AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(2): 3484–3522.
DOI: 10.3934/math.2023178
Received: 16 August 2022
Revised: 30 October 2022
Accepted: 10 November 2022
Published: 22 November 2022

Research article

A novel numerical dynamics of fractional derivatives involving singular and nonsingular kernels: designing a stochastic cholera epidemic model

Saima Rashid[1,*], Fahd Jarad[2,3,4,*], Hajid Alsubaie[5], Ayman A. Aly[5] and Ahmed Alotaibi[5]

[1] Department of Mathematics, Government College university, Faislabad, Pakistan
[2] Department of Mathematics, Çankaya University, Ankara, Turkey
[3] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan
[4] Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia
[5] Department of Mechnical Enginnering, College of Engineering, Taif University, Taif, Saudi Arabia

* **Correspondence:** Email: saimarashid@gcuf.edu.pk, fahd@cankaya.edu.tr.

Abstract: In this research, we investigate the direct interaction acquisition method to create a stochastic computational formula of cholera infection evolution via the fractional calculus theory. Susceptible people, infected individuals, medicated individuals, and restored individuals are all included in the framework. Besides that, we transformed the mathematical approach into a stochastic model since it neglected the randomization mechanism and external influences. The descriptive behaviours of systems are then investigated, including the global positivity of the solution, ergodicity and stationary distribution are carried out. Furthermore, the stochastic reproductive number for the system is determined while for the case $\mathbb{R}_0^s > 1$, some sufficient condition for the existence of stationary distribution is obtained. To test the complexity of the proposed scheme, various fractional derivative operators such as power law, exponential decay law and the generalized Mittag-Leffler kernel were used. We included a stochastic factor in every case and employed linear growth and Lipschitz criteria to illustrate the existence and uniqueness of solutions. So every case was numerically investigated, utilizing the newest numerical technique. According to simulation data, the main significant aspects of eradicating cholera infection from society are reduced interaction incidence, improved therapeutic rate, and hygiene facilities.

Keywords: cholera epidemic model; fractional derivative operator; numerical solutions; Itô derivative; ergodic and stationary distribution
Mathematics Subject Classification: 46S40, 47H10, 54H25

1. Introduction

Fractional calculus (FC), which is defined as the advancement or modification of conventional derivatives and integrals to non-integer order instances, has received considerable scholarly emphasis in recent times. According to the research, numerous scientific and other physical processes involve fractional derivatives, including ideological, aquifer, electrostatics, financing and hydrodynamics, to highlight a few [1, 2].

Several researchers have explored the simulated predictions owing to the availability and distinctiveness of fractional differential equation systems in diverse configurations, for example [3–6]. In FC, there are three types of fractional derivative/integral formulations: the Riemann-Liouville and Caputo derivatives, which are a concatenation of power law (PL) functions [1], including the first derivative; the Caputo-Fabrizio fractional derivative, which is a combination of the first derivative and the exponential decay (ED) law [7], including the Deta-Dirac feature; and the Atangana–Baleanu fractional derivative [8], which has a generalized Mittag-Leffler (GML) as a kernel [9–11]. The majority of the relevant and purposeful formulations in the various configuration domains of such fractional-order derivative formulations in the scope of FC, which includes numerical simulations modelled with formulae utilizing these three types of kernels, can be found in significant relevant scholarly studies in [12, 13] and the references cited therein.

Recently, ordinary differential equation (DE) and partial differential equation (PDE) solutions have often been deployed to generate modeling techniques for multiple connections in healthcare, biological, mechanical and technological procedures [14, 15]. The predominance and appropriateness of numerical method techniques offer the cornerstone for assessing precision and investigating pertinent natural occurrences. The investigation has indeed expanded to spatial complex issues [16], that comprise hereditary attributes of dynamic interplay in physics and technology. The processes of customary Itô-Doob form Markov decision DEs [17], stochastic PDEs [18], stochastic fractional DEs [19], and stochastic fractional PDEs in ambiguous contexts [20] were characterized in the framework of Itô-Doob type stochastic integral equations around 1960 for conspicuous supercomputing specific applications. The Wiener data transformation [21] addresses the effects of random environmental factors. This has been significantly explored by employing local stochastic formulae [22]. We believe that researchers are progressively pursuing advancements of vital constituent perceptions that devour the understanding of the initial estimate to its commensurate approximate analytical consideration. Based on the aforementioned contextualized formation of collaborative simulation and to incorporate auxiliary mapping and/or randomized perturbation characteristics of interplay into the quantitative equations highlighted by DEs.

Furthermore, newly developed randomized evaluation procedures for financial data (stock price) using a combination of classical computational models and randomized methodologies [23]. We need to inform the developing mathematical algorithms by deliberately providing key significant straightforwardly identified performance requirements or attributes to structure components in order to improve this approach to more incredibly complex processes in the scientific world operating under inherently functional and external randomized intervention. Rashid et al. [24] new estimates of a stochastic fractal-fractional immune effector response to viral infection via latently infectious tissues and [25] provided a detailed evaluation of the stochastic fractal-fractional TB model using the Mittag-Leffler kernel and white noise.

Cholera is a severe gastrointestinal disease induced by the consumption of contact sources of bodily fluids containing the bacteria Vibrio cholera. Only V. cholerae o1 and o139 are documented to result in cholera [26]. Highly virulent o1 (relatively significant, Vibrio cholera o139) gets through and endures the body's digestive enzyme resistance before penetrating the fluid coating that protects the colonic epithelium [27]. Even before they inhabit the esophageal stomach, they generate cholera toxin (that also provokes extracellular fluid volume efflux even by the small intestine's vascular endothelium), resulting in a plentiful, stress-free and mucilaginous gastroenteritis that can speedily cause serious hyponatremia and fatality if intervention is not offered immediately. The proportion of respondents who vomited in healthy participant studies was estimated to be $10^2 - 10^3$ [28]. Vibrio can be transmitted through human-to-human contact (e.g., faecal contamination) or through human-to-environment contact (e.g., consuming polluted food and beverages from the environment). These are the features of cholera, which belongs to the worldwide preventive pathogens as required by the World Health Organization (WHO), in addition to being part of a group of viral infections as specified by Chinese law for viral epidemic surveillance and prevention. Secretory alkaloids are used by pancreatic bacteria to penetrate or damage the intestine epithelium. They can manipulate human anatomy in a variety of ways, including intentionally causing illness, parthenogenesis, or encouraging other procedures such as manipulating innate immunity, escaping from the mitochondrion and penetrating human obstacles, to name a few.

Ecological phenomena have a substantial impact on the transmission of environmental infections. Since Vibrio cholerae may move through liquids, such a modification in hydrological mechanisms has had the ability to alter the dangerous poisoning in water. Climate change and storms, in addition to moisture and its periodicity, can also speed up or slow down the transmission pathway [29]. However, the environment where the virus's spread into epidemic domains [30, 31], along with informal transit and bacteria migrants on long journeys, (see Figure 1).

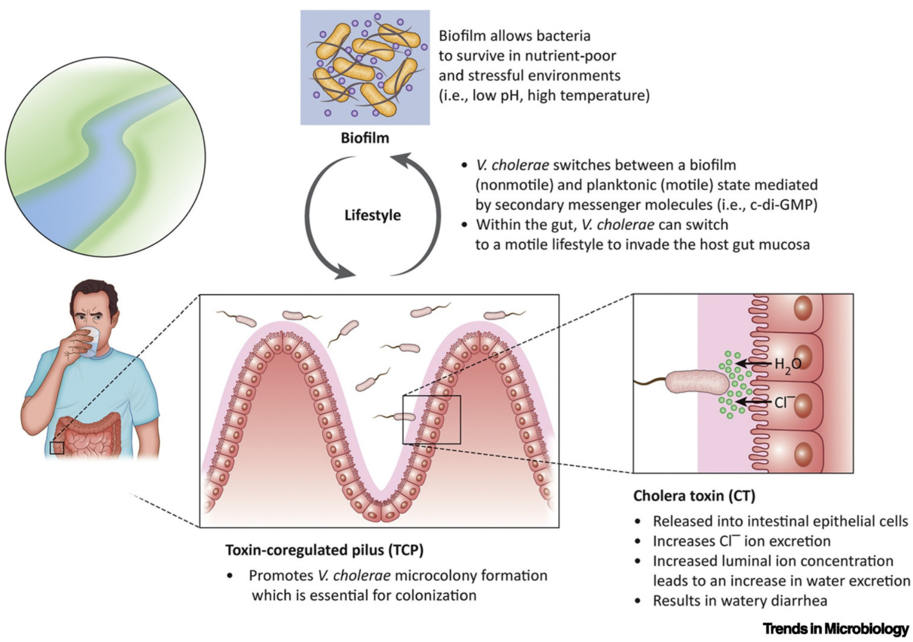


Figure 1. Primary penetration process of cholera disease.

The vibrionaceae group has a number of clinically important pathogens that can promote gastritis. Vibrio cholerae is the most well-known bacterium, transporting a wide range of compounds that cause sickness. This bacteria is the cause of cholera, a small intestinal illness characterized by acute gastroenteritis, nausea, and malnutrition. It is implicated in millions of instances and numerous fatalities every year throughout the globe.

Seafood and zooplankton were identified as having cholera germs. Individuals afflicted with cholera frequently experience diarrhea, and the risk of propagation is conceivable if this extremely watery feces, informally known as “rice-water,” pollutes groundwater consumed by others. Figure 2 show that a single intestinal cramping incident can multiply the amount of V. cholerae in the ecosystem by a million species.

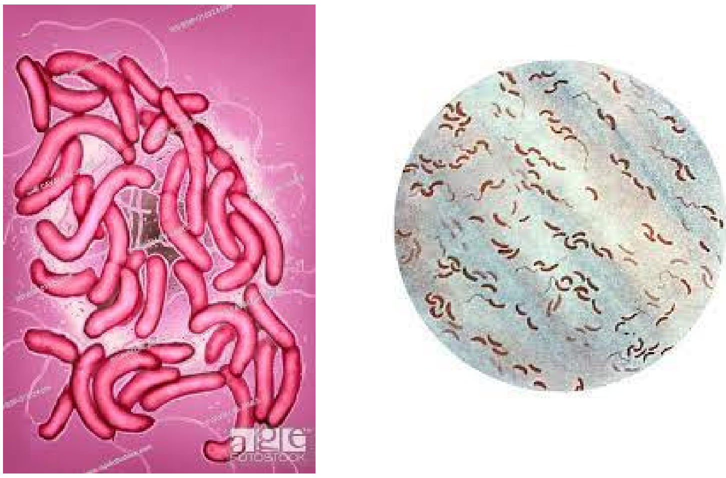


Figure 2. (a) Vibrio cholerae bacterium species. (b) Detecting cholera in a blood clot.

Susceptible people that relocate frequently potentially fall in touch with the virus in specific areas and spread it to other regularly inhabited communities, which might or might not be affected. Infected people may spread the illness by spewing microbes from their faces at a comparable time, especially if they exhibit no symptoms [32]. As a result, untreated individuals significantly increase pathogen intensities, which somehow spread within the hemodynamic network.

Personalized attempts to transmit deadly diseases are unacceptable. Professionals and perhaps various researchers employ numerical articulation and simulation tools to obtain contemporary documentation of the features and preventative strategies of infectious diseases and also to identify sensitive perceptions, fluctuations in attribution configurations and estimations [29, 33, 34]. The estimations concentrate on stratification for specific groups, such as vulnerability, contamination, treatment and recovery as well as specific therapeutic measures such as treatment, immunization, pasteurization and sanitation through enlightenment and immunization. When incidents occur, medicine appears to be increasingly recommended, i.e., a relatively brief strategy for managing and eradicating the sickness. Whenever it concerns disease avoidance and removal from the atmosphere, education is the most appropriate option [35–41]. The filtration technique, sewage removal, regulatory frameworks, hygiene practises and adequate sterilization are all aligned within the long-term pathogen prevention and elimination plan. It is apparent that certain people are aware of and adhere to the right cholera infection protection procedures, see; [42–46]. In the current state of contemporary knowledge, similar settlements are not recognized as a consistent framework of individuals from the perspective of vulnerable, polluted, and treated individual categories in response to monitoring, controlling, and extinguishing cholera behaviors. In this paper, we mainly concentrate on the simple transmission channel, despite the fact that the virus's stochastic fractional system dynamics.

The main aim is to demonstrate the control mechanism and analyze the cholera transmission dynamics provided by considerably using the stochastic FDEs strategy, taking into account the well-noted (Caputo, Caputo-Fabrizio, and Atangana-Baleanu derivative operators) are energized by the recent discussion. In addition, we have established the existence-uniqueness of the positive global solutions of the proposed model. Furthermore, the stochastic reproductive number and local stability of the system are carried out. Novel numerical findings are illustrated with the aid of fractional calculus. No one has previously utilized the idea of the randomized technique to fractional calculus in cholera intricacy previously, as stated in the study. This study takes environmental aspects into account and a randomized fractional framework of cholera evolution has been established. As a result, this represents a new methodology. The incorporation of untreated individuals and associated characterization in stochastic fractional DE will serve as a jumping-off point for researchers and theorists engaging in simulating techniques.

This work is structured as described in the following: Section 2 represents the basic formulations that can be utilized in the upcoming parts of the article. Section 3 configures the model and investigates the qualitative properties of the proposed model. Section 4 consists of the existence-uniqueness outcomes and integro-fractional procedure for the stochastic cholera epidemic model. These are illustrated via the well-known fractional differential operators and the Newton interpolation polynomial approach. In Section 6, we present our insights and conclusions.

2. Preliminaries

Here, we will examine several basic concepts and features of fractional calculus theory in this part, which will be essential in the subsequent subsections.

Definition 2.1. ([2]) The Caputo fractional derivative (CFD) is described as follows:

$$ {}_{0}^{c}\mathbf{D}_{\xi}^{\varphi}\mathbf{w}(\xi) = \begin{cases} \frac{1}{\Gamma(r-\varphi)} \int\limits_{0}^{\xi} \frac{\mathbf{w}^{(r)}(x_1)}{(\xi-x_1)^{\varphi+1-r}} dx_1, & r-1 < \varphi < r, \\ \frac{d^r}{d\xi^r}\mathbf{w}(\xi), & \varphi = r. \end{cases} \tag{2.1} $$

Definition 2.2. ([7]) The Caputo fractional derivative operator of order φ is described as follows:

$$ {}^{CF}\mathbf{D}_{\xi}^{\varphi}(\mathbf{w}(\xi)) = \frac{\mathbb{M}(\varphi)}{(1-\varphi)} \int\limits_{a_1}^{\xi} exp\Big(-\frac{\varphi(\xi-x_1)}{1-\varphi}\Big)\mathbf{w}'(\xi)d\xi, \tag{2.2} $$

where $\mathbf{w} \in \mathbf{H}^1(a_1, a_2)(Sobolev\ space)$, $a_1 < a_2, \varphi \in [0,1]$ and $\mathbb{M}(\varphi)$ signifies a normalization function as $\mathbb{M}(\varphi) = \mathbb{M}(0) = \mathbb{M}(1) = 1$.

Definition 2.3. ([7]) The fractional integral of the Caputo-Fabrizio operator is defined as:

$$ {}^{CF}\mathcal{I}_{\xi}^{\varphi}(\mathbf{w}(\xi)) = \frac{1-\varphi}{\mathbb{M}(\varphi)}\mathbf{w}(\xi) + \frac{\varphi}{\mathbb{M}(\varphi)} \int\limits_{a_1}^{\varrho} \mathbf{w}(x_1)dx_1. \tag{2.3} $$

Definition 2.4. ([8]) Suppose $\mathbf{w} \in H^1(0, a_1)$ be the fractional Atangana-Baleanu derivative of mapping $\mathbf{w}$ is stated by the subsequent integral:

$$ {}^{ABC}\mathbf{D}^{\varphi}_{0,\xi}\mathbf{w}(\xi) = \frac{\mathbb{ABC}(\varphi)}{1-\varphi} \int_0^{\xi} \mathbf{w}'(\xi) E_{\varphi}\Big(-\varphi \frac{(\xi - s_1)^{\varphi}}{1-\varphi} \Big) ds_1 \tag{2.4} $$

having $\varphi \in (0, 1)$, $\mathbb{ABC}(\varphi)$ is normalizing positive mapping admitting $\mathbb{ABC}(0) = \mathbb{ABC}(1) = 1$ and E_{φ} is the Mittag-Leffler mapping [8].

Definition 2.5. ([8]) The Atangana-Baleanu fractional integral form of the function $\mathbf{w} \in \mathbb{C}^1(a_1, b_1)$ is defined as:

$$ {}^{AB}_{a_1}\mathcal{I}^{\varphi}_{\xi}\mathbf{w}(\xi) = \frac{1-\varphi}{\mathbb{ABC}(\varphi)}\mathbf{w}(\xi) + \frac{\varphi}{\mathbb{ABC}(\varphi)\Gamma(\varphi)} \int_{a_1}^{\xi} \mathbf{w}(x_1)(\xi - x_1)^{\varphi-1} dx_1. \tag{2.5} $$

3. Model configuration and analysis

The mechanism takes into account the total population (($\mathbf{P}(\xi)$) and categorizes it into four groups: susceptibility ($\mathbf{S}(\xi)$), infectious ($\mathbf{I}(\xi)$), treatment ($\mathbf{T}(\xi)$), and restored ($\mathbf{R}(\xi)$). Susceptible people ($\mathbf{S}(\xi)$) are these kinds of people who can not be contaminated. However, in the long run, it may get polluted. Highly infectious individuals ($\mathbf{I}(\xi)$) who have developed infectious disease symptoms and have the potential to transmit the illnesses, intervention category ($\mathbf{T}(\xi)$) involve all who take medication after being colonized by infection for $\xi > 0$, and the healed cohort ($\mathbf{R}(\xi)$) includes those who have recovered from infection.

By gaining temporary immunity, the populace in the susceptibility group will enhance the evolution of ϕ and alongside a preservation rate π from the restored compartment. Furthermore, its population declines due to natural factors, resulting in a fatality incidence of ψ, as well as a translocation to the contaminated cohort at a pace of δ. The prevalence of association of δ will boost the population in the infected segment, in comparison to the randomly generated fatality incidence of ψ, epidemic resulting mortality incidence of η, and transfer to the therapy cohort with the therapy value of ζ will decrease the population. Assuming a ζ therapeutic incidence, a γ restoration speed, and a ψ spontaneous leading fatality rate. The restored cohort's community develops at a γ pace while shrinking due to natural risk factors at a ψ rate and resiliency decline at a ϕ rate.

The system is influenced by the respective assumptions: based on biological population density, the incidence and death are constant, but the incidence and death are not similar, all characteristics are non-negative, most people in the society are susceptible, restorative intervention is implemented on extremely infectious individuals, medicated people (people on therapies) do not communicate infection symptoms to the susceptible global populace, there is transient leniency, and there is the illness on restoration.

Considering the aforementioned explanations and premises, our system is informally depicted in Figure 3. The specification and process model are used to generate the underlying stochastic mathematical description (3.1) of Figure 3:

$$\begin{cases} \dot{\mathbf{S}}(\xi) = \pi + \phi\mathbf{R} - (\delta\mathbf{I} + \psi)\mathbf{S}, \\ \dot{\mathbf{I}}(\xi) = \delta\mathbf{IS} - (\psi + \eta + \zeta)\mathbf{I}, \\ \dot{\mathbf{T}}(\xi) = \zeta\mathbf{I} - (\gamma + \psi + \varsigma)\mathbf{T}, \\ \dot{\mathbf{R}}(\xi) = \gamma\mathbf{T} - (\phi + \psi)\mathbf{R}, \end{cases} \tag{3.1}$$

supplemented with ICs $\mathbf{S}(0) = \mathbf{S}_0,\ \mathbf{I}(0) = \mathbf{I}_0,\ \mathbf{T}(0) = \mathbf{T}_0,\ \mathbf{R}(0) = \mathbf{r}_0$.

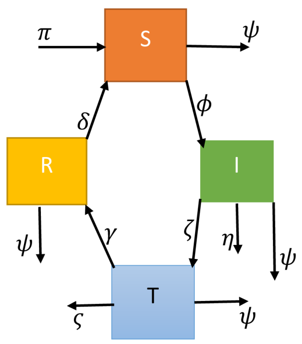


Figure 3. Schematic configuration of cholera model.

We converted the mathematical approach in formula (3.1) to a stochastic approach because it did not account for stochastic contextual parameters and did not have meaningful settings.

To expand, we multiplied formula (3.1) by $d\xi$ and add Brownian motion $B_J(\xi)$ and the strength of stochastic external conditions (ρ_J), So we achieve the desired stochastic system of cholera epidemic:

$$\begin{cases} d\mathbf{S}(\xi) = (\pi + \phi\mathbf{R} - (\delta\mathbf{I} + \psi)\mathbf{S})d\xi + \rho_1\mathbf{S}dB_1(\xi), \\ d\mathbf{I}(\xi) = (\delta\mathbf{IS} - (\psi + \eta + \zeta)\mathbf{I})d\xi + \rho_2\mathbf{I}dB_2(\xi), \\ d\mathbf{T}(\xi) = (\zeta\mathbf{I} - (\gamma + \psi + \varsigma)\mathbf{T})d\xi + \rho_3\mathbf{T}dB_3(\xi), \\ d\mathbf{R}(\xi) = (\gamma\mathbf{T} - (\phi + \psi)\mathbf{R})d\xi + \rho_4\mathbf{R}dB_4(\xi), \end{cases} \tag{3.2}$$

where $\rho_J \geq 0,\ J = 1,2,3,4$ represents the strength of Brownian motion and $B_J,\ J = 1,2,3,4$ are independent Brownian motion.

Suppose a complete probability space $(\Omega, \mathfrak{F}, \{\mathfrak{F}_\xi\}_{\xi\geq 0}, \mathbb{P})$ fulfilling the given assumptions (That are, it is nondecreasing and right continuous whilst $\mathfrak{F}_0$ have all empty sets $\mathbb{P}$), indicating $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_+^d = \{\mathbf{v} = (\mathbf{v}_1, ..., \mathbf{v}_d) \mid \mathbf{v}_i > 0,\ i \in [1, d]\}$. Also, there is an integral mapping $\mathfrak{F}_1(\xi)$ defined on $[0, \infty)$. introducing $\mathfrak{F}_1^u = \sup\{\mathfrak{F}_1(\xi) \mid \xi \geq 0\},\ \mathfrak{F}_1^l = \inf\{\mathfrak{F}_1(\xi) \mid \xi \geq 0\}$.

Next, we will examine at the d-dimensional stochastic DE

$$d\mathbf{X}(\xi) = \mathfrak{F}_1(\mathbf{X}(\xi), \xi)\mathrm{d}\xi + \mathbf{U}(\mathbf{X}(\xi), \xi)dB(\xi), \quad t \geq t_0,$$

subject to intial condition $\mathbf{X}(0) = X_0 \in \mathbb{R}^d$, $B(\xi)$ denotes a d-dimensional standard Brownian motion presented on the complete probability space $(\Omega, \mathfrak{F}, \{\mathfrak{F}_\xi\}_{\xi\geq 0}, \mathbb{P})$. Suppose $\mathbb{C}^{2,1}(\mathbb{R}^d \times [t_0, \infty]; \mathbb{R}_+)$ the collection of all positive $\mathfrak{I}(\mathbf{v},\xi)$ on $\mathbb{R}^d \times [t_0,\infty]$ such that continuous twice differentiable in $\mathbf{X}$ and once in ξ. The differential operator $\mathbb{L}$ is proposed by [47]:

$$\mathbb{L} = \frac{\partial}{\partial \xi} + \sum_{\iota=1}^{d_1} f_\iota(\mathbf{X},\xi)\frac{\partial}{\partial X_\iota} + \frac{1}{2}\sum_{\iota,\kappa=1}^{d_1} [\mathbf{U}^{\mathbf{T}}(\mathbf{X},\xi)\mathbf{U}(\mathbf{X},\xi)]_{\iota\kappa}\frac{\partial^2}{\partial X_\iota \partial X_\kappa}.$$

Now $\mathbb{L}$ imposed on a mapping $\mathfrak{I} \in \mathbb{C}^{2,1}(\mathbb{R}^d \times [t_0,\infty];\mathbb{R}_+)$, we have

$$\mathbb{L}\mathfrak{I}(\mathbf{X},\xi) = \mathfrak{I}_\xi(\mathbf{X},\xi) + \mathfrak{I}_{\mathbf{v}}(\mathbf{X},\xi)\mathfrak{F}_1(\mathbf{X},\xi) + \frac{1}{2}trac[\mathbf{U}^{\mathbf{T}}(\mathbf{X},\xi)\mathfrak{I}_{\mathbf{vv}}\mathbf{U}(\mathbf{X},\xi)],$$

where $\mathfrak{I}_\xi = \frac{\partial \mathfrak{I}}{\partial \xi}$, $\mathfrak{I}_{\mathbf{v}} = (\frac{\partial \mathfrak{I}}{\partial \mathbf{v}_1},...,\frac{\partial \mathfrak{I}}{\partial \mathbf{v}_d})$, $\mathfrak{I}_{\mathbf{vv}} = (\frac{\partial^2 \mathfrak{I}}{\partial \mathbf{v}_\iota \partial \mathbf{v}_\kappa})_{d_1 \times d_1}$. By the Itô technique, if $\mathbf{X}(\xi) \in \mathbb{R}^{d_1}$, then

$$\mathrm{d}\mathfrak{I}(\mathbf{X}(\xi),\xi) = \mathbb{L}\mathfrak{I}(\mathbf{X}(\xi),\xi)\mathrm{d}\xi + \mathfrak{I}_{\mathbf{v}}(\mathbf{X}(\xi),\xi)\mathbf{U}(\mathbf{X}(\xi),\xi)\mathrm{d}B(\xi).$$

3.1. Global positivity of the solutions of the cholera model

In this section, we will demonstrate that our system acknowledges a unique positive solution that is also bounded. For the sake of brevity, we define the stochastic model associated with the novel cholera epidemic model as assuming it is biologically well-posed. It is important to note that the confirmation illustrations for this portion and the entirety of the paper can be discovered in the literature; however, we modify the formalism with our novel model here [48, 49].

