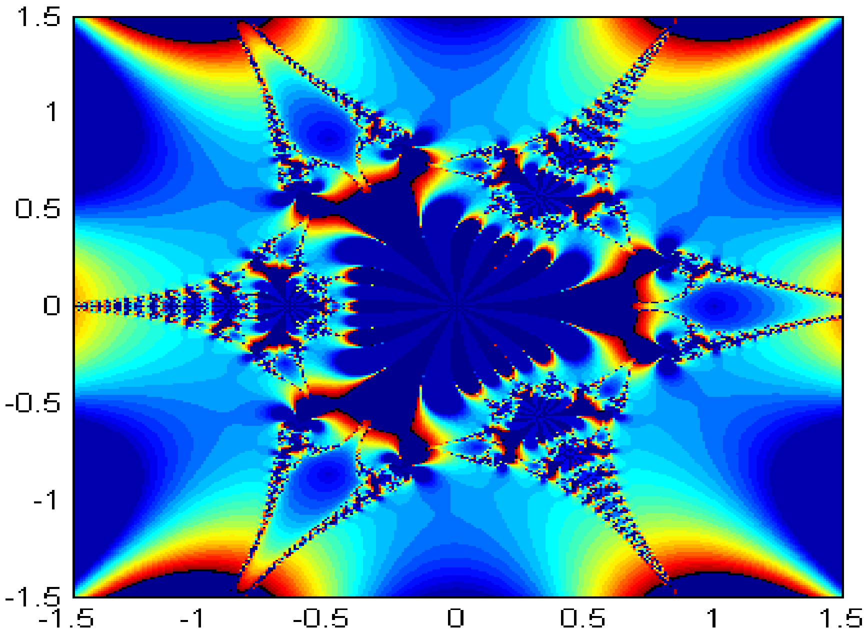

Figure 28 Basins Fractal of $F(z)=(z^3-1)(z^3+9)e^{z^6}$ for $n=6$

We Have Five Roots $1, \quad -\frac{1}{2}+\frac{\sqrt{3}}{2}i, \quad -\frac{1}{2}-\frac{\sqrt{3}}{2}i, \quad \pm 3$

Example 7. Some finite basins area fractals of different n .

