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A numerical method for distributed-order time fractional 2D Sobolev equation

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ABSTRACT

In this work, the distributed-order time fractional 2D Sobolev equation is introduced. The orthonormal Bernoulli polynomials, as a renowned family of basis functions, are employed to solve this problem. To effectively use of these polynomials in constructing a suitable methodology for this equation, some operational matrices regarding the ordinary and fractional derivative of them are derived. In the developed method, by approximating the unknown solution by means of these polynomials and using the mentioned matrices, as well as applying the collocation technique, a system of algebraic equations (in which the unknowns are the expansion coefficients of the solution function) is obtained, which by solving it, a solution for the main problem is obtained. By providing four test problems, the capability and accuracy of the scheme are studied.

Introduction

The importance of fractional derivatives is increasing daily due to their high capabilities in modeling fundamental problems of the real world. In fact, from a mathematical point of view, their greater degree of freedom compared to ordinary derivatives makes it possible to provide more accurate and suitable models for various problems [1]. Usually, these types of derivatives have memory retention property [1]. This means that in a period of time, the next behavior of a dynamical system will be affected by the behavior of the entire previous period. It should be noted that various forms of fractional derivatives have been presented. Distributed-order (DO) fractional derivatives are a special form of fractional derivatives that are obtained by integrating ordinary fractional derivatives with respect to their fractional order in a specific range [2,3]. In fact, ordinary fractional derivatives are a special case of these derivatives. This type of fractional derivatives has been widely considered in better modeling of various problems. Applications of these derivatives have been used in signal processing [4], electrochemistry [5], viscoelastic [6], diffusion [7], control [8], etc. We remind that the most important challenge in facing the fractional problems involving this type of derivatives is to find their solution, which in most cases is not possible with analytical methods. For this reason, in recent years, researchers have proposed numerical methods as a useful tool for this category of problems. Some of the numerical techniques proposed in recent years to solve such problems are: the fractional Taylor wavelets method [9], Polynomial-Sinc collocation method [10], Petrov–Galerkin method [11], Chebyshev wavelets method [12], Müntz–Legender polynomials method [13], Jacobi polynomials method [14], Galerkin spectral method [15], finite difference scheme [16], Legendre polynomials method [17,18], Chebyshev cardinal polynomials method [19] and piecewise Jacobi functions method [20].

The Sobolev equation (as a renowned partial differential equation) possesses a wide range of applications in important problems, such as the problem of humidity movement in the soil, fluid flow through fractured rocks, heat flow through different materials, propagation of long waves, etc. [21]. In recent years, various algorithms have been applied to solve the fractional forms of this equation. Some of these methods are: Crank–Nicolson finite element method [22], Crank–Nicolson finite volume element method [23], finite difference method [24], local discontinuous Galerkin method [25], a hybrid technique based on the Müntz–Legender wavelets and Müntz–Legender functions [26] and discrete Legendre polynomials method [27].

Due to the wide applications of the Sobolev equation in the mathematical modeling of various problems and with the knowledge of the high capabilities of DO fractional derivatives, in this study, we introduce a DO fractional form of this equation and develop a suitable computational method to solve it. So, we focus on the below equation:

$$\int_0^1 \mu(\gamma)\,{}_0^C D_t^{\gamma}\varphi(x,y,t)d\gamma - \lambda\Delta\varphi_t(x,y,t) - \sigma\Delta\varphi(x,y,t) = g(x,y,t),$$

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$$(x, y, t) \in [0, 1] \times [0, 1] \times [0, T], \tag{1}$$

where φ is the unknown solution (which is assumed to be continuous), $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian operator, λ and σ are given constants, ${}_{0}^{C}D_t^{\gamma}\varphi$ is the fractional differentiation of order γ with respect to temporal variable of φ in the Caputo form [1]. Here, $\mu : [0, 1] \longrightarrow \mathbb{R}^+$ is the distribution function that possesses the property $0 < \int_0^1 \mu(\gamma)d\gamma < \infty$ [28].

In numerical approaches developed using basis functions for differential and integral equations, polynomials-based approaches are often more efficient. For instance, see [29–32]. In particular, in the case of fractional equations, the calculation of their fractional derivatives is usually straightforward, and with appropriate methods, the matrix forms of their fractional derivatives can be obtained and used in related numerical approaches.

A family of polynomial basis functions that have been employed in the past decades to solve diverse problems is the Bernoulli polynomials (BPs). They are successfully employed to solve different, such as ordinary integral equations [33], stochastic integral equations [34], fractional delay differential equations [35], fractional coupled Boussinesq-Burger's equations [36], fractional reaction–advection–diffusion equation [37], fractional Benjamin–Bona–Mahony equation [38], fractional partial integro-differential equations [39] and fractional Lane–Emden equation [40]. We remind that if the solution of the problem under consideration is sufficiently smooth, the numerical results obtained by these polynomials are very accurate.

In this study, we use the orthonormal BPs to solve the above DO fractional Sobolev equation. First of all, we derive some formulas for computing operational matrices regarding ordinary and fractional derivatives of these polynomials. The established approach works in such a way that first the unknown solution is approximated by these polynomials. Then, using the derivative matrices, approximations for the derivatives in the equation are provided. Next, by inserting these approximations into the equation and employing the collocation technique, we generate an algebraic system of equations. At the end, by solving this system, a solution for the DO fractional equation is obtained. Several test problems are considered to verify the formulation of the approach as well as its accuracy.

The remainder of this study is as follows: Some prerequisites are collected in Section "Preliminaries". The orthonormal BPs and their operational matrices are given respectively in Sections "Orthonormal Bernoulli polynomials" and "Operational matrices". The proposed scheme and test problems are given respectively in Sections " The proposed method" and "Test problems". The conclusion of this work is reviewed in Section "Conclusion".

Preliminaries

Here, we have reviewed a few preparations that will be used in this study.

Definition 1 (*[1]*). Suppose that f is a differentiable function in its domain and $0 < \gamma \leq 1$ is a given constant. The Caputo fractional differentiation of order γ of this function is defined as

$${}_{0}^{C}D_t^{\gamma} f(t) = \begin{cases} \dfrac{1}{\Gamma(1-\gamma)} \displaystyle\int_0^t (t-s)^{-\gamma} f'(s)ds, & 0 < \gamma < 1, \\ f'(t), & \gamma = 1. \end{cases} \tag{2}$$

Note that for $\gamma = 0$, we have ${}_{0}^{C}D_t^{0} f(t) = f(t)$.

Corollary 1 (*[1]*). *For $k \in \mathbb{N} \cup \{0\}$, we achieve*

$${}_{0}^{C}D_t^{\gamma} t^k = \begin{cases} 0, & k = 0, \\ \dfrac{k!}{\Gamma(k-\gamma+1)} t^{k-\gamma}, & k \geq 1. \end{cases} \tag{3}$$

Definition 2 (*[41,42]*). An $(\hat{N}+1)$-point Legendre Gauss–Lobatto quadrature integration can be defined over $[0, 1]$ as follows:

$$\int_0^1 h(t)dt \simeq \frac{1}{2} \sum_{i=0}^{\hat{N}} \bar{\omega}_i h\left(\frac{1}{2}\left(\bar{t}_i + 1\right)\right), \tag{4}$$

where $\bar{t}_0 = -1$, $\bar{t}_{\hat{N}} = 1$ and $\bar{t}_i\,(i = 1, 2, \ldots, \hat{N}-1)$ are the zeros of $L'_{\hat{N}}$ (where $L_{\hat{N}}$ is the $\hat{N}$th Legendre polynomial), and

$$\bar{\omega}_i = \frac{2}{\hat{N}(\hat{N}+1)} \frac{1}{\left(L_{\hat{N}}\left(\bar{t}_i\right)\right)^2}. \tag{5}$$

In this work, we set $\hat{N} = 25$ in all computations.