Theorem 3.1. *Let the initial value* $(\mathbf{S}(0),\mathbf{I}(0),\mathbf{T}(0),\mathbf{R}(0))^T \in \mathbb{R}^4_+$, *then the solution* $(\mathbf{S}(\xi),\mathbf{I}(\xi),\mathbf{T}(\xi),\mathbf{R}(\xi))^T$ *for all strict positive time of the model* (3.2) *exists, and is positive almost surely (a.s).*

Proof. Surmise that $\mathcal{K}_0$ assumed to be the largest integer. Also, if $(\mathbf{S}_0,\mathbf{I}_0,\mathbf{T}_0,\mathbf{R}_0) \in \mathbb{R}^4_+$, then each factor of $(\mathbf{S}_0,\mathbf{I}_0,\mathbf{T}_0,\mathbf{R}_0)$ stays in $[\frac{1}{\Bbbk_0},1]$. For every integer $\mathcal{K} \geq \Bbbk_0$. Stopping time can be stated as follows:

$$\begin{aligned}\Lambda_{\Bbbk} &= \inf\Big\{\xi \in [0,\Lambda_\epsilon] : \mathbf{S}(\xi) \leq \frac{1}{\Bbbk},\ or\ \mathbf{I}(\xi) \leq \frac{1}{\Bbbk},\ \mathbf{T}(\xi) \leq \frac{1}{\Bbbk},\ \mathbf{R}(\xi) \leq \frac{1}{\Bbbk}\Big\},\\ \Lambda_\infty &= \inf\Big\{\xi \in [0,\Lambda_\epsilon] : \mathbf{S}(\xi) \leq 0,\ or\ \mathbf{I}(\xi) \leq 0,\ \mathbf{T}(\xi) \leq 0,\ \mathbf{R}(\xi) \leq 0\Big\}.\end{aligned} \tag{3.3}$$

Our goal is to demonstrate that $\mathbf{P}(\Lambda = \infty)$, i.e., $\mathbf{P}(\Lambda < \mathbf{Q})$, $for\ \mathbf{Q} > 0$, in order to reveal $\lim_{\xi \mapsto \infty} \sup \mathbf{P}(\Lambda_{\Bbbk} < 0) = 0$.

Introduce a Lyapunov candidate $\mathcal{U}$ as

$$\mathcal{U}(\xi) = \ln\frac{1}{\mathbf{S}(\xi)} + \ln\frac{1}{\mathbf{I}(\xi)} + \ln\frac{1}{\mathbf{T}(\xi)} + \ln\frac{1}{\mathbf{R}(\xi)}.$$

Using the fact of Itô formula, for $\mathbf{Q} > 0$, $\xi \in [0,\mathbf{Q} \wedge \Lambda_{\Bbbk}]$ to $\mathbf{X}(\xi) = (\mathbf{S}(\xi),\mathbf{I}(\xi),\mathbf{T}(\xi),\mathbf{R}(\xi))$, we find

$$d\mathcal{U}(\xi) = -\Big(\frac{1}{\mathbf{S}(\xi)}d\mathbf{S}(\xi) + \frac{1}{\mathbf{I}(\xi)}d\mathbf{I}(\xi) + \frac{1}{\mathbf{T}(\xi)}d\mathbf{T}(\xi) + \frac{1}{\mathbf{R}(\xi)}d\mathbf{R}(\xi) - \frac{1}{\mathbf{S}^2(\xi)}d\mathbf{S}^2(\xi) - \frac{1}{\mathbf{I}^2(\xi)}d\mathbf{I}^2(\xi)\Big).$$

Thus, we have

$$d\mathcal{U}(\xi) \quad = -\frac{1}{\mathbf{S}(\xi)}\Big\{(\pi + \phi\mathbf{R}(\xi) - \delta\mathbf{S}(\xi)\mathbf{I}(\xi) - \psi\mathbf{S}(\xi))d\xi + \rho_1\mathbf{S}(\xi)dB_1(\xi)\Big\}$$

$$
\begin{aligned}
&-\frac{1}{\mathbf{I}(\xi)}\Big\{(\delta\mathbf{S}(\xi)\mathbf{I}(\xi)-(\psi+\zeta+\eta)\mathbf{I}(\xi))d\xi+\rho_2\mathbf{I}(\xi)dB_2(\xi)\Big\}\\
&-\frac{1}{\mathbf{T}(\xi)}\Big\{(\zeta\mathbf{I}(\xi)-(\psi+\gamma+\varsigma)\mathbf{T}(\xi))d\xi+\rho_3\mathbf{T}(\xi)dB_3(\xi)\Big\}\\
&-\frac{1}{\mathbf{R}(\xi)}\Big\{(\gamma\mathbf{T}(\xi)-(\psi+\phi)\mathbf{R}(\xi))d\xi+\rho_4\mathbf{R}(\xi)dB_4(\xi)\Big\} \qquad (3.4)\\
&-\frac{1}{\mathbf{S}^2(\xi)}\Big\{(\pi+\phi\mathbf{R}(\xi)-\delta\mathbf{S}(\xi)\mathbf{I}(\xi)-\psi\mathbf{S}(\xi))d\xi+\rho_1\mathbf{S}(\xi)dB_1(\xi)\Big\}^2\\
&+\frac{1}{\mathbf{I}^2(\xi)}\Big\{(\delta\mathbf{S}(\xi)\mathbf{I}(\xi)-(\psi+\zeta+\eta)\mathbf{I}(\xi))d\xi+\rho_2\mathbf{I}(\xi)dB_2(\xi)\Big\}^2. \qquad (3.5)
\end{aligned}
$$

Setting $\mathcal{A}_1 = \pi+\phi\mathbf{R}(\xi)-\delta\mathbf{S}(\xi)\mathbf{I}(\xi)-\psi\mathbf{S}(\xi)$, $\mathcal{A}_2 = \rho_1\mathbf{S}(\xi)$, $\mathcal{A}_3 = \delta\mathbf{S}(\xi)\mathbf{I}(\xi)-(\psi+\eta+\zeta)\mathbf{I}(\xi)$ and $\mathcal{A}_4 = \rho\mathbf{S}(\xi)\mathbf{I}(\xi)$, then (3.4) reduces to

$$
\begin{aligned}
d\mathcal{U}(\xi) \quad &= -\frac{1}{\mathbf{S}(\xi)}\Big\{(\pi+\phi\mathbf{R}(\xi)-\delta\mathbf{S}(\xi)\mathbf{I}(\xi)-\psi\mathbf{S}(\xi))d\xi+\rho_1\mathbf{S}(\xi)dB_1(\xi)\Big\}\\
&-\frac{1}{\mathbf{I}(\xi)}\Big\{(\delta\mathbf{S}(\xi)\mathbf{I}(\xi)-(\psi+\zeta+\eta)\mathbf{I}(\xi))d\xi+\rho_2\mathbf{I}(\xi)dB_2(\xi)\Big\}\\
&-\frac{1}{\mathbf{T}(\xi)}\Big\{(\zeta\mathbf{I}(\xi)-(\psi+\gamma+\varsigma)\mathbf{T}(\xi))d\xi+\rho_3\mathbf{T}(\xi)dB_3(\xi)\Big\}\\
&-\frac{1}{\mathbf{R}(\xi)}\Big\{(\gamma\mathbf{T}(\xi)-(\psi+\phi)\mathbf{R}(\xi))d\xi+\rho_4\mathbf{R}(\xi)dB_4(\xi)\Big\}\\
&-\frac{1}{\mathbf{S}^2(\xi)}\Big\{\mathcal{A}_1^2\mathcal{A}_3^2\xi+\mathcal{A}_1\mathcal{A}_2d\xi dB_1(\xi)+\mathcal{A}_2^2\mathcal{A}_3^2dB_1(\xi)\Big\}\\
&+\frac{1}{\mathbf{I}^2(\xi)}\Big\{\epsilon^2\mathbf{I}(\xi)\epsilon\xi+\epsilon\mathcal{A}_4d\xi dB_2(\xi)+\mathcal{A}_4dB_2(\xi)\Big\}.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
d\mathcal{U}(\xi) \quad &= -\Big\{\Big(\frac{\pi}{\mathbf{S}(\xi)}+\frac{\phi\mathbf{R}(\xi)}{\mathbf{S}(\xi)}-\delta\mathbf{I}(\xi)-\psi\Big)d\xi+\rho_1dB_1(\xi)\Big\}\\
&-\Big\{(\delta\mathbf{S}(\xi)-(\psi+\zeta+\eta))d\xi+\rho_2\mathbf{I}(\xi)dB_2(\xi)\Big\}\\
&-\Big\{\Big(\frac{\zeta\mathbf{I}(\xi)}{\mathbf{T}(\xi)}-(\psi+\gamma+\varsigma))d\xi+\rho_3dB_3(\xi)\Big\}\\
&-\Big\{\Big(\frac{\gamma\mathbf{T}(\xi)}{\mathbf{R}(\xi)}-(\psi+\phi)\Big)d\xi+\rho_4dB_4(\xi)\Big\}\\
&-\Big\{\rho_1^2+\rho_2^2+\rho_3^2+\rho_4^2\Big\}d\xi\\
&-\Big\{\rho_1dB_1(\xi)+\rho_2dB_2(\xi)+\rho_3dB_3(\xi)+\rho_4dB_4(\xi)\Big\}.
\end{aligned}
$$

After simplification, the aforesaid equation reduced to

$$
d\mathcal{U}(\xi) = \mathcal{H}\mathcal{U}d\xi - \Big\{\rho_1dB_1(\xi)+\rho_2dB_2(\xi)+\rho_3dB_3(\xi)+\rho_4dB_4(\xi)\Big\},
$$

where

$$\mathcal{HU} = -\frac{\pi}{\mathbf{S}(\xi)} - \frac{\phi\mathbf{R}(\xi)}{\mathbf{S}(\xi)} + \delta\mathbf{I}(\xi) + \psi - \delta\mathbf{S}(\xi) + (\psi+\zeta+\eta) - \frac{\zeta\mathbf{I}(\xi)}{\mathbf{T}(\xi)} + (\psi+\gamma+\varsigma) - \frac{\gamma\mathbf{T}(\xi)}{\mathbf{R}(\xi)} + (\psi+\phi) + \rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4^2 = \complement.$$

As a consequence, we get

$$d\mathcal{U}(\xi) \leq \complement d\xi - \Big\{\rho_1 dB_1(\xi) + \rho_2 dB_2(\xi) + \rho_3 dB_3(\xi) + \rho_4 dB_4(\xi)\Big\}.$$

Performing integration from 0 *to* $\Lambda_{\Bbbk\wedge\mathbf{Q}}$, it can be deduced that

$$\begin{aligned}\int_0^{\Lambda_{\Bbbk\wedge\mathbf{Q}}} d\mathcal{U}(\mathbf{X}(\xi)) \quad &\leq \int_0^{\Lambda_{\Bbbk\wedge\mathbf{Q}}} \complement d\xi - \Big\{ \int_0^{\Lambda_{\Bbbk\wedge\mathbf{Q}}} \rho_1 dB_1(\xi) + \int_0^{\Lambda_{\Bbbk\wedge\mathbf{Q}}} \rho_2 dB_2(\xi) + \int_0^{\Lambda_{\Bbbk\wedge\mathbf{Q}}} \rho_3 dB_3(\xi) \\ &\quad + \int_0^{\Lambda_{\Bbbk\wedge\mathbf{Q}}} \rho_4 dB_4(\xi)\Big\},\end{aligned}$$

using the fact that $\Lambda_{\Bbbk\wedge\mathbf{Q}} = \min\{\Lambda_{\mathrm{n}}, \xi\}$. Performing the expectation on the aforementioned versions results in

$$\begin{aligned}\mathcal{U}(\mathbf{X}(\Lambda_{\Bbbk\wedge\mathbf{Q}})) \quad &\leq \mathcal{U}(\mathbf{X}(0)) + \complement \int_0^{\Lambda_{\Bbbk\wedge\mathbf{Q}}} d\xi - \Big\{ \int_0^{\Lambda_{\Bbbk\wedge\mathbf{Q}}} \rho_1 dB_1(\xi) + \int_0^{\Lambda_{\Bbbk\wedge\mathbf{Q}}} \rho_2 dB_2(\xi) + \int_0^{\Lambda_{\Bbbk\wedge\mathbf{Q}}} \rho_3 dB_3(\xi) \\ &\quad + \int_0^{\Lambda_{\Bbbk\wedge\mathbf{Q}}} \rho_4 dB_4(\xi)\Big\}.\end{aligned}$$

This implies that

$$\mathcal{E}\mathcal{U}(\mathbf{X}(\Lambda_{\Bbbk\wedge\mathbf{Q}})) \quad \leq \mathcal{U}(\mathbf{X}(0)) + \complement\mathcal{E} \leq \mathcal{U}(\mathbf{X}(0)) + \complement\mathbf{Q}. \tag{3.6}$$

As $\mathcal{U}(\mathbf{X}(\Lambda_{\Bbbk\wedge\mathbf{Q}})) > 0$, then

$$\begin{aligned}\mathcal{E}\mathcal{U}(\mathbf{X}(\Lambda_{\Bbbk\wedge\mathbf{Q}})) \quad &= \mathcal{E}[\mathcal{U}(\mathbf{X}(\Lambda_{\Bbbk\wedge\mathbf{Q}}))_{x_{1(\Lambda_{\Bbbk}\leq\mathbf{Q})}}] + \mathcal{E}[\mathcal{U}(\mathbf{X}(\Lambda_{\Bbbk\wedge\mathbf{Q}}))_{x_{1(\Lambda_{\Bbbk}>\mathrm{T})}}] \\ &\geq \mathcal{E}[\mathcal{U}(\mathbf{X}(\Lambda_{\Bbbk\wedge\mathbf{Q}}))_{x_{1(\Lambda_{\Bbbk}\leq\mathbf{Q})}}].\end{aligned} \tag{3.7}$$

Further, for $\Lambda_{\Bbbk}$, since certain factors of $\mathbf{X}(\Lambda_{\Bbbk})$, *say* $(\mathbf{S}(\Lambda_{\Bbbk}))$ including $0 < \mathbf{S}(\Lambda_{\Bbbk}) \leq \frac{1}{\Bbbk} < 1$.

Thus, $\mathcal{U}(\mathbf{X}(\Lambda_{\Bbbk})) \geq -\ln\left(\frac{1}{\Bbbk}\right)$, this allow us to write $\mathcal{U}(\mathbf{X}(\Lambda_{\Bbbk})) = \ln(\mathbf{S}(\Lambda_{\Bbbk})) \leq \ln\left(\frac{1}{\Bbbk}\right)$.

As a result, from (3.7) and the previous expression, we have

$$\begin{aligned}\mathcal{E}\mathcal{U}(\mathbf{X}(\Lambda_{\Bbbk\wedge\mathbf{Q}})) &\geq \mathcal{E}[\mathcal{U}(\mathbf{X}(\Lambda_{\Bbbk\wedge\mathbf{Q}}))_{x_1(\Lambda_{\Bbbk}\leq\mathbf{Q})}] \\ &\geq \Big\{-\ln\Big(\frac{1}{\Bbbk}\Big)\Big\}. \end{aligned} \tag{3.8}$$

Combining (3.6)–(3.8), we have

$$\mathcal{E}\mathcal{U}(\mathbf{X}(\Lambda_{\Bbbk\wedge\mathbf{Q}})) \geq -\ln\Big(\frac{1}{\Bbbk}\Big)\mathbf{P}(\Lambda_{\Bbbk\wedge\mathbf{Q}}). \tag{3.9}$$

It follows that

$$\begin{aligned}\mathbf{P}(\Lambda_{\Bbbk\wedge\mathbf{Q}}) &\leq \frac{\mathcal{E}\mathcal{U}(\mathbf{X}(\Lambda_{\Bbbk\wedge\mathbf{Q}}))}{\ln\Bbbk} \\ &\leq \frac{\mathcal{U}(\mathbf{X}(0)) + \complement\mathbf{Q}}{\ln\Bbbk}.\end{aligned}$$

Taking limit sup $\Bbbk \mapsto \infty$ on (3.9), $\forall\ \mathbf{Q} > 0$, we find

$$\mathbf{P}(\Lambda_{\Bbbk\wedge\mathbf{Q}}) \leq 0 \quad \Longrightarrow \lim_{\xi\mapsto\infty}\mathbf{P}(\Lambda_{\Bbbk\wedge\mathbf{Q}}) = 0.$$

This is the intended outcome.

3.2. Ergodicity and stationary distribution (ESD)

Now, we review and examine the model's (3.2) stationary distribution outcomes, which show that the infections are eliminated or enduring.

Suppose there be a regular Markov technique in $\mathbb{R}^{\mathfrak{n}}_{+}$ for which the behaviour is as below:

$$d\Phi(\xi) = b_1(\Phi)d\xi + \sum_{r_1}^{\mathbf{u}} \zeta_{r_1} d\Psi_{r_1}(\xi).$$

The diffusion matrix takes the form

$$\mathbb{A}(\Phi) = [a_{ij}(\varrho)], \ \ a_{ij}(\varrho) = \sum_{r_1=1}^{\mathbf{u}} \zeta^{i}_{r_1}(\varrho)\zeta^{r_1}_{j}(\varrho).$$

Lemma 3.1. *([50]) Suppose there is a unique stationary distribution technique $\Phi(\xi)$. If there is a bounded region involving regular boundary such that $S, \bar{S} \in \mathbb{R}^d \bar{S}$ closure $\bar{S} \in \mathbb{R}^d$ satisfies the following:*

(a) *The smallest eigenvalue for $\mathbb{A}(\xi)$ is bounded away from $(0,0)$ for the open region S having neighbourhood.*

(b) *For $\sigma \in \mathbb{R}^d S$, the mean time τ is bounded and for every compact subset $K \subset \mathbb{R}^{\mathfrak{n}}$, $\sup_{\sigma\in\Bbbk} S^{\sigma}\tau < \infty$. Therefore, $f_1(.)$ is an integrable mapping containing measure π, then*

$$P\Big(\lim_{\mathbb{T}\mapsto\infty}\frac{1}{\mathbb{T}}\int_0^{\mathbb{T}} f_1(\Phi_\sigma(\xi))d\xi = \int_{\mathbb{R}^d} f_1(\sigma)\pi(d\sigma)\Big) = 1, \ \forall\ \sigma \in \mathbb{R}^d.$$

Let us describe another threshold value for forthcoming use:

$$\mathbb{R}_0^s = \frac{\pi\delta\zeta}{(\psi - \frac{\rho_1^2}{2})(\psi + \eta + \zeta - \frac{\rho_2^2}{2})(\gamma + \psi + \varsigma - \frac{\rho_3^2}{2})}.$$

Theorem 3.2. *For* $\mathbb{R}_0^s > 1$, *then the model* (3.2) $(\mathbf{S}(\xi), \mathbf{I}(\xi), \mathbf{T}(\xi), \mathbf{R}(\xi))$ *is ergodic. Also, there exist a unique stationary distribution* $\eta(.)$.

Proof. We must first demonstrate the specified requirement M_1 of Lemma 3.1 in order to verify the Theorem, we state a non-negative $\mathbb{C}^2$-function $H_1 : \mathbb{R}_+^4 \mapsto \mathbb{R}_+$ of the form

$$H_1 = \mathbf{S} + \mathbf{I} + \mathbf{T} + \mathbf{R} - \wp_1 \ln \mathbf{S} - \wp_2 \ln \mathbf{I} - \wp_3 \ln \mathbf{T}, \tag{3.10}$$

here the positive constants must be calculated afterwards. These constants will have to be discovered later on. To interact with (3.10), we must first implement Itô strategy to the developed framework (3.2) to obtain

$$\mathcal{L}(\mathbf{S} + \mathbf{I} + \mathbf{T} + \mathbf{R}) = \pi - \psi\mathbf{N} - \eta\mathbf{I} - \varsigma\mathbf{T}. \tag{3.11}$$

It follows that

$$\begin{aligned}
\mathcal{L}(-\ln \mathbf{S}) &= -\frac{\pi}{\mathbf{S}} + (\delta\mathbf{I} + \psi) - \frac{\phi\mathbf{R}}{\mathbf{S}} - \frac{\rho_1^2}{2},\\
\mathcal{L}(-\ln \mathbf{I}) &= -\delta\mathbf{S} + (\psi + \eta + \zeta)\mathbf{I} - \frac{\rho_2^2}{2},\\
\mathcal{L}(-\ln \mathbf{T}) &= -\frac{\zeta\mathbf{I}}{\mathbf{T}} + (\gamma + \psi + \varsigma) - \frac{\rho_3^2}{2},\\
\mathcal{L}(-\ln \mathbf{R}) &= -\frac{\gamma\mathbf{T}}{\mathbf{R}} + (\phi + \psi) - \frac{\rho_4^2}{2}.
\end{aligned} \tag{3.12}$$

Further, we express

$$\mathcal{L}(H_1) = \mathcal{L}(\mathbf{S} + \mathbf{I} + \mathbf{T} + \mathbf{R}) - \wp_1\mathcal{L}(\ln \mathbf{S}) - \wp_2\mathcal{L}(\ln \mathbf{I}) - \wp_3\mathcal{L}(\ln \mathbf{T}). \tag{3.13}$$

The replacement of values in the preceding equation generates the following formula

$$\begin{aligned}
\mathcal{L}(H_1) \quad &= \pi - \psi\mathbf{N} - \eta\mathbf{I} - \varsigma\mathbf{T} + \wp_1\frac{\pi}{\mathbf{S}} - \wp_1(\delta\mathbf{I} + \psi) + \wp_1\frac{\phi\mathbf{R}}{\mathbf{S}} + \wp_1\frac{\rho_1^2}{2} + \wp_2\delta\mathbf{S} - \wp_2(\psi + \eta + \zeta)\mathbf{I}\\
&\quad + \wp_2\frac{\rho_2^2}{2} + \wp_3\frac{\zeta\mathbf{I}}{\mathbf{T}} - \wp_3(\gamma + \psi + \varsigma) + \wp_3\frac{\rho_3^2}{2} - \frac{\gamma\mathbf{T}}{\mathbf{R}} + (\phi + \psi) - \frac{\rho_4^2}{2}\\
&\le -4\Big(\wp_1\frac{\pi}{\mathbf{S}}\wp_2\delta\mathbf{S}\zeta\wp_3(\pi + \phi + \psi)\Big)^{1/4} + (\pi + \phi + \psi) - \wp_1\Big(\psi - \frac{\rho_1^2}{2}\Big) - \wp_2\Big(\psi + \eta + \zeta - \frac{\rho_2^2}{2}\Big)\\
&\quad - \wp_3\Big(\gamma + \psi + \varsigma - \frac{\rho_3^2}{2}\Big) - \psi\mathbf{N} - \eta\mathbf{I} - \varsigma\mathbf{T} - \wp_1(\delta\mathbf{I}) + \wp_1\frac{\phi\mathbf{R}}{\mathbf{S}} + \wp_2\delta\mathbf{S} + \wp_3\frac{\zeta\mathbf{I}}{\mathbf{T}} - \frac{\gamma\mathbf{T}}{\mathbf{R}}.
\end{aligned}$$

Now, we suppose that

$$\pi + \phi + \psi = \wp_1(\psi - \frac{\rho_1^2}{2}) = \wp_2(\psi + \eta + \zeta - \frac{\rho_2^2}{2}) = \wp_3(\gamma + \psi + \varsigma - \frac{\rho_3^2}{2}),$$

where

$$\wp_1 = \frac{\pi+\phi+\psi}{(\psi-\frac{\rho_1^2}{2})},\ \wp_2 = \frac{\pi+\phi+\psi}{(\psi+\eta+\zeta-\frac{\rho_2^2}{2})},\ \ \wp_3 = \frac{\pi+\phi+\psi}{(\gamma+\psi+\varsigma-\frac{\rho_3^2}{2})}.$$

Consequently, we have

$$\begin{aligned}\mathcal{L}(H_1) \quad &\leq -4\Big(\frac{\pi\delta\zeta(\pi+\phi+\psi)^4}{(\psi-\frac{\rho_1^2}{2})(\psi+\eta+\zeta-\frac{\rho_2^2}{2})(\gamma+\psi+\varsigma-\frac{\rho_3^2}{2})} - 4(\pi+\phi+\psi)^4\Big)^{1/4}\\ &\quad +\wp_1\Big(\delta\mathbf{I}-\frac{\phi\mathbf{R}}{\mathbf{S}}\Big)+\wp_2\delta\mathbf{S}+\wp_3\frac{\zeta\mathbf{I}}{\mathbf{T}}\\ &\leq -4(\pi+\phi+\psi)[(\mathbf{R}_0^s)^{1/4}-1]+\wp_1\Big(\delta\mathbf{I}-\frac{\phi\mathbf{R}}{\mathbf{S}}\Big).\end{aligned}$$

Furthermore, one can achieve that

$$\begin{aligned}H_2 \quad &= \wp_4(\mathbf{S}+\mathbf{I}+\mathbf{T}+\mathbf{R}-\wp_1\ln\mathbf{S}-\wp_2\ln\mathbf{I}-\wp_3\ln\mathbf{T})-\ln\mathbf{S}-\ln\mathbf{R}-\ln\mathbf{T}\\ &\quad +\mathbf{S}+\mathbf{I}+\mathbf{T}+\mathbf{R}\\ &= (\wp_4+1)(\mathbf{S}+\mathbf{I}+\mathbf{T}+\mathbf{R})-(\wp_1\wp_4+1)\ln\mathbf{S}-\wp_4\wp_2\ln\mathbf{I}-\ln\mathbf{R}-\wp_4\wp_3\ln\mathbf{T},\end{aligned}$$

here $\wp_4 > 0$ is a constant that will be find afterwards. It is important to present that

$$\lim \inf_{(\mathbf{S},\mathbf{I},\mathbf{T},\mathbf{R})\in\mathbb{R}_+^4\setminus\mathcal{U}_\kappa} H_2(\mathbf{S},\mathbf{I},\mathbf{T},\mathbf{R}) = +\infty,\ as\ \kappa\mapsto\infty,$$

here $\mathcal{U}_\kappa = (\frac{1}{\kappa},\kappa)\times(\frac{1}{\kappa},\kappa)\times(\frac{1}{\kappa},\kappa)\times(\frac{1}{\kappa},\kappa)$. The upcoming process is to illustrate that $H_2(\mathbf{S},\mathbf{I},\mathbf{T},\mathbf{R})$ has one and only one minimum value $H_2(\mathbf{S}(0),\mathbf{I}(0),\mathbf{T}(0),\mathbf{R}(0))$.

