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Research article

Certain midpoint-type Fejér and Hermite-Hadamard inclusions involving fractional integrals with an exponential function in kernel

Thongchai Botmart[1], Soubhagya Kumar Sahoo[2,*], Bibhakar Kodamasingh[2], Muhammad Amer Latif[3], Fahd Jarad[4,5,*] and Artion Kashuri[6]

[1] Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand

[2] Department of Mathematics, Institute of Technical Education and Research, Siksha 'O' Anusandhan University, Bhubaneswar 751030, India

[3] Department of Basic Sciences, Deanship of Preparatory Year, King Faisal University, Hofuf 31982, Al-Hasa, Saudi Arabia

[4] Department of Mathematics, Çankaya University 06790, Ankara, Turkey.

[5] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan

[6] Department of Mathematics, Faculty of Technical and Natural Sciences, University "Ismail Qemali", 9400 Vlora, Albania

* **Correspondence:** Email: soubhagyakumarsahoo@soa.ac.in, fahd@cankaya.edu.tr.

Abstract: In this paper, using positive symmetric functions, we offer two new important identities of fractional integral form for convex and harmonically convex functions. We then prove new variants of the Hermite-Hadamard-Fejér type inequalities for convex as well as harmonically convex functions via fractional integrals involving an exponential kernel. Moreover, we also present improved versions of midpoint type Hermite-Hadamard inequality. Graphical representations are given to validate the accuracy of the main results. Finally, applications associated with matrices, q-digamma functions and modifed Bessel functions are also discussed.

Keywords: Hermite-Hadamard-Fejér inequalities; convex function; harmonically convex function; fractional integral operators; matrices; q-digamma functions; modifed Bessel functions
Mathematics Subject Classification: 26A51, 26A33, 26D07, 26D10, 26D15

List of Abbreviations

The following abbreviations are used in this manuscript:

H-H	Hermite-Hadamard
H-H-M	Hermite-Hadamard-Mercer
H-H-F	Hermite-Hadamard-Fejér

1. Introduction and preliminaries

In science, convex functions have a long and distinguished history, and they have been the focus of study for almost a century. The rapid growth of convexity theory and applications of fractional calculus has kept the interest of a number of researchers on integral inequalities. Inequalities such as the H-H type, the Ostrowski-type, the H-H-M type, the Opial type, and other types, by using convex functions have been the focus of research for many years. The H-H inequality given in [1] has piqued the curiosity of most academics among all of these integral inequalities. Dragomir et al. [2] and Kirmaci et al. [3] presented some trapezoidal type inequalities and also some applications to special means. Following these articles, several mathematicians proposed new refinements of the Hermite-Hadamard inequality for various classes of convex functions and mappings such as quasi convex function [4], convex functions [5], m-convex functions [6], s-type convex functions of Raina type [7], σ-s-convex function [8] and harmonically convex functions [9]. Recently, this inequality was also investigated via different fractional integral operators, like Riemann-Liouville [10], ψ-Riemann-Liouville [11], Proportional fractional [12, 13], k-Riemann-Liouville [14], Caputo-Fabrizio [15, 16], generalized Atangana-Baleanu operator [17] to name a few.