Orthonormal Bernoulli polynomials

For a given number $m_1 \in \mathbb{Z}^+$, a set containing $(m_1 + 1)$ elements of the orthonormal BPs can be defined over $[0, 1]$ by the formula [36]:

$$B_{m_1,i}(x) = \sum_{k=0}^{i} \varrho_{ik} x^k, \qquad i = 0, 1, \ldots, m_1, \tag{6}$$

where

$$\varrho_{ik} = (-1)^{i+k} \sqrt{2i+1} \binom{i}{i-k} \binom{i+k}{k}. \tag{7}$$

The orthonormal property of these functions allows us to represent any function $\bar{\psi} \in L^2[0, 1]$ as follows:

$$\bar{\psi}(x) \simeq \sum_{i=0}^{m_1} \bar{\psi}_i B_{m_1,i}(x) \triangleq \boldsymbol{\Psi}_{m_1}^{\intercal} \mathbf{B}_{m_1}(x), \tag{8}$$

where

$$\boldsymbol{\Psi}_{m_1} = \left[\bar{\psi}_0 \ \bar{\psi}_1 \ \ldots \ \bar{\psi}_{m_1}\right]^{\intercal},$$

with

$$\bar{\psi}_i = \int_0^1 \bar{\psi}(x) B_{m_1,i}(x) dx, \tag{9}$$

and

$$\mathbf{B}_{m_1}(x) = \left[B_{m_1,0}(x) \ B_{m_1,1}(x) \ \ldots \ B_{m_1,m_1}(x)\right]^{\intercal}. \tag{10}$$

Also, for given numbers $m_1, m_2 \in \mathbb{Z}^+$, we can define the 2D orthonormal BPs as follows:

$$\bar{B}_{m_1 m_2,ij}(x, y) = B_{m_1,i}(x) B_{m_2,j}(y), \qquad i = 0, 1, \ldots, m_1, \ j = 0, 1, \ldots, m_2. \tag{11}$$

Moreover, for any two variables function $\bar{\psi} \in L^2([0, 1] \times [0, 1])$, we can consider the following representation:

$$\bar{\psi}(x, y) \simeq \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} \bar{\psi}_{ij} \bar{B}_{m_1 m_2,ij}(x, y) \triangleq \bar{\boldsymbol{\Psi}}_{m_1 m_2}^{\intercal} \bar{\mathbf{B}}_{m_1 m_2}(x, y), \tag{12}$$

where

$$\bar{\boldsymbol{\Psi}}_{m_1 m_1} = \left[\bar{\psi}_{00} \ \bar{\psi}_{01} \ \ldots \ \bar{\psi}_{0m_2} | \bar{\psi}_{10} \ \bar{\psi}_{11} \ \ldots \ \bar{\psi}_{1m_2} | \ldots | \bar{\psi}_{m_1 0} \ \bar{\psi}_{m_1 1} \ \ldots \ \bar{\psi}_{m_1 m_2}\right]^{\intercal},$$

with

$$\bar{\psi}_{ij} = \int_0^1 \int_0^1 \bar{\psi}(x, y) \bar{B}_{m_1 m_2,ij}(x, y) dx dy.$$

and

$$\begin{aligned} \bar{\mathbf{B}}_{m_1 m_2}(x, y) = \Big[& \bar{B}_{m_1 m_2,00}(x, y) \ \bar{B}_{m_1 m_2,01}(x, y) \\ & \ldots \ \bar{B}_{m_1 m_2,0m_2}(x, y) | \bar{B}_{m_1 m_2,10}(x, y) \ \bar{B}_{m_1 m_2,11}(x, y) \\ & \ldots \ \bar{B}_{m_1 m_2,1m_2}(x, y) | \ldots | \bar{B}_{m_1 m_2,m_1 0}(x, y) \ \bar{B}_{m_1 m_2,m_1 1}(x, y) \\ & \ldots \ \bar{B}_{m_1 m_2,m_1 m_2}(x, y) \Big]^{\intercal}. \end{aligned} \tag{13}$$

Note that for simplicity we can rewrite (12) as

$$\bar{\psi}(x,y) \simeq \sum_{r=0}^{(m_1+1)(m_2+1)-1} \tilde{\psi}_r \tilde{B}_{m_1 m_2,r}(x,y) \triangleq \bar{\boldsymbol{\Psi}}^{\intercal}_{m_1 m_1} \bar{\mathbf{B}}_{m_1 m_2}(x,y), \tag{14}$$

where $\tilde{\psi}_r = \bar{\psi}_{ij}$ and $\tilde{B}_{m_1 m_2,r}(x,y) = \bar{B}_{m_1 m_2,ij}(x,y)$ with $r = \left(m_2+1\right) i + j$ for $i = 0,1,\dots,m_1$ and $j = 0,1,\dots,m_2$. Similarly, we can approximate any three variables function $\bar{\varphi} \in L^2\left([0,1]\times[0,1]\times[0,T]\right)$ by the orthonormal BPs as follows:

$$\bar{\varphi}(x,y,t) \simeq \sum_{r=0}^{(m_1+1)(m_2+1)-1} \sum_{l=0}^{m_3} \tilde{\varphi}_{rl} \tilde{B}_{m_1 m_2,r}(x,y) \hat{B}_{m_3,l}(t) \triangleq \bar{\mathbf{B}}^{\intercal}_{m_1 m_2}(x,y) \bar{\boldsymbol{\Phi}}_{m_1 m_2 m_3} \hat{\mathbf{B}}_{m_3}(t), \tag{15}$$

where

$$\bar{\boldsymbol{\Phi}}_{m_1 m_2 m_3} = \begin{pmatrix} \tilde{\varphi}_{00} & \tilde{\varphi}_{01} & \dots & \tilde{\varphi}_{0m_3} \\ \tilde{\varphi}_{10} & \tilde{\varphi}_{11} & \dots & \tilde{\varphi}_{1m_3} \\ \vdots & \vdots & \dots & \vdots \\ \tilde{\varphi}_{(m_1 m_2+m_1+m_2)0} & \tilde{\varphi}_{(m_1 m_2+m_1+m_2)1} & \dots & \tilde{\varphi}_{(m_1 m_2+m_1+m_2)m_3} \end{pmatrix},$$

with

$$\tilde{\varphi}_{rl} = \int_0^T \int_0^1 \int_0^1 \bar{\varphi}(x,y,t) \tilde{B}_{m_1 m_2,r}(x,y) \hat{B}_{m_3,l}(t) dx\,dy\,dt,$$

and

$$\hat{\mathbf{B}}_{m_3}(t) = \left[\hat{B}_{m_3,0}(t)\ \hat{B}_{m_3,1}(t)\ \dots\ \hat{B}_{m_3,m_3}(t)\right]^{\intercal}, \tag{16}$$

where for a given $m_3 \in \mathbb{Z}^+$, the functions $\hat{B}_{m_3,l}(t)$ $(l = 0,1,\dots,m_3)$ are defined in [36] as

$$B_{m_3,l}(t) = \sum_{k=0}^{l} \hat{\varrho}_{lk} t^k, \tag{17}$$

such that

$$\hat{\varrho}_{lk} = (-1)^{l+k} \sqrt{\frac{(2l+1)}{T}} \frac{1}{T^k} \binom{l}{l-k} \binom{l+k}{k}. \tag{18}$$

Operational matrices

This section is dedicated to deriving some operational matrices for orthonormal BPs.