The partial derivative of $H_2(\mathbf{S},\mathbf{I},\mathbf{T},\mathbf{R})$ regarding to $\mathbf{S},\mathbf{I},\mathbf{T},\mathbf{R}$ is as follows

$$\begin{aligned}\frac{\partial H_2(\mathbf{S},\mathbf{I},\mathbf{T},\mathbf{R})}{\partial\mathbf{S}} &= 1+\wp_4-\frac{1+\wp_1\wp_4}{\mathbf{S}},\\ \frac{\partial H_2(\mathbf{S},\mathbf{I},\mathbf{T},\mathbf{R})}{\partial\mathbf{I}} &= 1+\wp_4-\frac{\wp_2\wp_4}{\mathbf{I}},\\ \frac{\partial H_2(\mathbf{S},\mathbf{I},\mathbf{T},\mathbf{R})}{\partial\mathbf{T}} &= 1+\wp_4-\frac{\wp_3\wp_4}{\mathbf{T}},\\ \frac{\partial H_2(\mathbf{S},\mathbf{I},\mathbf{T},\mathbf{R})}{\partial\mathbf{R}} &= 1+\wp_4-\frac{1}{\mathbf{R}}.\end{aligned} \tag{3.14}$$

It is simple to demonstrate that H_2 has a unique stagnation point, which is determined by the mentioned calculation:

$$(\mathbf{S},\mathbf{I},\mathbf{T},\mathbf{R}) = \Big(\frac{1+\wp_1\wp_4}{1+\wp_4},\frac{\wp_2\wp_4}{1+\wp_4},\frac{\wp_2\wp_4}{1+\wp_4},\frac{1}{1+\wp_4}\Big). \tag{3.15}$$

Also, the Hessian matrix of $H_2(\mathbf{S},\mathbf{I},\mathbf{T},\mathbf{R})$ at $(\mathbf{S}(0),\mathbf{I}(0),\mathbf{T}(0),\mathbf{R}(0))$ is presented by the following

$$\mathbf{B} = \begin{bmatrix}\frac{1+\wp_1\wp_4}{\mathbf{S}^2} & 0 & 0 & 0\\ 0 & \frac{\wp_2\wp_4}{\mathbf{I}^2} & 0 & 0\\ 0 & 0 & \frac{\wp_3\wp_4}{\mathbf{T}^2} & 0\\ 0 & 0 & 0 & \frac{1}{\mathbf{R}^2}\end{bmatrix}. \tag{3.16}$$

The aforesaid connection clearly shows that $\mathbf{B}$ is a positive definite matrix. So, $H_2(\mathbf{S},\mathbf{I},\mathbf{T},\mathbf{R})$ has lowest value $(\mathbf{S}(0),\mathbf{I}(0),\mathbf{T}(0),\mathbf{R}(0))$. Consequently, we can deduce from Lemma 3.1 and the continuity of $H_2(\mathbf{S},\mathbf{I},\mathbf{T},\mathbf{R})$ that it has a unique minimum value around $(\mathbf{S}(0),\mathbf{I}(0),\mathbf{T}(0),\mathbf{R}(0))$ in the interior of $\mathbb{R}^4_+$. Again, we introduce a non-negative $\mathbb{C}^2 : \mathbb{R}^4_+ \mapsto \mathbb{R}_+$ as follow:

$$H_1(\mathbf{S},\mathbf{I},\mathbf{T},\mathbf{R}) = H_2(\mathbf{S},\mathbf{I},\mathbf{T},\mathbf{R}) - H_2(\mathbf{S}(0),\mathbf{I}(0),\mathbf{T}(0),\mathbf{R}(0)). \tag{3.17}$$

The implementation of Itô scheme and framework (3.2) will provide us with

$$\begin{aligned}\mathcal{L}H_1 \quad &\leq \wp_4\Big\{-4(\pi+\phi+\psi)[(\mathbb{R}_0^s)^{1/4}-1]+\wp_1\Big(\delta\mathbf{I}-\frac{\phi\mathbf{R}}{\mathbf{S}}\Big)\Big\}-\frac{\pi}{\mathbf{S}}+(\delta\mathbf{I}+\psi)-\frac{\phi\mathbf{R}}{\mathbf{S}}-\frac{\rho_1^2}{2}\\ &-\delta\mathbf{S}+(\psi+\eta+\zeta)\mathbf{I}-\frac{\rho_2^2}{2}-\frac{\zeta\mathbf{I}}{\mathbf{T}}+(\gamma+\psi+\varsigma)-\frac{\rho_3^2}{2}\\ &-\frac{\gamma\mathbf{T}}{\mathbf{R}}+(\phi+\psi)-\frac{\rho_4^2}{2},\end{aligned} \tag{3.18}$$

or finally we can express

$$\begin{aligned}\mathcal{L}H_1 \quad &\leq -\wp_4\wp_5+(\wp_1\wp_4-1)\frac{\phi\mathbf{R}}{\mathbf{S}}-\frac{\pi}{\mathbf{S}}+(\gamma+\psi+\varsigma)-(\phi+\psi)++\delta\mathbf{I}+\psi-\frac{\phi\mathbf{R}}{\mathbf{S}}-\frac{\rho_1^2}{2}\\ &-\delta\mathbf{S}+(\psi+\eta+\zeta)\mathbf{I}-\frac{\zeta\mathbf{I}}{\mathbf{T}}+(\gamma+\psi+\varsigma)-\frac{\gamma\mathbf{T}}{\mathbf{R}}+(\phi+\psi)-\frac{\rho_1^2\vee\rho_2^2\vee\rho_3^2\vee\rho_4^2}{2},\end{aligned} \tag{3.19}$$

where $\wp_5 = 4(\pi+\psi+\phi)[(\mathbb{R}_0^s)^{1/4}-1] > 0$.

The description of a set is provided by

$$\mathcal{D} = \{\mathbf{S}\in[\varepsilon_1,\frac{1}{\varepsilon_2}],\mathbf{I}\in[\varepsilon_1,\frac{1}{\varepsilon_2}],\mathbf{T}\in[\varepsilon_1,\frac{1}{\varepsilon_2}],\mathbf{R}\in[\varepsilon_1,\frac{1}{\varepsilon_2}]\Big\}, \tag{3.20}$$

where ε_ι, $\iota = 1,2$, are constants that are incredibly small and must be determined later. The region $\mathbb{R}^4_+ \setminus \mathcal{D}$ is divided into ten regions as follows:

$$\begin{aligned}\mathcal{D}_1 &= \Big\{(\mathbf{S},\mathbf{I},\mathbf{T},\mathbf{R})\in\mathbb{R}^4_+, 0<\mathbf{S}\leq\varepsilon_1\Big\},\\ \mathcal{D}_2 &= \Big\{(\mathbf{S},\mathbf{I},\mathbf{T},\mathbf{R})\in\mathbb{R}^4_+, 0<\mathbf{I}\leq\varepsilon_2,\ \mathbf{S}>\varepsilon_2\Big\},\\ \mathcal{D}_3 &= \Big\{(\mathbf{S},\mathbf{I},\mathbf{T},\mathbf{R})\in\mathbb{R}^4_+, 0<\mathbf{T}\leq\varepsilon_1,\ \mathbf{I}>\varepsilon_2\Big\},\\ \mathcal{D}_4 &= \Big\{(\mathbf{S},\mathbf{I},\mathbf{T},\mathbf{R})\in\mathbb{R}^4_+, 0<\mathbf{R}\leq\varepsilon_1,\ \mathbf{T}>\varepsilon_2\Big\},\\ \mathcal{D}_5 &= \Big\{(\mathbf{S},\mathbf{I},\mathbf{T},\mathbf{R})\in\mathbb{R}^4_+, \mathbf{S}\geq\frac{1}{\varepsilon_2}\Big\},\\ \mathcal{D}_6 &= \Big\{(\mathbf{S},\mathbf{I},\mathbf{T},\mathbf{R})\in\mathbb{R}^4_+, \mathbf{I}\geq\frac{1}{\varepsilon_2}\Big\},\\ \mathcal{D}_7 &= \Big\{(\mathbf{S},\mathbf{I},\mathbf{T},\mathbf{R})\in\mathbb{R}^4_+, \mathbf{T}\geq\frac{1}{\varepsilon_2}\Big\},\\ \mathcal{D}_8 &= \Big\{(\mathbf{S},\mathbf{I},\mathbf{T},\mathbf{R})\in\mathbb{R}^4_+, \mathbf{R}\geq\frac{1}{\varepsilon_2}\Big\}.\end{aligned} \tag{3.21}$$

Clearly, $\mathbb{R}^4_+ \setminus \mathcal{D} = \bigcup_{\iota=1}^{8} \mathcal{D}_\iota,\ \iota = 1, ..., 8$. As a result, we will check $H_1(\mathbf{S}, \mathbf{I}, \mathbf{T}, \mathbf{R})$ for each $(\mathbf{S}, \mathbf{I}, \mathbf{T}, \mathbf{R}) \in \mathbb{R}^4_+ \setminus \mathcal{D}$. As a result of (3.19), it is not hard to conclude that

$$\mathcal{L}H_1(\mathbf{S}, \mathbf{I}, \mathbf{T}, \mathbf{R}),\ for\ (\mathbf{S}, \mathbf{I}, \mathbf{T}, \mathbf{R}) \in \mathbb{R}^4_+ \setminus \mathcal{D} = \bigcup_{\iota=1}^{8} \mathcal{D}_\iota,\ \iota = 1, ..., 8.$$

Case I. If $(\mathbf{S}, \mathbf{I}, \mathbf{T}, \mathbf{R}) \in \mathcal{D}_1$, then by (3.19), we have

$$\begin{aligned}\mathcal{L}H_1 \quad & \leq -\wp_4\wp_5 + (\wp_1\wp_4 - 1)\frac{\phi\mathbf{R}}{\mathbf{S}} - \frac{\pi}{\mathbf{S}} + (\gamma + \psi + \varsigma) - (\phi + \psi) + +\delta\mathbf{I} + \psi - \frac{\phi\mathbf{R}}{\mathbf{S}} - \frac{\rho_1^2}{2} \\ & -\delta\mathbf{S} + (\psi + \eta + \zeta)\mathbf{I} - \frac{\zeta\mathbf{I}}{\mathbf{T}} + (\gamma + \psi + \varsigma) - \frac{\gamma\mathbf{T}}{\mathbf{R}} + (\phi + \psi) - \frac{\rho_1^2 \vee \rho_2^2 \vee \rho_3^2 \vee \rho_4^2}{2} \\ & \leq -\wp_4\wp_5 + (\wp_2\wp_4 + 1)\frac{\phi}{\varepsilon_1} - \frac{\pi}{\varepsilon_1} \leq -1.\end{aligned}$$

Case II. If $(\mathbf{S}, \mathbf{I}, \mathbf{T}, \mathbf{R}) \in \mathcal{D}_2$, then by (3.19), we have

$$\begin{aligned}\mathcal{L}H_1 \quad & \leq -\wp_4\wp_5 + (\wp_1\wp_4 - 1)\frac{\phi\mathbf{R}}{\mathbf{S}} - \frac{\pi}{\mathbf{S}} + (\gamma + \psi + \varsigma) - (\phi + \psi) + +\delta\mathbf{I} + \psi - \frac{\phi\mathbf{R}}{\mathbf{S}} - \frac{\rho_1^2}{2} \\ & -\delta\mathbf{S} + (\psi + \eta + \zeta)\mathbf{I} - \frac{\zeta\mathbf{I}}{\mathbf{T}} + (\gamma + \psi + \varsigma) - \frac{\gamma\mathbf{T}}{\mathbf{R}} + (\phi + \psi) - \frac{\rho_1^2 \vee \rho_2^2 \vee \rho_3^2 \vee \rho_4^2}{2} \\ & \leq -\wp_4\wp_5 + (\wp_2\wp_4 + 1)\frac{\phi}{\varepsilon_1} - \delta\varepsilon_2 \leq -1.\end{aligned}$$

Case III. If $(\mathbf{S}, \mathbf{I}, \mathbf{T}, \mathrm{U}) \in \mathcal{D}_3$, then by (3.19), we have

$$\begin{aligned}\mathcal{L}H_1 \quad & \leq -\wp_4\wp_5 + (\wp_1\wp_4 - 1)\frac{\phi\mathbf{R}}{\mathbf{S}} - \frac{\pi}{\mathbf{S}} + (\gamma + \psi + \varsigma) - (\phi + \psi) + +\delta\mathbf{I} + \psi - \frac{\phi\mathbf{R}}{\mathbf{S}} - \frac{\rho_1^2}{2} \\ & -\delta\mathbf{S} + (\psi + \eta + \zeta)\mathbf{I} - \frac{\zeta\mathbf{I}}{\mathbf{T}} + (\gamma + \psi + \varsigma) - \frac{\gamma\mathbf{T}}{\mathbf{R}} + (\phi + \psi) - \frac{\rho_1^2 \vee \rho_2^2 \vee \rho_3^2 \vee \rho_4^2}{2} \\ & \leq -\wp_4\wp_5 - \gamma\varepsilon_1 - \frac{\zeta\varepsilon_1}{\varepsilon_2} \leq -1.\end{aligned}$$

Case IV. If $(\mathbf{S}, \mathbf{I}, \mathbf{T}, \mathbf{U}) \in \mathcal{D}_4$, then by (3.19), we have

$$\begin{aligned}\mathcal{L}H_1 \quad & \leq -\wp_4\wp_5 + (\wp_1\wp_4 - 1)\frac{\phi\mathbf{R}}{\mathbf{S}} - \frac{\pi}{\mathbf{S}} + (\gamma + \psi + \varsigma) - (\phi + \psi) + +\delta\mathbf{I} + \psi - \frac{\phi\mathbf{R}}{\mathbf{S}} - \frac{\rho_1^2}{2} \\ & -\delta\mathbf{S} + (\psi + \eta + \zeta)\mathbf{I} - \frac{\zeta\mathbf{I}}{\mathbf{T}} + (\gamma + \psi + \varsigma) - \frac{\gamma\mathbf{T}}{\mathbf{R}} + (\phi + \psi) - \frac{\rho_1^2 \vee \rho_2^2 \vee \rho_3^2 \vee \rho_4^2}{2} \\ & \leq -\wp_4\wp_5 - \frac{\varepsilon_2}{\varepsilon_1} \leq -1.\end{aligned}$$

Case V. If $(\mathbf{S}, \mathbf{I}, \mathbf{T}, \mathrm{U}) \in \mathcal{D}_5$, then by (3.19), we have

$$\begin{aligned}\mathcal{L}H_1 \quad & \leq -\wp_4\wp_5 + (\wp_1\wp_4 - 1)\frac{\phi\mathbf{R}}{\mathbf{S}} - \frac{\pi}{\mathbf{S}} + (\gamma + \psi + \varsigma) - (\phi + \psi) + +\delta\mathbf{I} + \psi - \frac{\phi\mathbf{R}}{\mathbf{S}} - \frac{\rho_1^2}{2} \\ & -\delta\mathbf{S} + (\psi + \eta + \zeta)\mathbf{I} - \frac{\zeta\mathbf{I}}{\mathbf{T}} + (\gamma + \psi + \varsigma) - \frac{\gamma\mathbf{T}}{\mathbf{R}} + (\phi + \psi) - \frac{\rho_1^2 \vee \rho_2^2 \vee \rho_3^2 \vee \rho_4^2}{2}\end{aligned}$$

$$\leq -\wp_4\wp_5 - \frac{\phi}{\varepsilon_2} \leq -1.$$

Case VI. If $(\mathbf{S}, \mathbf{I}, \mathbf{T}, \mathbf{U}) \in \mathcal{D}_6$, then by (3.19), we have

$$\begin{aligned}\mathcal{L}H_1 \quad & \leq -\wp_4\wp_5 + (\wp_1\wp_4 - 1)\frac{\phi\mathbf{R}}{\mathbf{S}} - \frac{\pi}{\mathbf{S}} + (\gamma + \psi + \varsigma) - (\phi + \psi) + +\delta\mathbf{I} + \psi - \frac{\phi\mathbf{R}}{\mathbf{S}} - \frac{\rho_1^2}{2} \\ & -\delta\mathbf{S} + (\psi + \eta + \zeta)\mathbf{I} - \frac{\zeta\mathbf{I}}{\mathbf{T}} + (\gamma + \psi + \varsigma) - \frac{\gamma\mathbf{T}}{\mathbf{R}} + (\phi + \psi) - \frac{\rho_1^2 \vee \rho_2^2 \vee \rho_3^2 \vee \rho_4^2}{2} \\ & \leq -\wp_4\wp_5 - \frac{\zeta\varepsilon_2}{\mathbf{T}} \leq -1.\end{aligned}$$

Case VII. If $(\mathbf{S}, \mathbf{I}, \mathbf{T}, \mathbf{U}) \in \mathcal{D}_7$, then by (3.19), we have

$$\begin{aligned}\mathcal{L}H_1 \quad & \leq -\wp_4\wp_5 + (\wp_1\wp_4 - 1)\frac{\phi\mathbf{R}}{\mathbf{S}} - \frac{\pi}{\mathbf{S}} + (\gamma + \psi + \varsigma) - (\phi + \psi) + +\delta\mathbf{I} + \psi - \frac{\phi\mathbf{R}}{\mathbf{S}} - \frac{\rho_1^2}{2} \\ & -\delta\mathbf{S} + (\psi + \eta + \zeta)\mathbf{I} - \frac{\zeta\mathbf{I}}{\mathbf{T}} + (\gamma + \psi + \varsigma) - \frac{\gamma\mathbf{T}}{\mathbf{R}} + (\phi + \psi) - \frac{\rho_1^2 \vee \rho_2^2 \vee \rho_3^2 \vee \rho_4^2}{2} \\ & \leq -\wp_4\wp_5 - \frac{\gamma\varepsilon_2}{\mathbf{R}} \leq -1.\end{aligned}$$

Case VIII. If $(\mathbf{S}, \mathbf{I}, \mathbf{T}, \mathbf{U}) \in \mathcal{D}_8$, then by (3.19), we have

$$\begin{aligned}\mathcal{L}H_1 \quad & \leq -\wp_4\wp_5 + (\wp_1\wp_4 - 1)\frac{\phi\mathbf{R}}{\mathbf{S}} - \frac{\pi}{\mathbf{S}} + (\gamma + \psi + \varsigma) - (\phi + \psi) + +\delta\mathbf{I} + \psi - \frac{\phi\mathbf{R}}{\mathbf{S}} - \frac{\rho_1^2}{2} \\ & -\delta\mathbf{S} + (\psi + \eta + \zeta)\mathbf{I} - \frac{\zeta\mathbf{I}}{\mathbf{T}} + (\gamma + \psi + \varsigma) - \frac{\gamma\mathbf{T}}{\mathbf{R}} + (\phi + \psi) - \frac{\rho_1^2 \vee \rho_2^2 \vee \rho_3^2 \vee \rho_4^2}{2} \\ & \leq -\wp_4\wp_5 - \frac{\gamma\mathbf{T}}{\varepsilon_2} \leq -1.\end{aligned}$$

In end, all of the preceding scenarios show that a positive constant $\mathbf{B}$ exists, so $\mathcal{L}H_1(\mathbf{S}, \mathbf{I}, \mathbf{T}, \mathbf{U}) < -\mathbf{B} < 0 \ \forall \ (\mathbf{S}, \mathbf{I}, \mathbf{T}, \mathbf{U}) \in \mathbb{R}_+^4 \setminus \mathcal{D}$. Hence

$$\begin{aligned}dH_1(\mathbf{S}, \mathbf{I}, \mathbf{T}, \mathbf{U}) \quad & < -\mathbf{B}d\xi + [(\wp_4 + 1)\mathbf{S} - (\wp_1\wp_4 + 1)\rho_1]dB_1(\xi) \\ & +[(\wp_4 + 1)\mathbf{I} - \wp_1\wp_4\rho_2]dB_2(\xi) + [(\wp_4 + 1)\mathbf{T} - \wp_3\wp_4\rho_3]dB_3(\xi) \\ & +[(\wp_4 + 1)\mathbf{U} - \rho_4]dB_4(\xi).\end{aligned} \tag{3.22}$$

Suppose $(\mathbf{S}, \mathbf{I}, \mathbf{T}, \mathbf{U}) = (u_1, u_2, u_3, u_4, u_5) = \bar{u} \in \mathbb{R}_+^4 \setminus \mathcal{D}$, the time $\chi^{\bar{u}}$, at which a path beginning at $\bar{u}$ leads to the set $\mathcal{D}$, $\chi^{\mathfrak{n}} = \inf\{\xi : |\mathbf{X}(\xi)| = \mathfrak{n}\}$ and $\chi^{n}(\xi) = \min\{\chi_{\bar{u}}, \xi, \chi^{\mathfrak{n}}\}$. Applying integration on (3.22) from 0 to $\phi^{(\mathfrak{n})}(\xi)$, considering expectation and Dynkins technique, we deduce that

$$\begin{aligned}& \mathbb{E}H_1(\mathbf{S}(\chi^{(\mathfrak{n})}(\xi)), \mathbf{I}(\chi^{(\mathfrak{n})}(\xi)), \mathbf{T}(\chi^{(\mathfrak{n})}(\xi)), \mathbf{R}(\chi^{(\mathfrak{n})}(\xi))) - H_1(\bar{u}) \\ & = \mathbb{E} \int_0^{\chi^{(\mathfrak{n})}(\xi)} H_1(\mathbf{S}(u_1), \mathbf{I_B}(u_1), \mathbf{T}(u_1), \mathbf{R}(u_1))du_1 \\ & \leq \mathbb{E} \int_0^{\chi^{(\mathfrak{n})}(\xi)} -\mathbf{B}du_1 = -\mathbf{B}\mathbb{E}\chi^{(\mathfrak{n})}(\xi).\end{aligned} \tag{3.23}$$

As $H(\bar{u})$ is positive, thus

$$\mathbb{E}\chi^{(\text{n})}(\xi) \leq \frac{H_1(\bar{u})}{\mathbf{B}}. \tag{3.24}$$

As a result, $\mathbf{P}\{\chi_\varepsilon = \infty\} = 1$ and we can confirm that the suggested approach (3.2) is regular. Following Fatou's well-known lemma, we need

$$\mathbb{E}\chi^{(\text{n})}(\xi) \leq \frac{H_1(\bar{u})}{\mathbf{B}} < \infty. \tag{3.25}$$

Clearly, $\sup_{\bar{u}\in\mathcal{K}} \mathbb{E}\chi^{\bar{u}} < \infty$, here $\mathcal{K}$ is a compact subset from $\mathbb{R}^4_+$. As a result, the second criterion of Lemma 3.1 is fulfilled.

Moreover, the diffusion matrix of the model (3.2) is

$$\mathbf{B} = \begin{bmatrix} \rho_1^2\mathbf{S}^2 & 0 & 0 & 0 \\ 0 & \rho_2^2\mathbf{I}^2 & 0 & 0 \\ 0 & 0 & \rho_3^2\mathbf{T}^2 & 0 \\ 0 & 0 & 0 & \rho_4^2\mathbf{U}^2 \end{bmatrix}. \tag{3.26}$$

Selecting $M_1 = \min_{(\mathbf{S},\mathbf{I},\mathbf{T},\mathbf{U})} \in \mathcal{D} \in \mathbb{R}^4_+\{\rho_1^2\mathbf{S}^2, \rho_2^2\mathbf{I}^2, \rho_3^2\mathbf{T}^2, \rho_4^2\mathbf{U}^2\}$, we illustrate

$$\sum_{\iota,\jmath=1}^{4} a_{\iota\jmath}(\mathbf{S},\mathbf{I},\mathbf{T},\mathbf{U})\ell_\iota\ell_\jmath = \rho_1^2\mathbf{S}^2\ell^2 + \rho_2^2\mathbf{I}^2\ell^2 + \rho_3^2\mathbf{T}^2\ell^2 + \rho_4^2\mathbf{U}^2\ell^2 \geq M_1|\ell|^2, \quad (\mathbf{S},\mathbf{I},\mathbf{T},\mathbf{U}) \in \bar{\mathcal{D}},$$

where $\ell = (\ell_1, \ell_2, \ell_3, \ell_4) \in \mathbb{R}^4_+$.

As a result, the **(a)** of Lemma 3.1 is satisfied. As a consequence of Lemma 3.1, the suggested stochastic framework is ergodic and has a unique stationary distribution.