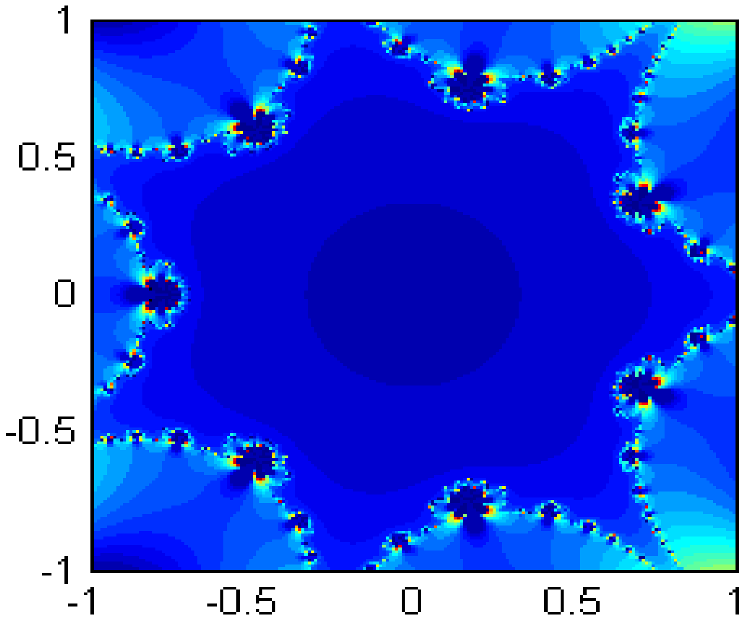

Figure 29 Basins Fractal of $F(z) = ze^{z^7}$ for $n = 7$

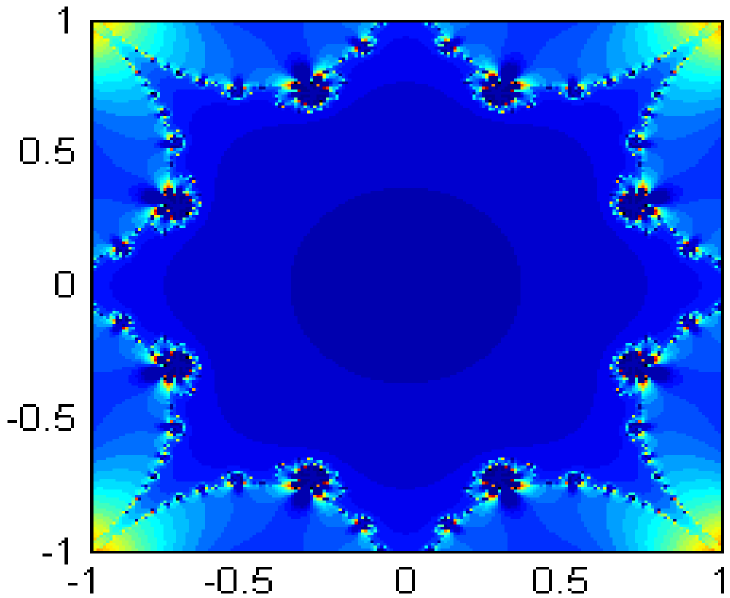

Figure 30 Basins Fractal of $F(z) = ze^{z^8}$ for $n = 8$

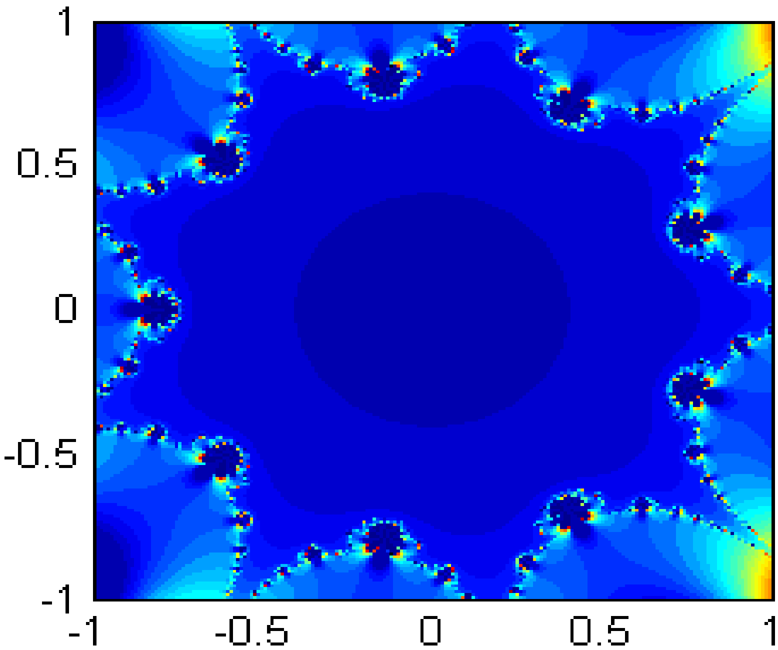

Figure 31 Basins Fractal of $F(z) = ze^{z^9}$ for $n = 9$

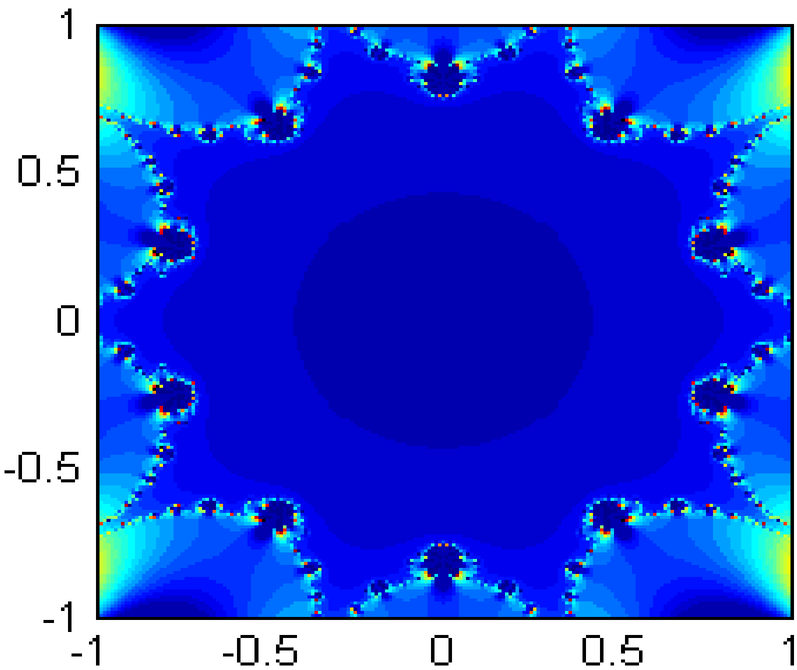

Figure 32 Basins Fractal of $F(z) = ze^{z^{10}}$ for $n = 10$

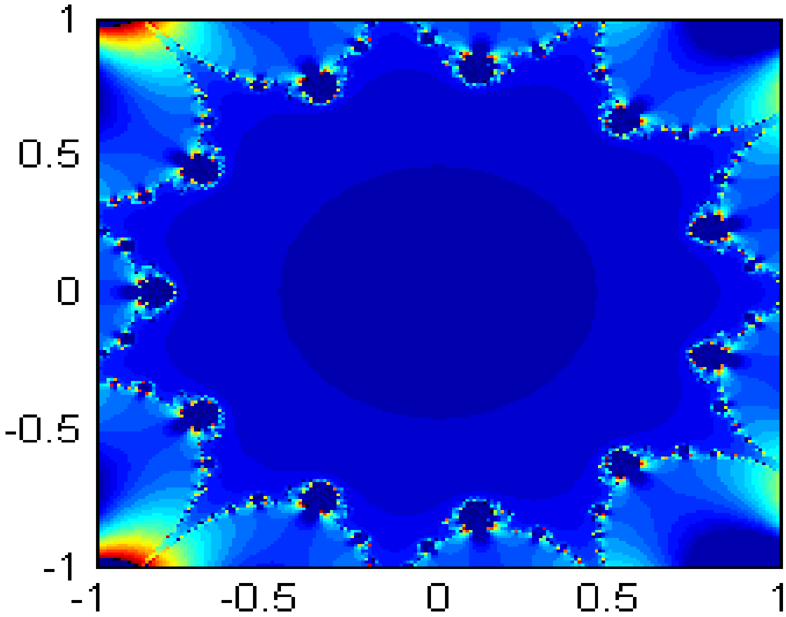

Figure 33 Basins Fractal of $F(z) = ze^{z^{11}}$ for $n = 11$

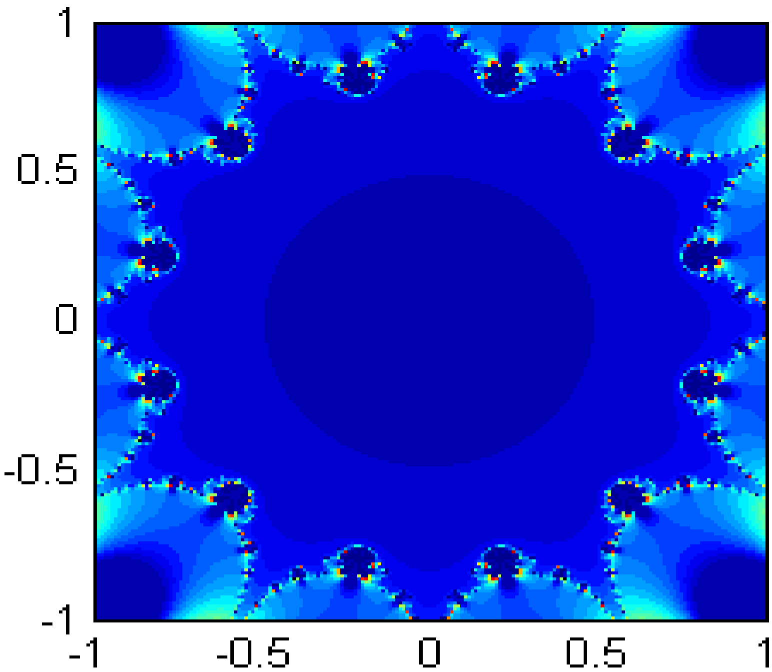

Figure 34 Basins Fractal of $F(z) = ze^{z^{12}}$ for $n = 12$

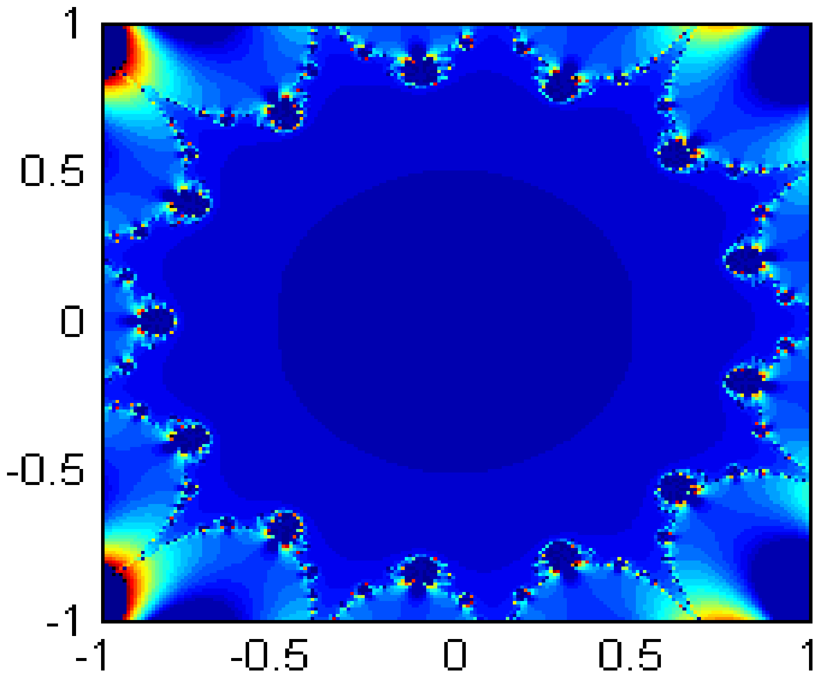

Figure 35 Basins Fractal of $F(z) = ze^{z^{13}}$ for $n = 13$

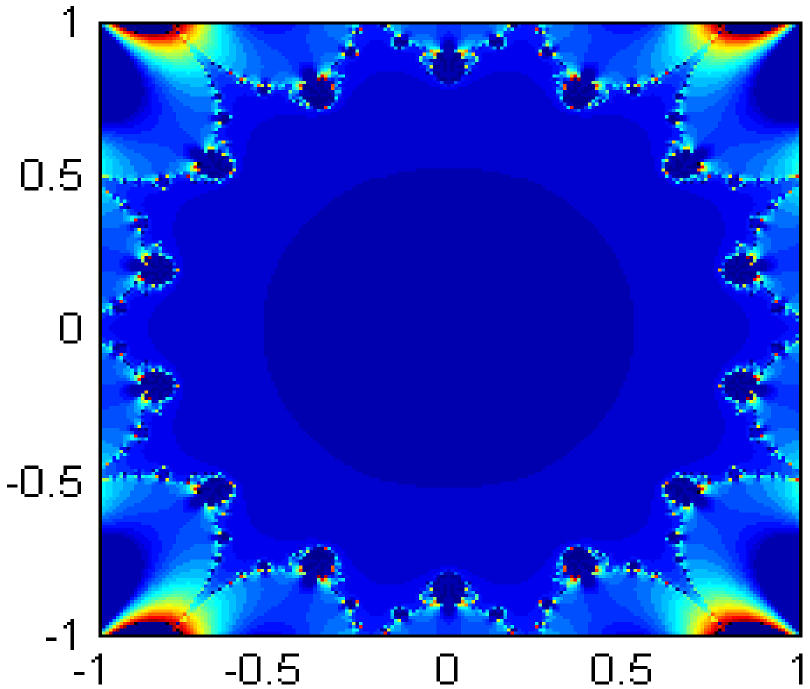

Figure 36 Basins Fractal of $F(z) = ze^{z^{14}}$ for $n = 14$

Iterative Number	Value of RGB			Color
	R	G	B	
1	1	0	0	
2	1	0.09375	0	
