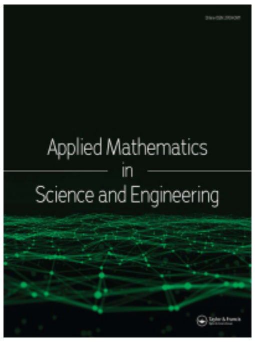

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Analysis of the family of integral equation involving incomplete types of I and $\bar{I}$ -functions

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ABSTRACT
The present article introduces and studies the Fredholm-type integral equation with an incomplete I-function (IIF) and an incomplete $\bar{I}$-function (I$\bar{I}$F) in its kernel. First, using fractional calculus and the Mellin transform principle, we solve an integral problem involving IIF. The idea of the Mellin transform and fractional calculus is then used to analyse an integral equation using the incomplete $\bar{I}$-function. This is followed by the discovery and investigation of several important exceptional cases. This article's general discoveries may yield new integral equations and solutions. The desired outcomes seem to be very helpful in resolving many real-world problems whose solutions represent different physical phenomena. And also, findings help solve introdifferential, fractional differential, and extended integral equation problems.

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1. Introduction

Over the last four decades, mathematicians and scientists have been attracted to fractional calculus and special functions because of their wide range of applications and significance in fields such as medical science, biological science, computer science, communication theory, fluid dynamics, viscoelasticity, diffusive transport, electrical finance networks, signal processing, probability theory, control theory, ecology, environmental science, and so on [1–4].

The scope of special functions is extensive, yet it constantly expands due to the development of new issues in engineering and applied science fields. In addition, the development of the H-function and the I-function is facilitated by dissemination. Jangid et al. [5] proposed the incomplete I-function and developed various integral transformations for it. A few applications are also presented [6, 7]. The integral equation has been observed in various response-related problems, including diffusion, queuing theory, reaction-diffusion,

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quantum mechanics and a variety of other areas of physics, biology, and probability theory, to name a few [8–10].

Fractional calculus is an augmentation of integer-order calculus and provides more accurate results than classical calculus. Therefore, it is widely used in the mathematical modelling of almost all science and engineering, medicine, and education areas [11–13]. Several fractional operators are available to deal with real-world problems, such as the Riemann–Liouville integral, Caputo derivative, Caputo–Fabrizio derivatives, Weyl integral, Weyl derivatives, Atangana–Balneau derivatives, Atangana–Balneau fractional integral, Hilfer fractional derivatives, and many others.

Methods for obtaining solutions to integral equations (IE) are typically beneficial in science and engineering. That is why we choose Fredholm IE, which consists of incomplete I-function, which are extensions and generalizations of higher transcendental functions. The most commonly used functions in mathematics, physics, engineering, and mathematical biology are special cases of the incomplete I-function.

The Fredholm IE, which incorporates special functions like Hypergeometric functions, Legendre functions, and Fox H-functions, is presented and explored by many authors [14–20]. We present the integral equation of the Fredholm type involving the IIF and I$\bar{I}$F in the kernel, which was inspired by a recent research endeavour on fractional calculus and special functions.

2. Mathematical preliminaries

The present part defines some basic definitions of the special functions and fractional operators.

Incomplete Gamma Function: The usual incomplete Gamma functions $\gamma(\mathfrak{c}, s)$ and $\Gamma(\mathfrak{c}, s)$ represented by Chaudhry and Zubai [21]

$$\gamma(\mathfrak{c}, s) := \int_0^s \theta^{\mathfrak{c}-1} e^{-\theta} \mathrm{d}\theta, \quad (\Re(\mathfrak{c}) > 0; s \geq 0), \tag{1}$$

and

$$\Gamma(\mathfrak{c}, s) := \int_s^\infty \theta^{\mathfrak{c}-1} e^{-\theta} \mathrm{d}\theta, \quad (\Re(\mathfrak{c}) > 0; s \geq 0), \tag{2}$$

satisfy the subsequent rule of decomposition:

$$\gamma(\mathfrak{c}, s) + \Gamma(\mathfrak{c}, s) := \Gamma(\mathfrak{c}), \quad (\Re(\mathfrak{c}) > 0), \tag{3}$$

where $\Re(\mathfrak{c})$ stands for real part of the parameter $\mathfrak{c}$.

Moreover, if we set $s = 0$, then we have $\Gamma(\mathfrak{c}, s) = \Gamma(\mathfrak{c})$.

I-Function: Rathie [22] discovered the I-function in 1997, which is defined as follows by the Mellin–Barnes kind contour integral:

$$I_{r,s}^{u,v}(\mathscr{V}) = I_{r,s}^{u,v}\left[\mathscr{V} \,\middle|\, \begin{matrix} (\Psi_1, \zeta_1; A_1), \cdots, (\Psi_r, \zeta_r; A_r) \\ (\Phi_1, \beta_1; B_1), \cdots, (\Phi_s, \beta_s; B_s) \end{matrix}\right] = \frac{1}{2\pi i} \int_{\$} \Psi(w) \mathscr{V}^w \mathrm{d}w, \tag{4}$$

where

$$\Psi(w) = \frac{\prod_{i=1}^{u} \{\Gamma(\Phi_i - \beta_i w)\}^{B_i} \prod_{i=1}^{v} \{\Gamma(1 - \Psi_i + \zeta_i w)\}^{A_i}}{\prod_{i=v+1}^{r} \{\Gamma(\Psi_i - \zeta_i w)\}^{A_i} \prod_{i=u+1}^{s} \{\Gamma(1 - \Phi_i + \beta_i w)\}^{B_i}}. \tag{5}$$

The appropriate conditions for the \$ contour convergence described in (4) and other representations, in addition to documentation about the I-function, can be seen in [22].

The Incomplete I-Functions: Now, we present a family of the incomplete I-functions [23] ${}^{\gamma}I_{r,s}^{u,v}(\mathscr{V})$ and ${}^{\Gamma}I_{r,s}^{u,v}(\mathscr{V})$, which leads to a natural generalization of a variety of I-functions:

$$\begin{aligned} {}^{\gamma}I_{r,s}^{u,v}(\mathscr{V}) &= {}^{\gamma}I_{r,s}^{u,v}\left[\mathscr{V}\,\middle|\, \begin{matrix} (\Psi_1,\zeta_1;A_1:Y),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r) \\ (\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s) \end{matrix}\right] \\ &= \frac{1}{2\pi i}\int_{\$}\Psi(w,Y)\mathscr{V}^{w}\mathrm{d}w, \end{aligned} \tag{6}$$

and

$$\begin{aligned} {}^{\Gamma}I_{r,s}^{u,v}(\mathscr{V}) &= {}^{\Gamma}I_{r,s}^{u,v}\left[\mathscr{V}\,\middle|\, \begin{matrix} (\Psi_1,\zeta_1;A_1:Y),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r) \\ (\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s) \end{matrix}\right] \\ &= \frac{1}{2\pi i}\int_{\$}\Phi(w,Y)\mathscr{V}^{w}\mathrm{d}w, \end{aligned} \tag{7}$$

for all $\mathscr{V}\neq 0$, where

$$\Psi(w,Y)=\frac{\{\gamma(1-\Psi_1+\zeta_1 w,Y)\}^{A_1}\prod_{i=1}^{u}\{\Gamma(\Phi_i-\beta_i w)\}^{B_i}\prod_{i=2}^{v}\{\Gamma(1-\Psi_i+\zeta_i w)\}^{A_i}}{\prod_{i=v+1}^{r}\{\Gamma(\Psi_i-\zeta_i w)\}^{A_i}\prod_{i=u+1}^{s}\{\Gamma(1-\Phi_i+\beta_i w)\}^{B_i}}, \tag{8}$$

and

$$\Phi(w,Y)=\frac{\{\Gamma(1-\Psi_1+\zeta_1 w,Y)\}^{A_1}\prod_{i=1}^{u}\{\Gamma(\Phi_i-\beta_i w)\}^{B_i}\prod_{i=2}^{v}\{\Gamma(1-\Psi_i+\zeta_i w)\}^{A_i}}{\prod_{i=v+1}^{r}\{\Gamma(\Psi_i-\zeta_i w)\}^{A_i}\prod_{i=u+1}^{s}\{\Gamma(1-\Phi_i+\beta_i w)\}^{B_i}}. \tag{9}$$

The following division relation is immediately produced by the definitions (6) and (7) for the value of $A_1 = 1$:

$${}^{\gamma}I_{r,s}^{u,v}[\mathscr{V}]+{}^{\Gamma}I_{r,s}^{u,v}[\mathscr{V}]=I_{r,s}^{u,v}[\mathscr{V}],\quad (\text{for } A_1=1). \tag{10}$$