4. Intgro-fractional stochastic cholera epidemic model

In this paper, we transform the aforementioned framework to a stochastic process in order to represent the randomness related to infectious ailment of cholera transmission. To accomplish this, we add a stochastic element to the initial formulation, as well as the fractional derivative.

$$\begin{aligned}
\mathbf{S}(\xi) - \mathbf{S}(0) &= \frac{1}{\Gamma(\varphi)}\int_0^\xi (\xi-\theta)^{\varphi-1}\mathbf{F}_1(\mathbf{S},\mathbf{I},\mathbf{T},\mathbf{R},\theta)d\theta + \frac{1}{\Gamma(\varphi)}\int_0^\xi (\xi-\theta)^{\varphi-1}\mathcal{G}_1(\mathbf{S},\theta)dB_1'(\theta),\\
\mathbf{I}(\xi) - \mathbf{I}(0) &= \frac{1}{\Gamma(\varphi)}\int_0^\xi (\xi-\theta)^{\varphi-1}\mathbf{F}_2(\mathbf{S},\mathbf{I},\mathbf{T},\mathbf{R},\theta)d\theta + \frac{1}{\Gamma(\varphi)}\int_0^\xi (\xi-\theta)^{\varphi-1}\mathcal{G}_2(\mathbf{I},\theta)dB_2'(\theta),\\
\mathbf{T}(\xi) - \mathbf{T}(0) &= \frac{1}{\Gamma(\varphi)}\int_0^\xi (\xi-\theta)^{\varphi-1}\mathbf{F}_3(\mathbf{S},\mathbf{I},\mathbf{T},\mathbf{R},\theta)d\theta + \frac{1}{\Gamma(\varphi)}\int_0^\xi (\xi-\theta)^{\varphi-1}\mathcal{G}_3(\mathbf{T},\theta)dB_3'(\theta),\\
\mathbf{R}(\xi) - \mathbf{R}(0) &= \frac{1}{\Gamma(\varphi)}\int_0^\xi (\xi-\theta)^{\varphi-1}\mathbf{F}_4(\mathbf{S},\mathbf{I},\mathbf{T},\mathbf{R},\theta)d\theta + \frac{1}{\Gamma(\varphi)}\int_0^\xi (\xi-\theta)^{\varphi-1}\mathcal{G}_4(\mathbf{R},\theta)dB_4'(\theta).
\end{aligned} \tag{4.1}$$

4.1. Existence-uniqueness outcomes for the proposed model

Now, the system's (3.1) existence-uniqueness criteria are described. To provide it, we demonstrate that for all $\jmath \in 1,2,3,4$

(a) $\left|\mathbf{F}_\jmath(\chi_\jmath,\theta)\right|^2$ and $\left|\mathcal{G}_\jmath(\chi_\jmath,\theta)\right|^2 \leq \Omega(1+\left|\chi_\jmath\right|^2)$, this is the criterion of linear growth.

(b) $\left|\mathbf{F}_\jmath(\chi_\jmath^1,\theta)-\mathbf{F}_\jmath(\chi_\jmath^2,\theta)\right| \leq \bar{\Omega}\left|\chi_\jmath^1-\chi_\jmath^2\right|^2\left|\mathcal{G}_\jmath(\chi_\jmath^1,\theta)-\mathcal{G}_\jmath(\chi_\jmath^2,\theta)\right|^2 \leq \bar{\Omega}\left|\chi_\jmath^1-\chi_\jmath^2\right|^2$.

For $\mathbf{F}_1(\mathbf{S},\xi)=\pi+\phi\mathbf{R}-(\delta\mathbf{I}+\psi)\mathbf{S}, \quad \mathcal{G}_1(\mathbf{S},\xi)=\rho_1\mathbf{S}$.

Furthermore, we have

$$\begin{aligned}
\left|\mathbf{F}_1(\mathbf{S},\xi)\right|^2 &= \left|\pi+\phi\mathbf{R}-(\delta\mathbf{I}+\psi)\mathbf{S}\right|^2\\
&\leq 3(\pi^2+\phi^2\left|\mathbf{R}\right|^2+\delta^2\left|\mathbf{I}\right|^2\left|\mathbf{S}\right|^2+\psi^2\left|\mathbf{S}\right|^2)\\
&\leq 3(\pi^2+\phi^2\sup_{\xi\in[0,v_1]}\left|\mathbf{R}\right|^2+\delta^2\sup_{\xi\in[0,v_1]}\left|\mathbf{I}\right|^2\left|\mathbf{S}\right|^2+\psi^2\left|\mathbf{S}\right|^2)\\
&\leq 3\Big(\pi^2+\phi^2\left\|\mathbf{R}\right\|^2\Big)\Big(1+\frac{(\delta^2\left\|\mathbf{I}\right\|_\infty^2+\psi^2)\left|\mathbf{S}\right|^2}{\pi^2+\phi^2\left\|\mathbf{R}\right\|^2}\Big),
\end{aligned}$$

under the assumption $\Big(1+\frac{(\delta^2\left\|\mathbf{I}\right\|_\infty^2+\psi^2)}{\pi^2+\phi^2\left\|\mathbf{R}\right\|^2}\Big)<1$, then

$$\left|\mathbf{F}_1(\mathbf{S},\xi)\right|^2 \leq \Omega_1(1+\left|\mathbf{S}\right|^2). \tag{4.2}$$

It is clear that

$$\left|\mathcal{G}_1(\mathbf{S},\xi)\right|^2 \leq \rho_1^2(1+\left|\mathbf{S}\right|^2) \leq \Omega_1(1+\left|\mathbf{S}\right|^2). \tag{4.3}$$

Again, we have $\mathbf{F}_2(\mathbf{I},\xi)=\delta\mathbf{IS}-(\psi+\eta+\zeta)\mathbf{I}, \quad \mathcal{G}_2(\mathbf{I},\xi)=\rho_2\mathbf{I}$.

Also, we have

$$\begin{aligned}
\left|\mathbf{F}_2(\mathbf{I},\xi)\right|^2 &= \left|\delta\mathbf{IS}-(\psi+\eta+\zeta)\mathbf{I}\right|^2\\
&\leq 2\delta^2\left|\mathbf{S}\right|^2\left|\mathbf{I}\right|^2+2(\psi+\eta+\zeta)^2\left|\mathbf{I}\right|^2\\
&\leq 2\delta^2\sup_{\xi\in[0,v_1]}\left|\mathbf{S}\right|^2\left|\mathbf{I}\right|^2+2(\psi+\eta+\zeta)^2\left|\mathbf{I}\right|^2\\
&\leq 2\delta^2\left\|\mathbf{S}\right\|_\infty^2\left|\mathbf{I}\right|^2+2(\psi+\eta+\zeta)^2\left|\mathbf{I}\right|^2\\
&\leq 2\delta^2\left\|\mathbf{S}\right\|_\infty^2\Big(1+\frac{(\psi+\eta+\zeta)^2}{\delta^2\left\|\mathbf{S}\right\|_\infty^2}\Big)\left|\mathbf{I}\right|^2,
\end{aligned} \tag{4.4}$$

under the supposition $\frac{(\psi+\eta+\zeta)^2}{\delta^2\left\|\mathbf{S}\right\|_\infty^2}<1$, then

$$\left|\mathbf{F}_2(\mathbf{I},\xi)\right|^2 \leq \Omega_2(1+\left|\mathbf{I}\right|^2). \tag{4.5}$$

It is clear that

$$\left|\mathcal{G}_2(\mathbf{I},\xi)\right|^2 \leq \rho_2^2(1+\left|\mathbf{I}\right|^2) \leq \Omega_2(1+\left|\mathbf{I}\right|^2). \tag{4.6}$$

Moreover, we have $\mathbf{F}_3(\mathbf{T},\xi) = \varrho\mathbf{I} - (\psi+\gamma+\varsigma)\mathbf{T}$, $\quad \mathcal{G}_3(\mathbf{T},\xi) = \rho_3\mathbf{T}$.

Also, we have

$$\begin{aligned}\left|\mathbf{F}_3(\mathbf{T},\xi)\right|^2 &= \left|\varrho\mathbf{I} - (\psi+\gamma+\varsigma)\mathbf{T}\right|^2\\ &\leq 2\varrho^2\left|\mathbf{I}\right|^2 + 2(\psi+\gamma+\varsigma)^2\left|\mathbf{T}\right|^2\\ &\leq 2\varrho^2 \sup_{\xi\in[0,\nu_1]}\left|\mathbf{I}\right|^2 + 2(\psi+\gamma+\varsigma)^2\left|\mathbf{T}\right|^2\\ &\leq 2\varrho^2\left\|\mathbf{I}\right\|_\infty^2 + 2(\psi+\gamma+\varsigma)^2\left|\mathbf{T}\right|^2\\ &\leq 2\varrho^2\left\|\mathbf{I}\right\|_\infty^2\Big(1+\frac{(\psi+\gamma+\varsigma)^2\left|\mathbf{T}\right|^2}{\varrho^2\left\|\mathbf{I}\right\|_\infty^2}\Big),\end{aligned} \tag{4.7}$$

under the supposition $\frac{(\psi+\gamma+\varsigma)^2}{\varrho^2\left\|\mathbf{I}\right\|_\infty^2} < 1$, then

$$\left|\mathbf{F}_3(\mathbf{T},\xi)\right|^2 \leq \Omega_3(1+\left|\mathbf{T}\right|^2). \tag{4.8}$$

It is clear that

$$\left|\mathcal{G}_3(\mathbf{T},\xi)\right|^2 \leq \rho_3^2(1+\left|\mathbf{T}\right|^2) \leq \Omega_3(1+\left|\mathbf{T}\right|^2). \tag{4.9}$$

Furthermore, we have $\mathbf{F}_4(\mathbf{R},\xi) = \gamma\mathbf{T} - (\psi+\phi)\mathbf{R}$, $\quad \mathcal{G}_4(\mathbf{R},\xi) = \rho_4\mathbf{R}$.

Also, we have

$$\begin{aligned}\left|\mathbf{F}_4(\mathbf{R},\xi)\right|^2 &= \left|\gamma\mathbf{T} - (\psi+\phi)\mathbf{R}\right|^2\\ &\leq 2\gamma^2\left|\mathbf{T}\right|^2 + 2(\psi+\phi)^2\left|\mathbf{R}\right|^2\\ &\leq 2\gamma^2 \sup_{\xi\in[0,\nu_1]}\left|\mathbf{T}\right|^2 + 2(\psi+\phi)^2\left|\mathbf{R}\right|^2\\ &\leq 2\gamma^2\left\|\mathbf{T}\right\|_\infty^2 + 2(\psi+\phi)^2\left|\mathbf{R}\right|^2\\ &\leq 2\gamma^2\left\|\mathbf{T}\right\|_\infty^2\Big(1+\frac{(\psi+\phi)^2\left|\mathbf{R}\right|^2}{\gamma^2\left\|\mathbf{T}\right\|_\infty^2}\Big),\end{aligned} \tag{4.10}$$

under the supposition $\frac{(\psi+\phi)^2}{\gamma^2\left\|\mathbf{T}\right\|_\infty^2} < 1$, then

$$\left|\mathbf{F}_4(\mathbf{R},\xi)\right|^2 \leq \Omega_4(1+\left|\mathbf{R}\right|^2). \tag{4.11}$$

It is clear that

$$\left|\mathcal{G}_4(\mathbf{R},\xi)\right|^2 \leq \rho_4^2(1+\left|\mathbf{R}\right|^2) \leq \Omega_4(1+\left|\mathbf{R}\right|^2). \tag{4.12}$$

As a result, if the criteria of linear growth is satisfied,

$$\min\left\{\Big(1+\frac{(\delta^2\left\|\mathbf{I}\right\|_\infty^2+\psi^2)}{\pi^2+\phi^2\left\|\mathbf{R}\right\|^2}\Big), \frac{(\psi+\eta+\zeta)^2}{2\delta^2\left\|\mathbf{S}\right\|_\infty^2}, \frac{(\psi+\gamma+\varsigma)^2}{\varrho^2\left\|\mathbf{I}\right\|_\infty^2}, \frac{(\psi+\phi)^2}{\gamma^2\left\|\mathbf{T}\right\|_\infty^2}\right\} < 1. \tag{4.13}$$

The evidence described herein generally applies to fractional derivative and fractal-fractional derivative models. Whenever the systems have categorized derivatives, fractional derivatives, and fractal-fractional derivatives, we now describe the mathematical results.

4.2. Numerical approximations

In this subsection, we illustrate numerical approximations utilizing the Newton Polynomial technique [51].

Case-I (**Classical case**). Let us begin utilizing the classical differentiation. The randomized processes are assumed to be differentiable in the sense that

$$dB_J(\xi) = B_J(\xi)d\xi. \tag{4.14}$$

For this, we have

$$\begin{aligned}
\mathbf{S}(\xi_{n+1}) \quad &= \mathbf{S}(\xi_n) + \frac{23}{12}\Delta\xi \mathbf{F}_1(\xi_n, \mathbf{S}_n, \mathbf{I}_n, \mathbf{R}_n) + \frac{5}{12}\Delta\xi \mathbf{F}_1(\xi_{n-2}, \mathbf{S}_{n-2}, \mathbf{I}_{n-2}, \mathbf{R}_{n-2}) \\
&-\frac{4}{3}\Delta\xi \mathbf{F}_1(\xi_{n-1}, \mathbf{S}_{n-1}, \mathbf{I}_{n-1}, \mathbf{R}_{n-1}) + \frac{23}{12}\Delta\xi\rho_1 \mathbf{S}(\xi_n)\frac{B_1'(\xi_{n+1}) - B_1'(\xi_n)}{\Delta\xi} \\
&+\frac{5}{12}\Delta\xi\rho_1 \mathbf{S}(\xi_{n-1})\frac{B_1'(\xi_n) - B_1'(\xi_{n-1})}{\Delta\xi} - \frac{4}{3}\Delta\xi\rho_1 \mathbf{S}(\xi_{n-2})\frac{B_1'(\xi_{n-1}) - B_1'(\xi_{n-2})}{\Delta\xi},
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
\mathbf{I}(\xi_{n+1}) \quad &= \mathbf{I}(\xi_n) + \frac{23}{12}\Delta\xi \mathbf{F}_2(\xi_n, \mathbf{S}_n, \mathbf{I}_n) + \frac{5}{12}\Delta\xi \mathbf{F}_2(\xi_{n-2}, \mathbf{S}_{n-2}, \mathbf{I}_{n-2}) \\
&-\frac{4}{3}\Delta\xi \mathbf{F}_2(\xi_{n-1}, \mathbf{S}_{n-1}, \mathbf{I}_{n-1}) + \frac{23}{12}\Delta\xi\rho_2 \mathbf{I}(\xi_n)\frac{B_2'(\xi_{n+1}) - B_2'(\xi_n)}{\Delta\xi} \\
&+\frac{5}{12}\Delta\xi\rho_2 \mathbf{I}(\xi_{n-1})\frac{B_2'(\xi_n) - B_2'(\xi_{n-1})}{\Delta\xi} - \frac{4}{3}\Delta\xi\rho_2 \mathbf{I}(\xi_{n-2})\frac{B_2'(\xi_{n-1}) - B_2'(\xi_{n-2})}{\Delta\xi},
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
\mathbf{T}(\xi_{n+1}) \quad &= \mathbf{T}(\xi_n) + \frac{23}{12}\Delta\xi \mathbf{F}_3(\xi_n, \mathbf{I}_n, \mathbf{T}_n) + \frac{5}{12}\Delta\xi \mathbf{F}_3(\xi_{n-2}, \mathbf{I}_{n-2}, \mathbf{T}_{n-2}) \\
&-\frac{4}{3}\Delta\xi \mathbf{F}_3(\xi_{n-1}, \mathbf{I}_{n-1}, \mathbf{T}_{n-1}) + \frac{23}{12}\Delta\xi\rho_3 \mathbf{T}(\xi_n)\frac{B_3'(\xi_{n+1}) - B_3'(\xi_n)}{\Delta\xi} \\
&+\frac{5}{12}\Delta\xi\rho_3 \mathbf{T}(\xi_{n-1})\frac{B_3'(\xi_n) - B_3'(\xi_{n-1})}{\Delta\xi} - \frac{4}{3}\Delta\xi\rho_3 \mathbf{T}(\xi_{n-2})\frac{B_3'(\xi_{n-1}) - B_3'(\xi_{n-2})}{\Delta\xi}
\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}
\mathbf{R}(\xi_{n+1}) \quad &= \mathbf{R}(\xi_n) + \frac{23}{12}\Delta\xi \mathbf{F}_4(\xi_n, \mathbf{T}_n, \mathbf{R}_n) + \frac{5}{12}\Delta\xi \mathbf{F}_4(\xi_{n-2}, \mathbf{T}_{n-2}, \mathbf{R}_{n-2}) \\
&-\frac{4}{3}\Delta\xi \mathbf{F}_4(\xi_{n-1}, \mathbf{T}_{n-1}, \mathbf{R}_{n-1}) + \frac{23}{12}\Delta\xi\rho_4 \mathbf{R}(\xi_n)\frac{B_4'(\xi_{n+1}) - B_4'(\xi_n)}{\Delta\xi} \\
&+\frac{5}{12}\Delta\xi\rho_4 \mathbf{R}(\xi_{n-1})\frac{B_4'(\xi_n) - B_4'(\xi_{n-1})}{\Delta\xi} - \frac{4}{3}\Delta\xi\rho_4 \mathbf{R}(\xi_{n-2})\frac{B_4'(\xi_{n-1}) - B_4'(\xi_{n-2})}{\Delta\xi},
\end{aligned} \tag{4.18}$$

Case-II (**Caputo derivative operator**). Here, we surmise the case when Caputo's fractional derivative operator based on the power law kernel is utilized as

$$
\begin{aligned}
\mathbf{S}(\xi_{\mathfrak{n}+1}) &= \mathbf{S}(0) + \frac{1}{\Gamma(\varphi)}\sum_{\mathbf{q}=0}^{\mathfrak{n}}\int_{\xi_{\mathbf{q}}}^{\xi_{\mathbf{q}+1}}(\xi_{\mathfrak{n}+1}-\theta)^{\varphi-1}\mathbf{F}_1(\mathbf{S},\mathbf{I},\mathbf{R},\theta)d\theta \\
&\quad + \frac{1}{\Gamma(\varphi)}\sum_{\mathbf{q}=0}^{\mathfrak{n}}\int_{\xi_{\mathbf{q}}}^{\xi_{\mathbf{q}+1}}(\xi_{\mathfrak{n}+1}-\theta)^{\varphi-1}\mathcal{G}_1(\mathbf{S},\theta)\rho_1 B_1'(\mathbf{S},\theta)d\theta, \\
\mathbf{I}(\xi_{\mathfrak{n}+1}) &= \mathbf{I}(0) + \frac{1}{\Gamma(\varphi)}\sum_{\mathbf{q}=0}^{\mathfrak{n}}\int_{\xi_{\mathbf{q}}}^{\xi_{\mathbf{q}+1}}(\xi_{\mathfrak{n}+1}-\theta)^{\varphi-1}\mathbf{F}_2(\mathbf{S},\mathbf{I},\theta)d\theta \\
&\quad + \frac{1}{\Gamma(\varphi)}\sum_{\mathbf{q}=0}^{\mathfrak{n}}\int_{\xi_{\mathbf{q}}}^{\xi_{\mathbf{q}+1}}(\xi_{\mathfrak{n}+1}-\theta)^{\varphi-1}\mathcal{G}_2(\mathbf{I},\theta)\rho_1 B_2'(\mathbf{I},\theta)d\theta, \\
\mathbf{T}(\xi_{\mathfrak{n}+1}) &= \mathbf{T}(0) + \frac{1}{\Gamma(\varphi)}\sum_{\mathbf{q}=0}^{\mathfrak{n}}\int_{\xi_{\mathbf{q}}}^{\xi_{\mathbf{q}+1}}(\xi_{\mathfrak{n}+1}-\theta)^{\varphi-1}\mathbf{F}_3(\mathbf{I},\mathbf{T},\theta)d\theta \\
&\quad + \frac{1}{\Gamma(\varphi)}\sum_{\mathbf{q}=0}^{\mathfrak{n}}\int_{\xi_{\mathbf{q}}}^{\xi_{\mathbf{q}+1}}(\xi_{\mathfrak{n}+1}-\theta)^{\varphi-1}\mathcal{G}_3(\mathbf{T},\theta)\rho_1 B_3'(\mathbf{T},\theta)d\theta, \\
\mathbf{R}(\xi_{\mathfrak{n}+1}) &= \mathbf{R}(0) + \frac{1}{\Gamma(\varphi)}\sum_{\mathbf{q}=0}^{\mathfrak{n}}\int_{\xi_{\mathbf{q}}}^{\xi_{\mathbf{q}+1}}(\xi_{\mathfrak{n}+1}-\theta)^{\varphi-1}\mathbf{F}_4(\mathbf{T},\mathbf{R},\theta)d\theta \\
&\quad + \frac{1}{\Gamma(\varphi)}\sum_{\mathbf{q}=0}^{\mathfrak{n}}\int_{\xi_{\mathbf{q}}}^{\xi_{\mathbf{q}+1}}(\xi_{\mathfrak{n}+1}-\theta)^{\varphi-1}\mathcal{G}_4(\mathbf{R},\theta)\rho_1 B_4'(\mathbf{R},\theta)d\theta.
\end{aligned}
$$

As the scheme suggested by Atangana and Touffik [52], the mappings $\mathbf{F}_1$ and $\mathcal{G}_1$ are estimated employing Lagrange polynomials. This results in

$$
\mathbf{S}(\xi_{\mathfrak{n}+1}) = \mathbf{S}(0) + \left\{
\begin{array}{l}
\frac{(\Delta)^{\varphi}}{\Gamma(\varphi+1)}\sum\limits_{\mathbf{q}=2}^{\mathfrak{n}}\mathbf{F}_1(\mathbf{S}_{\mathbf{q}-2},\mathbf{I}_{\mathbf{q}-2},\mathbf{R}_{\mathbf{q}-2},\xi_{\mathbf{q}-2})(\mathfrak{n}-\mathbf{q}+1)^{\varphi}-(\mathfrak{n}-\mathbf{q})^{\varphi} \\
\quad +\frac{(\Delta\xi)^{\varphi}}{\Gamma(\varphi+2)}\sum\limits_{\mathbf{q}=2}^{\mathfrak{n}}(\mathbf{F}_1(\mathbf{S}_{\mathbf{q}-1},\mathbf{I}_{\mathbf{q}-1},\mathbf{R}_{\mathbf{q}-1},\xi_{\mathbf{q}-1})-\mathbf{F}_1(\mathbf{S}_{\mathbf{q}-2},\mathbf{I}_{\mathbf{q}-2},\mathbf{R}_{\mathbf{q}-2},\xi_{\mathbf{q}-2})) \\
\quad \times\Big((\mathfrak{n}-\mathbf{q}+1)^{\varphi}(\mathfrak{n}-\mathbf{q}+2\varphi+3)-(\mathfrak{n}-\mathbf{q})^{\varphi}(\mathfrak{n}-\mathbf{q}+3\varphi+3)\Big) \\
\quad +\frac{(\Delta\xi)^{\varphi}}{2\Gamma(\varphi+3)}\sum\limits_{\mathbf{q}=2}^{\mathfrak{n}}(\mathbf{F}_1(\mathbf{S}_{\mathbf{q}},\mathbf{I}_{\mathbf{q}},\mathbf{R}_{\mathbf{q}},\xi_{\mathbf{q}})-2\mathbf{F}_1(\mathbf{S}_{\mathbf{q}-1},\mathbf{I}_{\mathbf{q}-1},\mathbf{R}_{\mathbf{q}-1},\xi_{\mathbf{q}-1}) \\
\quad +\mathbf{F}_1(\mathbf{S}_{\mathbf{q}-2},\mathbf{I}_{\mathbf{q}-2},\mathbf{R}_{\mathbf{q}-2},\xi_{\mathbf{q}-2})) \\
\quad \times\Big\{(\mathfrak{n}-\mathbf{q}+1)^{\varphi}\Big[2(\mathfrak{n}-\mathbf{q})^2+(3\varphi+10)(\mathfrak{n}-\mathbf{q})+2\varphi^2+9\varphi+12\Big] \\
\quad -(\mathfrak{n}-\mathbf{q})^{\varphi}\Big[2(\mathfrak{n}-\mathbf{q})^2+(5\varphi+10)(\mathfrak{n}-\mathbf{q})+6\varphi^2+18\varphi+12\Big]\Big\}
\end{array}
\right.
$$

$$+\left\{\begin{array}{l}\frac{(\Delta\xi)^{\varphi-1}}{\Gamma(\varphi+1)}\sum_{\mathbf{q}=2}^{\mathfrak{n}}(\mathcal{G}_1(\mathbf{S}_{\mathbf{q}-2},\mathbf{I}_{\mathbf{q}-2},\mathbf{R}_{\mathbf{q}-2},\xi_{\mathbf{q}-2})\rho_1\{B_1'(\xi_{\mathbf{q}-1})-B_1'(\xi_{\mathbf{q}-2})\}[(\mathfrak{n}-\mathbf{q}+1)^{\varphi}-(\mathfrak{n}-\mathbf{q})^{\varphi}]\\ \quad+\frac{(\Delta\xi)^{\varphi-1}}{\Gamma(\varphi+2)}\sum_{\mathbf{q}=2}^{\mathfrak{n}}(\mathcal{G}_1(\mathbf{S}_{\mathbf{q}-1},\mathbf{I}_{\mathbf{q}-1},\mathbf{R}_{\mathbf{q}-1},\xi_{\mathbf{q}-1})\rho_1\{B_1'(\xi_{\mathbf{q}-1})-B_1'(\xi_{\mathbf{q}-2})\}\\ \quad-\mathcal{G}_1(\mathbf{S}_{\mathbf{q}-2},\mathbf{I}_{\mathbf{q}-2},\mathbf{R}_{\mathbf{q}-2},\xi_{\mathbf{q}-2})\rho_1\{B_1'(\xi_{\mathbf{q}-1})-B_1'(\xi_{\mathbf{q}-2})\})\\ \quad\times\Big((\mathfrak{n}-\mathbf{q}+1)^{\varphi}(\mathfrak{n}-\mathbf{q}+2\varphi+3)-(\mathfrak{n}-\mathbf{q})^{\varphi}(\mathfrak{n}-\mathbf{q}+3\varphi+3)\Big)\\ \quad+\frac{(\Delta\xi)^{\varphi-1}}{2\Gamma(\varphi+3)}\sum_{\mathbf{q}=2}^{\mathfrak{n}}(\mathcal{G}_1(\mathbf{S}_{\mathbf{q}},\mathbf{I}_{\mathbf{q}},\mathbf{R}_{\mathbf{q}},\xi_{\mathbf{q}})\rho_1\{B_1'(\xi_{\mathbf{q}})-B_1'(\xi_{\mathbf{q}+1})\}\\ \quad-2\mathcal{G}_1(\mathbf{S}_{\mathbf{q}-1},\mathbf{I}_{\mathbf{q}-1},\mathbf{R}_{\mathbf{q}-1},\xi_{\mathbf{q}-1})\rho_1\{B_1'(\xi_{\mathbf{q}})-B_1'(\xi_{\mathbf{q}-1})\}\\ \quad+\mathcal{G}_1(\mathbf{S}_{\mathbf{q}-2},\mathbf{I}_{\mathbf{q}-2},\mathbf{R}_{\mathbf{q}-2},\xi_{\mathbf{q}-2})\rho_1\{B_1'(\xi_{\mathbf{q}-1})-B_1'(\xi_{\mathbf{q}-2})\})\\ \quad\times\Big\{(\mathfrak{n}-\mathbf{q}+1)^{\varphi}\Big[2(\mathfrak{n}-\mathbf{q})^2+(3\varphi+10)(\mathfrak{n}-\mathbf{q})+2\varphi^2+9\varphi+12\Big]\\ \quad-(\mathfrak{n}-\mathbf{q})^{\varphi}\Big[2(\mathfrak{n}-\mathbf{q})^2+(5\varphi+10)(\mathfrak{n}-\mathbf{q})+6\varphi^2+18\varphi+12\Big]\Big\},\end{array}\right.$$

