Research Article

Humaira Kalsoom, Muhammad Amer Latif, Saima Rashid, Dumitru Baleanu, and Yu-Ming Chu*

New (p, q)-estimates for different types of integral inequalities via (α, m)-convex mappings

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Abstract: In the article, we present a new (p, q)-integral identity for the first-order (p, q)-differentiable functions and establish several new (p, q)-quantum error estimations for various integral inequalities via (α, m)-convexity. We also compare our results with the previously known results and provide two examples to show the superiority of our obtained results.

Keywords: (p,q)-quantum calculus, Hermite-Hadamard inequality, Simpson's type inequality, (α,m)-convex functions

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1 Introduction

Integral inequalities are considered a fabulous tool for constructing the qualitative and quantitative properties in the field of pure and applied mathematics [1–20]. A continuous growth of interest has been occurring to meet the requirements for the wide applications of these inequalities. These applications are closely related to the convex functions and have been studied by many researchers using various techniques [21–32].

Now, we recall the definition of convex function as follows.

Let $K \subseteq \mathbb{R}$ be an interval. Then a real-valued function $g : K \to \mathbb{R}$ is said to be convex if the inequality

$$g(\lambda\phi + (1 - \lambda)\psi) \leq \lambda g(\phi) + (1 - \lambda)g(\psi)$$

holds for all $\phi, \psi \in K$ and $\lambda \in [0, 1]$.

For convex functions, many inequalities have been established by many authors, for example, Jensen inequality [33], Ostrowski inequality [34], hypergeometric function inequality [35], elliptic integral inequal-

*** Corresponding author: Yu-Ming Chu,** Department of Mathematics, Huzhou University, Huzhou 313000, China; School of Mathematics and Statistics, Changsha University of Science & Technology, Changsha 410014, China, e-mail: chuyuming2005@126.com
Humaira Kalsoom: Department of Mathematics, Zhejiang Normal University, Jinhua 321004, P. R. China, e-mail: humaira87@zju.edu.cn, humairazju@yahoo.com
Muhammad Amer Latif: Department of Basic Sciences, Deanship of Preparatory Year, King Faisal University, Hofuf Al-Hasa 31982, Saudi Arabia, e-mail: m_amer_latif@hotmail.com
Saima Rashid: Department of Mathematics, Government College University, Faisalabad 38000, Pakistan, e-mail: saimarashid@gcuf.edu.pk
Dumitru Baleanu: Department of Mathematics, Çankaya University, Ankara 06530, Turkey, e-mail: dumitru@cankaya.edu.tr

ities [36–41] and so on. But the most celebrated and significant inequality is the Hermite-Hadamard inequality [42,43], which is stated as follows.

Let $\phi < \psi$ and $g : [\phi, \psi] \to \mathbb{R}$ be a convex function. Then the double inequality

$$g\left(\frac{\phi + \psi}{2}\right) \le \frac{1}{\psi - \phi} \int_{\phi}^{\psi} g(\lambda)\mathrm{d}\lambda \le \frac{g(\phi) + g(\psi)}{2} \tag{1.1}$$

The Hermite-Hadamard inequality (1.1) has been extensively discussed because it is essential in developing a connection between the theory of convex functions and integral inequalities. A number of researchers have dedicated their efforts to extend, generalize and refine the Hermite-Hadamard inequality (1.1) for different classes of convex functions and mappings. Some recent results on inequality (1.1) can be found in the literature [44–46].

Let $g : [\phi, \psi] \to \mathbb{R}$ be a four times continuous and differentiable mapping on the interval $[\phi, \psi]$ such that $\|g^{(4)}\|_\infty = \sup_{z\in(\phi,\psi)} |g^{(4)}(z)| < \infty$. Then Simpson's inequality [47]

$$\left|\frac{1}{3}\left[\frac{g(\phi) + g(\psi)}{2} + 2g\left(\frac{\phi + \psi}{2}\right) - \frac{1}{\phi - \psi}\int_{\phi}^{\psi} g(z)\mathrm{d}z\right]\right| \le \frac{(\psi - \phi)^4}{2{,}880}\|g^{(4)}\|_\infty$$

holds.

Quantum calculus is the study of calculus without limits and is also known as q-calculus [48]. In q-calculus, we obtain the initial mathematical formulas as q approaches 1. The commencement of the analysis of q-calculus can be dated back to the era of Euler (1707–1783), who first initiated the q-calculus in the tracks of Newton's work on infinite series. Subsequently, Jackson [49] launched the concept of q-integrals and studied it in a systematic way. The aforementioned results lead to an intensive investigation on q-calculus in the twentieth century. The idea of q-calculus is used in numerous areas in mathematics and physics, especially in orthogonal polynomials, number theory, hypergeometric functions, mechanics and relativity theory. The concept of q-derivatives over the definite interval $[\phi, \psi]$ of $\mathbb{R}$ is introduced by Tariboon et al. [50,51], and they addressed several problems on quantum analogs such as Hölder inequality, Ostrowski inequality, Cauchy-Schwarz inequality, Grüss-Chebyshev inequality, Grüss inequality and other integral inequalities by classical convexity.

From the last few years, q-calculus has become an interesting topic for many researchers and several new results have been established in the literature [52–58]. Furthermore, Tunç and Göv [59,60] derived the notion of (p, q)-calculus on the intervals $[\phi, \psi]$ of $\mathbb{R}$, found the formulae for (p, q)-derivative and (p, q)-integral and established their several fundamental properties. The results that depend on (p, q)-calculus are the Minkowski inequality, Hölder inequality, Grüss and Grüss-Chebyshev inequality and many others. Kunt et al. [61] gave the generalized (p, q)-Hermite-Hadamard inequalities on the finite interval and some important results which are connected with (p, q)-midpoint-type inequality. Recently, (p, q)-calculus has been the subject of intensive research, and its refinements and generalizations can be found in the literature [61,62].

Now, we recall the definitions and theorems for (p, q)-derivative and (p, q)-integral.

Definition 1.1. [59] Let $0 < q < p \le 1$ and $g : [\phi, \psi] \to \mathbb{R}$ be a continuous function. Then the (p, q)-derivative of g at $\lambda \in [\phi, \psi]$ is defined by

$${}_\phi D_{p,q} g(\lambda) = \frac{g(p\lambda + (1 - p)\phi) - g(q\lambda + (1 - q)\phi)}{(p - q)(\lambda - \phi)} \quad (\lambda \ne \phi)$$

and

$${}_\phi D_{p,q} g(\phi) = \lim_{\lambda\to\phi} D_{p,q} g(\lambda).$$

Example 1.2. Define the function $g : [\phi, \psi] \to \mathbb{R}$ by $g(\lambda) = 2\lambda^2 + 1$ with $0 < q < p \leq 1$. Then for $\lambda \neq \phi$ we have

$$\begin{aligned}
{}_{\phi}D_{p,q}(2\lambda^2 + 1) &= \frac{(2(p\lambda + (1-p)\phi)^2 + 1) - (2(q\lambda + (1-q)\phi)^2 + 1)}{(p-q)(\lambda - \phi)} \\
&= \frac{2[2]_{p,q}\lambda^2 + 4\phi\lambda[1 - [2]_{p,q}] + 2\phi^2[[2]_{p,q} - 2]}{(\lambda - \phi)} \\
&= \frac{2\lambda[2]_{p,q}(\lambda - \phi) - 2\phi[2]_{p,q}(\lambda - \phi) + 4\phi(\lambda - \phi)}{(\lambda - \phi)} \\
&= 2[2]_{p,q}(\lambda - \phi) + 4\phi.
\end{aligned}$$

Definition 1.3. [59] Let $0 < q < p \leq 1$ and $g : [\phi, \psi] \to \mathbb{R}$ be a continuous function. Then the (p, q)-integral on $[\phi, \psi]$ is defined by

$$\int_{\phi}^{\lambda} g(x)\,{}_{\phi}d_{p,q}x = (p-q)(\lambda - \phi)\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} g\left(\frac{q^n}{p^{n+1}}\lambda + \left(1 - \frac{q^n}{p^{n+1}}\right)\phi\right)$$

for $\lambda \in [\phi, \psi]$.

If $c \in (\phi, \lambda)$, then the (p, q)-definite integral on $[c, \lambda]$ can be expressed as

$$\int_{c}^{\lambda} g(x)\,{}_{\phi}d_{p,q}x = \int_{\phi}^{\lambda} g(x)\,{}_{\phi}d_{p,q}x - \int_{\phi}^{c} g(x)\,{}_{\phi}d_{p,q}x.$$

Example 1.4. Define the function $g : [\phi, \psi] \to \mathbb{R}$ by $g(x) = 4x + 1$ with $0 < q < p \leq 1$. Then one has

$$\begin{aligned}
\int_{\phi}^{\lambda} (4x + 1)\,{}_{\phi}d_{p,q}x &= (p-q)(\lambda - \phi)\left(4\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^{n+1}}\lambda + \left(1 - \frac{q^n}{p^{n+1}}\right)\phi\right) + \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}}\right) \\
&= \frac{(\lambda - \phi)[4(\lambda - \phi(1 - p - q)) + [2]_{p,q}]}{[2]_{p,q}}.
\end{aligned}$$

Theorem 1.5. [61] *Let* $0 < q < p \leq 1$ *and* $g : [\phi, \psi] \to \mathbb{R}$ *be a convex differentiable function on* $[\phi, \psi]$. *Then*

$$g\left(\frac{q\phi + p\psi}{[2]_{p,q}}\right) \leq \frac{1}{p(\psi - \phi)} \int_{\phi}^{p\psi + (1-p)\phi} g(x)\,{}_{\phi}d_{p,q}x \leq \frac{qg(\phi) + pg(\psi)}{[2]_{p,q}}.$$

The definition of (α, m)-convex function was presented by Miheşan in [63] and is stated as follows.

Definition 1.6. Let $\alpha, m \in (0, 1]$ and $\psi^* > 0$. Then the function $g : [0, \psi^*] \to \mathbb{R}$ is said to be (α, m)-convex if the inequality

$$g(x\lambda + m(1 - \lambda)y) \leq \lambda^{\alpha} g(x) + m(1 - \lambda^{\alpha})g(y)$$

holds for all $x, y \in [0, \psi^*]$ and $\lambda \in [0, 1]$.

Zhang et al. [64] investigated some inequalities about q-differentiable convex and quasi-convex functions which are linked with the different types of inequalities in q-calculus.

Lemma 1.7. [64] *Let* $0 < q < 1$ *and* $g : [\phi, \psi] \to \mathbb{R}$ *be a* q*-differentiable function on* (ϕ, ψ) *such that* ${}_{\phi}D_q g$ *is continuous and integrable on* $[\phi, \psi]$. *Then*

$$\gamma[\mu g(\psi) + (1-\mu)g(\phi)] + (1-\gamma)g(\mu\psi + (1-\mu)\phi) - \frac{1}{\psi-\phi}\int_{\phi}^{\psi} g(x)_{\phi}d_q x$$
$$= (\psi-\phi)\left[\int_0^{\mu} (q\lambda + \gamma\mu - \gamma)_{\phi}D_q g(\lambda\psi + (1-\lambda)\phi)_0 d_q\lambda + \int_{\mu}^{1} (q\lambda + \gamma\mu - 1)_{\phi}D_q g(\lambda\psi + (1-\lambda)\phi)_0 d_q\lambda\right]$$

for all $\gamma, \mu \in [0, 1]$.

The main purpose of the article is to provide an identity, which is the generalization of an identity presented in Lemma 1.7, and establish the (p, q)-analogues of different types of integral inequalities via the (p, q)-differentiable (α, m)-convex functions. By using the new identity with distinct parameters we obtain some new (p, q)-quantum error estimations for different types of inequalities such as the midpoint-type, the Simpson-type, the average of midpoint-trapezoid-type and the trapezoid-type inequalities via (α, m)-convexity.

2 Auxiliary results

In order to obtain different types of integral inequalities through (p, q)-differentiable (α, m)-convex functions, we need several lemmas which we present in this section.

Lemma 2.1. *Let* $0 < q < p \leq 1$ *and* $g : [\phi, \psi] \to \mathbb{R}$ *be a* (p, q)*-differentiable function on* (ϕ, ψ) *such that* ${}_{\phi}D_{p,q}g$ *is continuous and integrable on* $[\phi, \psi]$*. Then*

$$\gamma[p\mu g(\psi) + (1-p\mu)g(\phi)] + (1-\gamma)g(p\mu\psi + (1-p\mu)\phi) - \frac{1}{p(\psi-\phi)}\int_{\phi}^{p\psi+(1-p)\phi} g(x)_{\phi}d_{p,q}x$$
$$= (\psi-\phi)\left[\int_0^{p\mu} (q\lambda + \gamma p\mu - \gamma)_{\phi}D_{p,q}g(\lambda\psi + (1-\lambda)\phi)_0 d_{p,q}\lambda + \int_{p\mu}^{1} (q\lambda + \gamma p\mu - 1)_{\phi}D_{p,q}g(\lambda\psi + (1-\lambda)\phi)_0 d_{p,q}\lambda\right]$$

for all $\gamma, \mu \in [0, 1]$.