The Incomplete $\bar{I}$-Function: When we fixed $B_1 = B_2 = \cdots = B_u = 1$ and $A_{v+1} = A_{v+2} = \cdots = A_r = 1$, then define the following new incomplete $\bar{I}$-function (I$\bar{I}$F) [5]:

$$\begin{aligned} {}^{\gamma}\bar{I}_{r,s}^{u,v}(\mathscr{V}) = {}^{\gamma}\bar{I}_{r,s}^{u,v}\Bigg[\mathscr{V}\,\Bigg|\, & \begin{matrix} (\Phi_1,\zeta_1;A_1:Y),(\Phi_2,\zeta_2;A_2),\cdots,(\Phi_v,\zeta_v;A_v), \\ (\Phi_1,\beta_1;1),(\Phi_2,\beta_2;1),\cdots,(\Phi_u,\beta_u;B_u), \end{matrix} \\ & \begin{matrix} (\Phi_{v+1},\zeta_{v+1};1),\cdots,(\Psi_r,\zeta_r;1) \\ (\Phi_{u+1},\beta_{u+1};B_{u+1}),\cdots,(\Phi_s,\beta_s;B_s) \end{matrix}\Bigg] = \frac{1}{2\pi i}\int_{\$}\bar{\Psi}(w,Y)\mathscr{V}^{w}\mathrm{d}w, \end{aligned} \tag{11}$$

and

$$\begin{aligned} {}^{\Gamma}\bar{I}_{r,s}^{u,v}(\mathscr{V}) = {}^{\Gamma}\bar{I}_{r,s}^{u,v}\Bigg[\mathscr{V}\,\Bigg|\, & \begin{matrix} (\Phi_1,\zeta_1;A_1:Y),(\Phi_2,\zeta_2;A_2),\cdots,(\Phi_v,\zeta_v;A_v), \\ (\Phi_1,\beta_1;1),(\Phi_2,\beta_2;1),\cdots,(\Phi_u,\beta_u;B_u), \end{matrix} \\ & \begin{matrix} (\Phi_{v+1},\zeta_{v+1};1),\cdots,(\Psi_r,\zeta_r;1) \\ (\Phi_{u+1},\beta_{u+1};B_{u+1}),\cdots,(\Phi_s,\beta_s;B_s) \end{matrix}\Bigg] = \frac{1}{2\pi i}\int_{\$}\bar{\Phi}(w,Y)\mathscr{V}^{w}\mathrm{d}w, \end{aligned} \tag{12}$$

for all $\mathscr{V} \neq 0$, where

$$\bar{\Psi}(w,Y) = \frac{\{\gamma(1-\Psi_1+\zeta_1 w, Y)\}^{A_1}\prod_{i=1}^{u}\{\Gamma(\Phi_i-\beta_i w)\}^1\prod_{i=2}^{v}\{\Gamma(1-\Psi_i+\zeta_i w)\}^{A_i}}{\prod_{i=v+1}^{r}\{\Gamma(\Psi_i-\zeta_i w)\}^1\prod_{i=u+1}^{s}\{\Gamma(1-\Phi_i+\beta_i w)\}^{B_i}}, \tag{13}$$

and

$$\bar{\Phi}(w,Y) = \frac{\{\Gamma(1-\Psi_1+\zeta_1 w, Y)\}^{A_1}\prod_{i=1}^{u}\{\Gamma(\Phi_i-\beta_i w)\}^1\prod_{i=2}^{v}\{\Gamma(1-\Psi_i+\zeta_i w)\}^{A_i}}{\prod_{i=v+1}^{r}\{\Gamma(\Psi_i-\zeta_i w)\}^1\prod_{i=u+1}^{s}\{\Gamma(1-\Phi_i+\beta_i w)\}^{B_i}}. \tag{14}$$

It we put $Y = 0$ in (12), then we get familiar $\bar{I}$ suggest by Rathie [22].

The IIF ${}^{\gamma}I_{r,s}^{u,v}(\mathscr{V})$ and ${}^{\Gamma}I_{r,s}^{u,v}(\mathscr{V})$ identified in (6) and (7) appear for $Y \geq 0$, according to the family of restrictions provided by Rathie [22], such as

$$\delta > 0, |\arg(\mathscr{V})| < \frac{1}{2}\delta\pi,$$

where

$$\delta = \sum_{i=1}^{u} B_i\beta_i - \sum_{i=u+1}^{s} B_i\beta_i + \sum_{i=1}^{v} A_i\zeta_i - \sum_{i=v+1}^{r} A_i\zeta_i. \tag{15}$$

Remark 2.1: Setting $Y = 0$, in (6) and (7) gives the I-Function determined by Rathie

$$\begin{aligned}{}^{\Gamma}I_{r,s}^{u,v}&\left[\mathscr{V}\,\middle|\,\begin{matrix}(\Psi_1,\zeta_1;A_1:0),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r)\\(\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s)\end{matrix}\right]\\&= I_{r,s}^{u,v}\left[\mathscr{V}\,\middle|\,\begin{matrix}(\Psi_1,\zeta_1;A_1),\cdots,(\Psi_r,\zeta_r;A_r)\\(\Phi_1,\beta_1;B_1),\cdots,(\Phi_s,\beta_s;B_s)\end{matrix}\right].\end{aligned} \tag{16}$$

Remark 2.2: When $A_i = 1, B_j = 1 (i = 1,2,\ldots,r, j = 1,2,\ldots,s)$ is set to (6) and (7), it becomes the Incomplete H-Function proposed by Srivastava

$$\begin{aligned}{}^{\gamma}I_{r,s}^{u,v}&\left[\mathscr{V}\,\middle|\,\begin{matrix}(\Psi_1,\zeta_1;1:Y),(\Psi_2,\zeta_2;1),\cdots,(\Psi_r,\zeta_r;1)\\(\Phi_1,\beta_1;1),(\Phi_2,\beta_2;1),\cdots,(\Phi_s,\beta_s;1)\end{matrix}\right]\\&= \gamma_{r,s}^{u,v}\left[\mathscr{V}\,\middle|\,\begin{matrix}(\Psi_1,\zeta_1:Y),(\Psi_2,\zeta_2),\cdots,(\Psi_r,\zeta_r)\\(\Phi_1,\beta_1),(\Phi_2,\beta_2),\cdots,(\Phi_s,\beta_s)\end{matrix}\right],\end{aligned} \tag{17}$$

and

$$\begin{aligned}{}^{\Gamma}I_{r,s}^{u,v}&\left[\mathscr{V}\,\middle|\,\begin{matrix}(\Phi_1,\zeta_1;1:Y),(\Phi_2,\zeta_2;1),\cdots,(\Phi_r,\zeta_r;1)\\(\Phi_1,\beta_1;1),(\Phi_2,\beta_2;1),\cdots,(\Phi_s,\beta_s;1)\end{matrix}\right]\\&= \Gamma_{r,s}^{u,v}\left[\mathscr{V}\,\middle|\,\begin{matrix}(\Psi_1,\zeta_1:Y),(\Psi_2,\zeta_2),\cdots,(\Psi_r,\zeta_r)\\(\Phi_1,\beta_1),(\Phi_2,\beta_2),\cdots,(\Phi_s,\beta_s)\end{matrix}\right].\end{aligned} \tag{18}$$

Remark 2.3: Next, we take $Y = 0, A_i = 1, B_j = 1 (i = 1,2,\ldots,r, j = 1,2,\ldots,s)$ in (6). The IIF is reduced to the well-known Fox H-function, defined and illustrated as follows

$$\begin{aligned}{}^{\Gamma}I_{r,s}^{u,v}&\left[\mathscr{V}\,\middle|\,\begin{matrix}(\Psi_1,\zeta_1;1:0),(\Psi_2,\zeta_2;1),\cdots,(\Psi_r,\zeta_r;1)\\(\Phi_1,\beta_1;1),(\Phi_2,\beta_2;1),\cdots,(\Phi_s,\beta_s;1)\end{matrix}\right]\\&= H_{r,s}^{u,v}\left[\mathscr{V}\,\middle|\,\begin{matrix}(\Psi_1,\zeta_1),(\Psi_2,\zeta_2),\cdots,(\Psi_r,\zeta_r)\\(\Phi_1,\beta_1),(\Phi_2,\beta_2),\cdots,(\Phi_s,\beta_s)\end{matrix}\right].\end{aligned} \tag{19}$$

Weyl Fractional Integral: The standard definition of the Weyl fractional integral (WFI) as given by Miller and Ross [24]:

$$\mathscr{W}^{-u}\phi(w) = \frac{1}{\Gamma(u)}\int_w^\infty (\xi - w)^{u-1}\phi(\xi)\mathrm{d}\xi, \quad (\Re(u) > 0). \tag{20}$$

Mellin Transform: The standard definition of the Mellin transform (MT) is given as follows [25, 26]:

$$\mathscr{M}[\phi(w);\mathcal{P}] = \hat{\phi}(\mathcal{P}) = \int_0^\infty w^{\mathcal{P}-1}\phi(w)\mathrm{d}w, \tag{21}$$

and

$$\mathscr{M}^{-1}[\hat{\phi}(\mathcal{P}); w] = \phi(w) = \frac{1}{2\pi\iota}\int_{\nabla-\iota\infty}^{\nabla+\iota\infty} w^{-\mathcal{P}}\hat{\phi}(\mathcal{P})\mathrm{d}\mathcal{P}, \tag{22}$$

where $\Re(\mathcal{P}) > 0$ and ∇ is constant.