$$\mathbf{I}(\xi_{\mathfrak{n}+1})=\mathbf{I}(0)+\left\{\begin{array}{l}\frac{(\Delta)^{\varphi}}{\Gamma(\varphi+1)}\sum_{\mathbf{q}=2}^{\mathfrak{n}}\mathbf{F}_2(\mathbf{S}_{\mathbf{q}-2},\mathbf{I}_{\mathbf{q}-2},\xi_{\mathbf{q}-2})(\mathfrak{n}-\mathbf{q}+1)^{\varphi}-(\mathfrak{n}-\mathbf{q})^{\varphi}\\ \quad+\frac{(\Delta\xi)^{\varphi}}{\Gamma(\varphi+2)}\sum_{\mathbf{q}=2}^{\mathfrak{n}}(\mathbf{F}_2(\mathbf{S}_{\mathbf{q}-1},\mathbf{I}_{\mathbf{q}-1},\xi_{\mathbf{q}-1})-\mathbf{F}_2(\mathbf{S}_{\mathbf{q}-2},\mathbf{I}_{\mathbf{q}-2},\xi_{\mathbf{q}-2}))\\ \quad\times\Big((\mathfrak{n}-\mathbf{q}+1)^{\varphi}(\mathfrak{n}-\mathbf{q}+2\varphi+3)-(\mathfrak{n}-\mathbf{q})^{\varphi}(\mathfrak{n}-\mathbf{q}+3\varphi+3)\Big)\\ \quad+\frac{(\Delta\xi)^{\varphi}}{2\Gamma(\varphi+3)}\sum_{\mathbf{q}=2}^{\mathfrak{n}}(\mathbf{F}_2(\mathbf{S}_{\mathbf{q}},\mathbf{I}_{\mathbf{q}},\xi_{\mathbf{q}})-2\mathbf{F}_2(\mathbf{S}_{\mathbf{q}-1},\mathbf{I}_{\mathbf{q}-1},\xi_{\mathbf{q}-1})+\mathbf{F}_2(\mathbf{S}_{\mathbf{q}-2},\mathbf{I}_{\mathbf{q}-2},\xi_{\mathbf{q}-2}))\\ \quad\times\Big\{(\mathfrak{n}-\mathbf{q}+1)^{\varphi}\Big[2(\mathfrak{n}-\mathbf{q})^2+(3\varphi+10)(\mathfrak{n}-\mathbf{q})+2\varphi^2+9\varphi+12\Big]\\ \quad-(\mathfrak{n}-\mathbf{q})^{\varphi}\Big[2(\mathfrak{n}-\mathbf{q})^2+(5\varphi+10)(\mathfrak{n}-\mathbf{q})+6\varphi^2+18\varphi+12\Big]\Big\}\end{array}\right.$$

$$+\left\{\begin{array}{l}\frac{(\Delta\xi)^{\varphi-1}}{\Gamma(\varphi+1)}\sum_{\mathbf{q}=2}^{\mathfrak{n}}(\mathcal{G}_2(\mathbf{S}_{\mathbf{q}-2},\mathbf{I}_{\mathbf{q}-2},\xi_{\mathbf{q}-2})\rho_2\{B_2'(\xi_{\mathbf{q}-1})-B_2'(\xi_{\mathbf{q}-2})\}[(\mathfrak{n}-\mathbf{q}+1)^{\varphi}-(\mathfrak{n}-\mathbf{q})^{\varphi}]\\ \quad+\frac{(\Delta\xi)^{\varphi-1}}{\Gamma(\varphi+2)}\sum_{\mathbf{q}=2}^{\mathfrak{n}}(\mathcal{G}_2(\mathbf{S}_{\mathbf{q}-1},\mathbf{I}_{\mathbf{q}-1},\xi_{\mathbf{q}-1})\rho_2\{B_2'(\xi_{\mathbf{q}-1})-B_2'(\xi_{\mathbf{q}-2})\}\\ \quad-\mathcal{G}_2(\mathbf{S}_{\mathbf{q}-2},\mathbf{I}_{\mathbf{q}-2},\xi_{\mathbf{q}-2})\rho_2\{B_2'(\xi_{\mathbf{q}-1})-B_2'(\xi_{\mathbf{q}-2})\})\\ \quad\times\Big((\mathfrak{n}-\mathbf{q}+1)^{\varphi}(\mathfrak{n}-\mathbf{q}+2\varphi+3)-(\mathfrak{n}-\mathbf{q})^{\varphi}(\mathfrak{n}-\mathbf{q}+3\varphi+3)\Big)\\ \quad+\frac{(\Delta\xi)^{\varphi-1}}{2\Gamma(\varphi+3)}\sum_{\mathbf{q}=2}^{\mathfrak{n}}(\mathcal{G}_2(\mathbf{S}_{\mathbf{q}},\mathbf{I}_{\mathbf{q}},\xi_{\mathbf{q}})\rho_2\{B_2'(\xi_{\mathbf{q}})-B_2'(\xi_{\mathbf{q}+1})\}\\ \quad-2\mathcal{G}_2(\mathbf{S}_{\mathbf{q}-1},\mathbf{I}_{\mathbf{q}-1},\xi_{\mathbf{q}-1})\rho_2\{B_2'(\xi_{\mathbf{q}})-B_2'(\xi_{\mathbf{q}-1})\}\\ \quad+\mathcal{G}_2(\mathbf{S}_{\mathbf{q}-2},\mathbf{I}_{\mathbf{q}-2},\xi_{\mathbf{q}-2})\rho_2\{B_2'(\xi_{\mathbf{q}-1})-B_2'(\xi_{\mathbf{q}-2})\})\\ \quad\times\Big\{(\mathfrak{n}-\mathbf{q}+1)^{\varphi}\Big[2(\mathfrak{n}-\mathbf{q})^2+(3\varphi+10)(\mathfrak{n}-\mathbf{q})+2\varphi^2+9\varphi+12\Big]\\ \quad-(\mathfrak{n}-\mathbf{q})^{\varphi}\Big[2(\mathfrak{n}-\mathbf{q})^2+(5\varphi+10)(\mathfrak{n}-\mathbf{q})+6\varphi^2+18\varphi+12\Big]\Big\},\end{array}\right.$$

$$\mathbf{T}(\xi_{\mathfrak{n}+1}) = \mathbf{T}(0) + \begin{cases} \frac{(\Delta)^{\varphi}}{\Gamma(\varphi+1)} \sum_{\mathbf{q}=2}^{\mathfrak{n}} \mathbf{F}_3(\mathbf{I}_{\mathbf{q}-2}, \mathbf{T}_{\mathbf{q}-2}, \xi_{\mathbf{q}-2})(\mathfrak{n}-\mathbf{q}+1)^{\varphi} - (\mathfrak{n}-\mathbf{q})^{\varphi} \\ \quad + \frac{(\Delta\xi)^{\varphi}}{\Gamma(\varphi+2)} \sum_{\mathbf{q}=2}^{\mathfrak{n}} \left(\mathbf{F}_3(\mathbf{I}_{\mathbf{q}-1}, \mathbf{T}_{\mathbf{q}-1}, \xi_{\mathbf{q}-1}) - \mathbf{F}_3(\mathbf{I}_{\mathbf{q}-2}, \mathbf{T}_{\mathbf{q}-2}, \xi_{\mathbf{q}-2})\right) \\ \quad \times\Big((\mathfrak{n}-\mathbf{q}+1)^{\varphi}(\mathfrak{n}-\mathbf{q}+2\varphi+3) - (\mathfrak{n}-\mathbf{q})^{\varphi}(\mathfrak{n}-\mathbf{q}+3\varphi+3)\Big) \\ \quad + \frac{(\Delta\xi)^{\varphi}}{2\Gamma(\varphi+3)} \sum_{\mathbf{q}=2}^{\mathfrak{n}} \left(\mathbf{F}_3(\mathbf{I}_{\mathbf{q}}, \mathbf{T}_{\mathbf{q}}, \xi_{\mathbf{q}}) - 2\mathbf{F}_3(\mathbf{I}_{\mathbf{q}-1}, \mathbf{T}_{\mathbf{q}-1}, \xi_{\mathbf{q}-1}) + \mathbf{F}_3(\mathbf{I}_{\mathbf{q}-2}, \mathbf{T}_{\mathbf{q}-2}, \xi_{\mathbf{q}-2})\right) \\ \quad \times\Big\{(\mathfrak{n}-\mathbf{q}+1)^{\varphi}\Big[2(\mathfrak{n}-\mathbf{q})^2 + (3\varphi+10)(\mathfrak{n}-\mathbf{q}) + 2\varphi^2 + 9\varphi + 12\Big] \\ \quad -(\mathfrak{n}-\mathbf{q})^{\varphi}\Big[2(\mathfrak{n}-\mathbf{q})^2 + (5\varphi+10)(\mathfrak{n}-\mathbf{q}) + 6\varphi^2 + 18\varphi + 12\Big]\Big\} \end{cases}$$

$$+ \begin{cases} \frac{(\Delta\xi)^{\varphi-1}}{\Gamma(\varphi+1)} \sum_{\mathbf{q}=2}^{\mathfrak{n}} (\mathcal{G}_3(\mathbf{I}_{\mathbf{q}-2}, \mathbf{T}_{\mathbf{q}-2}, \xi_{\mathbf{q}-2})\rho_3\{B_3'(\xi_{\mathbf{q}-1}) - B_3'(\xi_{\mathbf{q}-2})\}[(\mathfrak{n}-\mathbf{q}+1)^{\varphi} - (\mathfrak{n}-\mathbf{q})^{\varphi}] \\ \quad + \frac{(\Delta\xi)^{\varphi-1}}{\Gamma(\varphi+2)} \sum_{\mathbf{q}=2}^{\mathfrak{n}} (\mathcal{G}_3(\mathbf{I}_{\mathbf{q}-1}, \mathbf{T}_{\mathbf{q}-1}, \xi_{\mathbf{q}-1})\rho_3\{B_3'(\xi_{\mathbf{q}-1}) - B_3'(\xi_{\mathbf{q}-2})\} \\ \quad -\mathcal{G}_3(\mathbf{I}_{\mathbf{q}-2}, \mathbf{T}_{\mathbf{q}-2}, \xi_{\mathbf{q}-2})\rho_3\{B_3'(\xi_{\mathbf{q}-1}) - B_3'(\xi_{\mathbf{q}-2})\}) \\ \quad \times\Big((\mathfrak{n}-\mathbf{q}+1)^{\varphi}(\mathfrak{n}-\mathbf{q}+2\varphi+3) - (\mathfrak{n}-\mathbf{q})^{\varphi}(\mathfrak{n}-\mathbf{q}+3\varphi+3)\Big) \\ \quad + \frac{(\Delta\xi)^{\varphi-1}}{2\Gamma(\varphi+3)} \sum_{\mathbf{q}=2}^{\mathfrak{n}} (\mathcal{G}_3(\mathbf{I}_{\mathbf{q}}, \mathbf{T}_{\mathbf{q}}, \xi_{\mathbf{q}})\rho_3\{B_3'(\xi_{\mathbf{q}}) - B_3'(\xi_{\mathbf{q}+1})\} \\ \quad -2\mathcal{G}_3(\mathbf{I}_{\mathbf{q}-1}, \mathbf{T}_{\mathbf{q}-1}, \xi_{\mathbf{q}-1})\rho_3\{B_3'(\xi_{\mathbf{q}}) - B_3'(\xi_{\mathbf{q}-1})\} \\ \quad +\mathcal{G}_3(\mathbf{I}_{\mathbf{q}-2}, \mathbf{T}_{\mathbf{q}-2}, \xi_{\mathbf{q}-2})\rho_3\{B_3'(\xi_{\mathbf{q}-1}) - B_3'(\xi_{\mathbf{q}-2})\}) \\ \quad \times\Big\{(\mathfrak{n}-\mathbf{q}+1)^{\varphi}\Big[2(\mathfrak{n}-\mathbf{q})^2 + (3\varphi+10)(\mathfrak{n}-\mathbf{q}) + 2\varphi^2 + 9\varphi + 12\Big] \\ \quad -(\mathfrak{n}-\mathbf{q})^{\varphi}\Big[2(\mathfrak{n}-\mathbf{q})^2 + (5\varphi+10)(\mathfrak{n}-\mathbf{q}) + 6\varphi^2 + 18\varphi + 12\Big]\Big\}, \end{cases}$$

and

$$\mathbf{R}(\xi_{\mathfrak{n}+1}) = \mathbf{R}(0) + \begin{cases} \frac{(\Delta)^{\varphi}}{\Gamma(\varphi+1)} \sum_{\mathbf{q}=2}^{\mathfrak{n}} \mathbf{F}_4(\mathbf{T}_{\mathbf{q}-2}, \mathbf{R}_{\mathbf{q}-2}, \xi_{\mathbf{q}-2})(\mathfrak{n}-\mathbf{q}+1)^{\varphi} - (\mathfrak{n}-\mathbf{q})^{\varphi} \\ \quad + \frac{(\Delta\xi)^{\varphi}}{\Gamma(\varphi+2)} \sum_{\mathbf{q}=2}^{\mathfrak{n}} \left(\mathbf{F}_4(\mathbf{T}_{\mathbf{q}-1}, \mathbf{R}_{\mathbf{q}-1}, \xi_{\mathbf{q}-1}) - \mathbf{F}_4(\mathbf{I}_{\mathbf{q}-2}, \mathbf{T}_{\mathbf{q}-2}, \xi_{\mathbf{q}-2})\right) \\ \quad \times\Big((\mathfrak{n}-\mathbf{q}+1)^{\varphi}(\mathfrak{n}-\mathbf{q}+2\varphi+3) - (\mathfrak{n}-\mathbf{q})^{\varphi}(\mathfrak{n}-\mathbf{q}+3\varphi+3)\Big) \\ \quad + \frac{(\Delta\xi)^{\varphi}}{2\Gamma(\varphi+3)} \sum_{\mathbf{q}=2}^{\mathfrak{n}} \left(\mathbf{F}_4(\mathbf{T}_{\mathbf{q}}, \mathbf{R}_{\mathbf{q}}, \xi_{\mathbf{q}}) - 2\mathbf{F}_4(\mathbf{T}_{\mathbf{q}-1}, \mathbf{R}_{\mathbf{q}-1}, \xi_{\mathbf{q}-1}) + \mathbf{F}_4(\mathbf{T}_{\mathbf{q}-2}, \mathbf{R}_{\mathbf{q}-2}, \xi_{\mathbf{q}-2})\right) \\ \quad \times\Big\{(\mathfrak{n}-\mathbf{q}+1)^{\varphi}\Big[2(\mathfrak{n}-\mathbf{q})^2 + (3\varphi+10)(\mathfrak{n}-\mathbf{q}) + 2\varphi^2 + 9\varphi + 12\Big] \\ \quad -(\mathfrak{n}-\mathbf{q})^{\varphi}\Big[2(\mathfrak{n}-\mathbf{q})^2 + (5\varphi+10)(\mathfrak{n}-\mathbf{q}) + 6\varphi^2 + 18\varphi + 12\Big]\Big\} \end{cases}$$

$$+\begin{cases}\frac{(\Delta\xi)^{\varphi-1}}{\Gamma(\varphi+1)}\sum\limits_{\mathbf{q}=2}^{\mathfrak{n}}(\mathcal{G}_4(\mathbf{T}_{\mathbf{q}-2},\mathbf{R}_{\mathbf{q}-2},\xi_{\mathbf{q}-2})\rho_4\{B_4'(\xi_{\mathbf{q}-1})-B_4'(\xi_{\mathbf{q}-2})\}[(\mathfrak{n}-\mathbf{q}+1)^{\varphi}-(\mathfrak{n}-\mathbf{q})^{\varphi}]\\ +\frac{(\Delta\xi)^{\varphi-1}}{\Gamma(\varphi+2)}\sum\limits_{\mathbf{q}=2}^{\mathfrak{n}}(\mathcal{G}_4(\mathbf{T}_{\mathbf{q}-1},\mathbf{R}_{\mathbf{q}-1},\xi_{\mathbf{q}-1})\rho_4\{B_4'(\xi_{\mathbf{q}-1})-B_4'(\xi_{\mathbf{q}-2})\}\\ -\mathcal{G}_4(\mathbf{T}_{\mathbf{q}-2},\mathbf{R}_{\mathbf{q}-2},\xi_{\mathbf{q}-2})\rho_4\{B_4'(\xi_{\mathbf{q}-1})-B_4'(\xi_{\mathbf{q}-2})\})\\ \times\Big((\mathfrak{n}-\mathbf{q}+1)^{\varphi}(\mathfrak{n}-\mathbf{q}+2\varphi+3)-(\mathfrak{n}-\mathbf{q})^{\varphi}(\mathfrak{n}-\mathbf{q}+3\varphi+3)\Big)\\ +\frac{(\Delta\xi)^{\varphi-1}}{2\Gamma(\varphi+3)}\sum\limits_{\mathbf{q}=2}^{\mathfrak{n}}(\mathcal{G}_4(\mathbf{T}_{\mathbf{q}},\mathbf{R}_{\mathbf{q}},\xi_{\mathbf{q}})\rho_4\{B_4'(\xi_{\mathbf{q}})-B_4'(\xi_{\mathbf{q}+1})\}\\ -2\mathcal{G}_4(\mathbf{T}_{\mathbf{q}-1},\mathbf{R}_{\mathbf{q}-1},\xi_{\mathbf{q}-1})\rho_4\{B_4'(\xi_{\mathbf{q}})-B_4'(\xi_{\mathbf{q}-1})\}\\ +\mathcal{G}_4(\mathbf{T}_{\mathbf{q}-2},\mathbf{R}_{\mathbf{q}-2},\xi_{\mathbf{q}-2})\rho_4\{B_4'(\xi_{\mathbf{q}-1})-B_4'(\xi_{\mathbf{q}-2})\})\\ \times\Big\{(\mathfrak{n}-\mathbf{q}+1)^{\varphi}\Big[2(\mathfrak{n}-\mathbf{q})^2+(3\varphi+10)(\mathfrak{n}-\mathbf{q})+2\varphi^2+9\varphi+12\Big]\\ -(\mathfrak{n}-\mathbf{q})^{\varphi}\Big[2(\mathfrak{n}-\mathbf{q})^2+(5\varphi+10)(\mathfrak{n}-\mathbf{q})+6\varphi^2+18\varphi+12\Big]\Big\},\end{cases}$$

Case-III (**Caputo-Fabrizio derivative operator**). Here, we assume the case when Caputo-Fabrizio fractional derivative operator depend on the exponential decay kernel is utilized as

$$\begin{aligned}\mathbf{S}(\xi)-\mathbf{S}(0) &= \frac{1-\varphi}{\mathbb{M}(\varphi)}\mathbf{F}_1(\mathbf{S},\mathbf{I},\mathbf{R},\xi)+\frac{\varphi}{\mathbb{M}(\varphi)}\int_0^{\xi}\mathbf{F}_1(\mathbf{S},\mathbf{I},\mathbf{R},\theta)d\theta\\ &\quad+\frac{1-\varphi}{\mathbb{M}(\varphi)}\mathcal{G}_1(\mathbf{S},\mathbf{I},\mathbf{R},\xi)B_1'(\xi)+\frac{\varphi}{\mathbb{M}(\varphi)}\int_0^{\xi}\mathcal{G}_1(\mathbf{S},\mathbf{I},\mathbf{R},\theta)B_1'(\theta)d\theta,\\ \mathbf{I}(\xi)-\mathbf{I}(0) &= \frac{1-\varphi}{\mathbb{M}(\varphi)}\mathbf{F}_2(\mathbf{S},\mathbf{I},\xi)+\frac{\varphi}{\mathbb{M}(\varphi)}\int_0^{\xi}\mathbf{F}_2(\mathbf{S},\mathbf{I},\theta)d\theta\\ &\quad+\frac{1-\varphi}{\mathbb{M}(\varphi)}\mathcal{G}_2(\mathbf{S},\mathbf{I},\xi)B_1'(\xi)+\frac{\varphi}{\mathbb{M}(\varphi)}\int_0^{\xi}\mathcal{G}_2(\mathbf{S},\mathbf{I},\theta)B_1'(\theta)d\theta,\\ \mathbf{T}(\xi)-\mathbf{T}(0) &= \frac{1-\varphi}{\mathbb{M}(\varphi)}\mathbf{F}_3(\mathbf{I},\mathbf{T},\xi)+\frac{\varphi}{\mathbb{M}(\varphi)}\int_0^{\xi}\mathbf{F}_3(\mathbf{I},\mathbf{T},\theta)d\theta\\ &\quad+\frac{1-\varphi}{\mathbb{M}(\varphi)}\mathcal{G}_3(\mathbf{I},\mathbf{T},\xi)B_3'(\xi)+\frac{\varphi}{\mathbb{M}(\varphi)}\int_0^{\xi}\mathcal{G}_3(\mathbf{I},\mathbf{T},\theta)B_3'(\theta)d\theta,\\ \mathbf{R}(\xi)-\mathbf{R}(0) &= \frac{1-\varphi}{\mathbb{M}(\varphi)}\mathbf{F}_4(\mathbf{T},\mathbf{R},\xi)+\frac{\varphi}{\mathbb{M}(\varphi)}\int_0^{\xi}\mathbf{F}_4(\mathbf{T},\mathbf{R},\theta)d\theta\\ &\quad+\frac{1-\varphi}{\mathbb{M}(\varphi)}\mathcal{G}_4(\mathbf{T},\mathbf{R},\xi)B_4'(\xi)+\frac{\varphi}{\mathbb{M}(\varphi)}\int_0^{\xi}\mathcal{G}_4(\mathbf{T},\mathbf{R},\theta)B_4'(\theta)d\theta.\end{aligned}$$