Proof. By an identical transformation, we get

$$(\psi-\phi)\left[\int_0^{p\mu} (q\lambda + \gamma p\mu - \gamma)_{\phi}D_{p,q}g(\lambda\psi + (1-\lambda)\phi)_0 d_{p,q}\lambda + \int_{p\mu}^{1} (q\lambda + \gamma p\mu - 1)_{\phi}D_{p,q}g(\lambda\psi + (1-\lambda)\phi)_0 d_{p,q}\lambda\right]$$
$$= (\psi-\phi)\left[\int_0^{1} (q\lambda + \gamma p\mu - 1)_{\phi}D_{p,q}g(\lambda\psi + (1-\lambda)\phi)_0 d_{p,q}\lambda + (1-\gamma)\int_0^{p\mu} {}_{\phi}D_{p,q}g(\lambda\psi + (1-\lambda)\phi)_0 d_{p,q}\lambda\right]. \tag{2.1}$$

Applying Definitions 1.1 and 1.3, we have

$$\int_0^1 \lambda_{\phi}D_{p,q}g(\lambda\psi + (1-\lambda)\phi)_0 d_{p,q}\lambda$$
$$= \int_0^1 \frac{g(p\lambda\psi + (1-p\lambda)\phi) - g(q\lambda\psi + (1-q\lambda)\phi)}{(p-q)(\psi-\phi)}{}_0 d_{p,q}\lambda$$

$$
\begin{aligned}
&= \frac{1}{\psi-\phi}\left[\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}g\left(\frac{q^n}{p^n}\psi+\left(1-\frac{q^n}{p^n}\right)\phi\right)-\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}g\left(\frac{q^{n+1}}{p^{n+1}}\psi+\left(1-\frac{q^{n+1}}{p^{n+1}}\right)\phi\right)\right]\\
&= \frac{1}{\psi-\phi}\left[\frac{1}{p}\sum_{n=0}^{\infty}\frac{q^n}{p^n}g\left(\frac{q^n}{p^n}\psi+\left(1-\frac{q^n}{p^n}\right)\phi\right)-\frac{1}{q}\sum_{n=1}^{\infty}\frac{q^n}{p^n}g\left(\frac{q^n}{p^n}\psi+\left(1-\frac{q^n}{p^n}\right)\phi\right)\right]\\
&= \frac{1}{\psi-\phi}\left[\frac{1}{q}g(\psi)-\left(\frac{1}{q}-\frac{1}{p}\right)\times\sum_{n=0}^{\infty}\frac{q^n}{p^n}g\left(\frac{q^n}{p^n}\psi+\left(1-\frac{q^n}{p^n}\right)\phi\right)\right]\\
&= \frac{1}{q(\psi-\phi)}g(\psi)-\frac{1}{pq(\psi-\phi)^2}\int_{\phi}^{p\psi+(1-p)\phi} g(x)_{\phi}d_{p,q}x,
\end{aligned}
\tag{2.2}
$$

$$
\begin{aligned}
&\int_0^1 {}_{\phi}D_{p,q}g(\lambda\psi+(1-\lambda)\phi)_0 d_{p,q}\lambda\\
&\quad= \int_0^1 \frac{g(p\lambda\psi+(1-p\lambda)\phi)-g(q\lambda\psi+(1-q\lambda)\phi)}{\lambda(p-q)(\psi-\phi)}{}_0d_{p,q}\lambda\\
&\quad= \frac{1}{\psi-\phi}\left[\sum_{n=0}^{\infty}g\left(\frac{q^n}{p^n}\psi+\left(1-\frac{q^n}{p^n}\right)\phi\right)-\sum_{n=0}^{\infty}g\left(\frac{q^{n+1}}{p^{n+1}}\psi+\left(1-\frac{q^{n+1}}{p^{n+1}}\right)\phi\right)\right]\\
&\quad= \frac{g(\psi)-g(\phi)}{\psi-\phi},
\end{aligned}
\tag{2.3}
$$

$$
\begin{aligned}
&\int_0^{p\mu} {}_{\phi}D_{p,q}g(\lambda\psi+(1-\lambda)\phi)_0 d_{p,q}\lambda\\
&\quad= \int_0^{p\mu} \frac{g(p\lambda\psi+(1-p\lambda)\phi)-g(q\lambda\psi+(1-q\lambda)\phi)}{\lambda(p-q)(\psi-\phi)}{}_0d_{p,q}\lambda\\
&\quad= \frac{1}{\psi-\phi}\left[\sum_{n=0}^{\infty}g\left(\frac{q^n}{p^n}p\mu\psi+\left(1-\frac{q^n}{p^n}p\mu\right)\phi\right)-\sum_{n=0}^{\infty}g\left(\frac{q^{n+1}}{p^{n+1}}p\mu\psi+\left(1-\frac{q^{n+1}}{p^{n+1}}p\mu\right)\phi\right)\right]\\
&\quad= \frac{g(p\mu\psi+(1-p\mu)\phi)-g(\phi)}{\psi-\phi}.
\end{aligned}
\tag{2.4}
$$

Substituting (2.2), (2.3) and (2.4) into (2.1), we obtain the desired result. □

Remark 2.1. The following statements are true under the conditions of Lemma 2.1.

(1) If $\mu = 0$, then we get

$$
g(\phi)-\frac{1}{p(\psi-\phi)}\int_{\phi}^{p\psi+(1-p)\phi} g(x)_{\phi}d_{p,q}x = (\psi-\phi)\int_0^1 (q\lambda-1)_{\phi}D_{p,q}g(\lambda\psi+(1-\lambda)\phi)_0 d_{p,q}\lambda.
$$

(2) If $p = \mu = 1$, then we have

$$
g(\psi)-\frac{1}{p(\psi-\phi)}\int_{\phi}^{p\psi+(1-p)\phi} g(x)_{\phi}d_{p,q}x = (\psi-\phi)\int_0^1 q\lambda_{\phi}D_{p,q}g(\lambda\psi+(1-\lambda)\phi)_0 d_{p,q}\lambda.
$$

(3) If $\mu = 1/[2]_{p,q}$, then one has

$$
\gamma\frac{qg(\phi)+pg(\psi)}{[2]_{p,q}}+(1-\gamma)g\left(\frac{q\phi+p\psi}{[2]_{p,q}}\right)-\frac{1}{p(\psi-\phi)}\int_{\phi}^{p\psi+(1-p)\phi} g(x)_{\phi}d_{p,q}x
$$

$$= (\psi - \phi)\left[\int_0^{\frac{p}{[2]_{p,q}}} \left(q\lambda - \frac{\gamma q}{[2]_{p,q}}\right){}_\phi D_{p,q} g(\lambda\psi + (1-\lambda)\phi)_0 d_{p,q}\lambda + \int_{\frac{p}{[2]_{p,q}}}^1 \left(q\lambda + \frac{p(\gamma - 1) - q}{[2]_{p,q}}\right){}_\phi D_{p,q} g(\lambda\psi + (1-\lambda)\phi)_0 d_{p,q}\lambda\right].$$

Remark 2.2. If all the conditions of Lemma 2.1 are satisfied, then the following four statements are true:

(1) If $\gamma = 0$, then we get

$$g(p\mu\psi + (1-p\mu)\phi) - \frac{1}{p(\psi-\phi)}\int_\phi^{p\psi+(1-p)\phi} g(x)_\phi d_{p,q}x = (\psi - \phi)\left[\int_0^{p\mu} q\lambda_\phi D_{p,q} g(\lambda\psi + (1-\lambda)\phi)_0 d_{p,q}\lambda + \int_{p\mu}^1 (q\lambda - 1)_\phi D_{p,q} g(\lambda\psi + (1-\lambda)\phi)_0 d_{p,q}\lambda\right]. \tag{2.5}$$

Let $\mu = 1/[2]_{p,q}$ in (2.5). Then we acquire the midpoint-type integral identity

$$g\left(\frac{q\phi + p\psi}{[2]_{p,q}}\right) - \frac{1}{p(\psi-\phi)}\int_\phi^{p\psi+(1-p)\phi} g(x)_\phi d_{p,q}x = (\psi - \phi)\left[\int_0^{\frac{p}{[2]_{p,q}}} q\lambda_\phi D_{p,q} g(\lambda\psi + (1-\lambda)\phi)_0 d_{p,q}\lambda + \int_{[2]_{p,q}}^1 (q\lambda - 1)_\phi D_{p,q} g(\lambda\psi + (1-\lambda)\phi)_0 d_{p,q}\lambda\right], \tag{2.6}$$

which was proposed by Kunt et al. in [61], and equation (2.6) leads to Lemma 11 of [65] if $p = 1$.

(2) If $\gamma = 1/3$, then we get

$$\frac{1}{3}[p\mu g(\psi) + (1-p\mu)g(\phi) + 2g(p\mu\psi + (1-p\mu)\phi)] - \frac{1}{p(\psi-\phi)}\int_\phi^{p\psi+(1-p)\phi} g(x)_\phi d_{p,q}x = (\psi - \phi)\left[\int_0^{p\mu}\left(q\lambda + \frac{p\mu - 1}{3}\right){}_\phi D_{p,q} g(\lambda\psi + (1-\lambda)\phi)_0 d_{p,q}\lambda + \int_{p\mu}^1 \left(q\lambda + \frac{p\mu - 3}{3}\right){}_\phi D_{p,q} g(\lambda\psi + (1-\lambda)\phi)_0 d_{p,q}\lambda\right]. \tag{2.7}$$

In particular, if $\mu = 1/[2]_{p,q}$, then equation (2.7) leads to the Simpson-type integral identity

$$\frac{1}{3}\left[\frac{qg(\phi) + pg(\psi)}{[2]_{p,q}} + 2g\left(\frac{q\phi + p\psi}{[2]_{p,q}}\right)\right] - \frac{1}{p(\psi-\phi)}\int_\phi^{p\psi+(1-p)\phi} g(x)_\phi d_{p,q}x = (\psi - \phi)\left\{\int_0^{\frac{p}{[2]_{p,q}}}\left(q\lambda - \frac{q}{3[2]_{p,q}}\right){}_\phi D_{p,q} g(\lambda\psi + (1-\lambda)\phi)_0 d_{p,q}\lambda + \int_{\frac{p}{[2]_{p,q}}}^1 \left(q\lambda - \frac{2p + 3q}{3[2]_{p,q}}\right){}_\phi D_{p,q} g(\lambda\psi + (1-\lambda)\phi)_0 d_{p,q}\lambda\right\}.$$

(3) If $\gamma = 1/2$, then one has

$$\frac{1}{2}[p\mu g(\psi) + (1 - p\mu)g(\phi) + g(p\mu\psi + (1 - p\mu)\phi)] - \frac{1}{p(\psi - \phi)} \int_{\phi}^{p\psi+(1-p)\phi} g(x)_{\phi}d_{p,q}x$$
$$= (\psi - \phi)\left\{ \int_{0}^{p\mu} \left(q\lambda + \frac{p\mu - 1}{2}\right)_{\phi}D_{p,q}g(\lambda\psi + (1 - \lambda)\phi)_0 d_{p,q}\lambda \right. \tag{2.8}$$
$$\left. + \int_{p\mu}^{1} \left(q\lambda + \frac{p\mu - 2}{2}\right)_{\phi}D_{p,q}g(\lambda\psi + (1 - \lambda)\phi)_0 d_{p,q}\lambda \right\}.$$

In particular, if $\mu = 1/[2]_{p,q}$, then equation (2.8) gives the average of midpoint-trapezoid-type integral identity

$$\frac{1}{2}\left[\frac{qg(\phi) + pg(\psi)}{[2]_{p,q}} + g\left(\frac{q\phi + p\psi}{[2]_{p,q}}\right)\right] - \frac{1}{p(\psi - \phi)} \int_{\phi}^{p\psi+(1-p)\phi} g(x)_{\phi}d_{p,q}x$$
$$= (\psi - \phi)\left\{ \int_{0}^{\frac{p}{[2]_{p,q}}} \left(q\lambda - \frac{q}{2[2]_{p,q}}\right)_{\phi}D_{p,q}g(\lambda\psi + (1 - \lambda)\phi)_0 d_{p,q}\lambda \right.$$
$$\left. + \int_{\frac{p}{[2]_{p,q}}}^{1} \left(q\lambda - \frac{p + 2q}{2[2]_{p,q}}\right)_{\phi}D_{p,q}g(\lambda\psi + (1 - \lambda)\phi)_0 d_{p,q}\lambda \right\}.$$

(4) Let $\gamma = 1$. Then we get

$$p\mu g(\psi) + (1 - p\mu)g(\phi) - \frac{1}{p(\psi - \phi)} \int_{\phi}^{p\psi+(1-p)\phi} g(x)_{\phi}d_{p,q}x$$
$$= (\psi - \phi) \int_{0}^{1} (q\lambda + p\mu - 1)_{\phi}D_{p,q}g(\lambda\psi + (1 - \lambda)\phi)_0 d_{p,q}\lambda. \tag{2.9}$$

Let $\mu = 1/[2]_{p,q}$. Then equation (2.9) leads to the trapezoid-type integral identity

$$\frac{qg(\phi) + pg(\psi)}{[2]_{p,q}} - \frac{1}{p(\psi - \phi)} \int_{\phi}^{p\psi+(1-p)\phi} g(x)_{\phi}d_{p,q}x$$
$$= (\psi - \phi) \int_{0}^{1} \left(q\lambda - \frac{q}{[2]_{p,q}}\right)\phi D_{p,q}g(\lambda\psi + (1 - \lambda)\phi)_0 d_{p,q}\lambda, \tag{2.10}$$

which was proposed by Latif et al. in [66].