Next, we provide two significant results incorporated into our main findings.

(1) The MT of I*I*F [5] is defined as follows:

$$\mathscr{M}\left\{ {}^{\Gamma}I^{u,v}_{r,s}\left[\mathscr{V}\mathcal{V}^{\$}\left| \begin{matrix} (\varPsi_1,\zeta_1;A_1 : Y),(\varPsi_2,\zeta_2;A_2),\cdots,(\varPsi_r,\zeta_r;A_r) \\ (\varPhi_1,\beta_1;B_1),(\varPhi_2,\beta_2;B_2),\cdots,(\varPhi_s,\beta_s;B_s) \end{matrix} \right.\right];\mathcal{P}\right\}$$
$$= \frac{\mathscr{V}^{-\frac{\mathcal{P}}{\$}}}{\$}\Psi\left(-\frac{\mathcal{P}}{\$}, Y\right), \tag{23}$$

where $\Psi(-\frac{\mathcal{P}}{\$}, Y)$ is defined in Equation (8) and the terms are given in [5].

(2) The MT of I$\bar{I}$F is defined as follows:

$$\mathscr{M}\left\{ {}^{\Gamma}\bar{I}^{u,v}_{r,s}\left[\mathscr{V}\mathcal{V}^{\$}\left| \begin{matrix} (\varPsi_1,\zeta_1;A_1 : Y),(\varPsi_2,\zeta_2;A_2),\cdots,(\varPsi_v,\zeta_v;A_v), \\ (\varPhi_1,\beta_1;1),(\varPhi_2,\beta_2;1),\cdots,(\varPhi_u,\beta_u;B_u), \end{matrix} \right.\right.\right.$$
$$\left.\left.\begin{matrix} (\varPsi_{v+1},\zeta_{v+1};1),\cdots,(\varPsi_r,\zeta_r;1) \\ (\varPhi_{u+1},\beta_{u+1};B_{u+1}),\cdots,(\varPhi_s,\beta_s;B_s) \end{matrix}\right];\mathcal{P}\right\} = \frac{\mathscr{V}^{-\frac{\mathcal{P}}{\$}}}{\$}\bar{\Phi}\left(-\frac{\mathcal{P}}{\$}, Y\right), \tag{24}$$

where $\bar{\Phi}(-\frac{\mathcal{P}}{\$}, Y)$ is defined in Equation (14).

Assume that $\mathscr{A}$ is the space with whole functions $\mathcal{F}$ that satisfy and are properly characterized on $\mathbb{R}_{0+} = [0,\infty)$.

(a) $\mathcal{F} \in \mathscr{C}(\mathbb{R}_{0+})$;
(b) $\lim_{x\to\infty}\{x^{\mathfrak{a}}\mathcal{F}^{\mathfrak{N}}(x)\} = 0$, (For all $\mathfrak{a},\mathfrak{N} \in \mathcal{Z}_{0+}$); ($\mathcal{Z}_{0+} = 0,1,2,\ldots$), and
(c) $\mathcal{F}(x) = \mathrm{O}(1), x \to 0$.

See Lighthill's [27] work for more information on the space of suitable functions expressed on the entire real line $(-\infty,\infty)$.

3. Solution of an integral equation of Fredholm type utilizing I*I*F

In this section, using the MT method and the well-recognized WFI, we provide the solution to the Fredholm-type integral problem involving the I*I*F.

Lemma 3.1: *Let*

(A) $u, v, r, s \in \mathcal{Z}_{0+}$ *such that (s.t.)* $0 \leq v \leq r$ *and* $1 \leq u \leq s$,
(B) $\Re(\Lambda - \kappa) > 0$; $\Re(\kappa) + \$\Re(\frac{\Phi_i}{B_i}) > 0$, $(i = 1, 2, \dots, u)$,
(C) $Y \geq 0, \$ > 0$, *and* $\Lambda \in \mathbb{C}$,
(D) $|\arg(C)| < \frac{1}{2}\pi\delta$, *provided* δ *is characterized in the relation* (15).

Then,

$$\mathscr{W}^{\kappa-\Lambda}\left\{\mathcal{V}^{-\Lambda}{}^{\Gamma}I^{u,v}_{r,s}\left[C\left(\frac{X}{\mathcal{V}}\right)^{\$}\middle|\begin{matrix}(\Psi_1,\zeta_1;A_1:Y),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r)\\(\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s)\end{matrix}\right]\right\}$$
$$=\mathcal{V}^{-\kappa}{}^{\Gamma}I^{u,v+1}_{r+1,s+1}\left[C\left(\frac{X}{\mathcal{V}}\right)^{\$}\middle|\begin{matrix}(\Psi_1,\zeta_1;A_1:Y),(1-\kappa,\$;1),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r)\\(\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s),(1-\Lambda,\$;1)\end{matrix}\right]. \quad (25)$$

Proof: We first consider the integral contour framework of the I*I*F provided in relation (6) and then change the sequence of integrals to demonstrate the assertion along with (25). We next apply the WFI defined in Equation (20) (within the specified permitted conditions).

$$\mathscr{W}^{\kappa-\Lambda}\left\{\mathcal{V}^{-\Lambda}{}^{\Gamma}I^{u,v}_{r,s}\left[C\left(\frac{X}{\mathcal{V}}\right)^{\$}\middle|\begin{matrix}(\Psi_1,\zeta_1;A_1:Y),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r)\\(\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s)\end{matrix}\right]\right\}$$
$$=\frac{1}{\Gamma(\Lambda-\kappa)}\int_{\mathcal{V}}^{\infty}(U-\mathcal{V})^{\Lambda-\kappa-1}\left(U^{-\Lambda}\frac{1}{2\pi\iota}\int_{\$}C^{w}\left(\frac{X}{U}\right)^{\$w}\Phi(w,Y)\mathrm{d}w\right)\mathrm{d}U$$
$$=\frac{1}{2\pi\iota}\frac{1}{\Gamma(\Lambda-\kappa)}\int_{\$}C^{w}X^{\$w}\Phi(w,Y)\cdot\left(\int_{\mathcal{V}}^{\infty}(U-\mathcal{V})^{\Lambda-\kappa-1}U^{-\Lambda-\$w}\mathrm{d}U\right)\mathrm{d}w$$
$$=\mathcal{V}^{-\kappa}\frac{1}{2\pi\iota}\int_{\$}\Phi(w,Y)C^{w}\left(\frac{X}{\mathcal{V}}\right)^{\$w}\frac{\Gamma(\kappa+\$w)}{\Gamma(\Lambda+\$w)}\mathrm{d}w$$
$$=\mathcal{V}^{-\kappa}{}^{\Gamma}I^{u,v+1}_{r+1,s+1}\left[C\left(\frac{X}{\mathcal{V}}\right)^{\$}\middle|\begin{matrix}(\Psi_1,\zeta_1;A_1:Y),(1-\kappa,\$;1),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r)\\(\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s),(1-\Lambda,\$;1)\end{matrix}\right].$$

We can easily achieve the desired result by evaluating the power function at the WFI and thereupon trying to interpret the arising Mellin Barnes contour integral in terms of I*I*F. ■

Lemma 3.2: *Suppose*

(A) $u, v, r, s \in \mathcal{Z}_{0+}$ *s.t.* $1 \leq u \leq s$ *and* $0 \leq v \leq r$,
(B) $\Re(\Lambda - \kappa) > 0$; $\Re(\kappa) + \$\Re(\frac{\Phi_i}{B_i}) > 0$, $(i = 1, 2, \dots, u)$,
(C) $Y \geq 0, \$ > 0$, *and* $\Lambda \in \mathbb{C}$,
(D) $|\arg(C)| < \frac{1}{2}\pi\delta$, *provided* δ *is characterized in the relation* (15).