For $\xi=\xi_{\mathfrak{n}+1}$ and $\xi=\xi_{\mathfrak{n}}$, we have

$$\mathbf{S}(\xi_{n+1}) = \mathbf{S}(\xi_n) + \begin{cases} \frac{1-\varphi}{\mathbb{M}(\varphi)}\Big[\mathbf{F}_1(\mathbf{S}(\xi_n),\mathbf{I}(\xi_n),\mathbf{R}(\xi_n),(\xi_n)) - \mathbf{F}_1(\mathbf{S}(\xi_{n-1}),\mathbf{I}(\xi_{n-1}),\mathbf{R}(\xi_{n-1}),(\xi_{n-1}))\Big] \\ +\frac{\varphi}{\mathbb{M}(\varphi)}\Big\{\frac{23}{12}\Delta\xi\mathbf{F}_1(\mathbf{S}(\xi_n),\mathbf{I}(\xi_n),\mathbf{R}(\xi_n),(\xi_n)) \\ +\frac{5}{12}\Delta\xi\mathbf{F}_1(\mathbf{S}(\xi_{n-2}),\mathbf{I}(\xi_{n-2}),\mathbf{R}(\xi_{n-2}),(\xi_{n-2})) \\ -\frac{4}{3}\Delta\xi\mathbf{F}_1(\mathbf{S}(\xi_{n-1}),\mathbf{I}(\xi_{n-1}),\mathbf{R}(\xi_{n-1}),(\xi_{n-1}))\Big\} \end{cases}$$

$$+ \begin{cases} \frac{1-\varphi}{\mathbb{M}(\varphi)}\Big[\mathcal{G}_1(\mathbf{S}(\xi_n),\mathbf{I}(\xi_n),\mathbf{R}(\xi_n),(\xi_n))\frac{B_1'(\xi_{n+1})-B_1'(\xi_n)}{\Delta\xi} \\ -\mathcal{G}_1(\mathbf{S}(\xi_{n-1}),\mathbf{I}(\xi_{n-1}),\mathbf{R}(\xi_{n-1}),(\xi_{n-1}))\frac{B_1'(\xi_n)-B_1'(\xi_{n-1})}{\Delta\xi}\Big] \\ +\frac{\varphi}{\mathbb{M}(\varphi)}\Big\{\frac{23}{12}\Delta\xi\rho_1\mathcal{G}_1(\mathbf{S}(\xi_n),\mathbf{I}(\xi_n),\mathbf{R}(\xi_n),(\xi_n))\frac{B_1'(\xi_{n+1})-B_1'(\xi_n)}{\Delta\xi} \\ +\frac{5}{12}\Delta\xi\rho_1\mathcal{G}_1(\mathbf{S}(\xi_{n-2}),\mathbf{I}(\xi_{n-2}),\mathbf{R}(\xi_{n-2}),(\xi_{n-2}))\frac{B_1'(\xi_{n-1})-B_1'(\xi_{n-2})}{\Delta\xi} \\ -\frac{4}{3}\rho_1\Delta_1\xi\mathcal{G}_1(\mathbf{S}(\xi_{n-1}),\mathbf{I}(\xi_{n-1}),\mathbf{R}(\xi_{n-1}),(\xi_{n-1}))\frac{B_1'(\xi_n)-B_1'(\xi_{n-1})}{\Delta\xi}\Big\}, \end{cases}$$

$$\mathbf{I}(\xi_{n+1}) = \mathbf{I}(\xi_n) + \begin{cases} \frac{1-\varphi}{\mathbb{M}(\varphi)}\Big[\mathbf{F}_2(\mathbf{S}(\xi_n),\mathbf{I}(\xi_n),(\xi_n)) - \mathbf{F}_2(\mathbf{S}(\xi_{n-1}),\mathbf{I}(\xi_{n-1}),(\xi_{n-1}))\Big] \\ +\frac{\varphi}{\mathbb{M}(\varphi)}\Big\{\frac{23}{12}\Delta\xi\mathbf{F}_2(\mathbf{S}(\xi_n),\mathbf{I}(\xi_n),(\xi_n)) \\ +\frac{5}{12}\Delta\xi\mathbf{F}_2(\mathbf{S}(\xi_{n-2}),\mathbf{I}(\xi_{n-2}),(\xi_{n-2})) \\ -\frac{4}{3}\Delta\xi\mathbf{F}_2(\mathbf{S}(\xi_{n-1}),\mathbf{I}(\xi_{n-1}),(\xi_{n-1}))\Big\} \end{cases}$$

$$+ \begin{cases} \frac{1-\varphi}{\mathbb{M}(\varphi)}\Big[\mathcal{G}_2(\mathbf{S}(\xi_n),\mathbf{I}(\xi_n),(\xi_n))\frac{B_2'(\xi_{n+1})-B_2'(\xi_n)}{\Delta\xi} \\ -\mathcal{G}_2(\mathbf{S}(\xi_{n-1}),\mathbf{I}(\xi_{n-1}),(\xi_{n-1}))\frac{B_2'(\xi_n)-B_2'(\xi_{n-1})}{\Delta\xi}\Big] \\ +\frac{\varphi}{\mathbb{M}(\varphi)}\Big\{\frac{23}{12}\Delta\xi\rho_2\mathcal{G}_2(\mathbf{S}(\xi_n),\mathbf{I}(\xi_n),(\xi_n))\frac{B_2'(\xi_{n+1})-B_2'(\xi_n)}{\Delta\xi} \\ +\frac{5}{12}\Delta\xi\rho_2\mathcal{G}_2(\mathbf{S}(\xi_{n-2}),\mathbf{I}(\xi_{n-2}),(\xi_{n-2}))\frac{B_2'(\xi_{n-1})-B_2'(\xi_{n-2})}{\Delta\xi} \\ -\frac{4}{3}\rho_2\Delta_1\xi\mathcal{G}_2(\mathbf{S}(\xi_{n-1}),\mathbf{I}(\xi_{n-1}),(\xi_{n-1}))\frac{B_2'(\xi_n)-B_2'(\xi_{n-1})}{\Delta\xi}\Big\}, \end{cases}$$

$$\mathbf{T}(\xi_{n+1}) = \mathbf{T}(\xi_n) + \begin{cases} \frac{1-\varphi}{\mathbb{M}(\varphi)}\Big[\mathbf{F}_3(\mathbf{I}(\xi_n),\mathbf{T}(\xi_n),(\xi_n)) - \mathbf{F}_3(\mathbf{I}(\xi_{n-1}),\mathbf{T}(\xi_{n-1}),(\xi_{n-1}))\Big] \\ +\frac{\varphi}{\mathbb{M}(\varphi)}\Big\{\frac{23}{12}\Delta\xi\mathbf{F}_3(\mathbf{I}(\xi_n),\mathbf{T}(\xi_n),(\xi_n)) \\ +\frac{5}{12}\Delta\xi\mathbf{F}_3(\mathbf{I}(\xi_{n-2}),\mathbf{T}(\xi_{n-2}),(\xi_{n-2})) \\ -\frac{4}{3}\Delta\xi\mathbf{F}_3(\mathbf{I}(\xi_{n-1}),\mathbf{T}(\xi_{n-1}),(\xi_{n-1}))\Big\} \end{cases}$$

$$+ \begin{cases} \frac{1-\varphi}{\mathbb{M}(\varphi)}\Big[\mathcal{G}_3(\mathbf{I}(\xi_n),\mathbf{T}(\xi_n),(\xi_n))\frac{B_3'(\xi_{n+1})-B_3'(\xi_n)}{\Delta\xi} \\ -\mathcal{G}_3(\mathbf{I}(\xi_{n-1}),\mathbf{T}(\xi_{n-1}),(\xi_{n-1}))\frac{B_3'(\xi_n)-B_3'(\xi_{n-1})}{\Delta\xi}\Big] \\ +\frac{\varphi}{\mathbb{M}(\varphi)}\Big\{\frac{23}{12}\Delta\xi\rho_3\mathcal{G}_3(\mathbf{I}(\xi_n),\mathbf{T}(\xi_n),(\xi_n))\frac{B_3'(\xi_{n+1})-B_3'(\xi_n)}{\Delta\xi} \\ +\frac{5}{12}\Delta\xi\rho_3\mathcal{G}_3(\mathbf{I}(\xi_{n-2}),\mathbf{T}(\xi_{n-2}),(\xi_{n-2}))\frac{B_3'(\xi_{n-1})-B_3'(\xi_{n-2})}{\Delta\xi} \\ -\frac{4}{3}\rho_3\Delta_1\xi\mathcal{G}_3(\mathbf{I}(\xi_{n-1}),\mathbf{T}(\xi_{n-1}),(\xi_{n-1}))\frac{B_3'(\xi_n)-B_3'(\xi_{n-1})}{\Delta\xi}\Big\} \end{cases}$$

and

$$\mathbf{R}(\xi_{n+1}) = \mathbf{R}(\xi_n) + \begin{cases} \frac{1-\varphi}{\mathbb{M}(\varphi)}\Big[\mathbf{F}_4(\mathbf{T}(\xi_n), \mathbf{R}(\xi_n), (\xi_n)) - \mathbf{F}_4(\mathbf{T}(\xi_{n-1}), \mathbf{R}(\xi_{n-1}), (\xi_{n-1}))\Big] \\ +\frac{\varphi}{\mathbb{M}(\varphi)}\Big\{\frac{23}{12}\Delta\xi\mathbf{F}_4(\mathbf{T}(\xi_n), \mathbf{R}(\xi_n), (\xi_n)) \\ +\frac{5}{12}\Delta\xi\mathbf{F}_4(\mathbf{T}(\xi_{n-2}), \mathbf{R}(\xi_{n-2}), (\xi_{n-2})) \\ -\frac{4}{3}\Delta\xi\mathbf{F}_4(\mathbf{T}(\xi_{n-1}), \mathbf{R}(\xi_{n-1}), (\xi_{n-1}))\Big\} \end{cases}$$
$$+ \begin{cases} \frac{1-\varphi}{\mathbb{M}(\varphi)}\Big[\mathcal{G}_4(\mathbf{T}(\xi_n), \mathbf{R}(\xi_n), (\xi_n))\frac{B_4'(\xi_{n+1})-B_4'(\xi_n)}{\Delta\xi} \\ -\mathcal{G}_4(\mathbf{T}(\xi_{n-1}), \mathbf{R}(\xi_{n-1}), (\xi_{n-1}))\frac{B_4'(\xi_n)-B_4'(\xi_{n-1})}{\Delta\xi}\Big] \\ +\frac{\varphi}{\mathbb{M}(\varphi)}\Big\{\frac{23}{12}\Delta\xi\rho_4\mathcal{G}_4(\mathbf{T}(\xi_n), \mathbf{R}(\xi_n), (\xi_n))\frac{B_4'(\xi_{n+1})-B_4'(\xi_n)}{\Delta\xi} \\ +\frac{5}{12}\Delta\xi\rho_4\mathcal{G}_4(\mathbf{T}(\xi_{n-2}), \mathbf{R}(\xi_{n-2}), (\xi_{n-2}))\frac{B_4'(\xi_{n-1})-B_4'(\xi_{n-2})}{\Delta\xi} \\ -\frac{4}{3}\rho_4\Delta_1\xi\mathcal{G}_4(\mathbf{T}(\xi_{n-1}), \mathbf{R}(\xi_{n-1}), (\xi_{n-1}))\frac{B_4'(\xi_n)-B_4'(\xi_{n-1})}{\Delta\xi}\Big\}. \end{cases}$$

Case-IV (**Atangana-Baleanu derivative operator**). Here, we assume the case when Atangana-Baleanu fractional derivative operator depend on the Mittag-Leffler kernel is utilized as

$$\begin{aligned}
\mathbf{S}(\xi) - \mathbf{S}(0) &= \frac{1-\varphi}{\mathbb{M}(\varphi)}\mathbf{F}_1(\mathbf{S}, \mathbf{I}, \mathbf{R}, \xi) + \frac{\varphi}{\mathbb{ABC}(\varphi)\Gamma(\varphi)}\int_0^\xi (\xi-\theta)^\varphi \mathbf{F}_1(\mathbf{S}, \mathbf{I}, \mathbf{R}, \theta)d\theta \\
&+\frac{1-\varphi}{\mathbb{ABC}(\varphi)}\mathcal{G}_1(\mathbf{S}, \mathbf{I}, \mathbf{R}, \xi)B_1'(\xi) + \frac{\varphi}{\mathbb{ABC}(\varphi)\Gamma(\varphi)}\int_0^\xi (\xi-\theta)^{\varphi-1}\mathcal{G}_1(\mathbf{S}, \mathbf{I}, \mathbf{R}, \theta)\rho_1 B_1'(\mathbf{S}, \theta)d\theta, \\
\mathbf{I}(\xi) - \mathbf{I}(0) &= \frac{1-\varphi}{\mathbb{ABC}(\varphi)}\mathbf{F}_2(\mathbf{S}, \mathbf{I}, \xi) + \frac{\varphi}{\mathbb{ABC}(\varphi)\Gamma(\varphi)}\int_0^\xi (\xi-\theta)^\varphi \mathbf{F}_2(\mathbf{S}, \mathbf{I}, \theta)d\theta \\
&+\frac{1-\varphi}{\mathbb{ABC}(\varphi)}\mathcal{G}_2(\mathbf{S}, \mathbf{I}, \xi)B_2'(\xi) + \frac{\varphi}{\mathbb{ABC}(\varphi)\Gamma(\varphi)}\int_0^\xi (\xi-\theta)^{\varphi-1}\mathcal{G}_2(\mathbf{S}, \mathbf{I}, \theta)\rho_2 B_2'(\mathbf{I}, \theta)d\theta, \\
\mathbf{T}(\xi) - \mathbf{T}(0) &= \frac{1-\varphi}{\mathbb{ABC}(\varphi)}\mathbf{F}_3(\mathbf{I}, \mathbf{T}, \xi) + \frac{\varphi}{\mathbb{ABC}(\varphi)\Gamma(\varphi)}\int_0^\xi (\xi-\theta)^\varphi \mathbf{F}_3(\mathbf{I}, \mathbf{T}, \theta)d\theta \\
&+\frac{1-\varphi}{\mathbb{ABC}(\varphi)}\mathcal{G}_3(\mathbf{I}, \mathbf{T}, \xi)B_3'(\xi) + \frac{\varphi}{\mathbb{ABC}(\varphi)\Gamma(\varphi)}\int_0^\xi (\xi-\theta)^{\varphi-1}\mathcal{G}_3(\mathbf{I}, \mathbf{T}, \theta)\rho_3 B_3'(\mathbf{T}, \theta)d\theta, \\
\mathbf{R}(\xi) - \mathbf{R}(0) &= \frac{1-\varphi}{\mathbb{ABC}(\varphi)}\mathbf{F}_4(\mathbf{T}, \mathbf{R}, \xi) + \frac{\varphi}{\mathbb{ABC}(\varphi)\Gamma(\varphi)}\int_0^\xi (\xi-\theta)^\varphi \mathbf{F}_4(\mathbf{T}, \mathbf{R}, \theta)d\theta \\
&+\frac{1-\varphi}{\mathbb{ABC}(\varphi)}\mathcal{G}_4(\mathbf{T}, \mathbf{R}, \xi)B_4'(\xi) + \frac{\varphi}{\mathbb{ABC}(\varphi)\Gamma(\varphi)}\int_0^\xi (\xi-\theta)^{\varphi-1}\mathcal{G}_4(\mathbf{T}, \mathbf{R}, \theta)\rho_4 B_4'(\mathbf{R}, \theta)d\theta.
\end{aligned}$$

As the scheme suggested by Atangana and Touffik [52], the mappings $\mathbf{F}_1$ and $\mathcal{G}_1$ are estimated employing Lagrange polynomials. This results in

$$\mathbf{S}(\xi_{\mathfrak{n}+1}) = \mathbf{S}(0) + \begin{cases} \frac{1-\varphi}{\mathbb{ABC}(\varphi)}\mathbf{F}_1(\mathbf{S}(t_{\mathfrak{n}}), \mathbf{I}(t_{\mathfrak{n}}), \mathbf{R}(t_{\mathfrak{n}}), \xi_{\mathfrak{n}}) + \frac{\varphi(\Delta)^{\varphi}}{\mathbb{ABC}(\varphi)\Gamma(\varphi+1)} \sum_{\mathbf{q}=2}^{\mathfrak{n}} \mathbf{F}_1(\mathbf{S}_{\mathbf{q}-2}, \mathbf{I}_{\mathbf{q}-2}, \mathbf{R}_{\mathbf{q}-2}, \xi_{\mathbf{q}-2}) \\ \quad \times(\mathfrak{n} - \mathbf{q} + 1)^{\varphi} - (\mathfrak{n} - \mathbf{q})^{\varphi} + \frac{\varphi(\Delta\xi)^{\varphi}}{\mathbb{ABC}(\varphi)\Gamma(\varphi+2)} \sum_{\mathbf{q}=2}^{\mathfrak{n}} (\mathbf{F}_1(\mathbf{S}_{\mathbf{q}-1}, \mathbf{I}_{\mathbf{q}-1}, \mathbf{R}_{\mathbf{q}-1}, \xi_{\mathbf{q}-1}) \\ \quad -\mathbf{F}_1(\mathbf{S}_{\mathbf{q}-2}, \mathbf{I}_{\mathbf{q}-2}, \mathbf{R}_{\mathbf{q}-2}, \xi_{\mathbf{q}-2}))\Big((\mathfrak{n} - \mathbf{q} + 1)^{\varphi}(\mathfrak{n} - \mathbf{q} + 2\varphi + 3) \\ \quad -(\mathfrak{n} - \mathbf{q})^{\varphi}(\mathfrak{n} - \mathbf{q} + 3\varphi + 3)\Big) + \frac{\varphi(\Delta\xi)^{\varphi}}{2\mathbb{ABC}(\varphi)\Gamma(\varphi+3)} \sum_{\mathbf{q}=2}^{\mathfrak{n}} (\mathbf{F}_1(\mathbf{S}_{\mathbf{q}}, \mathbf{I}_{\mathbf{q}}, \mathbf{R}_{\mathbf{q}}, \xi_{\mathbf{q}}) \\ \quad -2\mathbf{F}_1(\mathbf{S}_{\mathbf{q}-1}, \mathbf{I}_{\mathbf{q}-1}, \mathbf{R}_{\mathbf{q}-1}, \xi_{\mathbf{q}-1}) + \mathbf{F}_1(\mathbf{S}_{\mathbf{q}-2}, \mathbf{I}_{\mathbf{q}-2}, \mathbf{R}_{\mathbf{q}-2}, \xi_{\mathbf{q}-2})) \\ \quad \times\Big\{(\mathfrak{n} - \mathbf{q} + 1)^{\varphi}\Big[2(\mathfrak{n} - \mathbf{q})^2 + (3\varphi + 10)(\mathfrak{n} - \mathbf{q}) + 2\varphi^2 + 9\varphi + 12\Big] \\ \quad -(\mathfrak{n} - \mathbf{q})^{\varphi}\Big[2(\mathfrak{n} - \mathbf{q})^2 + (5\varphi + 10)(\mathfrak{n} - \mathbf{q}) + 6\varphi^2 + 18\varphi + 12\Big]\Big\} \end{cases}$$

$$+ \begin{cases} \frac{1-\varphi}{\mathbb{ABC}(\varphi)}\mathcal{G}_1(\mathbf{S}(t_{\mathfrak{n}}), \mathbf{I}(t_{\mathfrak{n}}), \mathbf{R}(t_{\mathfrak{n}}), \xi_{\mathfrak{n}})\rho_1\{B_1'(\xi_{\mathfrak{n}+1}) - B_1'(\xi_{\mathfrak{n}})\} \\ \quad +\frac{\varphi(\Delta\xi)^{\varphi-1}}{\mathbb{ABC}(\varphi)\Gamma(\varphi+1)} \sum_{\mathbf{q}=2}^{\mathfrak{n}} (\mathcal{G}_1(\mathbf{S}_{\mathbf{q}-2}, \mathbf{I}_{\mathbf{q}-2}, \mathbf{R}_{\mathbf{q}-2}, \xi_{\mathbf{q}-2})\rho_1\{B_1'(\xi_{\mathbf{q}-1}) - B_1'(\xi_{\mathbf{q}-2})\} \\ \quad \times[(\mathfrak{n} - \mathbf{q} + 1)^{\varphi} - (\mathfrak{n} - \mathbf{q})^{\varphi}] + \frac{\varphi(\Delta\xi)^{\varphi-1}}{\mathbb{ABC}(\varphi)\Gamma(\varphi+2)} \sum_{\mathbf{q}=2}^{\mathfrak{n}} (\mathcal{G}_1(\mathbf{S}_{\mathbf{q}-1}, \mathbf{I}_{\mathbf{q}-1}, \mathbf{R}_{\mathbf{q}-1}, \xi_{\mathbf{q}-1}) \\ \quad \times\rho_1\{B_1'(\xi_{\mathbf{q}}) - B_1'(\xi_{\mathbf{q}-1})\} - \mathcal{G}_1(\mathbf{S}_{\mathbf{q}-2}, \mathbf{I}_{\mathbf{q}-2}, \mathbf{R}_{\mathbf{q}-2}, \xi_{\mathbf{q}-2})\rho_1\{B_1'(\xi_{\mathbf{q}-1}) - B_1'(\xi_{\mathbf{q}-2})\}) \\ \quad \times\Big((\mathfrak{n} - \mathbf{q} + 1)^{\varphi}(\mathfrak{n} - \mathbf{q} + 2\varphi + 3) - (\mathfrak{n} - \mathbf{q})^{\varphi}(\mathfrak{n} - \mathbf{q} + 3\varphi + 3)\Big) \\ \quad +\frac{\varphi(\Delta\xi)^{\varphi-1}}{2\mathbb{ABC}(\varphi)\Gamma(\varphi+3)} \sum_{\mathbf{q}=2}^{\mathfrak{n}} (\mathcal{G}_1(\mathbf{S}_{\mathbf{q}}, \mathbf{I}_{\mathbf{q}}, \mathbf{R}_{\mathbf{q}}, \xi_{\mathbf{q}})\rho_1\{B_1'(\xi_{\mathbf{q}+1}) - B_1'(\xi_{\mathbf{q}})\} \\ \quad -2\mathcal{G}_1(\mathbf{S}_{\mathbf{q}-1}, \mathbf{I}_{\mathbf{q}-1}, \mathbf{R}_{\mathbf{q}-1}, \xi_{\mathbf{q}-1})\rho_1\{B_1'(\xi_{\mathbf{q}}) - B_1'(\xi_{\mathbf{q}-1})\} \\ \quad +\mathcal{G}_1(\mathbf{S}_{\mathbf{q}-2}, \mathbf{I}_{\mathbf{q}-2}, \mathbf{R}_{\mathbf{q}-2}, \xi_{\mathbf{q}-2})\rho_1\{B_1'(\xi_{\mathbf{q}-1}) - B_1'(\xi_{\mathbf{q}-2})\}) \\ \quad \times\Big\{(\mathfrak{n} - \mathbf{q} + 1)^{\varphi}\Big[2(\mathfrak{n} - \mathbf{q})^2 + (3\varphi + 10)(\mathfrak{n} - \mathbf{q}) + 2\varphi^2 + 9\varphi + 12\Big] \\ \quad -(\mathfrak{n} - \mathbf{q})^{\varphi}\Big[2(\mathfrak{n} - \mathbf{q})^2 + (5\varphi + 10)(\mathfrak{n} - \mathbf{q}) + 6\varphi^2 + 18\varphi + 12\Big]\Big\}, \end{cases}$$

$$\mathbf{I}(\xi_{\mathfrak{n}+1}) = \mathbf{I}(0) + \begin{cases} \frac{1-\varphi}{\mathbb{ABC}(\varphi)}\mathbf{F}_2(\mathbf{S}(t_{\mathfrak{n}}), \mathbf{I}(t_{\mathfrak{n}}), \xi_{\mathfrak{n}}) + \frac{\varphi(\Delta)^{\varphi}}{\mathbb{ABC}(\varphi)\Gamma(\varphi+1)} \sum_{\mathbf{q}=2}^{\mathfrak{n}} \mathbf{F}_2(\mathbf{S}_{\mathbf{q}-2}, \mathbf{I}_{\mathbf{q}-2}, \xi_{\mathbf{q}-2}) \\ \quad \times(\mathfrak{n} - \mathbf{q} + 1)^{\varphi} - (\mathfrak{n} - \mathbf{q})^{\varphi} + \frac{\varphi(\Delta\xi)^{\varphi}}{\mathbb{ABC}(\varphi)\Gamma(\varphi+2)} \sum_{\mathbf{q}=2}^{\mathfrak{n}} (\mathbf{F}_2(\mathbf{S}_{\mathbf{q}-1}, \mathbf{I}_{\mathbf{q}-1}, \xi_{\mathbf{q}-1}) \\ \quad -\mathbf{F}_2(\mathbf{S}_{\mathbf{q}-2}, \mathbf{I}_{\mathbf{q}-2}, \xi_{\mathbf{q}-2}))\Big((\mathfrak{n} - \mathbf{q} + 1)^{\varphi}(\mathfrak{n} - \mathbf{q} + 2\varphi + 3) \\ \quad -(\mathfrak{n} - \mathbf{q})^{\varphi}(\mathfrak{n} - \mathbf{q} + 3\varphi + 3)\Big) + \frac{\varphi(\Delta\xi)^{\varphi}}{2\mathbb{ABC}(\varphi)\Gamma(\varphi+3)} \sum_{\mathbf{q}=2}^{\mathfrak{n}} (\mathbf{F}_2(\mathbf{S}_{\mathbf{q}}, \mathbf{I}_{\mathbf{q}}, \xi_{\mathbf{q}}) \\ \quad -2\mathbf{F}_2(\mathbf{S}_{\mathbf{q}-1}, \mathbf{I}_{\mathbf{q}-1}, \xi_{\mathbf{q}-1}) + \mathbf{F}_2(\mathbf{S}_{\mathbf{q}-2}, \mathbf{I}_{\mathbf{q}-2}, \xi_{\mathbf{q}-2})) \\ \quad \times\Big\{(\mathfrak{n} - \mathbf{q} + 1)^{\varphi}\Big[2(\mathfrak{n} - \mathbf{q})^2 + (3\varphi + 10)(\mathfrak{n} - \mathbf{q}) + 2\varphi^2 + 9\varphi + 12\Big] \\ \quad -(\mathfrak{n} - \mathbf{q})^{\varphi}\Big[2(\mathfrak{n} - \mathbf{q})^2 + (5\varphi + 10)(\mathfrak{n} - \mathbf{q}) + 6\varphi^2 + 18\varphi + 12\Big]\Big\} \end{cases}$$