Theorem 1. *The below relation is valid for the second order derivative of the vector $\mathbf{B}_{m_1}(x)$ defined in (10):*

$$\frac{d^2 \mathbf{B}_{m_1}(x)}{dx^2} = \mathbf{D}^{(2)}_{m_1} \mathbf{B}_{m_1}(x), \tag{19}$$

where

$$\mathbf{D}^{(2)}_{m_1} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ d^{(2,m_1)}_{20} & 0 & 0 & \dots & 0 & 0 & 0 \\ d^{(2,m_1)}_{30} & d^{(2,m_1)}_{31} & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ d^{(2,m_1)}_{(m_1-1)0} & d^{(2,m_1)}_{(m_1-1)1} & d^{(2,m_1)}_{(m_1-1)2} & \dots & 0 & 0 & 0 \\ d^{(2,m_1)}_{m_1 0} & d^{(2,m_1)}_{m_1 1} & d^{(2,m_1)}_{m_1 2} & \dots & d^{(2,m_1)}_{m_1(m_1-2)} & 0 & 0 \end{pmatrix},$$

and

$$d^{(2,m_1)}_{ij} = \begin{cases} a^{(2,m_1)}_{ij}, & 2 \le i \le m_1,\ 0 \le j \le i-2, \\ 0, & otherwise, \end{cases}$$

in which

$$a^{(2,m_1)}_{ij} = \sqrt{(2i+1)(2j+1)} \times \sum_{k=2}^{i} \sum_{r=0}^{j} \frac{(-1)^{i+j+k+r} k(k-1)}{k+r-1} \binom{i}{i-k} \binom{i+k}{k} \binom{j}{j-r} \binom{j+r}{r}.$$

Proof. From (6), we have

$$\frac{d^2 B_{m_1,i}(x)}{dx^2} = 0, \qquad i = 0,1, \tag{20}$$

and

$$\frac{d^2 B_{m_1,i}(x)}{dx^2} = \sum_{k=2}^{i} \varrho_{ik} k(k-1) x^{k-2}, \qquad i = 2,3,\dots,m_1, \tag{21}$$

where ϱ_{ik} is defined in (7). Expanding the results obtained in (21) by the orthonormal BPs and considering the orthogonal property of these functions, results in

$$\frac{d^2 B_{m_1,i}(x)}{dx^2} = \sum_{j=0}^{i-2} d^{(2,m_1)}_{ij} B_{m_1,j}(x), \tag{22}$$

where

$$d^{(2,m_1)}_{ij} = \int_0^1 \frac{d^2 B_{m_1,i}(x)}{dx^2} B_{m_1,j}(x) dx = \sum_{k=2}^{i} \varrho_{ik} k(k-1) \int_0^1 x^{k-2} B_{m_1,j}(x) dx. \tag{23}$$

From the definition regarding the functions $B_{m_1,j}(x)$ in (6), the outcomes extracted in (23) can be calculated as

$$d^{(2,m_1)}_{ij} = \sum_{k=2}^{i} \sum_{r=0}^{j} \varrho_{ik} \varrho_{jr} k(k-1) \int_0^1 x^{k+r-2} dx = \sum_{k=2}^{i} \sum_{r=0}^{j} \frac{\varrho_{ik} \varrho_{jr} k(k-1)}{k+r-1}. \tag{24}$$

Thus, from (22), we obtain

$$\underbrace{\begin{pmatrix} \frac{d^2 B_{m_1,0}(x)}{dx^2} \\ \frac{d^2 B_{m_1,1}(x)}{dx^2} \\ \frac{d^2 B_{m_1,2}(x)}{dx^2} \\ \vdots \\ \frac{d^2 B_{m_1,m_1-1}(x)}{dx^2} \\ \frac{d^2 B_{m_1,m_1}(x)}{dx^2} \end{pmatrix}}_{\frac{d^2 \mathbf{B}_{m_1}(x)}{dx^2}} = \underbrace{\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ d^{(2,m_1)}_{20} & 0 & 0 & \dots & 0 & 0 & 0 \\ d^{(2,m_1)}_{30} & d^{(2,m_1)}_{31} & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ d^{(2,m_1)}_{(m_1-1)0} & d^{(2,m_1)}_{(m_1-1)1} & d^{(2,m_1)}_{(m_1-1)2} & \dots & 0 & 0 & 0 \\ d^{(2,m_1)}_{m_1 0} & d^{(2,m_1)}_{m_1 1} & d^{(2,m_1)}_{m_1 2} & \dots & d^{(2,m_1)}_{m_1(m_1-2)} & 0 & 0 \end{pmatrix}}_{\mathbf{D}^{(2)}_{m_1}} \underbrace{\begin{pmatrix} B_{m_1,0}(x) \\ B_{m_1,1}(x) \\ B_{m_1,2}(x) \\ \vdots \\ B_{m_1,m_1-1}(x) \\ B_{m_1,m_1}(x) \end{pmatrix}}_{\mathbf{B}_{m_1}(x)}.$$

Hence, by applying the definition expressed in (7) for calculating ϱ_{ik} and ϱ_{jr}, and inserting them into (24), the desired result will be obtained. □

As a numerical example, we have

$$\mathbf{D}^{(2)}_{7} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12\sqrt{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 20\sqrt{21} & 0 & 0 & 0 & 0 & 0 & 0 \\ 120 & 0 & 84\sqrt{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 56\sqrt{33} & 0 & 36\sqrt{77} & 0 & 0 & 0 & 0 \\ 84\sqrt{13} & 0 & 72\sqrt{65} & 0 & 132\sqrt{13} & 0 & 0 & 0 \\ 0 & 324\sqrt{5} & 0 & 88\sqrt{105} & 0 & 52\sqrt{165} & 0 & 0 \end{pmatrix}.$$

Theorem 2. *The second order derivatives of the vector $\bar{\mathbf{B}}_{m_1 m_2}(x,y)$ defined in (13) can be given as follows:*

$$\frac{\partial^2 \bar{\mathbf{B}}_{m_1 m_2}(x,y)}{\partial x^2} = \mathbf{Q}^{(2)}_{m_1 m_2} \bar{\mathbf{B}}_{m_1 m_2}(x,y), \tag{25}$$