It is important to emphasise that Leibniz and L'Hospital are credited with developing the idea of fractional calculus (1695). Other mathematicians, such as Riemann, Liouville, Letnikov, Erdéli, Grünwald, and Kober, have made significant inputs to the field of fractional calculus and its numerous applications. Many physical and engineering experts are interested in fractional calculus because of its behaviour and capacity to address a wide range of practical issues. Fractional calculus is currently concerned with the study of so-called fractional order integral and derivative functions over real and complex domains, as well as its applications. In many cases, fractional analysis requires the use of arithmetic from classical analysis to produce more accurate conclusions. Numerous mathematical models can be handled by differential equations of fractional order. Fractional mathematical models have more conclusive and precise results than classical mathematical models because they are particular examples of fractional order mathematical models. In classical analysis, integer orders do not serve as an adequate representation of nature. By using mathematical modelling, it is possible to identify the endemics' unique transmission dynamics and get insight into how infection impacts a new population. To enhance the actual phenomena to a higher degree of precision and accuracy, non-integer order fractional differential equations (FDEs) are applied. Additionally, [18–22] use and reference their utilization of fractional calculus. Other interesting results for fractional calclus can be found in [23–25]. However, fractional computation enables us to consider any number of orders and formulate far more measurable objectives. In recent years, mathematicians have become more and more interested in presenting well-known inequalities using a variety of novel theories of fractional integral operators. There are several different integral inequality results for fractional integrals. For generalizing significant and well-known integral inequalities, these operators are helpful. The Hermite-Hadamard integral inequality is a particular type of integral inequality. It is frequently used in the literature and outlines the necessary and sufficient conditions for a function to be convex. The Hermite-Hadamard inequalities were generalized by Sarikaya et al. [10] using Riemann-Liouville fractional integrals. Işcan [26] expanded Sarikaya et al. [10] 's findings to include Hermite-Hadamard-Fejer-type inequalities. By utilizing the product of two convex functions, Chen [27] produced fractional Hermite-Hadamard-type integral inequalities. Ögülmüs et al. [28] incorporated the Hermite-Hadamard and Jensen-Mercer inequalities to present Hermite-Hadamard-Mercer type inequalities for Riemann-Liouville fractional integrals. Motivated by the above articles, Butt et al. (see [29]), presented new versions of Jensen and Jensen-Mercer type inequalities in the fractal sense. New fractional versions of Hermite-Hadamard-Mercer and Pachpatte-Mercer type inclusions are established for convex [30] and harmonically convex functions [31] respectively. Latif et al. [32] established Hermite-Hadamard-Fejér type inequalities for convex harmonic and a positive symmetric increasing function. New refinements of Hermite-Mercer type inequalities are presented in [33], Mercer-Ostrowski type inequalities are presented in [34]. Further, the Hermite-Hadamard inequality is also generalized for convexity and quasi convexity [35] and differentiable convex functions [36]. For further information on other fractional-order integral inequalities, see the papers [37–41].

Definition 1.1. (see [42]) Let $\mathscr{G} : \mathbb{X} \to \mathcal{R}$ be a function and $\mathbb{X}$ be a convex subset of a real vector space $\mathcal{R}$. Then we say that the function $\mathscr{G}$ is convex if and only if the following condition:

$$\mathscr{G}(\Phi\mathfrak{r} + (1-\Phi)\,\mathfrak{s}) \leq \Phi\mathscr{G}(\mathfrak{r}) + (1-\Phi)\,\mathscr{G}(\mathfrak{s}),$$

holds true for all $\mathfrak{r}, \mathfrak{s} \in \mathbb{X}$ and $\Phi \in [0, 1]$.

For further discussion, we first present the classical Hermite-Hadmard (H-H) inequality, which states that (see [1]):

If the function $\mathscr{G} : \mathbb{X} \subseteq \mathcal{R} \to \mathcal{R}$ is convex in $\mathbb{X}$ for $\mathfrak{r}, \mathfrak{s} \in \mathbb{X}$ and $\mathfrak{r} < \mathfrak{s}$, then

$$\mathscr{G}\left(\frac{\mathfrak{r}+\mathfrak{s}}{2}\right) \leq \frac{1}{\mathfrak{s}-\mathfrak{r}} \int_{\mathfrak{r}}^{\mathfrak{s}} \mathscr{G}(x)dx \leq \frac{\mathscr{G}(\mathfrak{r}) + \mathscr{G}(\mathfrak{s})}{2}. \tag{1.1}$$

Definition 1.2. (see [43]) Let there be a function $\mathscr{G} : [\mathfrak{r}, \mathfrak{s}] \to [0, \infty)$ and it is symmetric with respect to $\frac{\mathfrak{r}+\mathfrak{s}}{2}$, if

$$\mathscr{G}(\mathfrak{r} + \mathfrak{s} - \mathrm{x}) = \mathscr{G}(\mathrm{x}).$$

In 1906, Fejér [44] preposed the following weighted variant of Hermite-Hadamard inequality famously known as Hermite-Hadamard-Fejér inequality, given as