Then,

$$\mathscr{W}^{\kappa-\Lambda}\left\{\mathcal{V}^{-\Lambda\gamma}I_{r,s}^{u,v}\left[C\left(\frac{X}{\mathcal{V}}\right)^{\$}\middle|\begin{array}{c}(\Psi_1,\zeta_1;A_1:Y),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r)\\(\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s)\end{array}\right]\right\}$$
$$=\mathcal{V}^{-\kappa\gamma}I_{r+1,s+1}^{u,v+1}\left[C\left(\frac{X}{\mathcal{V}}\right)^{\$}\middle|\begin{array}{c}(\Psi_1,\zeta_1;A_1:Y),(1-\kappa,\$;1),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r)\\(\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s),(1-\Lambda,\$;1)\end{array}\right]. \quad (26)$$

Proof: We won't describe it in detail here because the proof is identical to Lemma 3.1. ■

Theorem 3.3: *Letting*

(A) $u,v,r,s\in\mathcal{Z}_{0+}$ *s.t.* $0\le v\le r$ *and* $1\le u\le s$,
(B) Ψ_r,β_s *are positive real numbers,*
(C) $\Re(\kappa)+\$(\frac{\Psi_i-1}{A_i})<0;\ \Re(\kappa)+\$\Re(\frac{\Phi_i}{B_i})>0$, $(i=1,2,\dots,u)$, $(i=1,2,\dots,v)$,
(D) $Y\ge 0,\$>0$, *and* $\Lambda\in\mathbb{C}$.

Consequently, the subsequent integral relationship formula holds:

$$\int_0^\infty {}^{\Gamma}I_{r+1,s+1}^{u,v+1}\left[C\left(\frac{X}{\mathcal{V}}\right)^{\$}\middle|\begin{array}{c}(\Psi_1,\zeta_1;A_1:Y),(1-\kappa,\$;1),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r)\\(\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s),(1-\Lambda,\$;1)\end{array}\right]$$
$$\times\mathcal{V}^{-\kappa}\phi(\mathcal{V})\mathrm{d}\mathcal{V}$$
$$=\int_0^\infty {}^{\Gamma}I_{r,s}^{u,v}\left[C\left(\frac{X}{\mathcal{V}}\right)^{\$}\middle|\begin{array}{c}(\Psi_1,\zeta_1;A_1:Y),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r)\\(\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s)\end{array}\right]$$
$$\times\mathcal{V}^{-\kappa}\mathscr{D}^{\kappa-\Lambda}\{\phi(\mathcal{V})\}\mathrm{d}\mathcal{V}, \quad (27)$$

with the result that $\phi\in\mathscr{A}$ *and* $X>0$.

Proof: Suppose $\mathscr{G}$ denotes the first component of the statement in Equation (27) of Theorem 3.3. Thereupon, using Lemma 3.1 and the definition in Equation (20), we obtain

$$\mathscr{G}=\int_0^\infty\phi(\mathcal{V})\left(\int_{\mathcal{V}}^\infty\frac{(U-\mathcal{V})^{\Lambda-\kappa-1}}{\Gamma(\Lambda-\kappa)}U^{-\Lambda}\right.$$
$$\left.{}^{\Gamma}I_{r,s}^{u,v}\left[C\left(\frac{X}{U}\right)^{\$}\middle|\begin{array}{c}(\Psi_1,\zeta_1;A_1:Y),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r)\\(\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s)\end{array}\right]\mathrm{d}U\right)\mathrm{d}\mathcal{V}.$$

Next, by altering the order of integration under the allowable circumstances, we obtain

$$\mathscr{G}=\int_0^\infty U^{-\Lambda}\,{}^{\Gamma}I_{r,s}^{u,v}\left[C\left(\frac{X}{U}\right)^{\$}\middle|\begin{array}{c}(\Psi_1,\zeta_1;A_1:Y),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r)\\(\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s)\end{array}\right]$$
$$\times\left(\int_0^U\frac{(U-\mathcal{V})^{\Lambda-\kappa-1}}{\Gamma(\Lambda-\kappa)}\phi(\mathcal{V})\mathrm{d}\mathcal{V}\right)\mathrm{d}U.$$

Moreover, using the widely used definition of the Riemann Liouville (RL) fractional derivative, we get

$$\mathscr{G} = \int_0^{\infty} {}^{\Gamma}I_{r,s}^{u,v}\left[C\left(\frac{X}{U}\right)^{\$}\middle|\begin{array}{c}(\Psi_1,\zeta_1;A_1:Y),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r)\\(\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s)\end{array}\right] \times U^{-\Lambda}\mathscr{D}^{\kappa-\Lambda}\{\phi(U)\}\mathrm{d}U,$$

which is the right-handed component of Equation (27). ■

Theorem 3.4: *Let*

(A) $u,v,r,s \in \mathcal{Z}_{0+}$ *s.t.* $1 \le u \le s$ *and* $0 \le v \le r$,
(B) Ψ_r, β_s *are positive real numbers,*
(C) $\Re(\kappa) + \$(\frac{\Psi_i - 1}{A_i}) < 0; (i = 1,2,\ldots,v), \Re(\kappa) + \$\Re(\frac{\Phi_i}{B_i}) > 0, (i = 1,2,\ldots,u)$,
(D) $Y \ge 0, \$ > 0$, *and* $\Lambda \in \mathbb{C}$.

Consequently, the subsequent integral relationship formula holds:

$$\int_0^{\infty} {}^{\gamma}I_{r+1,s+1}^{u,v+1}\left[C\left(\frac{X}{\mathcal{V}}\right)^{\$}\middle|\begin{array}{c}(\Psi_1,\zeta_1;A_1:Y),(1-\kappa,\$;1),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r)\\(\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s),(1-\Lambda,\$;1)\end{array}\right] \times \mathcal{V}^{-\kappa}\phi(\mathcal{V})\mathrm{d}\mathcal{V}$$
$$= \int_0^{\infty} {}^{\gamma}I_{r,s}^{u,v}\left[C\left(\frac{X}{\mathcal{V}}\right)^{\$}\middle|\begin{array}{c}(\Psi_1,\zeta_1;A_1:Y),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r)\\(\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s)\end{array}\right] \times \mathcal{V}^{-\kappa}\mathscr{D}^{\kappa-\Lambda}\{\phi(\mathcal{V})\}\mathrm{d}\mathcal{V}, \tag{28}$$

with the result that $\phi \in \mathscr{A}$ *and* $X > 0$.

Proof: We won't describe it in detail here because the proof is identical to Theorem 3.3. ■

Theorem 3.5: *Make the assumption that*

(A) $u,v,r,s \in \mathcal{Z}_{0+}$ *s.t.* $1 \le u \le s$ *and* $0 \le v \le r$,
(B) Ψ_r, β_s *are positive real numbers,*
(C) $\Re(\kappa) + \$(\frac{\Psi_i - 1}{A_i}) < 0; (i = 1,2,\ldots,v), \Re(\kappa) + \$\Re(\frac{\Phi_i}{B_i}) > 0, (i = 1,2,\ldots,u)$,
(D) $Y \ge 0, \$ > 0, \Lambda \in \mathbb{C}$, *and* $\phi, \psi \in \mathscr{A}$.

Then, the consequent IE:

$$\int_0^{\infty} \mathcal{V}^{-\kappa}\,{}^{\Gamma}I_{r,s}^{u,v}\left[C\left(\frac{X}{\mathcal{V}}\right)^{\$}\middle|\begin{array}{c}(\Psi_1,\zeta_1;A_1:Y),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r)\\(\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s)\end{array}\right] \times \phi(\mathcal{V})\mathrm{d}\mathcal{V} = \psi(X), \tag{29}$$

has now been solved by

$$\phi(X) = \frac{\$C^{\frac{\mathcal{P}}{\$}} X^{\Lambda-1}}{2\pi\iota} \int_{\$} X^{-\mathcal{P}} \left[\Phi\left(\frac{-\mathcal{P}}{\$}, Y\right)\right]^{-1} \Xi(\mathcal{P})\mathrm{d}\mathcal{P}, \tag{30}$$

where

$$\Xi(\mathcal{P}) = \int_0^\infty X^{\mathcal{P}-1}\psi(X)\mathrm{d}X,$$

and $\Phi(\frac{-\mathcal{P}}{\$}, Y)$ *is shown in Equation* (9).