3	1	0.1875	0	
4	1	0.28125	0	
5	1	0.375	0	
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
20	0.21875	1	0	
21	0.125	1	0	
29	0	1	0.625	
30	0	1	0.71875	
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
57	1	0	0.75	
58	1	0	0.65625	
59	1	0	0.5625	
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
64	0	0	0	

Table 2 Color Palette's Descriptions Finite Basins Area

CHAPTER 4

CONCLUSION

In this thesis, we gave a review of Haruta method and we used his results to determine the type of basins area when $n \geq 3$by applied the complex Newton method on exponential functions $F(z) = P(z)e^{Q(z)}$, where $P(z) = z$ and $Q(z) = z^n$. We simulate our methods with MATLAB, and we get the same results of [31, 38- 42]. Therefore, when n =1, 2, we get infinite area of the complex Newton's basin, and when $n \geq 3$ we obtain finite area of complex Newton's basin. Also, when we change $P(z) = z$ in the complex exponential function to another polynomial we get also the same results when $n \geq 3$, and $n \leq 2$, that means the area of the basin depend on the deg(Q) not on the $P(z)$. The basin lie in Julia set for all n. The basins for each root in Julia set is finite area of $n \geq 3$ and infinite area of $n \leq 2$. If n is even means the attracting petals of the fixed point zero and infinity are symmetrical about x-axis and y-axis. We also added Color Palette's Descriptions table of the finite and infinite area.

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APPENDICES A

THE MATLAB CODE

```
clc
clear all
close all

x = -1:0.01:1;
y = -1:0.01:1;

for k = 1:length(y)
    for m = 1:length(x),
        z = x0(k,m) + i*y0(k,m);
        for itr = 1:30
            z = z - ((z)*exp(z^7))./(exp(z^7) + 7*z^7*exp(z^7));
            if abs((z)*exp(z^7)) < 1e-3
                n(k,m) = itr;
                break
            end
        end
    end
end

for ic = 1:nc
    ip = find(n >= pc(ic) & n <= pc(ic+1));
    if ~isempty(ip)
        ih = ih + 1;
        h(ih) = plot(x0(ip),y0(ip),'.','markersize',2,'color',[cmap(nc-
ic+1,:)]);
        hold on
        pause(0.001)
    end
end
```

APPENDICES B

CURRICULUM VITAE

PERSONAL INFORMATION

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Date and Place of Birth: 20 June 1979, Diyala-Iraq

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EDUCATION

Degree	Institution	Year of Graduation
M.Sc.	Çankaya University, Mathematics and Computer Science	2015
B.Sc.	University of Mustansiriya	2002
High School	Tahreer High School	1999

WORK EXPERIENCE

Year	Place	Enrollment
2003-2008	Diyala University	Employee
2008-peresent	Kirkuk University	Employee

FOREIN LANGUAGES

English.

HOBBIES

Travel, Books, Sport.