where

$$\mathbf{Q}^{(2)}_{m_1 m_2} = \mathbf{D}^{(2)}_{m_1} \otimes \mathbf{I}_{m_2}$$

$$= \begin{pmatrix} \mathbf{O}_{m_2} & \mathbf{O}_{m_2} & \mathbf{O}_{m_2} & \cdots & \mathbf{O}_{m_2} & \mathbf{O}_{m_2} & \mathbf{O}_{m_2} \\ \mathbf{O}_{m_2} & \mathbf{O}_{m_2} & \mathbf{O}_{m_2} & \cdots & \mathbf{O}_{m_2} & \mathbf{O}_{m_2} & \mathbf{O}_{m_2} \\ d_{20}^{(2,m_1)}\mathbf{I}_{m_2} & \mathbf{O}_{m_2} & \mathbf{O}_{m_2} & \cdots & \mathbf{O}_{m_2} & \mathbf{O}_{m_2} & \mathbf{O}_{m_2} \\ d_{30}^{(2,m_1)}\mathbf{I}_{m_2} & d_{31}^{(2,m_1)}\mathbf{I}_{m_2} & \mathbf{O}_{m_2} & \cdots & \mathbf{O}_{m_2} & \mathbf{O}_{m_2} & \mathbf{O}_{m_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ d_{(m_1-1)0}^{(2,m_1)}\mathbf{I}_{m_2} & d_{(m_1-1)1}^{(2,m_1)}\mathbf{I}_{m_2} & d_{(m_1-1)2}^{(2,m_1)}\mathbf{I}_{m_2} & \cdots & \mathbf{O}_{m_2} & \mathbf{O}_{m_2} & \mathbf{O}_{m_2} \\ d_{m_1 0}^{(2,m_1)}\mathbf{I}_{m_2} & d_{m_1 1}^{(2,m_1)}\mathbf{I}_{m_2} & d_{m_1 2}^{(2,m_1)}\mathbf{I}_{m_2} & \cdots & d_{m_1(m_1-2)}^{(2,m_1)}\mathbf{I}_{m_2} & \mathbf{O}_{m_2} & \mathbf{O}_{m_2} \end{pmatrix},$$

in which $\mathbf{Q}^{(2)}_{m_1 m_2}$ *is an* $(m_1+1)(m_2+1)$*-order square matrix,* $\mathbf{D}^{(2)}_{m_1}$ *is the matrix obtained in* Theorem 1*,* $\otimes$ *denotes the tensor product,* $\mathbf{O}_{m_2}$ *is an* (m_2+1)*-order zero matrix and* $\mathbf{I}_{m_2}$ *is an* (m_2+1)*-order identity matrix, and*

$$\frac{\partial^2 \bar{\mathbf{B}}_{m_1 m_2}(x,y)}{\partial y^2} = \mathbf{P}^{(2)}_{m_1 m_2}\bar{\mathbf{B}}_{m_1 m_2}(x,y), \tag{26}$$

where $\mathbf{P}^{(2)}_{m_1 m_2}$ *is an* $(m_1+1)(m_2+1)$*-order square matrix in the form of*

$$\mathbf{P}^{(2)}_{m_1 m_2} = \begin{pmatrix} \mathbf{D}^{(2)}_{m_2} & \mathbf{O}_{m_2} & \mathbf{O}_{m_2} & \cdots & \mathbf{O}_{m_2} & \mathbf{O}_{m_2} \\ \mathbf{O}_{m_2} & \mathbf{D}^{(2)}_{m_2} & \mathbf{O}_{m_2} & \cdots & \mathbf{O}_{m_2} & \mathbf{O}_{m_2} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \mathbf{O}_{m_2} & \mathbf{O}_{m_2} & \mathbf{O}_{m_2} & \cdots & \mathbf{D}^{(2)}_{m_2} & \mathbf{O}_{m_2} \\ \mathbf{O}_{m_2} & \mathbf{O}_{m_2} & \mathbf{O}_{m_2} & \cdots & \mathbf{O}_{m_2} & \mathbf{D}^{(2)}_{m_2} \end{pmatrix}$$

in which $\mathbf{D}^{(2)}_{m_2}$ *is an* (m_2+1)*-order square matrix that can be obtained similar to* Theorem 1 *as*

$$\left[\mathbf{D}^{(2)}_{m_2}\right]_{ij} = \begin{cases} b_{ij}^{(2,m_2)}, & 2 \le i \le m_2,\ 0 \le j \le i-2, \\ 0, & otherwise, \end{cases}$$

for $0 \le i, j \le m_2$*, in which*

$$b_{ij}^{(2,m_2)} = \sqrt{(2i+1)(2j+1)} \times \sum_{k=2}^{i}\sum_{r=0}^{j} \frac{(-1)^{i+j+k+r}k(k-1)}{k+r-1}\binom{i}{i-k}\binom{i+k}{k}\binom{j}{j-r}\binom{j+r}{r}.$$

Proof. The proof is straightforward. So, we leave it to the reader. □

Theorem 3. *The first derivative of* $\hat{\mathbf{B}}_{m_3}(t)$ *given in* (16) *can be stated as*

$$\frac{d\hat{\mathbf{B}}_{m_3}(t)}{dt} = \mathbf{D}^{(1)}_{m_3}\hat{\mathbf{B}}_{m_3}(t), \tag{27}$$

where

$$\mathbf{D}^{(1)}_{m_3} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ d_{10}^{(1,m_3)} & 0 & 0 & \cdots & 0 & 0 & 0 \\ d_{20}^{(1,m_3)} & d_{21}^{(1,m_3)} & 0 & \cdots & 0 & 0 & 0 \\ d_{30}^{(1,m_3)} & d_{31}^{(1,m_3)} & d_{32}^{(1,m_3)} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ d_{(m_3-1)0}^{(1,m_3)} & d_{(m_3-1)1}^{(1,m_3)} & d_{(m_3-1)2}^{(1,m_3)} & \cdots & d_{(m_3-1)(m_3-2)}^{(1,m_3)} & 0 & 0 \\ d_{m_3 0}^{(1,m_3)} & d_{m_3 1}^{(1,m_3)} & d_{m_3 2}^{(1,m_3)} & \cdots & d_{m_3(m_3-2)}^{(1,m_3)} & d_{m_3(m_3-1)}^{(1,m_3)} & 0 \end{pmatrix},$$

and

$$d_{ij}^{(1,m_3)} = \begin{cases} c_{ij}^{(1,m_3)}, & 1 \le i \le m_3,\ 0 \le j \le i-1, \\ 0, & otherwise, \end{cases}$$

in which

$$c_{ij}^{(1,m_3)} = \frac{\sqrt{(2i+1)(2j+1)}}{T} \times \sum_{k=1}^{i}\sum_{r=0}^{j} \frac{(-1)^{i+j+k+r}k}{k+r}\binom{i}{i-k}\binom{i+k}{k}\binom{j}{j-r}\binom{j+r}{r}.$$

Proof. The proof method is similar to the one expressed for Theorem 1. So, we omit it. □

Theorem 4. *The fractional differentiation of order* $0 < \gamma \le 1$ *of* $\hat{\mathbf{B}}_{m_3}(t)$ *defined in* (16) *can be approximated as*

$${}_0^C D_t^{\gamma}\hat{\mathbf{B}}_{m_3}(t) \simeq \mathbf{S}^{(\gamma)}_{m_3}\hat{\mathbf{B}}_{m_3}(t), \tag{28}$$

where

$$\mathbf{S}^{(\gamma)}_{m_3} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ s_{10}^{(\gamma,m_3)} & s_{11}^{(\gamma,m_3)} & \cdots & s_{1(m_3-1)}^{(\gamma,m_3)} & s_{1m_3}^{(\gamma,m_3)} \\ s_{20}^{(\gamma,m_3)} & s_{21}^{(\gamma,m_3)} & \cdots & s_{2(m_3-1)}^{(\gamma,m_3)} & s_{2m_3}^{(\gamma,m_3)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ s_{m_3 0}^{(\gamma,m_3)} & s_{m_3 1}^{(\gamma,m_3)} & \cdots & s_{m_3(m_3-1)}^{(\gamma,m_3)} & s_{m_3 m_3}^{(\gamma,m_3)} \end{pmatrix},$$

and

$$s_{ij}^{(\gamma,m_3)} = \begin{cases} v_{ij}^{(\gamma,m_3)}, & 1 \le i \le m_3,\ 0 \le j \le m_3, \\ 0, & otherwise, \end{cases}$$

in which

$$v_{ij}^{(\gamma,m_3)} = \frac{\sqrt{(2i+1)(2j+1)}}{T^{\gamma}} \times \sum_{k=1}^{i}\sum_{r=0}^{j} \frac{(-1)^{i+j+k+r}k!}{(k+r-\gamma+1)\Gamma(k-\gamma+1)}\binom{i}{i-k}\binom{i+k}{k}\binom{j}{j-r}\binom{j+r}{r}.$$