Proof: ϕ is substituted for $\mathscr{D}^{\Lambda-\kappa}\phi$ in Equation (27) to determine the integral equation's (29) solution; we get

$$\int_0^\infty \mathcal{V}^{-\kappa}\,{}^{\Gamma}I^{u,v+1}_{r+1,s+1} \times \left[C\left(\frac{X}{\mathcal{V}}\right)^{\$} \middle| \begin{matrix} (\Psi_1,\zeta_1;A_1:Y),(1-\kappa,\$;1),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r) \\ (\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s),(1-\Lambda,\$;1) \end{matrix} \right] \times \mathscr{D}^{\Lambda-\kappa}\{\phi(\mathcal{V})\}\mathrm{d}\mathcal{V} = \psi(X).$$

By multiplying both sides by $X^{\mathcal{P}-1}$, integrating from 0 to ∞ with respect to X, and thereupon altering the order of integration together with the allowable circumstances, we get

$$\Xi(\mathcal{P}) = \int_0^\infty X^{\mathcal{P}-1}\psi(X)\mathrm{d}X = \int_0^\infty \mathcal{V}^{-\kappa}\mathscr{D}^{\Lambda-\kappa}\{\phi(\mathcal{V})\} \times \left(\int_0^\infty X^{\mathcal{P}-1}\right. \\ \left. {}^{\Gamma}I^{u,v+1}_{r+1,s+1}\left[C\left(\frac{X}{\mathcal{V}}\right)^{\$} \middle| \begin{matrix} (\Psi_1,\zeta_1;A_1:Y),(1-\kappa,\$;1),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r) \\ (\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s),(1-\Lambda,\$;1) \end{matrix} \right]\mathrm{d}X\right)\mathrm{d}\mathcal{V}.$$

Now, using Equation (23), we obtain

$$\Xi(\mathcal{P}) = \frac{\Gamma(\kappa-\mathcal{P})}{\$\Gamma(\Lambda-\mathcal{P})} C^{-\frac{\mathcal{P}}{\$}} \Phi\left(\frac{-\mathcal{P}}{\$}, Y\right) \int_0^\infty \mathcal{V}^{\mathcal{P}-\kappa}\mathscr{D}^{\Lambda-\kappa}\{\phi(\mathcal{V})\}\mathrm{d}\mathcal{V}.$$

Moreover, by using the Mellin inversion theorem, we obtain

$$\mathscr{D}^{\Lambda-\kappa}\{\phi(\mathcal{V})\} = \frac{\$}{2\pi\iota}\int_{\$} \mathcal{V}^{\kappa-\mathcal{P}-1}\frac{\Gamma(\Lambda-\mathcal{P})}{\Gamma(\kappa-\mathcal{P})} C^{\frac{\mathcal{P}}{\$}} \left[\Phi\left(\frac{-\mathcal{P}}{\$}, Y\right)\right]^{-1} \Xi(\mathcal{P})\mathrm{d}\mathcal{P}.$$

Next, by operating on each side with $\mathscr{D}^{\kappa-\Lambda}$, we obtain

$$\phi(\mathcal{V}) = \frac{\$}{2\pi\iota}\mathscr{D}^{\kappa-\Lambda}\left\{\int_{\$} \mathcal{V}^{\kappa-\mathcal{P}-1}\frac{\Gamma(\Lambda-\mathcal{P})}{\Gamma(\kappa-\mathcal{P})} C^{\frac{\mathcal{P}}{\$}} \left[\Phi\left(\frac{-\mathcal{P}}{\$}, Y\right)\right]^{-1} \Xi(\mathcal{P})\mathrm{d}\mathcal{P}\right\},$$

which eventually gives

$$\phi(X) = \frac{\$C^{\frac{\mathcal{P}}{\$}} X^{\Lambda-1}}{2\pi\iota} \int_{\$} X^{-\mathcal{P}} \left[\Phi\left(\frac{-\mathcal{P}}{\$}, Y\right)\right]^{-1} \Xi(\mathcal{P})\mathrm{d}\mathcal{P}.$$

■

Theorem 3.6: *Let*

(A) $u, v, r, s \in \mathcal{Z}_{0+}$ *s.t.* $1 \leq u \leq s$ *and* $0 \leq v \leq r$,
(B) Ψ_r, β_s *are positive real numbers,*
(C) $\Re(\kappa) + \$(\frac{\Psi_i - 1}{A_i}) < 0$; $(i = 1, 2, \ldots, v)$, $\Re(\kappa) + \$\Re(\frac{\Phi_i}{B_i}) > 0$, $(i = 1, 2, \ldots, u)$,
(D) $Y \geq 0, \$ > 0, \Lambda \in \mathbb{C}$, *and* $\phi, \psi \in \mathscr{A}$.

Whereupon, the subsequent IE:

$$\int_0^\infty \mathcal{V}^{-\kappa\gamma} I_{r,s}^{u,v}\left[C\left(\frac{X}{\mathcal{V}}\right)^{\$} \middle| \begin{matrix} (\Psi_1, \zeta_1; A_1 : Y), (\Psi_2, \zeta_2; A_2), \cdots, (\Psi_r, \zeta_r; A_r) \\ (\Phi_1, \beta_1; B_1), (\Phi_2, \beta_2; B_2), \cdots, (\Phi_s, \beta_s; B_s) \end{matrix}\right] \times \phi(\mathcal{V}) \mathrm{d}\mathcal{V} = \psi(X), \tag{31}$$

has a solution given by

$$\phi(X) = \frac{\$ C^{\frac{\mathcal{P}}{\$}} X^{\Lambda - 1}}{2\pi\iota} \int_{\$} X^{-\mathcal{P}} \left[\Psi\left(\frac{-\mathcal{P}}{\$}, Y\right)\right]^{-1} \Xi(\mathcal{P}) \mathrm{d}\mathcal{P}, \tag{32}$$

where

$$\Xi(\mathcal{P}) = \int_0^\infty X^{\mathcal{P}-1} \psi(X) \mathrm{d}X,$$

and $\Psi(\frac{-\mathcal{P}}{\$}, Y)$ *is shown in Equation* (8).

Proof: Since the proof is identical to that of Theorem 3.5, we will skip the details here. ■

4. Solution of an integral equation of Fredholm type utilizing I$\bar{I}$F

In this section, using the MT method and the well-recognized WFI, we provide the solution to the Fredholm-type integral problem implicating the I$\bar{I}$F.

Lemma 4.1: *Let*

(A) $u, v, r, s \in \mathcal{Z}_{0+}$ *s.t.* $0 \leq v \leq r$ *and* $1 \leq u \leq s$,
(B) $\Re(\Lambda - \kappa) > 0$; $\Re(\kappa) + \$\Re(\frac{\Phi_i}{B_i}) > 0$, $(i = 1, 2, \ldots, u)$,
(C) $Y \geq 0, \$ > 0$, *and* $\Lambda \in \mathbb{C}$,
(D) $|\arg(C)| < \frac{1}{2}\pi\delta$, *where* δ *is provided in Equation* (15).

Then,

$$\mathscr{W}^{\kappa - \Lambda}\left\{\mathcal{V}^{-\Lambda\Gamma} \bar{I}_{r,s}^{u,v}\left[C\left(\frac{X}{\mathcal{V}}\right)^{\$} \middle| \begin{matrix} (\Psi_1, \zeta_1; A_1 : Y), (\Psi_2, \zeta_2; A_2), \cdots, (\Psi_r, \zeta_r; A_r) \\ (\Phi_1, \beta_1; B_1), (\Phi_2, \beta_2; B_2), \cdots, (\Phi_s, \beta_s; B_s) \end{matrix}\right]\right\}$$
$$= \mathcal{V}^{-\kappa\Gamma} \bar{I}_{r+1,s+1}^{u,v+1} \times \left[C\left(\frac{X}{\mathcal{V}}\right)^{\$} \middle| \begin{matrix} (\Psi_1, \zeta_1; A_1 : Y), (1-\kappa, \$; 1), (\Psi_2, \zeta_2; A_2), \cdots, (\Psi_r, \zeta_r; A_r) \\ (\Phi_1, \beta_1; B_1), (\Phi_2, \beta_2; B_2), \cdots, (\Phi_s, \beta_s; B_s), (1-\Lambda, \$; 1) \end{matrix}\right]. \tag{33}$$

Proof: We first demonstrate the integral contour form of the I$\bar{I}$F provided in Equation (11), thereupon change the pattern of integrals to demonstrate the assertion in Equation (33). We next apply the WFI defined in Equation (20) (within the specified permitted conditions). We can easily achieve the desired result by evaluating the power function at the WFI and thereupon trying to interpret the arising Mellin Barnes contour integral in terms of I$\bar{I}$F. ■

Lemma 4.2: *Let*

(A) $u, v, r, s \in \mathcal{Z}_{0+}$ *s.t.* $0 \le v \le r$ *and* $1 \le u \le s$,
(B) $\Re(\Lambda - \kappa) > 0$; $\Re(\kappa) + \$\Re(\frac{\Phi_i}{B_i}) > 0$, $(i = 1, 2, \ldots, u)$,
(C) $Y \ge 0, \$ > 0$, *and* $\Lambda \in \mathbb{C}$,
(D) $|\arg(C)| < \frac{1}{2}\pi\delta$, *where* δ *is provided in Equation* (15).