In particular, Lemma 3.1 of [55] can be derived from equation (2.10) if we take $p = 1$.

The following Lemma 2.2 can be obtained immediately from Definition 1.3.

Lemma 2.2. *Let* $0 < q < p \leq 1$, $0 \leq \mu \leq 1$ *and* $\xi \in [0, \infty)$. *Then we have*

$$\int_{0}^{p\mu} \lambda^{\xi}{}_0 d_{p,q}\lambda = (p - q)\mu^{\xi+1} \sum_{n=0}^{\infty} \left(\frac{q}{p}\right)^{(\xi+1)n} = \frac{\mu^{\xi+1}(p - q)p^{\xi+1}}{p^{\xi+1} - q^{\xi+1}}$$

and

$$\int_0^{p\mu} (1-\lambda)^{\xi}\,{}_0d_{p,q}\lambda = (p-q)\mu \sum_{n=0}^{\infty} \frac{q^n}{p^n}\left(1-\frac{q^n}{p^n}\mu\right)^{\xi}.$$

Lemma 2.3. *Let* $0<q<p\leq 1$, $\gamma,\mu\in[0,1]$ *and* $\xi\in[0,\infty)$. *Then we get*

$$\begin{aligned} v_1(\gamma,p\mu,\xi) &= \int_0^{p\mu} \lambda^{\xi}|q\lambda-(\gamma-\gamma p\mu)|\,{}_0d_{p,q}\lambda \\ &= \begin{cases} \dfrac{\mu^{\xi+1}(p-q)p^{\xi+1}(\gamma-\gamma p\mu)}{p^{\xi+1}-q^{\xi+1}} - \dfrac{q\mu^{\xi+2}(p-q)p^{\xi+2}}{p^{\xi+2}-q^{\xi+2}}, & (\gamma+q)p\mu\leq\gamma, \\ \left[\dfrac{2(p-q)(\gamma-\gamma p\mu)^{\xi+2}(p^{\xi+2}-q^{\xi+2}-p^{\xi+1}+q^{\xi+1})}{q^{\xi+1}(p^{\xi+1}-q^{\xi+1})(p^{\xi+2}-q^{\xi+2})}\right. \\ \left. +\dfrac{q\mu^{\xi+2}(p-q)p^{\xi+2}}{p^{\xi+2}-q^{\xi+2}} - \dfrac{\mu^{\xi+1}(p-q)p^{\xi+1}(\gamma-\gamma p\mu)}{p^{\xi+1}-q^{\xi+1}}\right], & (\gamma+q)p\mu>\gamma \end{cases} \end{aligned}$$

and

$$\begin{aligned} v_2(\gamma,p\mu,\xi) &= \int_0^{p\mu} (1-\lambda)^{\xi}|(q\lambda-(\gamma-\gamma p\mu)|\,{}_0d_{p,q}\lambda \\ &= \begin{cases} (p-q)\mu\displaystyle\sum_{n=0}^{\infty}\frac{q^n}{p^n}\left(\gamma-\gamma p\mu-\frac{q^{n+1}}{p^n}\mu\right)\left(1-\frac{q^n}{p^n}\mu\right)^{\xi}, & (\gamma+q)p\mu\leq\gamma, \\ \left[2(p-q)(\gamma-\gamma p\mu)^2\displaystyle\sum_{n=0}^{\infty}\frac{q^{n-1}}{p^{n+1}}\left(1-\frac{q^n}{p^{n+1}}\right)\left(1-\frac{q^{n-1}}{p^{n+1}}(\gamma-\gamma p\mu)\right)^{\xi}\right. \\ \left. -(p-q)\mu\displaystyle\sum_{n=0}^{\infty}\frac{q^n}{p^n}\left(\gamma-\gamma p\mu-\frac{q^{n+1}}{p^n}\mu\right)\left(1-\frac{q^n}{p^n}\mu\right)^{\xi}\right], & (\gamma+q)p\mu>\gamma. \end{cases} \end{aligned}$$

Proof. If $(\gamma+q)p\mu\leq\gamma$, then it follows from Lemma 2.1 that

$$\begin{aligned} \int_0^{p\mu} \lambda^{\xi}|(q\lambda-(\gamma-\gamma p\mu)|\,{}_0d_{p,q}\lambda &= \int_0^{p\mu} [(\gamma-\gamma p\mu)\lambda^{\xi}-q\lambda^{\xi+1}]\,{}_0d_{p,q}\lambda \\ &= \frac{\mu^{\xi+1}(p-q)p^{\xi+1}(\gamma-\gamma p\mu)}{p^{\xi+1}-q^{\xi+1}} - \frac{q\mu^{\xi+2}(p-q)p^{\xi+2}}{p^{\xi+2}-q^{\xi+2}}. \end{aligned}$$

If $(\gamma+q)p\mu>\gamma$, then from Lemma 2.1 we get

$$\begin{aligned} \int_0^{p\mu} \lambda^{\xi}|q\lambda-(\gamma-\gamma p\mu)|\,{}_0d_{p,q}\lambda &= \int_0^{\frac{\gamma-\gamma p\mu}{q}} [(\gamma-\gamma p\mu)\lambda^{\xi}-q\lambda^{\xi+1}]\,{}_0d_{p,q}\lambda + \int_{\frac{\gamma-\gamma p\mu}{q}}^{p\mu} [q\lambda^{\xi+1}-(\gamma-\gamma p\mu)\lambda^{\xi}]\,{}_0d_{p,q}\lambda \\ &= 2\int_0^{\frac{\gamma-\gamma p\mu}{q}} [(\gamma-\gamma p\mu)\lambda^{\xi}-q\lambda^{\xi+1}]\,{}_0d_{p,q}\lambda + \int_0^{p\mu} [q\lambda^{\xi+1}-(\gamma-\gamma p\mu)\lambda^{\xi}]\,{}_0d_{p,q}\lambda \\ &= \frac{2(p-q)(\gamma-\gamma p\mu)^{\xi+2}(p^{\xi+2}-q^{\xi+2}-p^{\xi+1}+q^{\xi+1})}{q^{\xi+1}(p^{\xi+1}-q^{\xi+1})(p^{\xi+2}-q^{\xi+2})} \\ &\quad + \frac{q\mu^{\xi+2}(p-q)p^{\xi+2}}{p^{\xi+2}-q^{\xi+2}} - \frac{\mu^{\xi+1}(p-q)p^{\xi+1}(\gamma-\gamma p\mu)}{p^{\xi+1}-q^{\xi+1}}. \end{aligned}$$

Similarly, we also get

$$\int_0^{p\mu}(1-\lambda)^{\xi}|(q\lambda-(\gamma-\gamma p\mu)|_0 d_{p,q}\lambda=\begin{cases}(p-q)\mu\sum_{n=0}^{\infty}\frac{q^n}{p^n}\left(\gamma-\gamma p\mu-\frac{q^{n+1}}{p^n}\mu\right)\left(1-\frac{q^n}{p^n}\mu\right)^{\xi}, & (\gamma+q)p\mu\le\gamma,\\ \Bigg[2(p-q)(\gamma-\gamma p\mu)^2\sum_{n=0}^{\infty}\frac{q^{n-1}}{p^{n+1}}\left(1-\frac{q^n}{p^{n+1}}\right)\left(1-\frac{q^{n-1}}{p^{n+1}}(\gamma-\gamma p\mu)\right)^{\xi} & \\ -(p-q)\mu\sum_{n=0}^{\infty}\frac{q^n}{p^n}\left(\gamma-\gamma p\mu-\frac{q^{n+1}}{p^n}\mu\right)\left(1-\frac{q^n}{p^n}\mu\right)^{\xi}\Bigg], & (\gamma+q)p\mu>\gamma,\end{cases}$$

which completes the proof of Lemma 2.3. □

The following Lemmas 2.4–2.9 can be obtained by using the definition of q-integrals, we omit the details of their proofs.

Lemma 2.4. *Let* $0<q<p\le 1$, $\gamma,\mu\in[0,1]$ *and* $\xi\in[0,\infty)$. *Then we have*

$$\begin{aligned}v_3(\gamma,p\mu,\xi)&=\int_0^1\lambda^{\xi}|q\lambda-(1-\gamma p\mu)|_0 d_{p,q}\lambda\\&=\begin{cases}\frac{(p-q)p^{\xi+1}(1-\gamma p\mu)}{p^{\xi+1}-q^{\xi+1}}-\frac{q(p-q)p^{\xi+2}}{p^{\xi+2}-q^{\xi+2}}, & \gamma p\mu+q\le 1,\\ \Bigg[\frac{2(p-q)(1-\gamma p\mu)^{\xi+2}(p^{\xi+2}-q^{\xi+2}-p^{\xi+1}+q^{\xi+1})}{q^{\xi+1}(p^{\xi+1}-q^{\xi+1})(p^{\xi+2}-q^{\xi+2})} & \\ +\frac{q(p-q)p^{\xi+2}}{p^{\xi+2}-q^{\xi+2}}-\frac{(p-q)p^{\xi+1}(1-\gamma p\mu)}{p^{\xi+1}-q^{\xi+1}}\Bigg], & \gamma p\mu+q>1\end{cases}\end{aligned}$$

and

$$\begin{aligned}v_4(\gamma,p\mu,\xi)&=\int_0^1(1-\lambda)^{\xi}|q\lambda-(1-\gamma p\mu)|_0 d_{p,q}\lambda\\&=\begin{cases}(p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(1-\gamma p\mu-\frac{q^{n+1}}{p^{n+1}}\right)\left(1-\frac{q^n}{p^{n+1}}\right)^{\xi}, & \gamma p\mu+q\le 1,\\ \Bigg[2(p-q)(1-\gamma p\mu)^2\sum_{n=0}^{\infty}\frac{q^{n-1}}{p^{n+1}}\left(1-\frac{q^n}{p^{n+1}}\right)\left(1-\frac{q^{n-1}}{p^{n+1}}(1-\gamma p\mu)\right)^{\xi} & \\ -(p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(1-\gamma p\mu-\frac{q^{n+1}}{p^{n+1}}\right)\left(1-\frac{q^n}{p^{n+1}}\right)^{\xi}\Bigg], & \gamma p\mu+q>1.\end{cases}\end{aligned}$$

Lemma 2.5. *Let* $0<q<p\le 1$, $\gamma,\mu\in[0,1]$ *and* $\xi\in[0,\infty)$. *Then one has*

$$\begin{aligned}v_5(\gamma,p\mu,\xi)&=\int_0^{p\mu}\lambda^{\xi}|q\lambda-(1-\gamma p\mu)|_0 d_{p,q}\lambda\\&=\begin{cases}\frac{\mu^{\xi+1}(p-q)p^{\xi+1}(1-\gamma p\mu)}{p^{\xi+1}-q^{\xi+1}}-\frac{q\mu^{\xi+2}(p-q)p^{\xi+2}}{p^{\xi+2}-q^{\xi+2}}, & (\gamma+q)p\mu\le 1,\\ \Bigg[\frac{2(p-q)(1-\gamma p\mu)^{\xi+2}(p^{\xi+2}-q^{\xi+2}-p^{\xi+1}+q^{\xi+1})}{q^{\xi+1}(p^{\xi+1}-q^{\xi+1})(p^{\xi+2}-q^{\xi+2})} & \\ +\frac{q\mu^{\xi+2}(p-q)p^{\xi+2}}{p^{\xi+2}-q^{\xi+2}}-\frac{\mu^{\xi+1}(p-q)p^{\xi+1}(1-\gamma p\mu)}{p^{\xi+1}-q^{\xi+1}}\Bigg], & (\gamma+q)p\mu>1\end{cases}\end{aligned}$$

and

$$v_6(\gamma, p\mu, \xi) = \int_0^{p\mu} (1-\lambda)^{\xi} |q\lambda - (1-\gamma\mu)| {}_0d_{p,q}\lambda$$
$$= \begin{cases} (p-q)\mu \sum_{n=0}^{\infty} \frac{q^n}{p^n}\left(1 - \gamma p\mu - \frac{q^{n+1}}{p^n}\mu\right)\left(1 - \frac{q^n}{p^n}\mu\right)^{\xi}, & (\gamma+q)p\mu \le 1, \\ \Bigg[2(p-q)(1-\gamma\mu)^2 \sum_{n=0}^{\infty} \frac{q^{n-1}}{p^{n+1}}\left(1 - \frac{q^n}{p^{n+1}}\right)\left(1 - \frac{q^{n-1}}{p^{n+1}}(1-\gamma p\mu)\right)^{\xi} & \\ - (p-q)\mu \sum_{n=0}^{\infty} \frac{q^n}{p^n}\left(1 - \gamma p\mu - \frac{q^{n+1}}{p^n}\mu\right)\left(1 - \frac{q^n}{p^n}\mu)\right)^{\xi}\Bigg], & (\gamma+q)p\mu > 1. \end{cases}$$