Proof. From Corollary 1 and the formula expressed in (17), we have

$${}_0^C D_t^{\gamma}\hat{B}_{m_3,0}(t) = 0, \tag{29}$$

and

$${}_0^C D_t^{\gamma}\hat{B}_{m_3,i}(t) = \sum_{k=1}^{i}\frac{\hat{\varrho}_{ik}k!}{\Gamma(k-\gamma+1)}t^{k-\gamma}, \qquad i = 1,2,\dots,m_3. \tag{30}$$

Approximating the above result by the orthonormal BPs, gives

$${}_0^C D_t^{\gamma}\hat{B}_{m_3,i}(t) \simeq \sum_{j=0}^{m_3} s_{ij}^{(\gamma,m_3)}\hat{B}_{m_3,j}(t), \tag{31}$$

where

$$\begin{aligned} s_{ij}^{(\gamma,m_3)} &= \int_0^T \left({}_0^C D_t^{\gamma}\hat{B}_{m_3,i}(t)\right)\hat{B}_{m_3,j}(t)dt \\ &= \sum_{k=1}^{i}\frac{\hat{\varrho}_{ik}k!}{\Gamma(k-\gamma+1)}\int_0^T t^{k-\gamma}\hat{B}_{m_3,j}(t)dt \\ &= \sum_{k=1}^{i}\sum_{r=0}^{j}\frac{\hat{\varrho}_{ik}\hat{\varrho}_{jr}k!}{\Gamma(k-\gamma+1)}\int_0^T t^{k+r-\gamma}dt \\ &= \sum_{k=1}^{i}\sum_{r=0}^{j}\frac{\hat{\varrho}_{ik}\hat{\varrho}_{jr}k!T^{k+r-\gamma+1}}{(k+r-\gamma+1)\Gamma(k-\gamma+1)}. \end{aligned} \tag{32}$$

Hence, from (31), we get

$$\underbrace{\begin{pmatrix} {}_0^C D_t^{\gamma}\hat{B}_{m_3,0}(t) \\ {}_0^C D_t^{\gamma}\hat{B}_{m_3,1}(t) \\ {}_0^C D_t^{\gamma}\hat{B}_{m_3,2}(t) \\ \vdots \\ {}_0^C D_t^{\gamma}\hat{B}_{m_3,m_3}(t) \end{pmatrix}}_{{}_0^C D_t^{\gamma}\hat{\mathbf{B}}_{m_3}(t)} = \underbrace{\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ s_{10}^{(\gamma,m_3)} & s_{11}^{(\gamma,m_3)} & \cdots & s_{1(m_3-1)}^{(\gamma,m_3)} & s_{1m_3}^{(\gamma,m_3)} \\ s_{20}^{(\gamma,m_3)} & s_{21}^{(\gamma,m_3)} & \cdots & s_{2(m_3-1)}^{(\gamma,m_3)} & s_{2m_3}^{(\gamma,m_3)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ s_{m_3 0}^{(\gamma,m_3)} & s_{m_3 1}^{(\gamma,m_3)} & \cdots & s_{m_3(m_3-1)}^{(\gamma,m_3)} & s_{m_3 m_3}^{(\gamma,m_3)} \end{pmatrix}}_{\mathbf{S}^{(\gamma)}_{m_3}} \underbrace{\begin{pmatrix} \hat{B}_{m_3,0}(t) \\ \hat{B}_{m_3,1}(t) \\ \hat{B}_{m_3,2}(t) \\ \vdots \\ \hat{B}_{m_3,m_3}(t) \end{pmatrix}}_{\hat{\mathbf{B}}_{m_3}(t)}.$$

Finally, by considering the definitions of $\hat{\varrho}_{ik}$ and $\hat{\varrho}_{jr}$ from (18) and inserting them into (32), the expressed assertion will be proved. □

Remark 1. In the case of $\gamma = 1$, the matrix $\mathbf{S}^{(\gamma)}_{m_3}$ will be equal to $\mathbf{D}^{(1)}_{m_3}$. Moreover, for $\gamma = 0$, we have ${}_0^C D_t^{\gamma}\hat{\mathbf{B}}_{m_3}(t) = \mathbf{I}_{m_3}\hat{\mathbf{B}}_{m_3}(t)$, where $\mathbf{I}_{m_3}$ is an (m_3+1)-order square identity matrix.

Corollary 2. *For $0 \le \gamma \le 1$, we have*

$$ {}_0^C D_t^{\gamma} \hat{\mathbf{B}}_{m_3}(t) \simeq \bar{\mathbf{S}}_{m_3}^{(\gamma)} \hat{\mathbf{B}}_{m_3}(t), \tag{33} $$

where

$$ \bar{\mathbf{S}}_{m_3}^{(\gamma)} = \begin{pmatrix} \bar{s}_{10}^{(\gamma,m_3)} & 0 & \dots & 0 & 0 \\ \bar{s}_{10}^{(\gamma,m_3)} & \bar{s}_{11}^{(\gamma,m_3)} & \dots & \bar{s}_{1(m_3-1)}^{(\gamma,m_3)} & \bar{s}_{1m_3}^{(\gamma,m_3)} \\ \bar{s}_{20}^{(\gamma,m_3)} & \bar{s}_{21}^{(\gamma,m_3)} & \dots & \bar{s}_{2(m_3-1)}^{(\gamma,m_3)} & \bar{s}_{2m_3}^{(\gamma,m_3)} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \bar{s}_{m_30}^{(\gamma,m_3)} & \bar{s}_{m_31}^{(\gamma,m_3)} & \dots & \bar{s}_{m_3(m_3-1)}^{(\gamma,m_3)} & \bar{s}_{m_3m_3}^{(\gamma,m_3)} \end{pmatrix}, $$

and

$$ \bar{s}_{ij}^{(\gamma,m_3)} = \begin{cases} \begin{cases} 1, & i=j, \\ 0, & otherwise, \end{cases} & \gamma = 0, \\ \left[\mathbf{S}_{m_3}^{(\gamma)}\right]_{ij}, & 0 < \alpha \le 1, \end{cases} $$

Theorem 5. *The distributed-order fractional differentiation of $\hat{\mathbf{B}}_{m_3}(t)$ given in (16) can be approximated as*

$$ \int_0^1 \mu(\gamma)\, {}_0^C D_t^{\gamma} \hat{\mathbf{B}}_{m_3}(t) d\gamma \simeq \mathbf{Z}_{m_3} \hat{\mathbf{B}}_{m_3}(t), \tag{34} $$

where

$$ \mathbf{Z}_{m_3} = \begin{pmatrix} z_{00} & z_{01} & \cdots & z_{0m_3} \\ z_{10} & z_{11} & \cdots & z_{1m_3} \\ \vdots & \vdots & \cdots & \vdots \\ z_{m_30} & z_{m_31} & \cdots & z_{m_3m_3} \end{pmatrix}, \tag{35} $$

and $z_{ij} = \frac{1}{2}\sum_{r=0}^{\hat{N}} \bar{\omega}_r \mu\left(\frac{1}{2}(\bar{t}_r+1)\right) \bar{s}_{ij}^{\left(\frac{1}{2}(\bar{t}_r+1),m_3\right)}$, in which $\bar{s}_{ij}^{(\gamma,m_3)}$ is expressed in Corollary 2.