Then,

$$\mathscr{W}^{\kappa-\Lambda}\left\{\mathcal{V}^{-\Lambda\gamma}\bar{I}^{u,v}_{r,s}\left[C\left(\frac{X}{\mathcal{V}}\right)^{\$}\,\middle|\,\begin{matrix}(\Psi_1,\zeta_1;A_1:Y),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r)\\(\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s)\end{matrix}\right]\right\}$$
$$= \mathcal{V}^{-\kappa\gamma}\bar{I}^{u,v+1}_{r+1,s+1}$$
$$\times\left[C\left(\frac{X}{\mathcal{V}}\right)^{\$}\,\middle|\,\begin{matrix}(\Psi_1,\zeta_1;A_1:Y),(1-\kappa,\$;1),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r)\\(\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s),(1-\Lambda,\$;1)\end{matrix}\right]. \quad (34)$$

Proof: We first demonstrate the integral contour form of the I$\bar{I}$F provided in Equation (12), then change the process of integrals to demonstrate the assertion in Equation (34). We next apply the WFI defined in Equation (20) (within the specified permitted conditions). We can easily achieve the desired result by evaluating the power function at the WFI and thereupon trying to interpret the arising Mellin Barnes contour integral in terms of I$\bar{I}$F. ■

Theorem 4.3: *Let*

(A) $u, v, r, s \in \mathcal{Z}_{0+}$ *s.t.* $0 \le v \le r$ *and* $1 \le u \le s$,
(B) Ψ_r, β_s *are positive real numbers,*
(C) $\Re(\kappa) + \$(\frac{\Psi_i - 1}{A_i}) < 0$; $(i = 1, 2, \ldots, v)$, $\Re(\kappa) + \$\Re(\frac{\Phi_i}{B_i}) > 0$, $(i = 1, 2, \ldots, u)$,
(D) $Y \ge 0, \$ > 0$, *and* $\Lambda \in \mathbb{C}$.

Consequently, the subsequent integral relationship holds:

$$\int_0^\infty {}^{\Gamma}\bar{I}^{u,v+1}_{r+1,s+1}\left[C\left(\frac{X}{\mathcal{V}}\right)^{\$}\,\middle|\,\begin{matrix}(\Psi_1,\zeta_1;A_1:Y),(1-\kappa,\$;1),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r)\\(\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s),(1-\Lambda,\$;1)\end{matrix}\right]$$
$$\times\,\mathcal{V}^{-\kappa}\phi(\mathcal{V})\mathrm{d}\mathcal{V}$$

$$= \int_0^{\infty} {}^{\Gamma}\bar{I}_{r,s}^{u,v}\left[C\left(\frac{X}{\mathcal{V}}\right)^{\$}\ \middle|\ \begin{matrix}(\Psi_1,\zeta_1;A_1:Y),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r)\\(\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s)\end{matrix}\right]$$
$$\times\ \mathcal{V}^{-\kappa}\mathscr{D}^{\kappa-\Lambda}\{\phi(\mathcal{V})\}\mathrm{d}\mathcal{V}, \tag{35}$$

with the result that $\phi \in \mathscr{A}$ and $X > 0$.

Proof: Suppose $\bar{\mathscr{G}}$ denotes the first component of the statement in Equation (35) of Theorem 4.3. Thereupon, using Lemma 4.1 and the definition in Equation (20), we have

$$\bar{\mathscr{G}} = \int_0^{\infty}\phi(\mathcal{V})\left(\int_{\mathcal{V}}^{\infty}\frac{(U-\mathcal{V})^{\Lambda-\kappa-1}}{\Gamma(\Lambda-\kappa)}U^{-\Lambda}\right.$$
$$\left. {}^{\Gamma}\bar{I}_{r,s}^{u,v}\left[C\left(\frac{X}{U}\right)^{\$}\ \middle|\ \begin{matrix}(\Psi_1,\zeta_1;A_1:Y),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r)\\(\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s)\end{matrix}\right]\mathrm{d}U\right)\mathrm{d}\mathcal{V}.$$

Next, by altering the order of integration under the allowable circumstances, we obtain

$$\bar{\mathscr{G}} = \int_0^{\infty} U^{-\Lambda}\,{}^{\Gamma}\bar{I}_{r,s}^{u,v}\left[C\left(\frac{X}{U}\right)^{\$}\ \middle|\ \begin{matrix}(\Psi_1,\zeta_1;A_1:Y),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r)\\(\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s)\end{matrix}\right]$$
$$\times\left(\int_0^{U}\frac{(U-\mathcal{V})^{\Lambda-\kappa-1}}{\Gamma(\Lambda-\kappa)}\phi(\mathcal{V})\mathrm{d}\mathcal{V}\right)\mathrm{d}U.$$

Moreover, using the widely used definition of the RL fractional derivative, we get

$$\bar{\mathscr{G}} = \int_0^{\infty} {}^{\Gamma}\bar{I}_{r,s}^{u,v}\left[C\left(\frac{X}{U}\right)^{\$}\ \middle|\ \begin{matrix}(\Psi_1,\zeta_1;A_1:Y),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r)\\(\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s)\end{matrix}\right]$$
$$\times\ U^{-\Lambda}\mathscr{D}^{\kappa-\Lambda}\{\phi(U)\}\mathrm{d}U,$$

which is the right-handed component of Equation (35). ■

Theorem 4.4: *Let*

(A) $u, v, r, s \in \mathcal{Z}_{0+}$ *s.t.* $1 \le u \le s$ *and* $0 \le v \le r$,
(B) Ψ_r, β_s *are positive real numbers,*
(C) $\Re(\kappa) + \$(\frac{\Psi_i - 1}{A_i}) < 0$; $(i = 1, 2, \ldots, v)$, $\Re(\kappa) + \$\Re(\frac{\Phi_i}{B_i}) > 0$, $(i = 1, 2, \ldots, u)$,
(D) $Y \ge 0, \$ > 0$, *and* $\Lambda \in \mathbb{C}$.

Consequently, the subsequent integral relation holds true:

$$\int_0^{\infty} {}^{\gamma}\bar{I}_{r+1,s+1}^{u,v+1}\left[C\left(\frac{X}{\mathcal{V}}\right)^{\$}\ \middle|\ \begin{matrix}(\Psi_1,\zeta_1;A_1:Y),(1-\kappa,\$;1),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r)\\(\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s),(1-\Lambda,\$;1)\end{matrix}\right]$$
$$\times\ \mathcal{V}^{-\kappa}\phi(\mathcal{V})\mathrm{d}\mathcal{V}$$

$$= \int_0^\infty {}^{\gamma}\bar{I}_{r,s}^{u,v}\left[C\left(\frac{X}{\mathcal{V}}\right)^{\$}\middle|\begin{array}{c}(\Psi_1,\zeta_1;A_1:Y),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r)\\(\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s)\end{array}\right]$$
$$\times \mathcal{V}^{-\kappa}\mathscr{D}^{\kappa-\Lambda}\{\phi(\mathcal{V})\}\mathrm{d}\mathcal{V}, \tag{36}$$

provided that $\phi \in \mathscr{A}$ *and* $X > 0$*.*

Proof: Since the proof is identical to that of Theorem 4.3, we will skip the details here. ■

Theorem 4.5: *Considering*

(A) $u, v, r, s \in \mathcal{Z}_{0+}$ *s.t.* $1 \le u \le s$ *and* $0 \le v \le r$,
(B) Ψ_r, β_s *are positive real numbers,*
(C) $\Re(\kappa) + \$(\frac{\Psi_i - 1}{A_i}) < 0$; $(i = 1, 2, \ldots, v)$, $\Re(\kappa) + \$\Re(\frac{\Phi_i}{B_i}) > 0$, $(i = 1, 2, \ldots, u)$,
(D) $Y \ge 0, \$ > 0, \Lambda \in \mathbb{C}$, *and* $\phi, \psi \in \mathscr{A}$.