Theorem 1.1. *Let there be a convex function $\mathscr{G} : [\mathfrak{r}, \mathfrak{s}] \subseteq \mathcal{R} \setminus \{0\} \to \mathcal{R}$ with $\mathfrak{r} < \mathfrak{s}$. If $\mathscr{D} : [\mathfrak{r}, \mathfrak{s}] \subseteq \mathcal{R} \setminus \{0\} \to \mathcal{R}$ be a convex symmetric and integrable function with respect to $\frac{\mathfrak{r}+\mathfrak{s}}{2}$. Then*

$$\mathscr{G}\left(\frac{\mathfrak{r}+\mathfrak{s}}{2}\right)\int_{\mathfrak{r}}^{\mathfrak{s}} \mathscr{D}(x)dx \leq \frac{1}{\mathfrak{s}-\mathfrak{r}} \int_{\mathfrak{r}}^{\mathfrak{s}} \mathscr{G}(x)\mathscr{D}(x)dx \leq \frac{\mathscr{G}(\mathfrak{r}) + \mathscr{G}(\mathfrak{s})}{2} \int_{\mathfrak{r}}^{\mathfrak{s}} \mathscr{D}(x)dx, \tag{1.2}$$

holds true.

Definition 1.3. (see [9]) Let $\mathscr{G} : \mathbb{X} \to \mathcal{R}$ be a function and $\mathbb{X}$ be a subset of a real vector space $\mathcal{R}$. Then we say that the function $\mathscr{G}$ is harmonically convex if and only if the following condition

$$\mathscr{G}\left(\frac{\mathfrak{r}\mathfrak{s}}{\Phi\mathfrak{r} + (1-\Phi)\,\mathfrak{s}}\right) \le \Phi\mathscr{G}(\mathfrak{s}) + (1-\Phi)\,\mathscr{G}(\mathfrak{r}),$$

holds true for all $\mathfrak{r}, \mathfrak{s} \in \mathbb{X}$ and $\Phi \in [0, 1]$.

For further discussion, we first present the classical Hermite-Hadmard (H-H) inequality, which states that (see [9]):

If the function $\mathscr{G} : \mathbb{X} \subseteq \mathcal{R} \to \mathcal{R}$ is harmonically convex in $\mathbb{X}$ for $\mathfrak{r}, \mathfrak{s} \in \mathbb{X}$ and $\mathfrak{r} < \mathfrak{s}$, then

$$\mathscr{G}\left(\frac{2\mathfrak{r}\mathfrak{s}}{\mathfrak{r}+\mathfrak{s}}\right) \le \frac{\mathfrak{r}\mathfrak{s}}{\mathfrak{s}-\mathfrak{r}} \int_{\mathfrak{r}}^{\mathfrak{s}} \frac{\mathscr{G}(x)}{x^2} dx \le \frac{\mathscr{G}(\mathfrak{r}) + \mathscr{G}(\mathfrak{s})}{2}. \tag{1.3}$$

Definition 1.4. (see [45]) Let there be a function $\mathscr{G} : [\mathfrak{r}, \mathfrak{s}] \to [0, \infty)$ and it is harmonically symmetric with respect to $\frac{2\mathfrak{r}\mathfrak{s}}{\mathfrak{r}+\mathfrak{s}}$, if

$$\mathscr{G}(\Phi) = \mathscr{G}\left(\frac{1}{\frac{1}{\mathfrak{r}} + \frac{1}{\mathfrak{s}} - \frac{1}{\Phi}}\right).$$

In the year 2014, Chen and Wu [46] proposed the following weighted variant of Hermite-Hadamard inequality for harmonically convex function, given as

Theorem 1.2. *Let there be a convex function $\mathscr{G} : [\mathfrak{r}, \mathfrak{s}] \subseteq \mathcal{R} \setminus \{0\} \to \mathcal{R}$ with $\mathfrak{r} < \mathfrak{s}$. If $\mathscr{D} : [\mathfrak{r}, \mathfrak{s}] \subseteq \mathcal{R} \setminus \{0\} \to \mathcal{R}$ be a convex symmetric and integrable function with respect to $\frac{\mathfrak{r}+\mathfrak{s}}{2}$. Then,*

$$\mathscr{G}\left(\frac{2\mathfrak{r}\mathfrak{s}}{\mathfrak{r}+\mathfrak{s}}\right)\int_{\mathfrak{r}}^{\mathfrak{s}} \frac{\mathscr{D}(x)}{x^2} dx \le \frac{\mathfrak{r}\mathfrak{s}}{\mathfrak{s}-\mathfrak{r}} \int_{\mathfrak{r}}^{\mathfrak{s}} \frac{\mathscr{G}(x)\mathscr{D}(x)}{x^2} dx \le \frac{\mathscr{G}(\mathfrak{r}) + \mathscr{G}(\mathfrak{s})}{2} \int_{\mathfrak{r}}^{\mathfrak{s}} \frac{\mathscr{D}(x)}{x^2} dx, \tag{1.4}$$

holds true.