Lemma 2.6. *Let* $0 < q < p \le 1$ *and* $\gamma, \mu \in [0,1]$. *Then we have*

$$v_7(\gamma, p\mu) = \int_0^{p\mu} |q\lambda - (\gamma - \gamma p\mu)| {}_0d_{p,q}\lambda$$
$$= \begin{cases} \gamma p\mu(1-p\mu) - \frac{q\mu^2p^2}{[2]_{p,q}}, & (\gamma+q)p\mu \le \gamma, \\ \frac{2(\gamma-\gamma p\mu)^2([2]_{p,q}-1)}{q[2]_{p,q}} + \frac{q\mu^2p^2}{[2]_{p,q}} - \gamma p\mu(1-p\mu), & (\gamma+q)p\mu > \gamma. \end{cases}$$

Lemma 2.7. *Let* $0 < q < p \le 1$ *and* $\gamma, \mu \in [0,1]$. *Then we get*

$$v_8(\gamma, p\mu) = \int_0^{1} |q\lambda - (1 - \gamma p\mu)| {}_0d_{p,q}\lambda$$
$$= \begin{cases} \frac{p}{[2]_{p,q}} - \gamma p\mu, & \gamma p\mu + q \le 1, \\ \frac{2(1-\gamma p\mu)^2([2]_{p,q}-1)}{q[2]_{p,q}} + \gamma p\mu - \frac{p}{[2]_{p,q}}, & \gamma p\mu + q > 1. \end{cases}$$

Lemma 2.8. *Let* $0 < q < p \le 1$ *and* $\gamma, \mu \in [0,1]$. *Then one has*

$$v_9(\gamma, p\mu) = \int_0^{p\mu} |q\lambda - (1 - \gamma p\mu)| {}_0d_{p,q}\lambda = \begin{cases} p\mu(1-\gamma p\mu) - \frac{q\mu^2p^2}{[2]_{p,q}}, & (\gamma+q)p\mu \le 1, \\ \frac{2(1-\gamma p\mu)^2([2]_{p,q}-1)}{q[2]_{p,q}} + \frac{q\mu^2p^2}{[2]_{p,q}} - p\mu(1-\gamma p\mu)\mathrm{x}, & (\gamma+q)p\mu > 1. \end{cases}$$

Lemma 2.9. *Let* $0 < q < p \le 1$, $\gamma, \mu \in [0,1]$ *and* $\sigma \in [1, \infty)$. *Then we get*

$$v_{10}(\gamma, p\mu) = \int_0^{1} |q\lambda - (1 - \gamma p\mu)|^{\sigma} {}_0d_{p,q}\lambda = \begin{cases} (p-q) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}}\left(1 - \gamma p\mu - \frac{q^{n+1}}{p^{n+1}}\right)^{\sigma}, & 0 \le \gamma p\mu \le 1-q, \\ \Bigg[(p-q)(1-\gamma p\mu)^{\sigma+1} \sum_{n=0}^{\infty} \frac{q^{n-1}}{p^{n+1}}\left(1 - \frac{q^n}{p^{n+1}}\right)^{\sigma} & \\ +(p-q) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}}\left(\frac{q^{n+1}}{p^{n+1}} - 1 + \gamma p\mu\right)^{\sigma} & \\ -(p-q)(1-\gamma p\mu)^{\sigma+1} \sum_{n=0}^{\infty} \frac{q^{n-1}}{p^{n+1}}\left(\frac{q^n}{p^{n+1}} - 1\right)^{\sigma}\Bigg], & 1-q < \gamma p\mu \le 1. \end{cases}$$

3 Main results

Theorem 3.1. *Let* $0 \leq \phi < \psi < \infty$, $0 < q < p \leq 1$, $\alpha, m \in (0, 1]$ *and* $g : J \supset 0, \infty) \to \mathbb{R}$ *be a* (p, q)*-differentiable function on* J° *(the interior of* J*) such that* ${}_{\phi}D_{p,q}g$ *is continuous and integrable on* $\left[0, \frac{\psi}{m}\right]$ *and* $|{}_{\phi}D_{p,q}g|$ *is* (α, m)*-convex on* $\left[0, \frac{\psi}{m}\right]$*. Then the inequality*

$$\left|\gamma[p\mu g(\psi) + (1 - p\mu)g(\phi)] + (1 - \gamma)g(p\mu\psi + (1 - p\mu)\phi) - \frac{1}{p(\psi - \phi)}\int_{\phi}^{p\psi+(1-p)\phi} g(x)\,{}_{\phi}d_{p,q}x\right|$$
$$\leq \min[H_1(\gamma, p\mu, \alpha, m), H_2(\gamma, p\mu, \alpha, m)]$$

holds for all $\gamma, \mu \in [0, 1]$, *where*

$$H_1(\gamma, p\mu, \alpha, m) = (\psi - \phi)\left\{[v_1(\gamma, p\mu, \alpha) + v_3(\gamma, p\mu, \alpha) - v_5(\gamma, p\mu, \alpha)]|{}_{\phi}D_{p,q}g(\psi)| + m[v_7(\gamma, p\mu)\right.$$
$$\left. + v_8(\gamma, p\mu) - v_9(\gamma, p\mu) - v_1(\gamma, p\mu, \alpha) - v_3(\gamma, p\mu, \alpha) + v_5(\gamma, p\mu, \alpha)]\left|{}_{\phi}D_{p,q}g\left(\frac{\phi}{m}\right)\right|\right\},$$

$$H_2(\gamma, p\mu, \alpha, m) = (\psi - \phi)\left\{[v_2(\gamma, p\mu, \alpha) + v_4(\gamma, p\mu, \alpha) - v_6(\gamma, p\mu, \alpha)]|{}_{\phi}D_{p,q}g(\phi)| + m[v_7(\gamma, p\mu)\right.$$
$$\left. + v_8(\gamma, p\mu) - v_9(\gamma, p\mu) - v_2(\gamma, p\mu, \alpha) - v_4(\gamma, p\mu, \alpha) + v_6(\gamma, p\mu, \alpha)]\left|{}_{\phi}D_{p,q}g\left(\frac{\psi}{m}\right)\right|\right\}.$$

Proof. From Lemma 2.1, the property of the modulus and the (α, m)-convexity of $|{}_{\phi}D_{p,q}g|$ we have

$$\left|\gamma[p\mu g(\psi) + (1 - p\mu)g(\phi)] + (1 - \gamma)g(p\mu\psi + (1 - p\mu)\phi) - \frac{1}{p(\psi - \phi)}\int_{\phi}^{p\psi+(1-p)\phi} g(x)\,{}_{\phi}d_{p,q}x\right|$$
$$\leq (\psi - \phi)\left[\int_0^{p\mu} |q\lambda + \gamma p\mu - \gamma||{}_{\phi}D_{p,q}g(\lambda\psi + (1 - \lambda)\phi)|\,{}_0d_{p,q}\lambda\right.$$
$$\left. + \int_{p\mu}^{1} |q\lambda + \gamma p\mu - 1||{}_{\phi}D_{p,q}g(\lambda\psi + (1 - \lambda)\phi)|\,{}_0d_{p,q}\lambda\right]$$
$$\leq (\psi - \phi)\left[\int_0^{p\mu} |q\lambda - (\gamma - \gamma p\mu)|\left[\lambda^{\alpha}|{}_{\phi}D_{p,q}g(\psi)| + m(1 - \lambda^{\alpha})\left|{}_{\phi}D_{p,q}g\left(\frac{\phi}{m}\right)\right|\right]{}_0d_{p,q}\lambda\right.$$
$$\left. + \int_{p\mu}^{1} |q\lambda - (1 - \gamma p\mu)|\left[\lambda^{\alpha}|{}_{\phi}D_{p,q}g(\psi)| + m(1 - \lambda^{\alpha})\left|{}_{\phi}D_{p,q}g\left(\frac{\phi}{m}\right)\right|\right]{}_0d_{p,q}\lambda\right]$$

$$= (\psi - \phi)\left\{\left[\int_0^{p\mu} \lambda^{\alpha}|q\lambda - (\gamma - \gamma p\mu)|\,{}_0d_{p,q}\lambda + \int_0^{1} \lambda^{\alpha}|q\lambda - (1 - \gamma p\mu)|\,{}_0d_{p,q}\lambda\right.\right.$$
$$\left. - \int_0^{p\mu} \lambda^{\alpha}|q\lambda - (1 - \gamma p\mu)|\,{}_0d_{p,q}\lambda\right]|{}_{\phi}D_{p,q}g(\psi)| + m\left[\int_0^{p\mu} |q\lambda - (\gamma - \gamma p\mu)|\,{}_0d_{p,q}\lambda\right.$$

$$+\int_0^1 |q\lambda-(1-\gamma p\mu)|\,{}_0d_{p,q}\lambda-\int_0^{p\mu}|q\lambda-(1-\gamma p\mu)|\,{}_0d_{p,q}\lambda$$
$$-\int_0^{p\mu}\lambda^\alpha|q\lambda-(\gamma-\gamma p\mu)|\,{}_0d_{p,q}\lambda-\int_0^1\lambda^\alpha|q\lambda-(1-\gamma p\mu)|\,{}_0d_{p,q}\lambda$$
$$\left.\left.+\int_0^{p\mu}\lambda^\alpha|q\lambda-(1-\gamma p\mu)|\,{}_0d_{p,q}\lambda\right]\left|{}_\phi D_{p,q}g\left(\frac{\phi}{m}\right)\right|\right\}$$
$$\le(\psi-\phi)\left\{\left[\int_0^{p\mu}(1-\lambda)^\alpha|q\lambda-(\gamma-\gamma p\mu)|\,{}_0d_{p,q}\lambda+\int_0^1(1-\lambda)^\alpha\right.\right.$$
$$\left.\times|q\lambda-(1-\gamma p\mu)|\,{}_0d_{p,q}\lambda-\int_0^{p\mu}(1-\lambda)^\alpha|q\lambda-(1-\gamma p\mu)|\,{}_0d_{p,q}\lambda\right]$$
$$\times|{}_\phi D_{p,q}g(\phi)|+m\left[\int_0^{p\mu}|q\lambda-(\gamma-\gamma p\mu)|\,{}_0d_{p,q}\lambda+\int_0^1|q\lambda-(1-\gamma p\mu)|\,{}_0d_{p,q}\lambda\right.$$
$$-\int_0^{p\mu}|q\lambda-(1-\gamma p\mu)|\,{}_0d_{p,q}\lambda-\int_0^{p\mu}(1-\lambda)^\alpha|q\lambda-(\gamma-\gamma p\mu)|\,{}_0d_{p,q}\lambda-\int_0^1(1-\lambda)^\alpha$$
$$\left.\left.\times|q\lambda-(1-\gamma p\mu)|\,{}_0d_{p,q}\lambda+\int_0^{p\mu}(1-\lambda)^\alpha|q\lambda-(1-\gamma p\mu)|\,{}_0d_{p,q}\lambda\right]\left|{}_\phi D_{p,q}g\left(\frac{\psi}{m}\right)\right|\right\}.$$

Using Lemmas 2.3–2.8, we get the desired result. □

Corollary 3.1. *Let* $\mu=1/[2]_{p,q}$. *Then* Theorem 3.1 *leads to*

$$\left|\gamma\frac{qg(\phi)+pg(\psi)}{[2]_{p,q}}+(1-\gamma)g\left(\frac{q\phi+p\psi}{[2]_{p,q}}\right)-\frac{1}{p(\psi-\phi)}\int_\phi^{p\psi+(1-p)\phi}g(x)\,{}_\phi d_{p,q}x\right|$$
$$\le\min\left[H_1\left(\gamma,\frac{p}{[2]_{p,q}},\alpha,m\right),H_2\left(\gamma,\frac{p}{[2]_{p,q}},\alpha,m\right)\right].$$

Remark 3.1.