Hence, the subsequent IE:

$$\int_0^\infty \mathcal{V}^{-\kappa}\Gamma\bar{I}_{r,s}^{u,v}\left[C\left(\frac{X}{\mathcal{V}}\right)^{\$}\middle|\begin{array}{c}(\Psi_1,\zeta_1;A_1:Y),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r)\\(\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s)\end{array}\right]$$
$$\times \phi(\mathcal{V})\mathrm{d}\mathcal{V} = \psi(X), \tag{37}$$

has a solution, and it is provided by

$$\phi(X) = \frac{\$C^{\frac{\mathcal{P}}{\$}}X^{\Lambda-1}}{2\pi\iota}\int_{\$} X^{-\mathcal{P}}\left[\bar{\Phi}\left(\frac{-\mathcal{P}}{\$}, Y\right)\right]^{-1}\bar{\Xi}(\mathcal{P})\mathrm{d}\mathcal{P}, \tag{38}$$

where

$$\bar{\Xi}(\mathcal{P}) = \int_0^\infty X^{\mathcal{P}-1}\psi(X)\mathrm{d}X,$$

and $\bar{\Phi}(\frac{-\mathcal{P}}{\$}, Y)$ *is shown in Equation (9).*

Proof: ϕ is substituted for $\mathscr{D}^{\Lambda-\kappa}\phi$ in Equation (35) to determine the integral equation's (37) solution; we get

$$\int_0^\infty \mathcal{V}^{-\kappa}\Gamma\bar{I}_{r+1,s+1}^{u,v+1}$$
$$\times\left[C\left(\frac{X}{\mathcal{V}}\right)^{\$}\middle|\begin{array}{c}(\Psi_1,\zeta_1;A_1:Y),(1-\kappa,\$;1),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r)\\(\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s),(1-\Lambda,\$;1)\end{array}\right]$$
$$\times \mathscr{D}^{\Lambda-\kappa}\{\phi(\mathcal{V})\}\mathrm{d}\mathcal{V} = \psi(X).$$

By multiplying both sides by $X^{\mathcal{P}-1}$, integrating from 0 to ∞ in relation to X, and thereafter altering the order of integration along with the allowable circumstances, we get

$$\bar{\Xi}(\mathcal{P}) = \int_0^\infty X^{\mathcal{P}-1}\psi(X)\mathrm{d}X = \int_0^\infty \mathcal{V}^{-\kappa}\mathscr{D}^{\Lambda-\kappa}\{\phi(\mathcal{V})\} \times \left(\int_0^\infty X^{\mathcal{P}-1}\right.$$

$$\left.\Gamma\bar{I}^{u,v+1}_{r+1,s+1}\left[C\left(\frac{X}{\mathcal{V}}\right)^{\$}\middle| \begin{matrix}(\Psi_1,\zeta_1;A_1:Y),(1-\kappa,\$;1),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r)\\(\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s),(1-\Lambda,\$;1)\end{matrix}\right]\mathrm{d}X\right)\mathrm{d}\mathcal{V}.$$

Now, using Equation (24), we obtain

$$\bar{\Xi}(\mathcal{P}) = \frac{\Gamma(\kappa-\mathcal{P})}{\$\Gamma(\Lambda-\mathcal{P})}C^{-\frac{\mathcal{P}}{\$}}\bar{\Phi}\left(\frac{-\mathcal{P}}{\$},Y\right)\int_0^\infty \mathcal{V}^{\mathcal{P}-\kappa}\mathscr{D}^{\Lambda-\kappa}\{\phi(\mathcal{V})\}\mathrm{d}\mathcal{V}.$$

Moreover, by using the Mellin inversion theorem, we obtain

$$\mathscr{D}^{\Lambda-\kappa}\{\phi(\mathcal{V})\} = \frac{\$}{2\pi\iota}\int_{\$}\mathcal{V}^{\kappa-\mathcal{P}-1}\frac{\Gamma(\Lambda-\mathcal{P})}{\Gamma(\kappa-\mathcal{P})}C^{\frac{\mathcal{P}}{\$}}\left[\bar{\Phi}\left(\frac{-\mathcal{P}}{\$},Y\right)\right]^{-1}\bar{\Xi}(\mathcal{P})\mathrm{d}\mathcal{P}.$$

Next, by operating on each side with $\mathscr{D}^{\kappa-\Lambda}$, we obtain

$$\phi(\mathcal{V}) = \frac{\$}{2\pi\iota}\mathscr{D}^{\kappa-\Lambda}\left\{\int_{\$}\mathcal{V}^{\kappa-\mathcal{P}-1}\frac{\Gamma(\Lambda-\mathcal{P})}{\Gamma(\kappa-\mathcal{P})}C^{\frac{\mathcal{P}}{\$}}\left[\bar{\Phi}\left(\frac{-\mathcal{P}}{\$},Y\right)\right]^{-1}\bar{\Xi}(\mathcal{P})\mathrm{d}\mathcal{P}\right\},$$

which eventually gives

$$\phi(X) = \frac{\$C^{\frac{\mathcal{P}}{\$}}X^{\Lambda-1}}{2\pi\iota}\int_{\$}X^{-\mathcal{P}}\left[\bar{\Phi}\left(\frac{-\mathcal{P}}{\$},Y\right)\right]^{-1}\bar{\Xi}(\mathcal{P})\mathrm{d}\mathcal{P}.$$

■

Theorem 4.6: *Let*

(A) $u, v, r, \in \mathcal{Z}_{0+}$ *s.t.* $1 \le u \le s$ *and* $0 \le v \le r$,
(B) Ψ_r, β_s *are positive real numbers,*
(C) $\Re(\kappa) + \$(\frac{\Psi_i - 1}{A_i}) < 0$; $(i = 1, 2, \ldots, v)$, $\Re(\kappa) + \$\Re(\frac{\Phi_i}{B_i}) > 0$, $(i = 1, 2, \ldots, u)$,
(D) $Y \ge 0, \$ > 0, \Lambda \in \mathbb{C}$, *and* $\phi, \psi \in \mathscr{A}$.

Then, the preceding IE:

$$\int_0^\infty \mathcal{V}^{-\kappa\gamma}\bar{I}^{u,v}_{r,s}\left[C\left(\frac{X}{\mathcal{V}}\right)^{\$}\middle| \begin{matrix}(\Psi_1,\zeta_1;A_1:Y),(\Psi_2,\zeta_2;A_2),\cdots,(\Psi_r,\zeta_r;A_r)\\(\Phi_1,\beta_1;B_1),(\Phi_2,\beta_2;B_2),\cdots,(\Phi_s,\beta_s;B_s)\end{matrix}\right]$$
$$\times\,\phi(\mathcal{V})\mathrm{d}\mathcal{V} = \psi(X), \qquad (39)$$

has a solution given by

$$\phi(X) = \frac{\$C^{\frac{\mathcal{P}}{\$}}X^{\Lambda-1}}{2\pi\iota}\int_{\$}X^{-\mathcal{P}}\left[\bar{\Psi}\left(\frac{-\mathcal{P}}{\$},Y\right)\right]^{-1}\bar{\Xi}(\mathcal{P})\mathrm{d}\mathcal{P}, \qquad (40)$$

where

$$\bar{\Xi}(\mathcal{P}) = \int_0^\infty X^{\mathcal{P}-1}\psi(X)\mathrm{d}X,$$

and $\bar{\Psi}(\frac{-\mathcal{P}}{\$}, Y)$ *is shown in Equation (8).*

Proof: Since the proof is identical to that of Theorem 4.5, we will skip the details here. ■

5. Remarks

This section presents some of the already available findings, which directly follow our major findings. Later, we develop more fascinating outcomes achieved by focusing on the parameters of the I*I*F and the I$\bar{I}$F of our main result.

Remark 5.1: If setting $A_i = 1, B_j = 1 (i = 1, 2, \ldots, r, j = 1, 2, \ldots, s)$ in Equations (27) and (28), then the outcome is as reported by Bansal et al. [15].

Remark 5.2: If we consider setting $Y = 0, A_i = 1, B_j = 1 (i = 1, 2, \ldots, r, j = 1, 2, \ldots, s)$ in Equations (27) and (28), then the outcome is as reported by Srivastava et al. [10].

Now, we present some essential examples based on the main findings.