Proof. According to Corollary 2, we have

$$ \int_0^1 \mu(\gamma)\, {}_0^C D_t^{\gamma} \hat{\mathbf{B}}_{m_3}(t) d\alpha \simeq \left(\int_0^1 \mu(\gamma) \bar{\mathbf{S}}_{m_3}^{(\gamma)} d\gamma \right) \hat{\mathbf{B}}_{m_3}(t) \triangleq \mathbf{Z}_{m_3} \hat{\mathbf{B}}_{m_3}(t), $$

where $\mathbf{Z}_{m_3}$ is in the form expressed in (35), and its elements are obtained as follows:

$$ z_{ij} = \int_0^1 \mu(\gamma) \bar{s}_{ij}^{(\gamma,m_3)} d\gamma. \tag{36} $$

Hence, computing the integrals in (36) using an $(\hat{N}+1)$-point Legendre Gauss–Lobatto quadrature technique, results in

$$ z_{ij} = \frac{1}{2}\sum_{r=0}^{\hat{N}} \bar{\omega}_r \mu\left(\frac{1}{2}(\bar{t}_r+1)\right) \bar{s}_{ij}^{\left(\frac{1}{2}(\bar{t}_r+1),m_3\right)}, $$

which ends the proof. □

The proposed method

This section is dedicated to developing a computational technique for the equation introduced in (1) under the following conditions:

$$ \varphi(x,y,0) = \bar{\varphi}_0(x,y), \tag{37} $$

and

$$ \begin{aligned} &\varphi(0,y,t) = \bar{\varphi}_1(y,t), \quad \varphi(1,y,t) = \bar{\varphi}_2(y,t) \\ &\varphi(x,0,t) = \bar{\varphi}_3(x,t), \quad \varphi(x,1,t) = \bar{\varphi}_4(x,t), \end{aligned} \tag{38} $$

where $\bar{\varphi}_r$, $r = 0,1,\dots,4$ are given continuous functions. To this purpose, we approximate the unknown solution as follows:

$$ \varphi(x,y,t) \simeq \sum_{r=0}^{(m_1+1)(m_2+1)-1} \sum_{l=0}^{m_3} \varphi_{r,l} \tilde{B}_{m_1m_2,r}(x,y) B_{m_3,l}(t) \triangleq \bar{\mathbf{B}}_{m_1m_2}^{\intercal}(x,y) \boldsymbol{\Phi}_{m_1m_2m_3} \hat{\mathbf{B}}_{m_3}(t), \tag{39} $$

where

$$ \boldsymbol{\Phi}_{m_1m_2m_3} = \begin{pmatrix} \varphi_{00} & \varphi_{01} & \cdots & \varphi_{0m_3} \\ \varphi_{10} & \varphi_{11} & \cdots & \varphi_{1m_3} \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_{(m_1m_2+m_1+m_2)0} & \varphi_{(m_1m_2+m_1+m_2)1} & \cdots & \varphi_{(m_1m_2+m_1+m_2)m_3} \end{pmatrix}. $$

From (39) and Theorems 2 and 3, we get

$$ \begin{aligned} &\Delta\varphi(x,y,t) \simeq \bar{\mathbf{B}}_{m_1m_2}^{\intercal}(x,y)\left[\left(\mathbf{Q}_{m_1m_2}^{(2)}\right)^{\intercal} + \left(\mathbf{P}_{m_1m_2}^{(2)}\right)^{\intercal}\right] \boldsymbol{\Phi}_{m_1m_2m_3} \hat{\mathbf{B}}_{m_3}(t), \\ &\Delta\varphi_t(x,y,t) \simeq \bar{\mathbf{B}}_{m_1m_2}^{\intercal}(x,y)\left[\left(\mathbf{Q}_{m_1m_2}^{(2)}\right)^{\intercal} \left(\mathbf{P}_{m_1m_2}^{(2)}\right)^{\intercal}\right] \boldsymbol{\Phi}_{m_1m_2m_3} \mathbf{D}_{m_3}^{(1)} \hat{\mathbf{B}}_{m_3}(t). \end{aligned} \tag{40} $$

Also, using (39) and Theorem 5, we obtain

$$ \int_0^1 \mu(\gamma)\, {}_0^C D_t^{\gamma} \varphi(x,y,t) d\gamma \simeq \bar{\mathbf{B}}_{m_1m_2}^{\intercal}(x,y) \boldsymbol{\Phi}_{m_1m_2m_3} \mathbf{Z}_{m_3} \hat{\mathbf{B}}_{m_3}(t). \tag{41} $$

Substituting (40) and (41) into (1) yields to the following residual function:

$$ \begin{aligned} &\bar{\mathbf{B}}_{m_1m_2}^{\intercal}(x,y)\left[\boldsymbol{\Phi}_{m_1m_2m_3}\mathbf{Z}_{m_3} - \lambda\left[\left(\mathbf{Q}_{m_1m_2}^{(2)}\right)^{\intercal} + \left(\mathbf{P}_{m_1m_2}^{(2)}\right)^{\intercal}\right] \boldsymbol{\Phi}_{m_1m_2m_3} \mathbf{D}_{m_3}^{(1)} \right. \\ &\left. -\sigma\left[\left(\mathbf{Q}_{m_1m_2}^{(2)}\right)^{\intercal} + \left(\mathbf{P}_{m_1m_2}^{(2)}\right)^{\intercal}\right] \boldsymbol{\Phi}_{m_1m_2m_3}\right] \hat{\mathbf{B}}_{m_3}(t) - g(x,y,t) \triangleq R(x,y,t) \simeq 0. \end{aligned} \tag{42} $$

Moreover, from (37)–(39), we define

$$ \bar{\mathbf{B}}_{m_1m_2}^{\intercal}(x,y) \boldsymbol{\Phi}_{m_1m_2m_3} \hat{\mathbf{B}}_{m_3}(0) - \bar{\varphi}_0(x,y) \triangleq \Lambda_0(x,y) \simeq 0, \tag{43} $$

and

$$ \begin{aligned} &\bar{\mathbf{B}}_{m_1m_2}^{\intercal}(0,y) \boldsymbol{\Phi}_{m_1m_2m_3} \hat{\mathbf{B}}_{m_3}(t) - \bar{\varphi}_1(y,t) \triangleq \Lambda_1(y,t) \simeq 0, \\ &\bar{\mathbf{B}}_{m_1m_2}^{\intercal}(1,y) \boldsymbol{\Phi}_{m_1m_2m_3} \hat{\mathbf{B}}_{m_3}(t) - \bar{\varphi}_2(y,t) \triangleq \Lambda_2(y,t) \simeq 0 \\ &\bar{\mathbf{B}}_{m_1m_2}^{\intercal}(x,0) \boldsymbol{\Phi}_{m_1m_2m_3} \hat{\mathbf{B}}_{m_3}(t) - \bar{\varphi}_3(x,t) \triangleq \Lambda_3(x,t) \simeq 0, \\ &\bar{\mathbf{B}}_{m_1m_2}^{\intercal}(x,1) \boldsymbol{\Phi}_{m_1m_2m_3} \hat{\mathbf{B}}_{m_3}(t) - \bar{\varphi}_4(x,t) \triangleq \Lambda_4(x,t) \simeq 0. \end{aligned} \tag{44} $$