Definition 1.5. (see, for details, [10,47]; see also [48]) Let $\mathscr{G} \in \mathcal{L}[\mathfrak{r}, \mathfrak{s}]$. Then, the Riemann-Liouville fractional integrals of the order $\alpha > 0$, are defined as follows:

$$\mathtt{I}_{\mathfrak{r}^+}^{\alpha}\mathscr{G}(x) = \frac{1}{\Gamma(\alpha)} \int_{\mathfrak{r}}^{x} (x - \mathfrak{m})^{\alpha-1}\,\mathscr{G}(\mathfrak{m})\,d\mathfrak{m} \qquad (x > \mathfrak{r}),$$

and

$$\mathtt{I}_{\mathfrak{s}^-}^{\alpha}\mathscr{G}(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\mathfrak{s}} (\mathfrak{m} - x)^{\alpha-1}\,\mathscr{G}(\mathfrak{m})\,d\mathfrak{m} \qquad (x < \mathfrak{s}),$$

respectively, where $\Gamma(\alpha) = \int_0^{\infty} \Phi^{\alpha-1} e^{-\Phi} d\Phi$ is the Euler gamma function.

Definition 1.6. (see, [49] for details) Let $\mathscr{G} \in \mathcal{L}[\mathfrak{r}, \mathfrak{s}]$. Then, the new left and right fractional integrals $\mathtt{I}_{\mathfrak{r}^+}^{\alpha}$ and $\mathtt{I}_{\mathfrak{s}^-}^{\alpha}$ of order $\alpha > 0$ are defined as

$$\mathtt{I}_{\mathfrak{r}^+}^{\alpha}\mathscr{G}(x) := \frac{1}{\alpha} \int_{\mathfrak{r}}^{x} e^{-\frac{1-\alpha}{\alpha}(x-\mathfrak{m})}\,\mathscr{G}(\mathfrak{m})\,d\mathfrak{m} \qquad (0 \le \mathfrak{r} < x < \mathfrak{s}),$$

and

$$\mathtt{I}_{\mathfrak{s}^-}^{\alpha}\mathscr{G}(x) := \frac{1}{\alpha} \int_{x}^{\mathfrak{s}} e^{-\frac{1-\alpha}{\alpha}(\mathfrak{m}-x)}\,\mathscr{G}(\mathfrak{m})\,d\mathfrak{m} \qquad (0 \le \mathfrak{r} < x < \mathfrak{s}),$$

respectively.

It should be noted that

$$\lim_{\alpha\to 1} \mathtt{I}_{\mathfrak{r}^+}^{\alpha}\mathscr{G}(x) = \int_{\mathfrak{r}}^{x} \mathscr{G}(\mathfrak{m})\, d\mathfrak{m} \text{ and } \lim_{\alpha\to 1} \mathtt{I}_{\mathfrak{s}^-}^{\alpha}\mathscr{G}(x) = \int_{x}^{\mathfrak{s}} \mathscr{G}(\mathfrak{m})\, d\mathfrak{m}.$$

Sarikaya et al. [37], in their article proved some interesting mid-point type Hermite-Hadamard inequalities. Here, we present one of his main results as follows:

Theorem 1.3. (see [37]) Let $\mathscr{G} : [\mathfrak{r}, \mathfrak{s}] \longrightarrow \mathcal{R}$ be a convex function with $0 \le \mathfrak{r} \le \mathfrak{s}$. If $\mathscr{G} \in \mathcal{L}[\mathfrak{r}, \mathfrak{s}]$, then the following inequality for Riemann-Liouville fractional integral operator holds true:

$$\mathscr{G}\left(\frac{\mathfrak{r}+\mathfrak{s}}{2}\right) \le \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mathfrak{s}-\mathfrak{r})^{\alpha}} \left[\mathtt{I}^{\alpha}_{(\frac{\mathfrak{r}+\mathfrak{s}}{2})^+}\mathscr{G}(\mathfrak{s}) + \mathtt{I}^{\alpha}_{(\frac{\mathfrak{r}+\mathfrak{s}}{2})^-}\mathscr{G}(\mathfrak{r})\right] \le \frac{\mathscr{G}(\mathfrak{r}) + \mathscr{G}(\mathfrak{s})}{2}.$$

The major goal of this paper is to establish Fejér type fractional inequalities using differintegrals of the $\left(\frac{\mathfrak{r}+\mathfrak{s}}{2}\right)$ type for both convex and harmonically convex functions via a novel fractional integral operator. In order to derive those inequalities, first we prove two new lemmas i.e., Lemmas 2.1 and 3.1 for convex and harmonic convex functions respectively.