Example 5.3: If we substitute $u = 1, v = r,\ s = s + 1, C(\frac{X}{\mathcal{V}})^{\$} = -C(\frac{X}{\mathcal{V}})^{\$}, \beta_1 = 1, \Phi_1 = 0, \Psi_i = 1 - \Psi_i,\ \Phi_i = 1 - \Phi_i,\ A_i = 1, B_i = 1$ in Theorem 3.3, then we obtain the Fredholm integral equation solution that uses the incomplete Fox–Wright function ${}_r\Psi_s^{\Gamma}$.

$$\int_0^\infty {}_{r+1}\Psi_{s+1}^{\Gamma}\left[C\left(\frac{X}{\mathcal{V}}\right)^{\$}\,\middle|\,\begin{matrix}(\Psi_1, \zeta_1 : Y), (1-\kappa, \$), (\Psi_2, \zeta_2), \cdots, (\Psi_r, \zeta_r)\\ (\Phi_1, \beta_1), (\Phi_2, \beta_2), \cdots, (\Phi_s, \beta_s), (1-\Lambda, \$)\end{matrix}\right] \times \mathcal{V}^{-\kappa}\phi(\mathcal{V})\mathrm{d}\mathcal{V}. \tag{41}$$

Solution: Suppose $\mathscr{G}$ denotes the component of Equation (41) of Example 5.3. Thereupon, using Lemma 3.1 and the definition in Equation (20), we obtain

$$\mathscr{G} = \int_0^\infty \phi(\mathcal{V})\left(\int_{\mathcal{V}}^\infty \frac{(U-\mathcal{V})^{\Lambda-\kappa-1}}{\Gamma(\Lambda-\kappa)}U^{-\Lambda}\right.$$
$$\left. {}_r\Psi_s^{\Gamma}\left[C\left(\frac{X}{U}\right)^{\$}\,\middle|\,\begin{matrix}(\Psi_1, \zeta_1 : Y), (\Psi_2, \zeta_2), \cdots, (\Psi_r, \zeta_r)\\ (\Phi_1, \beta_1), (\Phi_2, \beta_2), \cdots, (\Phi_s, \beta_s)\end{matrix}\right]\mathrm{d}U\right)\mathrm{d}\mathcal{V}.$$

Next, by altering the order of integration under the allowable circumstances, we obtain

$$\mathscr{G} = \int_0^\infty U^{-\Lambda}\,{}_r\Psi_s^{\Gamma}\left[C\left(\frac{X}{U}\right)^{\$}\,\middle|\,\begin{matrix}(\Psi_1, \zeta_1 : Y), (\Psi_2, \zeta_2), \cdots, (\Psi_r, \zeta_r)\\ (\Phi_1, \beta_1), (\Phi_2, \beta_2), \cdots, (\Phi_s, \beta_s)\end{matrix}\right]$$
$$\times\left(\int_0^U \frac{(U-\mathcal{V})^{\Lambda-\kappa-1}}{\Gamma(\Lambda-\kappa)}\phi(\mathcal{V})\mathrm{d}\mathcal{V}\right)\mathrm{d}U.$$

Moreover, using the widely used definition of the RL fractional derivative, we get

$$\mathscr{G} = \int_0^\infty {}_r\Psi_s^\Gamma \left[C\left(\frac{X}{U}\right)^\$ \middle| \begin{matrix} (\Psi_1, \zeta_1 : Y), (\Psi_2, \zeta_2), \cdots, (\Psi_r, \zeta_r) \\ (\Phi_1, \beta_1), (\Phi_2, \beta_2), \cdots, (\Phi_s, \beta_s) \end{matrix} \right] \times U^{-\Lambda} \mathscr{D}^{\kappa-\Lambda}\{\phi(U)\}\mathrm{d}U.$$

Now, replacing $\mathcal{V} = U$, then we get the final solution of Equation (41).

$$\mathscr{G} = \int_0^\infty {}_r\Psi_s^\Gamma \left[C\left(\frac{X}{\mathcal{V}}\right)^\$ \middle| \begin{matrix} (\Psi_1, \zeta_1 : Y), (\Psi_2, \zeta_2), \cdots, (\Psi_r, \zeta_r) \\ (\Phi_1, \beta_1), (\Phi_2, \beta_2), \cdots, (\Phi_s, \beta_s) \end{matrix} \right] \times \mathcal{V}^{-\kappa} \mathscr{D}^{\kappa-\Lambda}\{\phi(\mathcal{V})\}\mathrm{d}\mathcal{V}. \tag{42}$$

Example 5.4: If we substitute $u = 1, v = r,\ s = s+1, C(\frac{X}{\mathcal{V}})^\$ = -C(\frac{X}{\mathcal{V}})^\$, \beta_1 = 1, \Phi_1 = 0, \Psi_i = 1 - \Psi_i, \Phi_i = 1 - \Phi_i, A_i = 1, B_i = 1,\ \zeta_i = 1, \beta_i = 1$ in Theorem 3.3, then we obtain the Fredholm integral equation solution that uses the incomplete generalized hypergeometric function ${}_r\Gamma_s$.

$$\int_0^\infty {}_{r+1}\Gamma_{s+1} \left[C\left(\frac{X}{\mathcal{V}}\right)^\$ \middle| \begin{matrix} (\Psi_1 : Y), (1-\kappa, \$), \Psi_2, \ldots, \Psi_r \\ \Phi_1, \Phi_2, \ldots, \Phi_s, (1-\Lambda, \$) \end{matrix} \right] \times \mathcal{V}^{-\kappa}\phi(\mathcal{V})\mathrm{d}\mathcal{V}. \tag{43}$$

Solution: Suppose $\mathscr{G}$ denotes the component of Equation (43) of Example 5.4. Thereupon, using Lemma 3.1 and the definition in Equation (20), we obtain

$$\mathscr{G} = \int_0^\infty \phi(\mathcal{V}) \left(\int_{\mathcal{V}}^\infty \frac{(U-\mathcal{V})^{\Lambda-\kappa-1}}{\Gamma(\Lambda-\kappa)} U^{-\Lambda} {}_r\Gamma_s \left[C\left(\frac{X}{U}\right)^\$ \middle| \begin{matrix} (\Psi_1 : Y), \Psi_2, \ldots, \Psi_r \\ \Phi_1, \Phi_2, \ldots, \Phi_s \end{matrix} \right] \mathrm{d}U \right) \mathrm{d}\mathcal{V}.$$

Next, by altering the order of integration under the allowable circumstances, we obtain

$$\mathscr{G} = \int_0^\infty U^{-\Lambda} {}_r\Gamma_s \left[C\left(\frac{X}{U}\right)^\$ \middle| \begin{matrix} (\Psi_1 : Y), \Psi_2, \ldots, \Psi_r \\ \Phi_1, \Phi_2, \ldots, \Phi_s \end{matrix} \right] \times \left(\int_0^U \frac{(U-\mathcal{V})^{\Lambda-\kappa-1}}{\Gamma(\Lambda-\kappa)} \phi(\mathcal{V})\mathrm{d}\mathcal{V} \right) \mathrm{d}U.$$

Moreover, using the widely used definition of the RL fractional derivative, we get

$$\mathscr{G} = \int_0^\infty {}_r\Gamma_s \left[C\left(\frac{X}{U}\right)^\$ \middle| \begin{matrix} (\Psi_1 : Y), \Psi_2, \ldots, \Psi_r \\ \Phi_1, \Phi_2, \ldots, \Phi_s \end{matrix} \right] \times U^{-\Lambda} \mathscr{D}^{\kappa-\Lambda}\{\phi(U)\}\mathrm{d}U.$$

Now, replacing $\mathcal{V} = U$, then we get the final solution of Equation (43).

$$\mathscr{G} = \int_0^\infty {}_r\Gamma_s \left[C\left(\frac{X}{\mathcal{V}}\right)^\$ \middle| \begin{matrix} (\Psi_1 : Y), \Psi_2, \ldots, \Psi_r \\ \Phi_1, \Phi_2, \ldots, \Phi_s \end{matrix} \right] \times \mathcal{V}^{-\kappa} \mathscr{D}^{\kappa-\Lambda}\{\phi(\mathcal{V})\}\mathrm{d}\mathcal{V}. \tag{44}$$

Similarly, as above examples, we find a new instance for Theorems 3.3, 3.4, 4.3 and 4.4.

6. Conclusions

This paper introduces the Fredholm-type IE involving the IIF and the I$\bar{I}$F in the kernel. After, we acquire the Mellin transformation of the incomplete I-function. By figuring out the precise values of the various parameters of the IIF and the I$\bar{I}$F, we also highlight some known outcomes. Given this observation, the results presented here, being of a general character, can yield numerous generating functions for a particular class of incomplete I-function and other special functions expressible in terms of I-functions. Our conclusions are crucial in many different fields. With their aid, a wide range of fascinating and useful fractional integral equations with applications in engineering, communication theory, probability theory, and science can be created. In the near future work, solutions to other differential and integral equations may be obtained by including an incomplete I-function in the kernel for more generalization to transcendental problems.

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No data were used for this study.

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