Now, from (42)–(44), we generate the below system:

$$ \begin{cases} R\left(x_i,y_j,t_l\right) = 0, & & i = 2,3,\dots,m_1,\ j = 2,3,\dots,m_2, \\ & & l = 2,3,\dots,m_3+1, \\ \Lambda_0\left(x_i,y_j\right) = 0, & & i = 1,2,\dots,m_1+1,\ j = 1,2,\dots,m_2+1, \\ \Lambda_r\left(y_j,t_l\right) = 0, & r = 1,2, & j = 1,2,\dots,m_2+1,\ l = 2,3,\dots,m_3+1, \\ \Lambda_r\left(x_i,t_l\right) = 0, & r = 3,4, & i = 2,3,\dots,m_1,\ l = 2,3,\dots,m_3+1, \end{cases} \tag{45} $$

where

$$ x_i = \frac{1}{2}\left(1 - \cos\left(\frac{(2i-1)\pi}{2\left(m_1+1\right)}\right)\right),\ y_j = \frac{1}{2}\left(1 - \cos\left(\frac{(2j-1)\pi}{2\left(m_2+1\right)}\right)\right), $$
$$ t_l = \frac{1}{2}\left(1 - \cos\left(\frac{(2l-1)\pi}{2\left(m_3+1\right)}\right)\right). $$

Finally, by solving the system generated in (45) and assigning $\boldsymbol{\Phi}_{m_1m_2m_3}$, we obtain a solution for the main distributed-fractional problem using (39). In this paper, we have used the "*fsolve*" command of Maple 18 (with precision 25 decimal digits) to solve the above system.

Test problems

Here, we inquire the reliability of the explained technique on four test problems. The below formulas are considered to measure the accuracy of the outcomes:

Table 1
The errors produced of the yielded outcomes with some values of (m_1, m_2, m_3) in Example 1.

(m_1, m_2, m_3)	$(4,3,3)$	$(5,4,4)$	$(6,5,5)$	$(7,6,6)$	$(8,7,7)$
E_φ^∞	3.8093×10^{-04}	1.3028×10^{-05}	1.0766×10^{-06}	1.3047×10^{-07}	7.1811×10^{-08}
E_φ^2	1.5748×10^{-04}	4.2093×10^{-06}	5.0174×10^{-07}	6.6659×10^{-08}	3.8147×10^{-08}

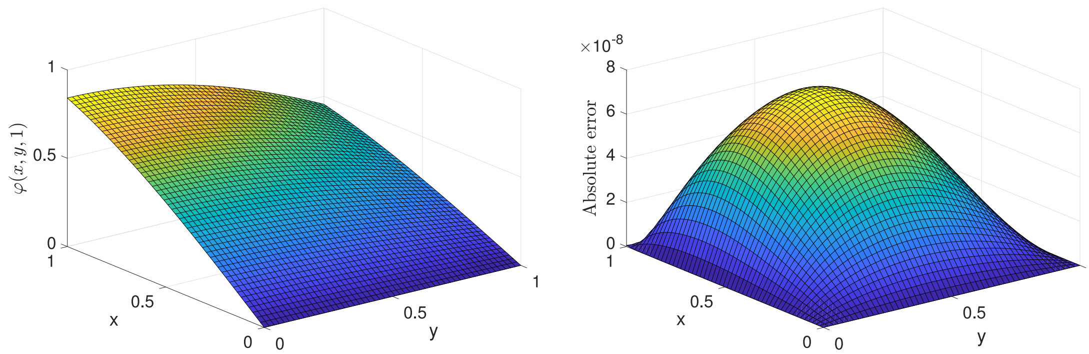

Fig. 1. The outcomes obtained for $\varphi(x, y, 1)$ (left) and related absolute error (right) with $(m_1 = 8, m_2 = m_3 = 7)$ in Example 1.

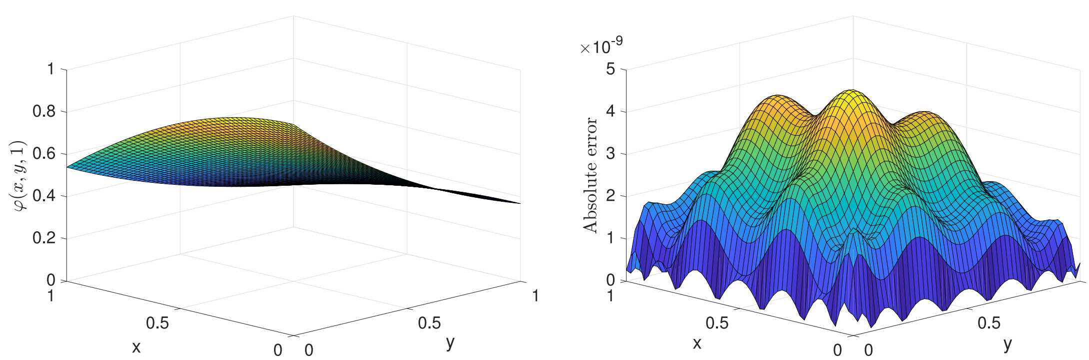

Fig. 2. The outcomes obtained for $\varphi(x, y, 1)$ (left) and related absolute error (right) with $(m_1 = m_2 = 7, m_3 = 8)$ in Example 2.

$$E_\varphi^\infty = \max_{(x,y)\in[0,1]\times[0,1]} |\varphi(x, y, T) - \tilde{\varphi}(x, y, T)|,$$

$$E_\varphi^2 = \left(\int_0^1 \int_0^1 (\varphi(x, y, T) - \tilde{\varphi}(x, y, T))^2 \, dx\, dy \right)^{1/2},$$

in which φ is the true solution and $\tilde{\varphi}$ is the extracted solution of the mentioned scheme.

Example 1. Consider Eq. (1) with $\mu(\gamma) = \Gamma(4 - \gamma)$, $\lambda = \sigma = 1$, $T = 1$ and

$$g(x, y, t) = \left(\frac{6t^2(t-1)}{\ln(t)} + 2t^3 + 6t^2 \right) \sin(x)\cos(y),$$

where the analytic solution is $\varphi(x, y, t) = t^3 \sin(x)\cos(y)$. Other information can be found using this solution. The methodology of Section "The proposed method" is used with some values of (m_1, m_2, m_3) for this example. The appeared errors are shown in Table 1. These results confirm that the numerical solutions converges to the analytic solution. In addition, with a small number of bases, high accuracy results can be obtained. We have provided Fig. 1 to illustrate the results obtained by $(m_1 = 8, m_2 = m_3 = 7)$.

Example 2. Consider the problem (1) where $\mu(\gamma) = \Gamma(5 - \gamma)$, $\lambda = 1$, $\sigma = 2$, $T = 1$ and

$$g(x, y, t) = \left(\frac{24t^3(t-1)}{\ln(t)} \right) \cos(x)e^{-y}.$$

The true solution is $\varphi(x, y, t) = t^4 \cos(x)e^{-y}$. We have utilized the expressed algorithm for this problem and provided the extracted outcomes in Table 2. The high capability of the stated method can be clearly seen from these results. The acquired outcomes for $(m_1 = m_2 = 7, m_3 = 8)$ are shown in Fig. 2.