$$+\begin{cases} \frac{1-\varphi}{\mathbb{ABC}(\varphi)}\mathcal{G}_2(\mathbf{S}(t_\mathfrak{n}),\mathbf{I}(t_\mathfrak{n}),\xi_\mathfrak{n})\rho_2\{B_2'(\xi_{\mathfrak{n}+1})-B_2'(\xi_\mathfrak{n})\} \\ \quad +\frac{\varphi(\Delta\xi)^{\varphi-1}}{\mathbb{ABC}(\varphi)\Gamma(\varphi+1)}\sum_{\mathbf{q}=2}^{\mathfrak{n}}(\mathcal{G}_2(\mathbf{S}_{\mathbf{q}-2},\mathbf{I}_{\mathbf{q}-2},\xi_{\mathbf{q}-2})\rho_2\{B_2'(\xi_{\mathbf{q}-1})-B_2'(\xi_{\mathbf{q}-2})\} \\ \quad \times[(\mathfrak{n}-\mathbf{q}+1)^\varphi-(\mathfrak{n}-\mathbf{q})^\varphi]+\frac{\varphi(\Delta\xi)^{\varphi-1}}{\mathbb{ABC}(\varphi)\Gamma(\varphi+2)}\sum_{\mathbf{q}=2}^{\mathfrak{n}}(\mathcal{G}_2(\mathbf{S}_{\mathbf{q}-1},\mathbf{I}_{\mathbf{q}-1},\xi_{\mathbf{q}-1}) \\ \quad \times\rho_2\{B_2'(\xi_\mathbf{q})-B_2'(\xi_{\mathbf{q}-1})\}-\mathcal{G}_2(\mathbf{S}_{\mathbf{q}-2},\mathbf{I}_{\mathbf{q}-2},\xi_{\mathbf{q}-2})\rho_2\{B_2'(\xi_{\mathbf{q}-1})-B_2'(\xi_{\mathbf{q}-2})\}) \\ \quad \times\Big((\mathfrak{n}-\mathbf{q}+1)^\varphi(\mathfrak{n}-\mathbf{q}+2\varphi+3)-(\mathfrak{n}-\mathbf{q})^\varphi(\mathfrak{n}-\mathbf{q}+3\varphi+3)\Big) \\ \quad +\frac{\varphi(\Delta\xi)^{\varphi-1}}{2\mathbb{ABC}(\varphi)\Gamma(\varphi+3)}\sum_{\mathbf{q}=2}^{\mathfrak{n}}(\mathcal{G}_2(\mathbf{S}_\mathbf{q},\mathbf{I}_\mathbf{q},\xi_\mathbf{q})\rho_2\{B_2'(\xi_{\mathbf{q}+1})-B_2'(\xi_\mathbf{q})\} \\ \quad -2\mathcal{G}_2(\mathbf{S}_{\mathbf{q}-1},\mathbf{I}_{\mathbf{q}-1},\xi_{\mathbf{q}-1})\rho_2\{B_2'(\xi_\mathbf{q})-B_2'(\xi_{\mathbf{q}-1})\} \\ \quad +\mathcal{G}_2(\mathbf{S}_{\mathbf{q}-2},\mathbf{I}_{\mathbf{q}-2},\xi_{\mathbf{q}-2})\rho_2\{B_2'(\xi_{\mathbf{q}-1})-B_2'(\xi_{\mathbf{q}-2})\}) \\ \quad \times\Big\{(\mathfrak{n}-\mathbf{q}+1)^\varphi\Big[2(\mathfrak{n}-\mathbf{q})^2+(3\varphi+10)(\mathfrak{n}-\mathbf{q})+2\varphi^2+9\varphi+12\Big] \\ \quad -(\mathfrak{n}-\mathbf{q})^\varphi\Big[2(\mathfrak{n}-\mathbf{q})^2+(5\varphi+10)(\mathfrak{n}-\mathbf{q})+6\varphi^2+18\varphi+12\Big]\Big\}, \end{cases}$$

$$\mathbf{T}(\xi_{\mathfrak{n}+1})=\mathbf{T}(0)+\begin{cases} \frac{1-\varphi}{\mathbb{ABC}(\varphi)}\mathbf{F}_3(\mathbf{I}(t_\mathfrak{n}),\mathbf{T}(t_\mathfrak{n}),\xi_\mathfrak{n})+\frac{\varphi(\Delta)^\varphi}{\mathbb{ABC}(\varphi)\Gamma(\varphi+1)}\sum_{\mathbf{q}=2}^{\mathfrak{n}}\mathbf{F}_3(\mathbf{I}_{\mathbf{q}-2},\mathbf{T}_{\mathbf{q}-2},\xi_{\mathbf{q}-2}) \\ \quad \times(\mathfrak{n}-\mathbf{q}+1)^\varphi-(\mathfrak{n}-\mathbf{q})^\varphi+\frac{\varphi(\Delta\xi)^\varphi}{\mathbb{ABC}(\varphi)\Gamma(\varphi+2)}\sum_{\mathbf{q}=2}^{\mathfrak{n}}(\mathbf{F}_3(\mathbf{I}_{\mathbf{q}-1},\mathbf{T}_{\mathbf{q}-1},\xi_{\mathbf{q}-1}) \\ \quad -\mathbf{F}_3(\mathbf{I}_{\mathbf{q}-2},\mathbf{T}_{\mathbf{q}-2},\xi_{\mathbf{q}-2}))\Big((\mathfrak{n}-\mathbf{q}+1)^\varphi(\mathfrak{n}-\mathbf{q}+2\varphi+3) \\ \quad -(\mathfrak{n}-\mathbf{q})^\varphi(\mathfrak{n}-\mathbf{q}+3\varphi+3)\Big)+\frac{\varphi(\Delta\xi)^\varphi}{2\mathbb{ABC}(\varphi)\Gamma(\varphi+3)}\sum_{\mathbf{q}=2}^{\mathfrak{n}}(\mathbf{F}_3(\mathbf{I}_\mathbf{q},\mathbf{T}_\mathbf{q},\xi_\mathbf{q}) \\ \quad -2\mathbf{F}_3(\mathbf{I}_{\mathbf{q}-1},\mathbf{T}_{\mathbf{q}-1},\xi_{\mathbf{q}-1})+\mathbf{F}_3(\mathbf{I}_{\mathbf{q}-2},\mathbf{T}_{\mathbf{q}-2},\xi_{\mathbf{q}-2})) \\ \quad \times\Big\{(\mathfrak{n}-\mathbf{q}+1)^\varphi\Big[2(\mathfrak{n}-\mathbf{q})^2+(3\varphi+10)(\mathfrak{n}-\mathbf{q})+2\varphi^2+9\varphi+12\Big] \\ \quad -(\mathfrak{n}-\mathbf{q})^\varphi\Big[2(\mathfrak{n}-\mathbf{q})^2+(5\varphi+10)(\mathfrak{n}-\mathbf{q})+6\varphi^2+18\varphi+12\Big]\Big\} \end{cases}$$

$$+\begin{cases} \frac{1-\varphi}{\mathbb{ABC}(\varphi)}\mathcal{G}_3(\mathbf{I}(t_\mathfrak{n}),\mathbf{T}(t_\mathfrak{n}),\xi_\mathfrak{n})\rho_3\{B_3'(\xi_{\mathfrak{n}+1})-B_3'(\xi_\mathfrak{n})\} \\ \quad +\frac{\varphi(\Delta\xi)^{\varphi-1}}{\mathbb{ABC}(\varphi)\Gamma(\varphi+1)}\sum_{\mathbf{q}=2}^{\mathfrak{n}}(\mathcal{G}_3(\mathbf{I}_{\mathbf{q}-2},\mathbf{T}_{\mathbf{q}-2},\xi_{\mathbf{q}-2})\rho_3\{B_3'(\xi_{\mathbf{q}-1})-B_3'(\xi_{\mathbf{q}-2})\} \\ \quad \times[(\mathfrak{n}-\mathbf{q}+1)^\varphi-(\mathfrak{n}-\mathbf{q})^\varphi]+\frac{\varphi(\Delta\xi)^{\varphi-1}}{\mathbb{ABC}(\varphi)\Gamma(\varphi+2)}\sum_{\mathbf{q}=2}^{\mathfrak{n}}(\mathcal{G}_3(\mathbf{I}_{\mathbf{q}-1},\mathbf{T}_{\mathbf{q}-1},\xi_{\mathbf{q}-1}) \\ \quad \times\rho_3\{B_3'(\xi_\mathbf{q})-B_3'(\xi_{\mathbf{q}-1})\}-\mathcal{G}_3(\mathbf{I}_{\mathbf{q}-2},\mathbf{T}_{\mathbf{q}-2},\xi_{\mathbf{q}-2})\rho_3\{B_3'(\xi_{\mathbf{q}-1})-B_3'(\xi_{\mathbf{q}-2})\}) \\ \quad \times\Big((\mathfrak{n}-\mathbf{q}+1)^\varphi(\mathfrak{n}-\mathbf{q}+2\varphi+3)-(\mathfrak{n}-\mathbf{q})^\varphi(\mathfrak{n}-\mathbf{q}+3\varphi+3)\Big) \\ \quad +\frac{\varphi(\Delta\xi)^{\varphi-1}}{2\mathbb{ABC}(\varphi)\Gamma(\varphi+3)}\sum_{\mathbf{q}=2}^{\mathfrak{n}}(\mathcal{G}_3(\mathbf{I}_\mathbf{q},\mathbf{T}_\mathbf{q},\xi_\mathbf{q})\rho_3\{B_3'(\xi_{\mathbf{q}+1})-B_3'(\xi_\mathbf{q})\} \\ \quad -2\mathcal{G}_3(\mathbf{I}_{\mathbf{q}-1},\mathbf{T}_{\mathbf{q}-1},\xi_{\mathbf{q}-1})\rho_3\{B_3'(\xi_\mathbf{q})-B_3'(\xi_{\mathbf{q}-1})\} \\ \quad +\mathcal{G}_3(\mathbf{I}_{\mathbf{q}-2},\mathbf{T}_{\mathbf{q}-2},\xi_{\mathbf{q}-2})\rho_3\{B_3'(\xi_{\mathbf{q}-1})-B_3'(\xi_{\mathbf{q}-2})\}) \\ \quad \times\Big\{(\mathfrak{n}-\mathbf{q}+1)^\varphi\Big[2(\mathfrak{n}-\mathbf{q})^2+(3\varphi+10)(\mathfrak{n}-\mathbf{q})+2\varphi^2+9\varphi+12\Big] \\ \quad -(\mathfrak{n}-\mathbf{q})^\varphi\Big[2(\mathfrak{n}-\mathbf{q})^2+(5\varphi+10)(\mathfrak{n}-\mathbf{q})+6\varphi^2+18\varphi+12\Big]\Big\}, \end{cases}$$

and

$$\mathbf{R}(\xi_{\mathfrak{n}+1}) = \mathbf{R}(0) + \begin{cases} \frac{1-\varphi}{\mathbb{ABC}(\varphi)}\mathbf{F}_4(\mathbf{T}(t_{\mathfrak{n}}), \mathbf{R}(t_{\mathfrak{n}}), \xi_{\mathfrak{n}}) + \frac{\varphi(\Delta)^{\varphi}}{\mathbb{ABC}(\varphi)\Gamma(\varphi+1)} \sum_{\mathbf{q}=2}^{\mathfrak{n}} \mathbf{F}_4(\mathbf{T}_{\mathbf{q}-2}, \mathbf{R}_{\mathbf{q}-2}, \xi_{\mathbf{q}-2}) \\ \times(\mathfrak{n} - \mathbf{q} + 1)^{\varphi} - (\mathfrak{n} - \mathbf{q})^{\varphi} + \frac{\varphi(\Delta\xi)^{\varphi}}{\mathbb{ABC}(\varphi)\Gamma(\varphi+2)} \sum_{\mathbf{q}=2}^{\mathfrak{n}} (\mathbf{F}_4(\mathbf{T}_{\mathbf{q}-1}, \mathbf{R}_{\mathbf{q}-1}, \xi_{\mathbf{q}-1}) \\ -\mathbf{F}_4(\mathbf{T}_{\mathbf{q}-2}, \mathbf{R}_{\mathbf{q}-2}, \xi_{\mathbf{q}-2}))\Big((\mathfrak{n} - \mathbf{q} + 1)^{\varphi}(\mathfrak{n} - \mathbf{q} + 2\varphi + 3) \\ -(\mathfrak{n} - \mathbf{q})^{\varphi}(\mathfrak{n} - \mathbf{q} + 3\varphi + 3)\Big) + \frac{\varphi(\Delta\xi)^{\varphi}}{2\mathbb{ABC}(\varphi)\Gamma(\varphi+3)} \sum_{\mathbf{q}=2}^{\mathfrak{n}} (\mathbf{F}_4(\mathbf{T}_{\mathbf{q}}, \mathbf{R}_{\mathbf{q}}, \xi_{\mathbf{q}}) \\ -2\mathbf{F}_4(\mathbf{T}_{\mathbf{q}-1}, \mathbf{R}_{\mathbf{q}-1}, \xi_{\mathbf{q}-1}) + \mathbf{F}_4(\mathbf{T}_{\mathbf{q}-2}, \mathbf{R}_{\mathbf{q}-2}, \xi_{\mathbf{q}-2})) \\ \times\Big\{(\mathfrak{n} - \mathbf{q} + 1)^{\varphi}\Big[2(\mathfrak{n} - \mathbf{q})^2 + (3\varphi + 10)(\mathfrak{n} - \mathbf{q}) + 2\varphi^2 + 9\varphi + 12\Big] \\ -(\mathfrak{n} - \mathbf{q})^{\varphi}\Big[2(\mathfrak{n} - \mathbf{q})^2 + (5\varphi + 10)(\mathfrak{n} - \mathbf{q}) + 6\varphi^2 + 18\varphi + 12\Big]\Big\} \end{cases}$$

$$+ \begin{cases} \frac{1-\varphi}{\mathbb{ABC}(\varphi)}\mathcal{G}_4(\mathbf{T}(t_{\mathfrak{n}}), \mathbf{R}(t_{\mathfrak{n}}), \xi_{\mathfrak{n}})\rho_4\{B_4'(\xi_{\mathfrak{n}+1}) - B_4'(\xi_{\mathfrak{n}})\} \\ +\frac{\varphi(\Delta\xi)^{\varphi-1}}{\mathbb{ABC}(\varphi)\Gamma(\varphi+1)} \sum_{\mathbf{q}=2}^{\mathfrak{n}} (\mathcal{G}_4(\mathbf{T}_{\mathbf{q}-2}, \mathbf{R}_{\mathbf{q}-2}, \xi_{\mathbf{q}-2})\rho_4\{B_4'(\xi_{\mathbf{q}-1}) - B_4'(\xi_{\mathbf{q}-2})\} \\ \times[(\mathfrak{n} - \mathbf{q} + 1)^{\varphi} - (\mathfrak{n} - \mathbf{q})^{\varphi}] + \frac{\varphi(\Delta\xi)^{\varphi-1}}{\mathbb{ABC}(\varphi)\Gamma(\varphi+2)} \sum_{\mathbf{q}=2}^{\mathfrak{n}} (\mathcal{G}_4(\mathbf{T}_{\mathbf{q}-1}, \mathbf{R}_{\mathbf{q}-1}, \xi_{\mathbf{q}-1}) \\ \times\rho_4\{B_4'(\xi_{\mathbf{q}}) - B_4'(\xi_{\mathbf{q}-1})\} - \mathcal{G}_4(\mathbf{I}_{\mathbf{q}-2}, \mathbf{T}_{\mathbf{q}-2}, \xi_{\mathbf{q}-2})\rho_4\{B_4'(\xi_{\mathbf{q}-1}) - B_4'(\xi_{\mathbf{q}-2})\}) \\ \times\Big((\mathfrak{n} - \mathbf{q} + 1)^{\varphi}(\mathfrak{n} - \mathbf{q} + 2\varphi + 3) - (\mathfrak{n} - \mathbf{q})^{\varphi}(\mathfrak{n} - \mathbf{q} + 3\varphi + 3)\Big) \\ +\frac{\varphi(\Delta\xi)^{\varphi-1}}{2\mathbb{ABC}(\varphi)\Gamma(\varphi+3)} \sum_{\mathbf{q}=2}^{\mathfrak{n}} (\mathcal{G}_4(\mathbf{T}_{\mathbf{q}}, \mathbf{R}_{\mathbf{q}}, \xi_{\mathbf{q}})\rho_4\{B_4'(\xi_{\mathbf{q}+1}) - B_4'(\xi_{\mathbf{q}})\} \\ -2\mathcal{G}_4(\mathbf{T}_{\mathbf{q}-1}, \mathbf{R}_{\mathbf{q}-1}, \xi_{\mathbf{q}-1})\rho_4\{B_4'(\xi_{\mathbf{q}}) - B_4'(\xi_{\mathbf{q}-1})\} \\ +\mathcal{G}_4(\mathbf{T}_{\mathbf{q}-2}, \mathbf{R}_{\mathbf{q}-2}, \xi_{\mathbf{q}-2})\rho_4\{B_4'(\xi_{\mathbf{q}-1}) - B_4'(\xi_{\mathbf{q}-2})\}) \\ \times\Big\{(\mathfrak{n} - \mathbf{q} + 1)^{\varphi}\Big[2(\mathfrak{n} - \mathbf{q})^2 + (3\varphi + 10)(\mathfrak{n} - \mathbf{q}) + 2\varphi^2 + 9\varphi + 12\Big] \\ -(\mathfrak{n} - \mathbf{q})^{\varphi}\Big[2(\mathfrak{n} - \mathbf{q})^2 + (5\varphi + 10)(\mathfrak{n} - \mathbf{q}) + 6\varphi^2 + 18\varphi + 12\Big]\Big\}. \end{cases}$$

5. Results and discussion

Simulation results for framework (3.2) are performed in this part utilizing classical, Caputo, Caputo-Fabrizio, and Atangana-Baleanu fractional derivatives in the Caputo sense, respectively. The input variables used in simulation studies are summarized in Table 1, with the attribute settings of φ and δ modified. When $\mathbb{R}_0^{\mathbf{S}} < 1$, the disease-free equilibrium point is shown to be robust, but unstable when $\mathbb{R}_0^{\mathbf{S}} > 1$. Numerical configurations are developed by applying the strategy of Atangana and Touffik [52].

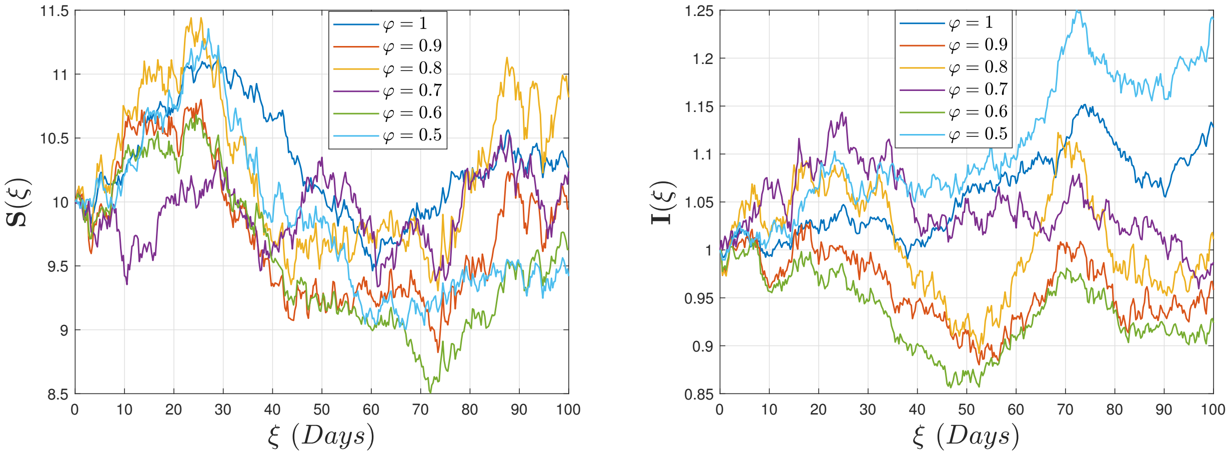
Table 1. Parameter's specifications and their explanations for cholera epidemic model.

Parameters	Explanation	Data estimated	References
π	Acquisition rate	0.0013	Estimated
δ	Interaction rate	0.011	[53]
ψ	Natural death rate	0.000025	[54]
ϕ	Immunity reduction level by the cured one	0.003	[55]
ς	Illness with a high mortality rate $\mathbf{T}(\xi)$	0.04	Estimated
ζ	Intervention rate of infectious persons	0.115	[56]
η	Illness with a high mortality rate $\mathbf{I}(\xi)$	0.015	[56]
γ	People cured have better recovery rate	0.2	[56].

Figures 4 and 5 show the various cohort stochastic solutions and cholera bacteria concentrations in an aqueous reservoir presented by the Caputo fractional derivative formulation. The overall number of people and the number of people who are secure have converged and are approaching the current population, whilst the number of susceptibility, contaminated, and restored people, as well as the cholera microbe density, has swiftly fallen and is now nonexistent. This suggests that the infection is eradicated when the incidence rate of susceptibility and restored people to the preserved group, as a result of sanitary adherence, cholera pathogen consumption, and interaction with immune-compromised victims, exceeds the number of recruits.

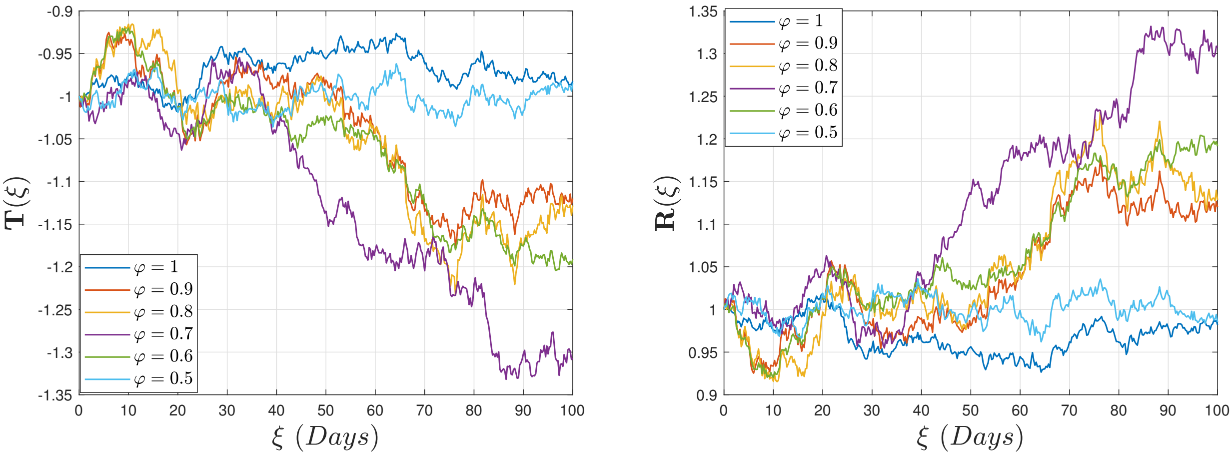
As with cholera illnesses, the best treatment necessitates a very strategic approach over the course of $\xi = 100$ days within the Caputo-Fabrizio derivative opertor, as illustrated in Figures 6 and 7. Furthermore, Figures 6 and 7 show that when an epidemic begins, a significant amount of hygiene is beneficial in reducing the outburst of transmission when fractional-orders decreases. Figures 6 and 7 also demonstrates that using extremely high measures of confinement at the start of temporarily is especially effective in preventing mortality in symptomatic infected communities. To protect susceptible groups from acquiring contamination, educational attainment is also maximized. Best scores on all handling functions for up to $\xi = 100$ days demonstrate the severity of the congenital cholera sickness in this occurrence. When the number of cholera leptospirosis cases is not too significant, the effectiveness of prevention increases. As shown in Figures 6 and 7, these three regulations affect highly contagious communities and bacterial populaces to decline considerably.

Furthermore, Figures 8 and 9 show how cholera sickness transmission can be detected after it has been suppressed for $\xi = 100$ days using the Atanagana-Baleanu fractional derivative in the Caputo sense. Figures 8 and 9 show that the proportion of the community that is susceptible has decreased since the earliest times when fractional-order decreases significantly. This is attributable to the large number of susceptible individuals that interacted with vibrio cholerae infested environments and hence acquired contaminated. Regulating literacy has also led to a considerable reduction in susceptible population densities. Figures 8 and 9 show that the community level of literacy has developed since the prehistoric period and is adversely proportionate to the statistical level of susceptible people. Moreover, according to preventive management, the prevalence of estimated incidence, particularly symptomatic, begins to plummet. When the level of preventive regulation is increased from start to end $\xi = 100$ days, more symptoms of infectious communities are quarantined than several susceptible people are contaminated. Incubation is efficacious in curing affected individuals and allowing them to recuperate swiftly. Due to the sheer number of infectious individuals that are recovering, it looks like the population proportion of restored people has been increasing since the start of time.



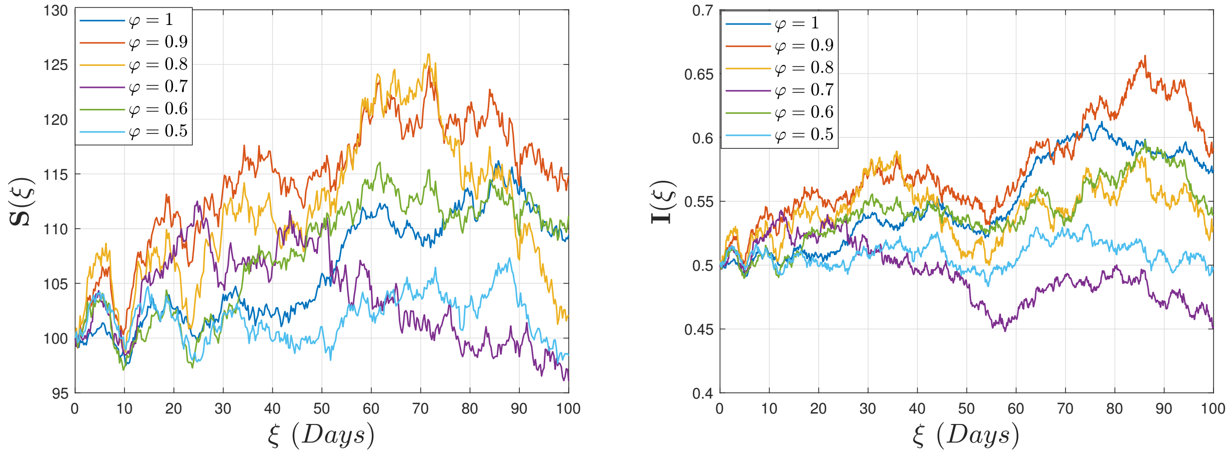
(a) **(b)**

Figure 4. (a) Stochastic behaviour of susceptible class $\mathbf{S}(\xi)$. (b) Stochastic behaviour of infected class $\mathbf{I}(\xi)$ for multiple fractional-orders using power-law kernel and small randomization density.