(1) Let $\gamma=0$. Then Corollary 3.1 gives the midpoint-type integral inequality

$$\left|g\left(\frac{q\phi+p\psi}{[2]_{p,q}}\right)-\frac{1}{p(\psi-\phi)}\int_\phi^{p\psi+(1-p)\phi}g(x)\,{}_\phi d_{p,q}x\right|\le\min\left[H_1\left(0,\frac{p}{[2]_{p,q}},\alpha,m\right),H_2\left(0,\frac{p}{[2]_{p,q}},\alpha,m\right)\right],$$

where

$$H_1\left(0,\frac{p}{[2]_{p,q}},\alpha,m\right)=(\phi-\psi)\left[\frac{[([2]_{p,q})^{\alpha+2}-(p^{\alpha+2}+q^{\alpha+2})](p-q)^2}{([2]_{p,q})^{\alpha+2}(p^{\alpha+1}-q^{\alpha+1})(p^{\alpha+2}-q^{\alpha+2})}|{}_\phi D_{p,q}g(\psi)|\right.$$
$$\left.+m\left[\frac{2qp^2}{([2]_{p,q})^3}-\frac{[([2]_{p,q})^{\alpha+2}-(p^{\alpha+2}+q^{\alpha+2})](p-q)^2}{([2]_{p,q})^{\alpha+2}(p^{\alpha+1}-q^{\alpha+1})(p^{\alpha+2}-q^{\alpha+2})}\right]\left|{}_\phi D_{p,q}g\left(\frac{\phi}{m}\right)\right|\right],$$

$$H_2\left(0, \frac{p}{[2]_{p,q}}, \alpha, m\right) = (\psi - \phi)\left\{\left[\nu_2\left(0, \frac{p}{[2]_{p,q}}, \alpha\right) + \nu_4\left(0, \frac{p}{[2]_{p,q}}, \alpha\right) - \nu_6\left(0, \frac{p}{[2]_{p,q}}, \alpha\right)\right]|{}_\phi D_{p,q}g(\phi)| + m\left[\frac{2qp^2}{([2]_{p,q})^3} - \nu_2\left(0, \frac{p}{[2]_{p,q}}, \alpha\right) - \nu_4\left(0, \frac{p}{[2]_{p,q}}, \alpha\right) + \nu_6\left(0, \frac{p}{[2]_{p,q}}, \alpha\right)\right]\left|{}_\phi D_{p,q}g\left(\frac{\psi}{m}\right)\right|\right\}.$$

In particular, if $\alpha = 1 = m$, then we obtain

$$\left|g\left(\frac{q\phi + p\psi}{[2]_{p,q}}\right) - \frac{1}{p(\psi - \phi)}\int_\phi^{p\psi+(1-p)\phi} g(x)_\phi d_{p,q}x\right| \le (\psi - \phi)q\left[\frac{3p^3}{([2]_{p,q})^3(p^2 + pq + q^2)}|{}_\phi D_{p,q}g(\psi)| + \frac{2p^4 + 2p^2q^2 + 2p^3q - 3p^3}{([2]_{p,q})^3(p^2 + pq + q^2)}|{}_\phi D_{p,q}g(\phi)|\right], \tag{3.1}$$

which was proposed by Kunt et al. in [61].

Let $p = 1$. Then equation (3.1) becomes Theorem 13 of [65].

(2) Taking $\gamma = 1/3$ and $\alpha = 1 = m$ in Corollary 3.1, we get the Simpson-type integral inequality

$$\left|\frac{1}{3}\left[\frac{qg(\phi) + pg(\psi)}{[2]_{p,q}} + 2g\left(\frac{q\phi + p\psi}{[2]_{p,q}}\right)\right] - \frac{1}{p(\psi - \phi)}\int_\phi^{p\psi+(1-p)\phi} g(x)_\phi d_{p,q}x\right| \le \min\left[H_1\left(\frac{1}{3}, \frac{p}{[2]_{p,q}}, 1, 1\right), H_2\left(\frac{1}{3}, \frac{p}{[2]_{p,q}}, 1, 1\right)\right].$$

If $q \to 1^-$ and $p = 1$, then we obtain

$$\left|\frac{1}{3}\left[\frac{g(\phi) + g(\psi)}{2} + 2g\left(\frac{\phi + \psi}{2}\right)\right] - \frac{1}{(\psi - \phi)}\int_a^\psi g(x)\mathrm{d}x\right| \le \frac{5(\psi - \phi)}{72}[|g'(\psi)| + |g'(\phi)|],$$

which was proposed by Alomari et al. in [67].

(3) Let $\gamma = 1/2$ and $\alpha = 1 = m$. Then we get the average of midpoint and trapezoid-type integral inequality

$$\left|\frac{1}{2}\left[\frac{qg(\phi) + pg(\psi)}{[2]_{p,q}} + g\left(\frac{q\phi + p\psi}{[2]_{p,q}}\right)\right] - \frac{1}{p(\psi - \phi)}\int_\phi^{p\psi+(1-p)\phi} g(x)_\phi d_{p,q}x\right| \le \min\left[H_1\left(\frac{1}{2}, \frac{p}{[2]_{p,q}}, 1, 1\right), H_2\left(\frac{1}{2}, \frac{p}{[2]_{p,q}}, 1, 1\right)\right].$$

If $q \to 1^-$ and $p = 1$, then we obtain

$$\left|\frac{1}{2}\left[\frac{g(\phi) + g(\psi)}{2} + g\left(\frac{\phi + \psi}{2}\right)\right] - \frac{1}{(\psi - \phi)}\int_\phi^\psi g(x)\mathrm{d}x\right| \le \frac{\psi - \phi}{16}[|g'(\psi)\mathrm{c}| + |g'(\phi)|],$$

which was proposed by Xi et al. in [68].

(4) Let $\gamma = 1$. Then we get the trapezoid-type integral inequality

$$\left|\frac{qg(\phi) + pg(\psi)}{[2]_{p,q}} - \frac{1}{p(\psi - \phi)}\int_\phi^{p\psi+(1-p)\phi} g(x)_\phi d_{p,q}x\right| \le \min\left[H_1\left(1, \frac{p}{[2]_{p,q}}, \alpha, m\right), H_2\left(1, \frac{p}{[2]_{p,q}}, \alpha, m\right)\right].$$

In particular, if $\alpha = p = m = 1$, then we obtain

$$\left|\frac{qg(\phi)+g(\psi)}{[2]_q} - \frac{1}{\psi-\phi}\int_{\phi}^{\psi} g(x)_\phi d_q x\right| \leq (\psi-\phi)q^2\left[\left[\frac{1+4q+q^2}{([2]_q)^4[3]_q}|_\phi D_q g(\psi)| + \frac{1+3q^2+2q^3}{([2]_q)^4[3]_q}{}_\phi D_q g(\phi)|\right]\right],$$

which was proposed by Sudsutad et al. in [55].

Theorem 3.2. *Let* $0 \leq \phi < \psi < \infty$, $r > 1$, $\alpha, m \in (0,1]$, $0 < q < p \leq 1$ *and* $g : J \supset [0,\infty) \to \mathbb{R}$ *be a* (p,q)*-differentiable function on* J° (*the interior of* J) *such that* ${}_\phi D_{p,q} g$ *is continuous and integrable on* $\left[0, \frac{\psi}{m}\right]$ *and* $|_\phi D_{p,q} g|^r$ *is* (α,m)*-convex on* $\left[0, \frac{\psi}{m}\right]$. *Then the inequality*

$$\left|\gamma[p\mu g(\psi) + (1-p\mu)g(\phi)] + (1-\gamma)g(p\mu\psi + (1-p\mu)\phi) - \frac{1}{p(\psi-\phi)}\int_{\phi}^{p\psi+(1-p)\phi} g(x)_\phi d_{p,q}x\right|$$
$$\leq (\psi-\phi)\min[T_1(\gamma, p\mu, \alpha, m, r), T_2(\gamma, p\mu, \alpha, m, r)]$$

holds for all $\gamma, \mu \in [0,1]$, *where*

$$T_1(\gamma, p\mu, \alpha, m, r) = v_8^{1-\frac{1}{r}}(\gamma, p\mu)\left[v_3(\gamma, p\mu, \alpha)|_\phi D_{p,q} g(\psi)|^r + m(v_8(\gamma, p\mu) - v_3(\gamma, p\mu, \alpha))\left|{}_\phi D_{p,q} g\left(\frac{\phi}{m}\right)\right|^r\right]^{\frac{1}{r}}$$
$$+ (1-\gamma)(p\mu)^{1-\frac{1}{r}}\left[\Gamma_1(p\mu, \alpha)|_\phi D_{p,q} g(\psi)|^r + m\Gamma_2(p\mu, \alpha)\left|{}_\phi D_{p,q} g\left(\frac{\phi}{m}\right)\right|^r\right]^{\frac{1}{r}},$$

$$T_2(\gamma, p\mu, \alpha, m, r) = v_8^{1-\frac{1}{r}}(\gamma, p\mu)[v_4(\gamma, p\mu, \alpha)|_\phi D_{p,q} g(\phi)|^r + m(v_8(\gamma, p\mu) - v_4(\gamma, p\mu, \alpha))\left|{}_\phi D_{p,q} g\left(\frac{\psi}{m}\right)\right|^r]^{\frac{1}{r}}$$
$$+ (p\mu)^{1-\frac{1}{r}}(1-\gamma)\left[\Gamma_3(p\mu, \alpha)|_\phi D_{p,q} g(\phi)|^r + m\Gamma_4(p\mu, \alpha)\left|{}_\phi D_{p,q} g\left(\frac{\psi}{m}\right)\right|^r\right]^{\frac{1}{r}}$$

and

$$\Gamma_1(p\mu, \alpha) = \int_0^{p\mu} \lambda^\alpha{}_0 d_{p,q}\lambda = \frac{\mu^{\alpha+1}(p-q)p^{\alpha+1}}{p^{\alpha+1} - q^{\alpha+1}}, \tag{3.2}$$

$$\Gamma_2(p\mu, \alpha) = \int_0^{p\mu} (1-\lambda^\alpha)_0 d_{p,q}\lambda = p\mu - \frac{\mu^{\alpha+1}(p-q)p^{\alpha+1}}{p^{\alpha+1} - q^{\alpha+1}}, \tag{3.3}$$

$$\Gamma_3(p\mu, \alpha) = \int_0^{p\mu} (1-\lambda)^\alpha{}_0 d_{p,q}\lambda = (p-q)\mu\sum_{n=0}^{\infty}\frac{q^n}{p^n}\left(1 - \frac{q^n}{p^n}\mu\right)^\alpha, \tag{3.4}$$

$$\Gamma_4(p\mu, \alpha) = \int_0^{p\mu} (1-(1-\lambda)^\alpha)_0 d_{p,q}\lambda = p\mu - (p-q)\mu\sum_{n=0}^{\infty}\frac{q^n}{p^n}\left(1 - \frac{q^n}{p^n}\mu\right)^\alpha \tag{3.5}$$

and $v_3(\gamma, p\mu, \alpha)$, $v_4(\gamma, p\mu, \alpha)$ *and* $v_8(\gamma, p\mu)$ *are defined in* Lemmas 2.4 and 2.7, *respectively.*

Proof. Using Lemma 2.1 and the power mean inequality, we have

$$\left|\gamma[p\mu g(\psi)+(1-p\mu)g(\phi)]+(1-\gamma)g(p\mu\psi+(1-p\mu)\phi)-\frac{1}{p(\psi-\phi)}\int_{\phi}^{p\psi+(1-p)\phi} g(x)_{\phi}d_{p,q}x\right|$$
$$\leq(\psi-\phi)\left\{\left(\int_0^1|q\lambda-(1-\gamma p\mu)|_0d_{p,q}\lambda\right)^{1-\frac{1}{r}}\times\left(\int_0^1|q\lambda-(1-\gamma p\mu)||_{\phi}D_{p,q}g(\lambda\psi+(1-\lambda)\phi)|^r{}_0d_{p,q}\lambda\right)^{\frac{1}{r}}\right.$$
$$\left.+(1-\gamma)\left(\int_0^{p\mu}1_0d_{p,q}\lambda\right)^{1-\frac{1}{r}}\left(\int_0^{p\mu}|_{\phi}D_{p,q}g(\lambda\psi+(1-\lambda)\phi)|^r{}_0d_{p,q}\lambda\right)^{\frac{1}{r}}\right\}. \tag{3.6}$$