Example 3. Consider the problem (1) where $\mu(\gamma) = \Gamma\left(\frac{11}{2} - \gamma\right)$, $\lambda = \sigma = 2$, $T = 1$ and

$$g(x, y, t) = \left(\frac{\Gamma\left(\frac{11}{2}\right) t^{\frac{7}{2}}(t-1)}{\ln(t)} - 18t^{\frac{7}{2}} - 4t^{\frac{9}{2}} \right) e^{x-y}.$$

The true solution is $\varphi(x, y, t) = t^{\frac{9}{2}} e^{x-y}$. The explained technique is applied for this test problem. The errors produced of the extracted outcomes with some values of (m_1, m_2, m_3) are provided in Table 3. We deduce from this table, very accurate results can be obtained using the stated method. The obtained outcomes for $(m_1 = m_2 = 8, m_3 = 9)$ are shown graphically in Fig. 3.

Example 4. Consider the problem (1) where $\mu(\gamma) = \Gamma(6 - \gamma)$, $\lambda = \sigma = 1/2$, $T = 2$ and

$$g(x, y, t) = \left(\frac{10t^4(t-1)}{\ln(t)} \right) e^{-x} \sin(y).$$

The true solution is $\varphi(x, y, t) = t^5 e^{-x} \sin(y)$. We have used the proposed method for this problem and listed the obtained outcomes in Table 4. The high accuracy of the method can be clearly observed from these results. The obtained outcomes for $(m_1 = m_2 = 8, m_3 = 9)$ are shown in Fig. 4.

Table 2
The errors produced of the yielded outcomes with some values of (m_1, m_2, m_3) in Example 2.

(m_1, m_2, m_3)	$(3, 3, 4)$	$(4, 4, 5)$	$(5, 5, 6)$	$(6, 6, 7)$	$(7, 7, 8)$
E_φ^∞	7.4884×10^{-04}	2.7261×10^{-05}	1.4295×10^{-06}	4.9140×10^{-08}	4.9824×10^{-09}
E_φ^2	3.0392×10^{-04}	1.0068×10^{-05}	4.7857×10^{-07}	1.9651×10^{-08}	2.7157×10^{-09}

Table 3
The errors produced of the yielded outcomes with some values of (m_1, m_2, m_3) in Example 3.

(m_1, m_2, m_3)	$(4, 4, 5)$	$(5, 5, 6)$	$(6, 6, 7)$	$(7, 7, 8)$	$(8, 8, 9)$
E_φ^∞	1.4700×10^{-04}	8.0105×10^{-06}	1.4971×10^{-06}	3.3446×10^{-07}	9.4196×10^{-08}
E_φ^2	4.7601×10^{-05}	2.7756×10^{-06}	6.7180×10^{-07}	1.4639×10^{-07}	4.5926×10^{-08}

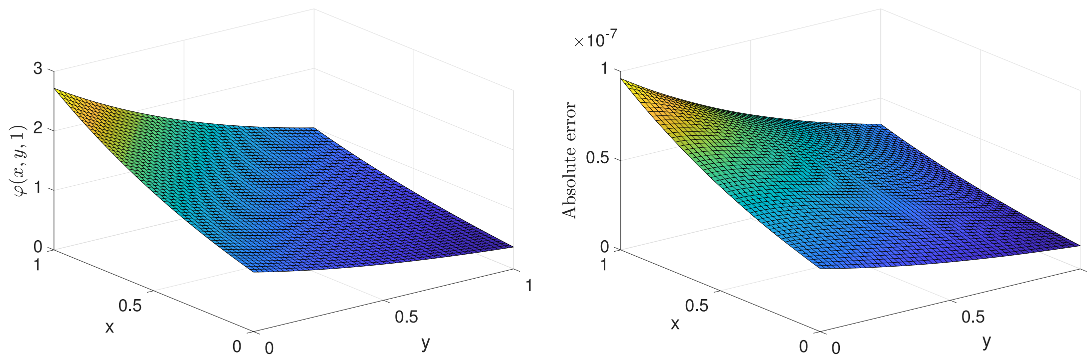

Fig. 3. The outcomes obtained for $\varphi(x, y, 1)$ (left) and related absolute error (right) with $(m_1 = m_2 = 8, m_3 = 9)$ in Example 1.

Table 4
The errors produced of the yielded outcomes with some values of (m_1, m_2, m_3) in Example 4.

(m_1, m_2, m_3)	$(4, 4, 5)$	$(5, 5, 6)$	$(6, 6, 7)$	$(7, 7, 8)$	$(8, 8, 9)$
E_φ^∞	9.5734×10^{-04}	2.2825×10^{-05}	1.0905×10^{-06}	6.2312×10^{-07}	9.4196×10^{-08}
E_φ^2	3.0603×10^{-04}	6.8508×10^{-06}	4.0792×10^{-07}	4.3150×10^{-07}	4.5926×10^{-08}

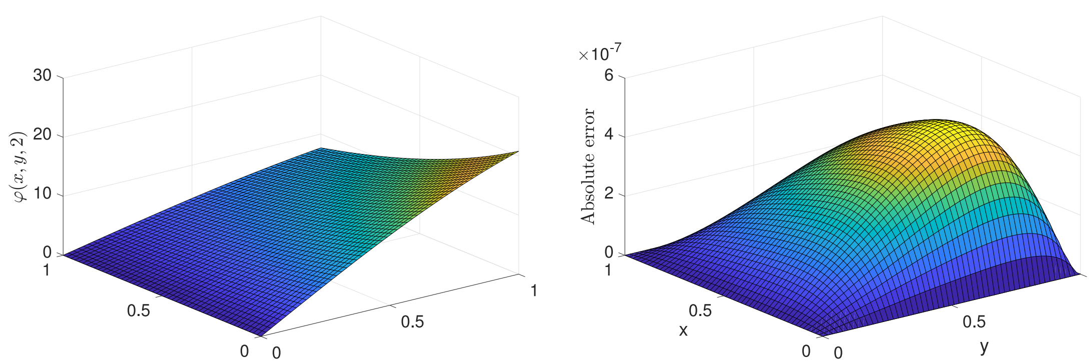

Fig. 4. The outcomes obtained for $\varphi(x, y, 2)$ (left) and related absolute error (right) with $(m_1 = m_2 = 8, m_3 = 9)$ in Example 1.

Conclusion

This paper introduced the distributed-order time fractional 2D Sobolev equation. The one variable and two variables Bernoulli polynomials were introduced to construct a numerical method for this equation. Several operational matrices regarding derivatives of these polynomials were obtained. In the developed method, by approximating the unknown solution using these polynomials and applying derived matrix relations, an algebraic system of equations was obtained, and by solving it, a solution for the main equation was obtained. By solving four test problems, it was proved that the introduced method for solving this problem is a suitable method with high accuracy. Considering that if the solution of a problem is sufficiently smooth, the numerical methods based on polynomials give numerical results with high accuracy, the method presented in this article can be developed for other fractional problems with smooth solutions. For problems which their solutions are in fractional form (not sufficiently smooth), the fractional form of Bernoulli functions can be used to obtain high-accuracy solutions.

CRediT authorship contribution statement

M.H. Heydari: Conceptualization, Methodology, Software, Validation, Visualization, Writing – original draft, Editing, Project administration. **S. Rashid:** Conceptualization, Validation, Methodology, Software, Editing. **F. Jarad:** Conceptualization, Validation, Methodology, Software, Editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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