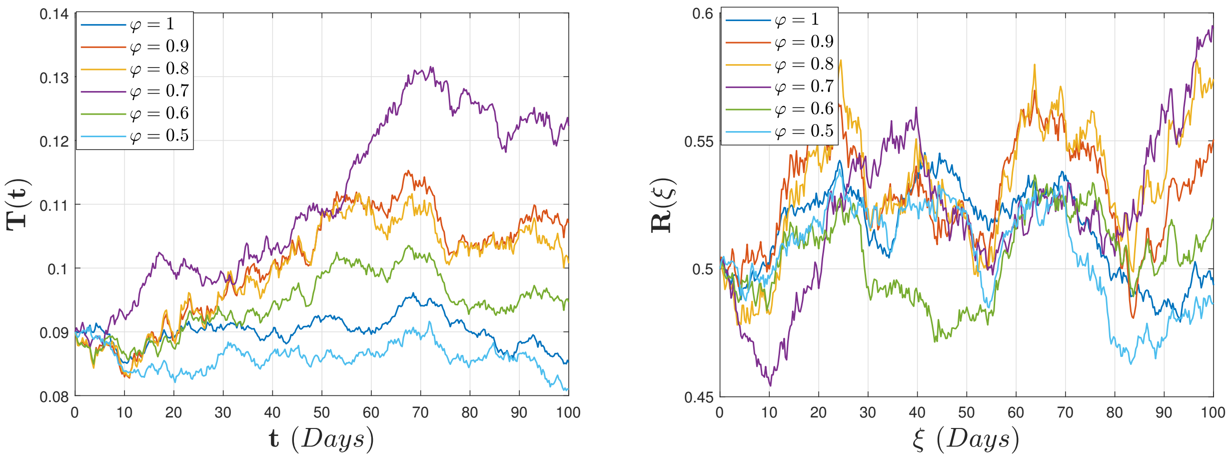
(a) **(b)**

Figure 5. (a) Stochastic behaviour of treated class $\mathbf{T}(\xi)$. (b) Stochastic behaviour of restored class $\mathbf{R}(\xi)$ for multiple fractional-orders using power-law kernel and small randomization density.



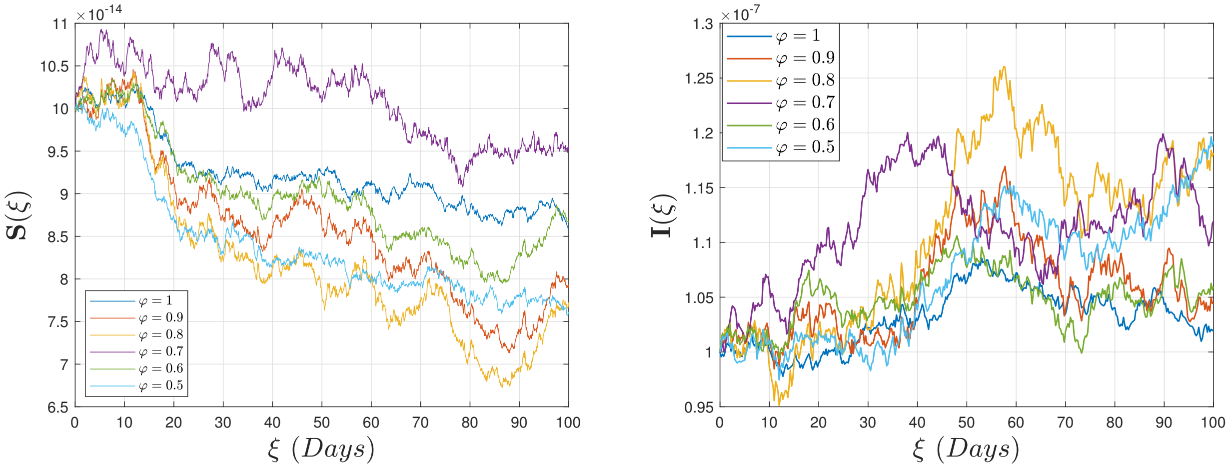
(a) **(b)**

Figure 6. (a) Stochastic behaviour of susceptible class $\mathbf{S}(\xi)$. (b) Stochastic behaviour of infected class $\mathbf{I}(\xi)$ for multiple fractional-orders using exponential decay kernel and small randomization density.



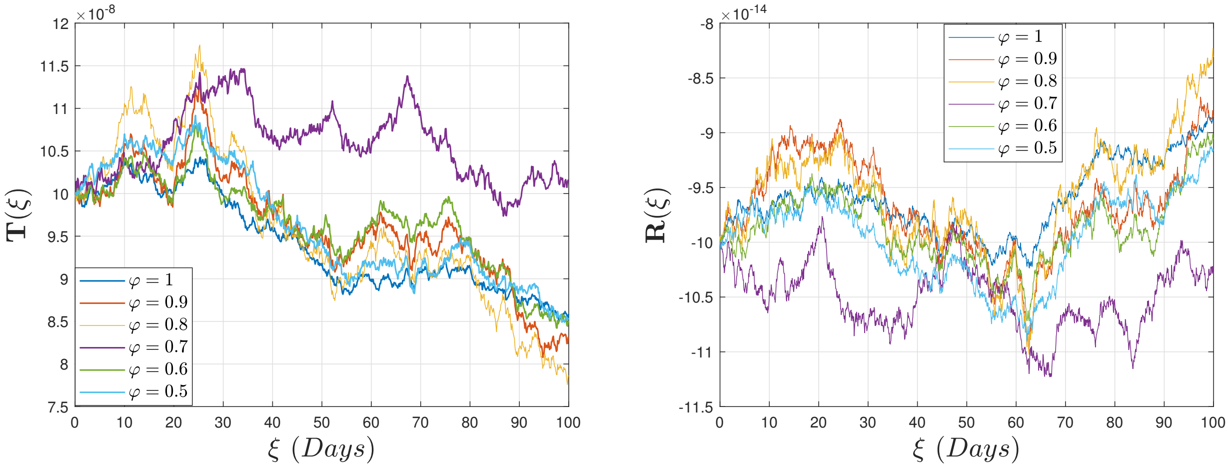
(a) **(b)**

Figure 7. (a) Stochastic behaviour of treated class $\mathbf{T}(\xi)$. (b) Stochastic behaviour of restored class $\mathbf{R}(\xi)$ for multiple fractional-orders using exponential decay kernel and small randomization density.



(a) **(b)**

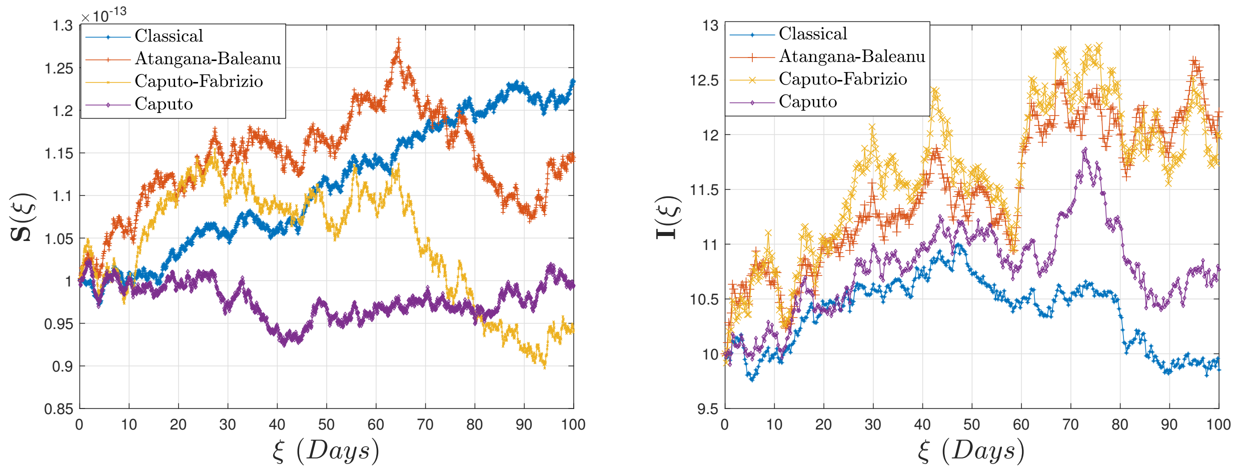
Figure 8. (a) Stochastic behaviour of susceptible class $\mathbf{S}(\xi)$. (b) Stochastic behaviour of infected class $\mathbf{I}(\xi)$ for multiple fractional-orders using generalized Mittag-Leffler kernel and small randomization density.



(a) **(b)**

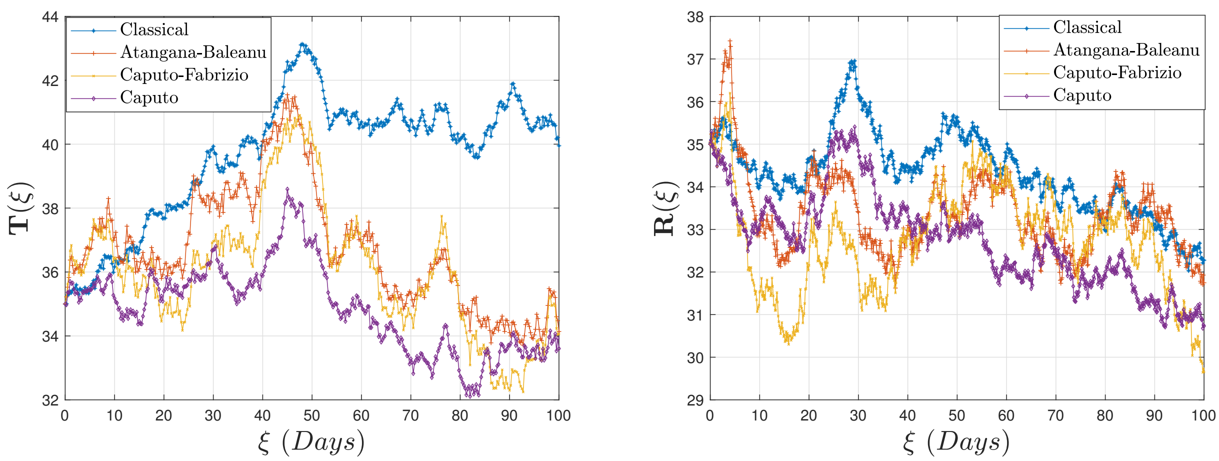
Figure 9. (a) Stochastic behaviour of treated class $\mathbf{T}(\xi)$. (b) Stochastic behaviour of restored class $\mathbf{R}(\xi)$ for multiple fractional-orders using generalized Mittag-Leffler kernel and small randomization density.

Figures 10 and 11 presented the simulation of various fractional-operators involving susceptible, infectious, treated, and recovered classes, which are used to be responsible for administering technical challenges with the goal of reducing the number of infectious people, pathogens, and the expense of purification, education, and incarceration. To illustrate the effectiveness of the specified setting, the numerical simulations are closed in harmony with the deterministic model investigated by [47]. According to the findings, the incidence of cholera individuals and bacteria surged dramatically at times during the case without control. The measures decreased the number of people infected with bacteria in the population, reducing the cholera endemic. This shows that cholera management techniques such as hygiene, awareness, and improved confinement can have a positive impact on reducing the disease's transmission.



(a) (b)

Figure 10. (a) Simulation behaviour of stochastic fractional model for the susceptible class $\mathbf{S}(\xi)$. (b) Simulation behaviour of stochastic fractional model for the infected class $\mathbf{I}(\xi)$ and small randomization density.



(a) (b)

Figure 11. (a) Simulation behaviour of stochastic fractional model for the treated class $\mathbf{T}(\xi)$. (b) Simulation behaviour of stochastic fractional model for the restored class $\mathbf{R}(\xi)$ and small randomization density.

6. Conclusions

In this article, we presented and extensively illustrated a cholera infection modelling technique incorporating fractional stochastic strategies. We discussed some fundamental properties of the model, including the boundedness and existence of a unique positive global solution of the system (3.2) which shows that the problem is well-behaved. Moreover, by constructing a suitable Lyapunov function and applying the Itô formula, we calculated the unique ergodic and stationary distribution. We also showed that whenever the white noise intensity is small enough and the reproduction number

$$\mathbb{R}_0^s = \frac{\pi\delta\zeta}{(\psi - \frac{\rho_1^2}{2})(\psi + \eta + \zeta - \frac{\rho_2^2}{2})(\gamma + \psi + \varsigma - \frac{\rho_3^2}{2})} > 1,$$

then the stochastic model (3.2) has a unique stationary distribution Since the stochastic model analyses stochastic environmental aspects or incorporates the randomization mechanism into account, it is generally more authentic (closer to the true solution) than the classical system [47]. By analyzing the classical and fractional stochastic approaches, the numerical simulation studies have been reviewed and assessed utilizing Matlab programming. The findings illustrate that the Atangana-Baleanu derivative operator is more realistic to the exact findings as compared to Caputo and Caputo-Fabrizo due to the inherited features of the kernel. We believe that the high intensity of cholera therapy leads substantially to the eradication of epidemic infections in the environment, as does streamlining processes. Further implementations are being developed to illustrate the significance and applicability of this technique. We attempt to explore a model formation under various time scales and fractal-fractional operator in our next ongoing project, which will include an instance involving $0 < \varphi < \frac{1}{2}$.

Acknowledgements

The authors would like to thank the Deanship of Scientific Research at Taif University for the grant received for this research. This research was supported by Taif university with a research grant (TURSP-2020/77).

Conflict of interest

The authors declare that they have no competing interests.

References

1. I. Podlubny, *Fractional differential equations*, San Diego: Academic Press, 1999.
2. S. N. Hajiseyedazizi, M. E. Samei, J. Alzabut, Y. M. Chu, On multi-step methods for singular fractional q-integro-differential equations, *Open Math.*, **19** (2021), 1378–1405. https://doi.org/10.1515/math-2021-0093
3. K. M. Owolabi, Z. Hammouch, Spatiotemporal patterns in the Belousov-Zhabotinskii reaction systems with Atangana-Baleanu fractional order derivative, *Phys. A*, **523** (2019), 1072–1090. https://doi.org/10.1016/j.physa.2019.04.017
4. S. Rashid, A. Khalid, S. Sultana, F. Jarad, K. M. Abualnaja, Y. S. Hamed, Novel numerical investigation of the fractional oncolytic effectiveness model with M1 virus via

generalized fractional derivative with optimal criterion, *Results Phys.*, **37** (2022), 105553. https://doi.org/10.1016/j.rinp.2022.105553

5. F. Z. Wang, M. N. Khan, I. Ahmad, H. Ahmad, H. Abu-Zinadah, Y. M. Chu, Numerical solution of traveling waves in chemical kinetics: time-fractional Fishers equations, *Fractals*, **30** (2022), 2240051. https://doi.org/10.1142/S0218348X22400515
6. O. A. Arqub, M. Al-Smadi, Atangana-Baleanu fractional approach to the solutions of BagleyTorvik and PainlevE¸ equations in Hilbert space, *Chaos Solitons Fract.*, **117** (2018), 161–167. https://doi.org/10.1016/j.chaos.2018.10.013
7. M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.*, **2** (2015), 73–85.
8. A. Atangana, D. Baleanu, New fractional derivatives with non-local and non-singular kernel theory and application to heat transfer model, *Therm. Sci.*, **20** (2016), 763–769.
9. S. Rashid, S. Sultana, Y. Karaca, A. Khalid, Y. M. Chu, Some further extensions considering discrete proportional fractional operators, *Fractals*, **30** (2022), 2240026. https://doi.org/10.1142/S0218348X22400266
10. S. Rashid, E. I. Abouelmagd, A. Khalid, F. B. Farooq, Y. M. Chu, Some recent developments on dynamical h-discrete fractional type inequalities in the frame of nonsingular and nonlocal kernels, *Fractals*, **30** (2022), 2240110. https://doi.org/10.1142/S0218348X22401107
11. S. Rashid, E. I. Abouelmagd, S. Sultana, Y. M. Chu, New developments in weighted n-fold type inequalities via discrete generalized h-proportional fractional operators, *Fractals*, **30** (2022), 2240056. https://doi.org/10.1142/S0218348X22400564
12. M. Al-Qurashi, S. Rashid, S. Sultana, F. Jarad, A. M. Alsharif, Fractional-order partial differential equations describing propagation of shallow water waves depending on power and Mittag-Leffler memory, *AIMS Math.*, **7** (2022), 12587–12619. https://doi.org/10.3934/math.2022697
13. S. Rashid, R. Ashraf, F. Jarad, Strong interaction of Jafari decomposition method with nonlinear fractional-order partial differential equations arising in plasma via the singular and nonsingular kernels, *AIMS Math.*, **7** (2022), 7936–7963. https://doi.org/10.3934/math.2022444
14. S. Rashid, F. Jarad, A. G. Ahmad, K. M. Abualnaja, New numerical dynamics of the heroin epidemic model using a fractional derivative with Mittag-Leffler kernel and consequences for control mechanisms, *Results Phys.*, **35** (2022), 105304. https://doi.org/10.1016/j.rinp.2022.105304
15. S. A. Iqbal, M. G. Hafez, Y. M. Chu, C. Park, Dynamical analysis of nonautonomous RLC circuit with the absence and presence of Atangana-Baleanu fractional derivativae, *J. Appl. Anal. Comput.*, **12** (2022), 770–789.
16. P. Waltman, *Deterministic threshold models in the theory of epidemics*, New York: Springer-Verlag, 1974. https://doi.org/10.1007/978-3-642-80820-3
17. L. Arnold, *Stochastic differential equations: theory and applications*, Wiley, 1974.
18. A. Friedman, *Stochastic differential equations and applications*, Vol. 1, Academic Press, 1975.
19. A. C. J. Luo, V. Afraimovich, *Long-range interactions, stochasticity and fractional dynamics: dedicated to George M. Zaslavsky (1935–2008)*, Springer, 2011.

20. A. G. Ladde, G. S. Ladde, Dynamic processes under random environment, *Bull. Marathwada Math. Soc.*, **8** (2007), 96–123.
21. R. J. Elliott, *Stochastic calculus and applications*, New York: Springer Verlag, 1982.
22. G. S. Ladde, L. Wu, Stochastic modeling and statistical analysis on the stock price processes, *Nonlinear Anal.: Theory Methods*, **71** (2009), 1203–1208.
23. G. S. Ladde, L. Wu, Development of nonlinear stochastic models by using stock price data and basic statistics, *Neural Parallel Sci. Comput.*, **18** (2010), 269–282.
24. S. Rashid, M. K. Iqbal, A. M. Alshehri, R. Ashraf, F. Jarad, A comprehensive analysis of the stochastic fractal-fractional tuberculosis model via Mittag-Leffler kernel and white noise, *Results Phys.*, **39** (2022), 105764. https://doi.org/10.1016/j.rinp.2022.105764
25. S. Rashid, R. Ashraf, Q. Ul-Ain Asif, F. Jarad, Novel dynamics of a stochastic fractal-fractional immune effector response to viral infection via latently infectious tissues, *Math. Biosci. Eng.*, **19** (2022), 11563–11594. https://doi.org/10.3934/mbe.2022539
26. P. S. Brachman, E. Abrutyn, *Bacterial infections of humans: epidemiology and control*, New York: Springer, 2009.
27. Centers for Disease Control and Prevention, Cholera–Vibrio cholerae infection, 2020. Available from: `https://www.cdc.gov/cholera/index.html`.
28. World Health Organization, C. L. Chaignat, 10 facts about cholera, 2017. Available from: `https://borgenproject.org/10-facts-cholera/`.
29. C. T. Codeco, Endemic and epidemic dynamics of cholera: the role of the aquatic reservoir, *BMC Infect. Dis.*, **1** (2001), 1–14.
30. E. Bertuzzo, S. Azaele, A. Maritan, M. Gatto, I. RodriguezIturbe, A. Rinaldo, On the space-time evolution of a cholera epidemic, *Water Resour. Res.*, **44** (2008), 1–8. https://doi.org/10.1029/2007WR006211
31. E. Bertuzzo, R. Casagrandi, M. Gatto, I. Rodriguez-Iturbe, A. Rinaldo, On spatially explicit models of cholera epidemics, *J. Roy. Soc. Interface*, **7** (2010), 321–333. https://doi.org/10.1098/rsif.2009.0204
32. L. Mari, E. Bertuzzo, L. Righetto, Modelling cholera epidemics: the role of waterways, human mobility and sanitation, *J. Roy. Soc. Interface*, **9** (2012), 376–388. https://doi.org/10.1098/rsif.2011.0304
33. A. A. Ayoade, M. O. Ibrahim, O. J. Peter, F. A. Oguntolu, On the global stability of cholera model with prevention and control, *Malays. J. Comput.*, **3** (2018), 28–36.
34. C. Ratchford, J. Wang, Modeling cholera dynamics at multiple scales: environmental evolution, between-host transmission, and within-host interaction, *Math. Biosci. Eng.*, **16** (2019), 782–812. https://doi.org/10.3934/mbe.2019037
35. A. Kumar, M. Kumar, Nilam, A study on the stability behavior of an epidemic model with ratio-dependent incidence and saturated treatment, *Theory Biosci.*, **139** (2020), 225–234. https://doi.org/10.1007/s12064-020-00314-6
36. M. Al-Adydah, A. Mwasa, J. M. Tchuenche, R. J. Smith, Modeling cholera disease with education and chlorination, *J. Biol. Syst.*, **21** (2013), 1340007. https://doi.org/10.1142/S021833901340007X

37. V. D. Nguyen, N. Sreenivasan, E. Lam, T. Ayers, D. Kargbo, F. Dafae, et al., Cholera epidemic associated with consumption of unsafe drinking water and street-vended water-eastern freetown, Sierra Leone, 2012, *Am. J. Trop. Med. Hyg.*, **90** (2014), 518–523. https://doi.org/10.4269/ajtmh.13-0567

38. J. G. Liu, X. J. Yang, Y. Y. Feng, L. L. Geng, Fundamental results to the weighted Caputo-type differential operator, *Appl. Math. Lett.*, **121** (2021), 107421. https://doi.org/10.1016/j.aml.2021.107421

39. T. H. Zhao, O. Castillo, H. Jahanshahi, A. Yusuf, M. O. Alassafi, F. E. Alsaadi, et al., A fuzzy-based strategy to suppress the novel coronavirus (2019-NCOV) massive outbreak, *Appl. Comput. Math.*, **20** (2021), 160–176.

40. T. H. Zhao, M. I. Khan, Y. M. Chu, Artificial neural networking (ANN) analysis for heat and entropy generation in flow of non-Newtonian fluid between two rotating disks, *Math. Methods Appl. Sci.*, 2021. https://doi.org/10.1002/mma.7310

41. J. G. Liu, X. J. Yang, L. L. Geng, Y. R. Fan, Group analysis of the time fractional (3 + 1)-dimensional KDV-type equation, *Fractals*, **29** (2021), 2150169. https://doi.org/10.1142/S0218348X21501693

42. S. Hussain, E. N. Madi, H. Khan, S. Etemad, S. Rezapour, T. Sitthiwirattham, et al., Investigation of the stochastic modeling of COVID-19 with environmental noise from the analytical and numerical point of view, *Mathematics*, **9** (2021), 3122. https://doi.org/10.3390/math9233122

43. K. Karthikeyan, P. Karthikeyan, H. M. Baskonus, K. Venkatachalam, Y. M. Chu, Almost sectorial operators on Ψ-Hilfer derivative fractional impulsive integro-differential equations, *Math. Methods Appl. Sci.*, **2021** (2021), 8045–8059. https://doi.org/10.1002/mma.7954

44. Y. M. Chu, U. Nazir, M. Sohail, M. M. Selim, J. R. Lee, Enhancement in thermal energy and solute particles using hybrid nanoparticles by engaging activation energy and chemical reaction over a parabolic surface via finite element approach, *Fractal Fract.*, **5** (2021), 119. https://doi.org/10.3390/fractalfract5030119

45. S. Hussain, E. N. Madi, H. Khan, H. Gulzar, S. Etemad, S. Rezapour, M. K. A. Kaabar, On the stochastic modeling of COVID-19 under the environmental white noise, *J. Function Spaces*, **2022** (2022), 4320865. https://doi.org/10.1155/2022/4320865

46. M. Ahmad, A. Zada, M. Ghaderi, R. George, S. Rezapour, On the Existence and stability of a neutral stochastic fractional differential system, *Fractal Fract.*, **6** (2022), 203. https://doi.org/10.3390/fractalfract6040203

47. G. T. Tilahun, W. A. Woldegerima, A. Wondifraw, Stochastic and deterministic mathematical model of cholera disease dynamics with direct transmission, *Adv. Differ. Equ.*, **2020** (2020), 670 https://doi.org/10.1186/s13662-020-03130-w

48. Y. Zhang, Y. Li, Q. Zhang, A. Li, Behavior of a stochastic SIR epidemic model with saturated incidence and vaccination rules, *Phys. A*, **501** (2018), 178–187. https://doi.org/10.1016/j.physa.2018.02.191

49. N. Sene, Analysis of the stochastic model for predicting the novel coronavirus disease, *Adv. Differ. Equ.*, **2020** (2020), 568. https://doi.org/10.1186/s13662-020-03025-w

50. R. Khasminskii, *Stochastic stability of differential equations*, Berlin: Springer, 2012.

51. A. Atangana, S. I. Araz, New numerical method for ordinary differential equations: Newton polynomial, *J. Comput. Appl. Math.*, **372** (2020), 112622. https://doi.org/10.1016/j.cam.2019.112622

52. M. Toufik, A. Atangana, New numerical approximation of fractional derivative with non-local and non-singular kernel: application to chaotic models, *Eur. Phys. J. Plus*, **132** (2017), 444. https://doi.org/10.1140/epjp/i2017-11717-0

53. E. A. Kamuhanda, S. Osman, M. Wainaina, Mathematical modelling and analysis of the dynamics of cholera, *Global J. Pure Appl. Math.*, **14** (2018), 1259–1275.

54. M. O. Beryl, L. O. George, N. O. Fredrick, Mathematical analysis of a cholera transmission model incorporating media coverage, *Int. J. Pure Appl. Math.*, **111** (2016), 219–231.

55. S. Fatima, I. Krishnarajah, M. Z. A. M. Jaffar, M. B. Adam, A mathematical model for the control of cholera in Nigeria, *Res. J. Environ. Earth Sci.*, **6** (2014), 321–325.

56. A. P. Lemos-Paiao, C. J. Silva, D. F. M. Torres, A cholera mathematical model with vaccination and the biggest outbreak of world's history, *AIMS Math.*, **3** (2018), 448–463. https://doi.org/10.3934/Math.2018.4.448