Utilizing the (α, m)-convexity of $|_{\phi}D_{p,q}g|^r$, we get

$$\begin{aligned}\int_0^1&|q\lambda-(1-\gamma p\mu)||_{\phi}D_{p,q}g(\lambda\psi+(1-\lambda)\phi)|^r{}_0d_{p,q}\lambda\\&\leq\int_0^1|q\lambda-(1-\gamma p\mu)|\left[\lambda^{\alpha}|_{\phi}D_{p,q}g(\psi)|^r+m(1-\lambda^{\alpha})\left|_{\phi}D_{p,q}g\left(\frac{\phi}{m}\right)\right|^r\right]{}_0d_{p,q}\lambda\\&=\left(\int_0^1\lambda^{\alpha}|q\lambda-(1-\gamma p\mu)|_0d_{p,q}\lambda\right)|_{\phi}D_{p,q}g(\psi)|^r\\&\quad+m\left(\int_0^1|q\lambda-(1-\gamma p\mu)|_0d_{p,q}\lambda-\int_0^1\lambda^{\alpha}|q\lambda-(1-\gamma p\mu)|_0d_{p,q}\lambda\right)\left|_{\phi}D_{p,q}g\left(\frac{\phi}{m}\right)\right|^r,\end{aligned} \tag{3.7}$$

$$\begin{aligned}\int_0^{p\mu}&|_{\phi}D_{p,q}g(\lambda\psi+(1-\lambda)\phi)|^r{}_0d_{p,q}\lambda\\&\leq\int_0^{p\mu}\left[\lambda^{\alpha}|_{\phi}D_{p,q}g(\psi)|^r+m(1-\lambda^{\alpha})\left|_{\phi}D_{p,q}g\left(\frac{\phi}{m}\right)\right|^r\right]{}_0d_{p,q}\lambda\\&=\left(\int_0^{p\mu}\lambda^{\alpha}{}_0d_{p,q}\lambda\right)|_{\phi}D_{p,q}g(\psi)|^r+m\left(\int_0^{p\mu}(1-\lambda^{\alpha})_0d_{p,q}\lambda\right)\left|_{\phi}D_{p,q}g\left(\frac{\phi}{m}\right)\right|^r.\end{aligned} \tag{3.8}$$

Using (3.7) and (3.8) in (3.6), we get

$$\begin{aligned}&\left|\gamma[p\mu g(\psi)+(1-p\mu)g(\phi)]+(1-\gamma)g(p\mu\psi+(1-p\mu)\phi)-\frac{1}{p(\psi-\phi)}\int_{\phi}^{p\psi+(1-p)\phi} g(x)_{\phi}d_{p,q}x\right|\\&\leq(\psi-\phi)\left\{\left[\int_0^1|q\lambda-(1-\gamma p\mu)|_0d_{p,q}\lambda\right]^{1-\frac{1}{r}}\left[\left(\int_0^1\lambda^{\alpha}|q\lambda-(1-\gamma p\mu)|_0d_{p,q}\lambda\right)|_{\phi}D_{p,q}g(\psi)|^r\right.\right.\\&\quad\left.+m\left(\int_0^1|q\lambda-(1-\gamma p\mu)|_0d_{p,q}\lambda-\int_0^1\lambda^{\alpha}|q\lambda-(1-\gamma p\mu)|_0d_{p,q}\lambda\right)\left|_{\phi}D_{p,q}g\left(\frac{\phi}{m}\right)\right|^r\right]^{\frac{1}{r}}\\&\quad\left.+(1-\gamma)(p\mu)^{1-\frac{1}{r}}\left[\left(\int_0^{p\mu}\lambda^{\alpha}{}_0d_{p,q}\lambda\right)|_{\phi}D_{p,q}g(\psi)|^r+m\left(\int_0^{p\mu}(1-\lambda^{\alpha})_0d_{p,q}\lambda\right)\left|_{\phi}D_{p,q}g\left(\frac{\phi}{m}\right)\right|^r\right]^{\frac{1}{r}}\right\}.\end{aligned} \tag{3.9}$$

Similarly, we get

$$\begin{aligned}
&\int_0^1 |q\lambda - (1-\gamma p\mu)||{}_\phi D_{p,q}g(\lambda\psi + (1-\lambda)\phi)|^r{}_0d_{p,q}\lambda \\
&\quad \le \int_0^1 |q\lambda - (1-\gamma p\mu)|\left[(1-\lambda)^\alpha|{}_\phi D_{p,q}g(\phi)|^r + m(1-(1-\lambda)^\alpha)\left|{}_\phi D_{p,q}g\left(\frac{\psi}{m}\right)\right|^r\right]{}_0d_{p,q}\lambda \\
&\quad = \left(\int_0^1 (1-\lambda)^\alpha|q\lambda - (1-\gamma p\mu)|{}_0d_{p,q}\lambda\right)|{}_\phi D_{p,q}g(\phi)|^r \\
&\qquad + m\left(\int_0^1 |q\lambda - (1-\gamma p\mu)|{}_0d_{p,q}\lambda - \int_0^1 (1-\lambda)^\alpha|q\lambda - (1-\gamma p\mu)|{}_0d_{p,q}\lambda\right)\left|{}_\phi D_{p,q}g\left(\frac{\psi}{m}\right)\right|^r.
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
&\int_0^{p\mu} |{}_\phi D_{p,q}g(\lambda\psi + (1-\lambda)\phi)|^r{}_0d_{p,q}\lambda \\
&\quad \le \int_0^{p\mu}\left[(1-\lambda)^\alpha|{}_\phi D_{p,q}g(\phi)|^r + m(1-(1-\lambda)^\alpha)\left|{}_\phi D_{p,q}g\left(\frac{\psi}{m}\right)D_{p,q}g\left(\frac{\psi}{m}\right)\right|^r\right]{}_0d_{p,q}\lambda \\
&\quad = \left(\int_0^{p\mu}(1-\lambda)^\alpha{}_0d_{p,q}\lambda\right)|{}_\phi D_{p,q}g(\phi)|^r + m\left(\int_0^{p\mu}(1-(1-\lambda)^\alpha){}_0d_{p,q}\lambda\right)\left|{}_\phi D_{p,q}g\left(\frac{\psi}{m}\right)\right|^r.
\end{aligned} \tag{3.11}$$

Using (3.10) and (3.11) in (3.6), we get

$$\begin{aligned}
&\left|\gamma[p\mu g(\psi) + (1-p\mu)g(\phi)] + (1-\gamma)g(p\mu b + (1-p\mu)a) - \frac{1}{p(\psi-\phi)}\int_\phi^{p\psi+(1-p)\phi} g(x){}_\phi d_{p,q}x\right| \\
&\le (\psi-\phi)\left\{\left[\int_0^1 |q\lambda-(1-\gamma p\mu)|{}_0d_{p,q}\lambda\right]^{1-\frac{1}{r}}\left[\left(\int_0^1(1-\lambda)^\alpha|q\lambda-(1-\gamma p\mu)|{}_0d_{p,q}\lambda\right)|{}_\phi D_{p,q}g(\phi)|^r\right.\right. \\
&\quad \left. + m\left(\int_0^1 |q\lambda-(1-\gamma p\mu)|{}_0d_{p,q}\lambda - \int_0^1(1-\lambda)^\alpha|q\lambda-(1-\gamma p\mu)|{}_0d_{p,q}\lambda\right)\left|{}_\phi D_{p,q}g\left(\frac{\psi}{m}\right)\right|^r\right]^{\frac{1}{r}} \\
&\quad \left. + (1-\gamma)(p\mu)^{1-\frac{1}{r}}\left[\left(\int_0^{p\mu}(1-\lambda)^\alpha{}_0d_{p,q}\lambda\right)|{}_\phi D_{p,q}g(\phi)|^r + m\left(\int_0^{p\mu}(1-(1-\lambda)^\alpha){}_0d_{p,q}\lambda\right)\left|{}_\phi D_{p,q}g\left(\frac{\psi}{m}\right)\right|^r\right]^{\frac{1}{r}}\right\}.
\end{aligned} \tag{3.12}$$

Therefore, the desired result follows from (3.9) and (3.12) together with Lemmas 2.4 and 2.7. □

Theorem 3.3. *Let $0 \le \phi < \psi < \infty$, $0 < q < p \le 1$, $r, s > 1$ with $r^{-1} + s^{-1} = 1$, $\alpha, m \in (0, 1]$ and $g : J \supset [0, \infty) \to \mathbb{R}$ be a (p, q)-differentiable function on J° (the interior of J) such that ${}_\phi D_{p,q}g$ is continuous and integrable on $\left[0, \frac{\psi}{m}\right]$ and $|{}_\phi D_{p,q}g|^r$ is (α, m)-convex on $\left[0, \frac{\psi}{m}\right]$. Then the inequality*

$$\begin{aligned}
&\left|\gamma[p\mu g(\psi) + (1-p\mu)g(\phi)] + (1-\gamma)g(p\mu\psi + (1-p\mu)\phi) - \frac{1}{p(\psi-\phi)}\int_\phi^{p\psi+(1-p)\phi} g(x){}_\phi d_{p,q}x\right| \\
&\quad \le (\psi-\phi)\min[K_1(\gamma, p\mu, \alpha, m), K_2(\gamma, p\mu, \alpha, m)]
\end{aligned}$$

holds for all $\gamma, \mu \in [0, 1]$, where

$$K_1(\gamma, p\mu, \alpha, m) = v_{10}^{\frac{1}{s}}(\gamma, p\mu)\left[\eta_2(\alpha)\,|{}_{\phi}D_{p,q}g(\psi)|^r + m(1-\eta_2(\alpha))\left|{}_{\phi}D_{p,q}g\left(\frac{\phi}{m}\right)\right|^r\right]^{\frac{1}{r}}$$
$$+ (1-\gamma)(p\mu)^{\frac{1}{s}}\left[\Gamma_1(p\mu,\alpha)\,|{}_{\phi}D_{p,q}g(\psi)|^r + m\Gamma_2(p\mu,\alpha)\left|{}_{\phi}D_{p,q}g\left(\frac{\phi}{m}\right)\right|^r\right]^{\frac{1}{r}},$$

$$K_2(\gamma, p\mu, \alpha, m) = v_{10}^{\frac{1}{s}}(\gamma, p\mu)\left[\eta_3(\alpha)\,|{}_{\phi}D_{p,q}g(\phi)|^r + m(1-\eta_3(\alpha))\left|{}_{\phi}D_{p,q}g\left(\frac{\psi}{m}\right)\right|^r\right]^{\frac{1}{r}}$$
$$+ (1-\gamma)(p\mu)^{\frac{1}{s}}\left[\Gamma_3(p\mu,\alpha)\,|{}_{\phi}D_{p,q}g(\phi)|^r + m\Gamma_4(p\mu,\alpha)\left|{}_{\phi}D_{p,q}g\left(\frac{\psi}{m}\right)\right|^r\right]^{\frac{1}{r}},$$

$$\eta_2(\alpha) = \int_0^1 \lambda^{\alpha}\,{}_0d_{p,q}\lambda = \frac{p-q}{p^{1+\alpha}-q^{1+\alpha}},$$
$$\eta_3(\alpha) = \int_0^1 (1-\lambda)^{\alpha}\,{}_0d_{p,q}\lambda = (p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(1-\frac{q^n}{p^{n+1}}\right)^{\alpha}$$

and $\Gamma_1(p\mu,\alpha)$, $\Gamma_2(p\mu,\alpha)$, $\Gamma_3(p\mu,\alpha)$ *and* $\Gamma_4(p\mu,\alpha)$ *are defined in* Theorem 3.2.

Proof. Using Lemma 2.1 and the Hölder inequality, we have

$$\left|\gamma[p\mu g(\psi) + (1-p\mu)g(\phi)] + (1-\gamma)g(p\mu\psi + (1-p\mu)\phi) - \frac{1}{p(\psi-\phi)}\int_{\phi}^{p\psi+(1-p)\phi} g(x)\,{}_{\phi}d_{p,q}x\right|$$
$$\le (\psi-\phi)\left\{\left(\int_0^1 |q\lambda-(1-\gamma p\mu)|^s\,{}_0d_{p,q}\lambda\right)^{\frac{1}{s}}\left(\int_0^1 |{}_{\phi}D_{p,q}g(\lambda\psi+(1-\lambda)\phi)|^r\,{}_0d_{p,q}\lambda\right)^{\frac{1}{r}}\right. \tag{3.13}$$
$$\left. + (1-\gamma)\left(\int_0^{p\mu} 1^s\,{}_0d_{p,q}\lambda\right)^{\frac{1}{s}}\left(\int_0^{p\mu} |{}_{\phi}D_{p,q}g(\lambda\psi+(1-\lambda)\phi)|^r\,{}_0d_{p,q}\lambda\right)^{\frac{1}{r}}\right\}.$$

Utilizing the (α, m)-convexity of $|{}_{\phi}D_{p,q}g|^r$, we get

$$\int_0^1 |{}_{\phi}D_{p,q}g(\lambda\psi+(1-\lambda)\phi)|^r\,{}_0d_{p,q}\lambda \le \int_0^1 \left[\lambda^{\alpha}|{}_{\phi}D_{p,q}g(\psi)|^r + m(1-\lambda^{\alpha})\left|{}_{\phi}D_{p,q}g\left(\frac{\phi}{m}\right)\right|^r\right]{}_0d_{p,q}\lambda$$
$$= \left(\int_0^1 \lambda^{\alpha}\,{}_0d_{p,q}\lambda\right)|{}_{\phi}D_{p,q}g(\psi)|^r + m\left(\int_0^1 (1-\lambda^{\alpha})\,{}_0d_{p,q}\lambda\right)\left|{}_{\phi}D_{p,q}g\left(\frac{\phi}{m}\right)\right|^r \tag{3.14}$$

and

$$\int_0^{p\mu} |{}_{\phi}D_{p,q}g(\lambda\psi+(1-\lambda)\phi)|^r\,{}_0d_{p,q}\lambda \le \int_0^{p\mu} \left[\lambda^{\alpha}\,|{}_{\phi}D_{p,q}g(\psi)|^r + m(1-\lambda^{\alpha})\left|{}_{\phi}D_{p,q}g\left(\frac{\phi}{m}\right)\right|^r\right]{}_0d_{p,q}\lambda$$
$$= \left(\int_0^{p\mu} \lambda^{\alpha}\,{}_0d_{p,q}\lambda\right)|{}_{\phi}D_{p,q}g(\psi)|^r + m\left(\int_0^{p\mu} (1-\lambda^{\alpha})\,{}_0d_{p,q}\lambda\right)\left|{}_{\phi}D_{p,q}g\left(\frac{\phi}{m}\right)\right|^r. \tag{3.15}$$

Using (3.14) and (3.15) in (3.13), we get

$$\left|\gamma[p\mu g(\psi)+(1-p\mu)g(\phi)]+(1-\gamma)g(p\mu\psi+(1-p\mu)\phi)-\frac{1}{p(\psi-\phi)}\int_{\phi}^{p\psi+(1-p)\phi}g(x)_{\phi}d_{p,q}x\right|$$
$$\leq(\psi-\phi)\left\{\left[\int_0^1|q\lambda-(1-\gamma p\mu)|^s{}_0d_{p,q}\lambda\right]^{\frac{1}{s}}\left[\left(\int_0^1\lambda^{\alpha}{}_0d_{p,q}\lambda\right)|_{\phi}D_{p,q}g(\psi)|^r+m\left(\int_0^1(1-\lambda^{\alpha})_0d_{p,q}\lambda\right)\left|_{\phi}D_{p,q}g\left(\frac{\phi}{m}\right)\right|^r\right]^{\frac{1}{r}}\right.$$
$$\left.+(1-\gamma)(p\mu)^{\frac{1}{s}}\left[\left(\int_0^{p\mu}\lambda^{\alpha}{}_0d_{p,q}\lambda\right)|_{\phi}D_{p,q}g(\psi)|^r+m\left(\int_0^{p\mu}(1-\lambda^{\alpha})_0d_{p,q}\lambda\right)\left|_{\phi}D_{p,q}g\left(\frac{\phi}{m}\right)\right|^r\right]^{\frac{1}{r}}\right\}. \tag{3.16}$$

Similarly, we get

$$\int_0^1|\phi D_{p,q}g(\lambda\psi+(1-\lambda)\phi)|^r{}_0d_{p,q}\lambda$$
$$\leq\int_0^1\left[(1-\lambda)^{\alpha}|\phi D_{p,q}g(\phi)|^r+m(1-(1-\lambda)^{\alpha})\left|\phi D_{p,q}g\left(\frac{\psi}{m}\right)\right|^r\right]{}_0d_{p,q}\lambda \tag{3.17}$$
$$=\left(\int_0^1(1-\lambda)^{\alpha}{}_0d_{p,q}\lambda\right)|\phi D_{p,q}g(\phi)|^r+m\left(\int_0^1(1-(1-\lambda)^{\alpha})_0d_{p,q}\lambda\right)\left|\phi D_{p,q}g\left(\frac{\psi}{m}\right)\right|^r$$

and

$$\int_0^{p\mu}|_{\phi}D_{p,q}g(\lambda\psi+(1-\lambda)\phi)|^r{}_0d_{p,q}\lambda$$
$$\leq\int_0^{p\mu}\left[(1-\lambda)^{\alpha}|_{\phi}D_{p,q}g(\phi)|^r+m(1-(1-\lambda)^{\alpha})\left|_{\phi}D_{p,q}f\left(\frac{\psi}{m}\right)\right|^r\right]{}_0d_{p,q}\lambda \tag{3.18}$$
$$=\left(\int_0^{p\mu}(1-\lambda)^{\alpha}{}_0d_{p,q}\lambda\right)|_{\phi}D_{p,q}g(\phi)|^r+m\left(\int_0^{p\mu}(1-(1-\lambda)^{\alpha})_0d_{p,q}\lambda\right)\left|_{\phi}D_{p,q}g\left(\frac{\psi}{m}\right)\right|^r.$$

Using (3.17) and (3.18) in (3.13), we get

$$\left|\gamma[\mu f(\psi)+(1-\mu)g(\phi)]+(1-\gamma)g(p\mu\psi+(1-p\mu)\phi)-\frac{1}{p(\psi-\phi)}\int_{\phi}^{p\psi+(1-p)\phi}g(x)_{\phi}d_{p,q}x\right|$$
$$\leq(\psi-\phi)\left\{\left[\int_0^1|q\lambda-(1-\gamma p\mu)|^s{}_0d_{p,q}\lambda\right]^{\frac{1}{s}}\left[\left(\int_0^1(1-\lambda)^{\alpha}{}_0d_{p,q}\lambda\right)|_{\phi}D_{p,q}g(\phi)|^r\right.\right.$$
$$\left.+m\left(\int_0^1(1-(1-\lambda)^{\alpha})_0d_{p,q}\lambda\right)\left|_{\phi}D_{p,q}g\left(\frac{\psi}{m}\right)\right|^r\right]^{\frac{1}{r}} \tag{3.19}$$
$$\left.+(1-\gamma)(p\mu)^{\frac{1}{s}}\left[\left(\int_0^{p\mu}(1-\lambda)^{\alpha}{}_0d_{p,q}\lambda\right)|_{\phi}D_{p,q}g(\phi)|^r+m\left(\int_0^{p\mu}(1-(1-\lambda)^{\alpha})_0d_{p,q}\lambda\right)\left|_{\phi}D_{p,q}g\left(\frac{\psi}{m}\right)\right|^r\right]^{\frac{1}{r}}\right\}.$$

Therefore, Theorem 3.3 follows from (3.2)–(3.5), (3.16) and (3.19) together with Lemmas 2.2 and 2.9. □

Remark 3.2. If we put $\gamma=0,\frac{1}{3},\frac{1}{2},1$ and $\mu=\frac{1}{[2]_{p,q}}$ in Theorems 3.2 and 3.3, then we can get the midpoint-type integral inequality, the Simpson-type integral inequality, average of midpoint and trapezoid-type integral inequality and the trapezoid-type integral inequality, respectively.

Next, we establish the (p, q)-integral inequalities involving the product of two (α, m)-convex functions.

Theorem 3.4. *Let* $0 \le \phi < \psi < \infty$, $0 < q < p \le 1$, $\alpha_1, \alpha_2, m \in (0, 1]$ *and* $f, g : J \supset [0, \infty) \to \mathbb{R}$ *be continuous and integrable on* $\left[0, \frac{\psi}{m}\right]$ *such that* f *and* g *are* (α_1, m)*-convex and* (α_2, m)*-convex on* $\left[0, \frac{\psi}{m}\right]$, *respectively. Then the inequality*

$$\frac{1}{\psi - \phi} \int_{\phi}^{\psi} f(x)g(x)\,{}_{\phi}d_{p,q}x \le \min\{L_1(\alpha_1, \alpha_2, m), L_2(\alpha_1, \alpha_2, m)\}$$

holds, where

$$\begin{aligned} L_1(\alpha_1, \alpha_2, m) = m^2&\left[\frac{p - q}{p^{\alpha_1+\alpha_2+1} - q^{\alpha_1+\alpha_2+1}} - \frac{p - q}{p^{\alpha_1+1} - q^{\alpha_1+1}} - \frac{p - q}{p^{\alpha_2+1} - q^{\alpha_2+1}} + 1\right] f\left(\frac{\phi}{m}\right) g\left(\frac{\phi}{m}\right) \\ &+ \frac{p - q}{p^{\alpha_1+\alpha_2+1} - q^{\alpha_1+\alpha_2+1}} f(\psi)g(\psi) + m\left[\frac{p - q}{p^{\alpha_2+1} - q^{\alpha_2+1}} - \frac{p - q}{p^{\alpha_1+\alpha_2+1} - q^{\alpha_1+\alpha_2+1}}\right] f\left(\frac{\phi}{m}\right) g(\psi) \\ &+ m\left[\frac{p - q}{p^{\alpha_1+1} - q^{\alpha_1+1}} - \frac{p - q}{p^{\alpha_1+\alpha_2+1} - q^{\alpha_1+\alpha_2+1}}\right] f(\psi) g\left(\frac{\phi}{m}\right), \end{aligned}$$

$$\begin{aligned} L_2(\alpha_1, \alpha_2, m) = m^2&[\Lambda(\alpha_1, \alpha_2) - \Lambda(\alpha_1) - \Lambda(\alpha_2) + 1] f\left(\frac{\psi}{m}\right) g\left(\frac{\psi}{m}\right) \\ &+ \Lambda(\alpha_1, \alpha_2) f(\phi)g(\phi) + m[\Lambda(\alpha_1) - \Lambda(\alpha_1, \alpha_2)]\, f(\phi) g\left(\frac{\psi}{m}\right) \\ &+ m[\Lambda(\alpha_2) - \Lambda(\alpha_1, \alpha_2)] f\left(\frac{\psi}{m}\right) g(\phi), \end{aligned}$$

$$\Lambda(\alpha_1, \alpha_2) = \int_0^1 (1 - \lambda)^{\alpha_1+\alpha_2}\,{}_0d_{p,q}\lambda = (p - q) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left(1 - \frac{q^n}{p^{n+1}}\right)^{\alpha_1+\alpha_2}$$

and

$$\Lambda(\alpha_i) = \int_0^1 (1 - \lambda)^{\alpha_i}\,{}_0d_{p,q}\lambda = (p - q) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left(1 - \frac{q^n}{p^{n+1}}\right)^{\alpha_i} \quad (i = 1, 2).$$

Proof. Let $\lambda \in [0, 1]$. Then it follows from the (α_1, m)-convexity of f and the (α_2, m)-convexity of g that

$$f(\lambda\psi + (1 - \lambda)\phi) \le \lambda^{\alpha_1} f(\psi) + m(1 - \lambda^{\alpha_1}) f\left(\frac{\phi}{m}\right) \tag{3.20}$$

and

$$g(\lambda\psi + (1 - \lambda)\phi) \le \lambda^{\alpha_2} g(\psi) + m(1 - \lambda^{\alpha_2}) g\left(\frac{\phi}{m}\right). \tag{3.21}$$

Multiplying (3.20) with (3.21), we get

$$\begin{aligned} f(\lambda\psi + (1 - \lambda)\phi)g(\lambda\psi + (1 - \lambda)\phi) \le \lambda^{\alpha_1+\alpha_2} f(\psi)g(\psi) + m^2(1 - \lambda^{\alpha_1})(1 - \lambda^{\alpha_2}) f\left(\frac{\phi}{m}\right) g\left(\frac{\phi}{m}\right) \\ + m\lambda^{\alpha_2}(1 - \lambda^{\alpha_1}) f\left(\frac{\phi}{m}\right) g(\psi) + m\lambda^{\alpha_1}(1 - \lambda^{\alpha_2}) f(\psi) g\left(\frac{\phi}{m}\right). \end{aligned} \tag{3.22}$$

Taking the (p, q)-integral for (3.22) with respect to λ on $(0, 1)$ and by using Lemma 2.2, we get

$$\begin{aligned}\int_0^1 & f(\lambda\psi + (1-\lambda)\phi)g(\lambda\psi + (1-\lambda)\phi)_0 d_{p,q}\lambda \\ &\leq m^2\left[\frac{p-q}{p^{\alpha_1+\alpha_2+1}-q^{\alpha_1+\alpha_2+1}} - \frac{p-q}{p^{\alpha_1+1}-q^{\alpha_1+1}} - \frac{p-q}{p^{\alpha_2+1}-q^{\alpha_2+1}} + 1\right] f\left(\frac{\phi}{m}\right) g\left(\frac{\phi}{m}\right) + \frac{p-q}{p^{\alpha_1+\alpha_2+1}-q^{\alpha_1+\alpha_2+1}} f(\psi)g(\psi) \\ &+ m\left[\frac{p-q}{p^{\alpha_2+1}-q^{\alpha_2+1}} - \frac{p-q}{p^{\alpha_1+\alpha_2+1}-q^{\alpha_1+\alpha_2+1}}\right] f\left(\frac{\phi}{m}\right) g(\psi) + m\left[\frac{p-q}{p^{\alpha_1+1}-q^{\alpha_1+1}} - \frac{p-q}{p^{\alpha_1+\alpha_2+1}-q^{\alpha_1+\alpha_2+1}}\right] f(\psi) g\left(\frac{\phi}{m}\right).\end{aligned} \tag{3.23}$$

Similarly, we have

$$\begin{aligned}\int_0^1 & f(\lambda\psi + (1-\lambda)\phi)g(\lambda\psi + (1-\lambda)\phi)_0 d_{p,q}\lambda \\ &\leq m^2\left(\int_0^1 (1-\lambda)^{\alpha_1+\alpha_2}\,{}_0d_{p,q}\lambda - \int_0^1 (1-\lambda)^{\alpha_1}\,{}_0d_{p,q}\lambda - \int_0^1 (1-\lambda)^{\alpha_2}\,{}_0d_{p,q}\lambda + 1\right) f\left(\frac{\psi}{m}\right) g\left(\frac{\psi}{m}\right) \\ &+ \left(\int_0^1 (1-\lambda)^{\alpha_1+\alpha_2}\,{}_0d_{p,q}\lambda\right) f(\phi)g(\phi) + m\left(\int_0^1 (1-\lambda)^{\alpha_1}\,{}_0d_{p,q}\lambda - \int_0^1 (1-\lambda)^{\alpha_1+\alpha_2}\,{}_0d_{p,q}\lambda\right) f(\phi) g\left(\frac{\psi}{m}\right) \\ &+ m\left(\int_0^1 (1-\lambda)^{\alpha_2}\,{}_0d_{p,q}\lambda - \int_0^1 (1-\lambda)^{\alpha_1+\alpha_2}\,{}_0d_{p,q}\lambda\right) f\left(\frac{\psi}{m}\right) g(\phi).\end{aligned} \tag{3.24}$$

Some simple calculations lead to

$$\int_0^1 f(\lambda\psi + (1-\lambda)\phi)g(\lambda\psi + (1-\lambda)\phi)_0 d_{p,q}\lambda = \frac{1}{\psi-\phi}\int_\phi^\psi f(x)g(x)_\phi d_{p,q}x. \tag{3.25}$$

Therefore, the desired result follows easily from (3.23) to (3.25). □

Corollary 3.2. *If we choose* $\alpha_1 = \alpha_2 = \alpha$ *in* Theorem 3.4, *then we obtain*

$$\frac{1}{\psi-\phi}\int_\phi^\psi f(x)g(x)_\phi \mathrm{d}_{p,q}x \leq \min\{L_1(\alpha, m), L_2(\alpha, m)\},$$

where

$$\begin{aligned}L_1(\alpha, m) = m^2&\left[\frac{p-q}{p^{2\alpha+1}-q^{2\alpha+1}} - \frac{2(p-q)}{p^{\alpha+1}-q^{\alpha+1}} + 1\right] f\left(\frac{\phi}{m}\right) g\left(\frac{\phi}{m}\right) \\ &+ \frac{p-q}{p^{2\alpha+1}-q^{2\alpha+1}} f(\psi)g(\psi) + m\left[\frac{q^{\alpha+1}(p-q)(p^\alpha - q^\alpha)}{(p^{\alpha+1}-q^{\alpha+1})(p^{2\alpha+1}-q^{2\alpha+1})}\right]\left[f\left(\frac{\phi}{m}\right)g(\psi) + f(\psi)g\left(\frac{\phi}{m}\right)\right]\end{aligned}$$

and

$$\begin{aligned}L_2(\alpha, m) = m^2&\left[(p-q)\sum_{n=0}^\infty \frac{q^n}{p^{n+1}}\left(1-\frac{q^n}{p^{n+1}}\right)^{2\alpha} - 2(p-q)\sum_{n=0}^\infty \frac{q^n}{p^{n+1}}\left(1-\frac{q^n}{p^{n+1}}\right)^{\alpha} + 1\right] f\left(\frac{\psi}{m}\right) g\left(\frac{\psi}{m}\right) \\ &+ (p-q)\sum_{n=0}^\infty \frac{q^n}{p^{n+1}}\left(1-\frac{q^n}{p^{n+1}}\right)^{2\alpha} f(\phi)g(\phi) + m\left[(p-q)\sum_{n=0}^\infty \frac{q^n}{p^{n+1}}\left(1-\frac{q^n}{p^{n+1}}\right)^{\alpha}\right. \\ &\left. - (p-q)\sum_{n=0}^\infty \frac{q^n}{p^{n+1}}\left(1-\frac{q^n}{p^{n+1}}\right)^{2\alpha}\right]\left[f(\phi)g\left(\frac{\psi}{m}\right) + f\left(\frac{\psi}{m}\right)g(\phi)\right].\end{aligned}$$

In particular, the special case of $p = \alpha = m = 1$ *for* Corollary 3.2 *was proved by Sudsutad et al. in* [55].

4 Examples

Example 4.1. Let $g : J \supset [0, \infty) \to \mathbb{R}$ be defined by $g(x) = 4x + 1$. Then it is $\left(1, \frac{1}{3}\right)$-differentiable function on J° (the interior of J) and ${}_1D_{1,\frac{1}{3}}g$ is continuous and integrable on $[0, 10]$, $0 \le 1 < 5 < \infty$ and $0 < \frac{1}{3} < 1 \le 1$. If $|\phi D_{1,\frac{1}{3}}g|$ is $\left(1, \frac{1}{2}\right)$-convex on $[0, 10]$ with $\gamma = 0$ and $\mu = \frac{3}{4}$, then all the assumptions of Theorem 3.1 are satisfied.

We clearly see that

$$\begin{aligned}
&\left|\gamma[p\mu g(\psi) + (1 - p\mu)g(\phi)] + (1 - \gamma)g(p\mu\psi + (1 - p\mu)\phi) - \frac{1}{p(\psi - \phi)} \int\limits_{\phi}^{p\psi+(1-p)\phi} g(x)_\phi d_{p,q}x\right| \\
&= \left|g(4) - \frac{1}{4}\int\limits_1^5 (4x + 1)_1 d_{1,\frac{1}{3}}\mathrm{d}x\right| = 17 - \frac{68}{4} = 0,
\end{aligned} \tag{4.1}$$

where

$$\int\limits_1^5 (4x + 1)_1 d_{1,\frac{1}{3}}\mathrm{d}x = 68.$$

On the other hand,

$$\begin{aligned}
&v_1(\gamma, p\mu, \alpha) = v_1\left(0, \frac{3}{4}, \frac{1}{2}\right) \approx 0.1157, \quad (\gamma + q)p\mu > \gamma, \\
&v_2(\gamma, p\mu, \alpha) = v_2\left(0, \frac{3}{4}, \frac{1}{2}\right) \approx 0.0625, \quad (\gamma + q)p\mu > \gamma, \\
&v_3(\gamma, p\mu, \alpha) = v_3\left(0, \frac{3}{4}, \frac{1}{2}\right) \approx 0.5881, \quad \gamma p\mu + q \le 1, \\
&v_4(\gamma, p\mu, \alpha) = v_4\left(0, \frac{3}{4}, \frac{1}{2}\right) = 0, \quad \gamma p\mu + q \le 1, \\
&v_5(\gamma, p\mu, \alpha) = v_5\left(0, \frac{3}{4}, \frac{1}{2}\right) \approx 0.4205, \quad (\gamma + q)p\mu \le 1, \\
&v_6(\gamma, p\mu, \alpha) = v_6\left(0, \frac{3}{4}, \frac{1}{2}\right) \approx 0.1875, \quad (\gamma + q)p\mu \le 1, \\
&v_7(\gamma, p\mu) = v_7\left(0, \frac{3}{4}\right) \approx 0.1406, \quad (\gamma + q)p\mu \ge \gamma, \\
&v_8(\gamma, p\mu) = v_8\left(0, \frac{3}{4}\right) \approx 0.75, \quad \gamma p\mu + q \le 1, \\
&v_9(\gamma, p\mu) = v_9\left(0, \frac{3}{4}\right) \approx 0.6094, \quad (\gamma + q)p\mu \le 1, \\
&v_{10}(\gamma, p\mu) = v_{10}\left(0, \frac{3}{4}\right) \approx 0.2963, \quad 0 \le \gamma p\mu \le 1 - q.
\end{aligned} \tag{4.2}$$

Also, we have

$$\begin{aligned}
&|{}_\phi D_{p,q}g(\psi)| = |{}_0D_{1,\frac{1}{3}}(4\psi + 1)| = 4, \\
&\left|{}_\phi D_{p,q}g\left(\frac{\phi}{m}\right)\right| = |{}_0D_{1,\frac{1}{3}}(8\phi + 1)| = 3, \\
&|{}_\phi D_{p,q}g(\phi)| = |{}_0D_{1,\frac{1}{3}}(4\phi + 1)| = 0, \\
&\left|{}_\phi D_{p,q}g\left(\frac{\psi}{m}\right)\right| = |{}_0D_{1,\frac{1}{3}}(8\psi + 1)| = 8.
\end{aligned} \tag{4.3}$$

Observe that

$$\begin{aligned} H_1(\gamma, p\mu, \alpha, m) = (\psi - \phi)\Big\{&[\nu_1(\gamma, p\mu, \alpha) + \nu_3(\gamma, p\mu, \alpha) - \nu_5(\gamma, p\mu, \alpha)]\,|_\phi D_{p,q} g(\psi)| \\ &+ m\,[\nu_7(\gamma, p\mu) + \nu_8(\gamma, p\mu) - \nu_9(\gamma, p\mu) - \nu_1(\gamma, p\mu, \alpha) \\ &- \nu_3(\gamma, p\mu, \alpha) + \nu_5(\gamma, p\mu, \alpha)]\left|_\phi D_{p,q} g\left(\frac{\phi}{m}\right)\right|\Big\}. \end{aligned} \tag{4.4}$$

Substituting (4.2) and (4.3) in (4.4), and simple computations yield

$$H_1\left(0, \frac{3}{4}, \frac{1}{2}, \frac{1}{2}\right) \approx 4.3167. \tag{4.5}$$

Analogously, we have

$$\begin{aligned} H_2(\gamma, p\mu, \alpha, m) = (\psi - \phi)\Big\{&[\nu_2(\gamma, p\mu, \alpha) + \nu_4(\gamma, p\mu, \alpha) - \nu_6(\gamma, p\mu, \alpha)]\,|_\phi D_{p,q} g(\phi)| \\ &+ m[\nu_7(\gamma, p\mu) + \nu_8(\gamma, p\mu) - \nu_9(\gamma, p\mu) - \nu_2(\gamma, p\mu, \alpha) \\ &- \nu_4(\gamma, p\mu, \alpha) + \nu_6(\gamma, p\mu, \alpha)]\left|_\phi D_{p,q} g\left(\frac{\psi}{m}\right)\right|\Big\}. \end{aligned}$$

After simplification, we have

$$H_2\left(0, \frac{3}{4}, \frac{1}{2}, \frac{1}{2}\right) \approx 6.5. \tag{4.6}$$

From (4.5) and (4.6), we get

$$\min[H_1(\gamma, p\mu, \alpha, m), H_2(\gamma, p\mu, \alpha, m)] = \min\{4.3167, 6.5\} \approx 4.3167, \tag{4.7}$$

which shows that the following implications hold in (4.1) and (4.7)

$$0 < 4.3167.$$

Example 4.2. Let $f, g : J \supset [0, \infty) \to \mathbb{R}$ be defined by $f(x) = g(x) = x$. Then these functions are (p, q)-differentiable functions on J° (the interior of J) and continuous and integrable on $[0, 10]$ with $0 \le 1 < 5 < \infty$. If f and g are $\left(1, \frac{1}{2}\right)$-convex on $[0, 10]$ with $\alpha = 1$ and $m = \frac{1}{2}$, then all assumptions of Corollary 3.2 are satisfied.

Clearly,

$$\frac{1}{\psi - \phi} \int_\phi^\psi f(x) g(x)\, {}_\phi d_{p,q} x = \frac{1}{4} \int_1^5 x_1^2 d_{p,q} x \tag{4.8}$$

follows from Definition 1.3.

On the other hand,

$$L_1\left(1, \frac{1}{2}\right) = \frac{[2]_{p,q} + (p^2 + pq + q^2)[[2]_{p,q} - 2] + 25([2]_{p,q}) + 10q^2}{[2]_{p,q}(p^2 + pq + q^2)} \tag{4.9}$$

and

$$L_2\left(1, \frac{1}{2}\right) = \frac{25([2]_{p,q} + (p^2 + pq + q^2)[[2]_{p,q} - 2]) + ([2]_{p,q}) + 10q^2}{[2]_{p,q}(p^2 + pq + q^2)}. \tag{4.10}$$

From (4.8) and (4.10), we get

$$\min\left[L_1\left(1, \frac{1}{2}\right), L_2\left(1, \frac{1}{2}\right)\right] = L_1\left(1, \frac{1}{2}\right), \tag{4.11}$$

which shows that the following implications hold in (4.8) and (4.11)

$$\frac{1}{4}\int_{1}^{5} x^2{}_1d_{p,q}x < L_1\left(1, \frac{1}{2}\right)$$

for every $0 < q < p \leq 1$.

Remark 4.1. Similar technique can be applied to Theorems 3.2 and 3.3 to get the immediate consequences.

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