In this study, we used a new fractional integral operator to achieve more generalized results. This is caused by the exponential function that makes up the kernel of this fractional operator. Our results differ from prior generalizations in that they do not lead to the aforementioned fractional integral inequalities. Numerous experts have suggested utilizing different fractional integral operators to extend the Hermite-Hadamard and Fejér type inequalities, however, none of their findings exhibit an exponential property. This study generated interest in using an exponential function as the kernel to create more generalized fractional inequalities. Furthermore, the application of symmetric and harmonically symmetric functions to the main results gives the study of inequalities a new path. For other generalization regarding exponential kernel interested reader can see e.g., on distributed-order fractional derivative in [50]. There are many research gaps to be filled for integral inequalities involving fractional calculus for different types of convex functions, despite the fact that there exist many different forms of research on the growth of fractional integral inequalities. As a result, the main purpose of this research is to find new Hermite-Hadamard and Fejér type inequalities for positive symmetric functions using fractional integral operators.

Our present investigation is structured as follows. In Sections 2 and 3, we discuss two additional characteristics of the relevant fractional operator before proving some enhanced versions (mid-point types) of the Fejér and Hermite-Hadamard type inequalities for convex and harmonically convex functions respectively. Some applications are also taken into consideration in Section 4 to determine whether the predetermined results are appropriate. Section 5 explores a brief conclusion and possible areas for additional research that is related to the findings in this paper are discussed in Section 6.

2. Improved Fejér type results for convex functions

In this section for simplicity, we denote $\rho_c = \frac{1-\alpha}{\alpha}(\mathfrak{s}-\mathfrak{r})$. If $\alpha \to 1$, then $\rho_c = \frac{1-\alpha}{\alpha}(\mathfrak{s}-\mathfrak{r}) \to 0$.

Lemma 2.1. *Let $\mathscr{D} : [\mathfrak{r}, \mathfrak{s}] \subseteq \mathcal{R} \setminus \{0\} \to \mathcal{R}$ be a symmetric convex function with respect to $\frac{\mathfrak{r}+\mathfrak{s}}{2}$. Then for $\alpha > 0$, the following equality holds true:*

$$\mathtt{I}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}+}\mathscr{D}(\mathfrak{s}) = \mathtt{I}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}-}\mathscr{D}(\mathfrak{r}) = \frac{1}{2}\left[\mathtt{I}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}-}\mathscr{D}(\mathfrak{r}) + \mathtt{I}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}+}\mathscr{D}(\mathfrak{s})\right].$$

Proof. Since $\mathscr{D} : [\mathfrak{r}, \mathfrak{s}] \subseteq \mathcal{R} \setminus \{0\} \to \mathcal{R}$ is integrable and symmetric to $\frac{\mathfrak{r}+\mathfrak{s}}{2}$ we have $\mathscr{D}(\mathfrak{r} + \mathfrak{s} - \mathrm{x}) = \mathscr{D}(\mathrm{x})$. Also, Setting $\Phi = \mathfrak{r} + \mathfrak{s} - \mathrm{x}$ and $d\Phi = -d\mathrm{x}$, we have

$$
\begin{aligned}
\mathrm{I}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}+}\mathscr{D}(\mathfrak{s}) &= \frac{1}{\alpha}\int_{\frac{\mathfrak{r}+\mathfrak{s}}{2}}^{\mathfrak{s}} e^{-\frac{1-\alpha}{\alpha}(\mathfrak{s}-\Phi)}\mathscr{D}(\Phi)d\Phi \\
&= -\frac{1}{\alpha}\int_{\frac{\mathfrak{r}+\mathfrak{s}}{2}}^{\mathfrak{r}} e^{-\frac{1-\alpha}{\alpha}(\mathfrak{s}-(\mathfrak{r}+\mathfrak{s}-\mathrm{x}))}\mathscr{D}(\mathfrak{r}+\mathfrak{s}-\mathrm{x})d\mathrm{x} \\
&= \frac{1}{\alpha}\int_{\mathfrak{r}}^{\frac{\mathfrak{r}+\mathfrak{s}}{2}} e^{-\frac{1-\alpha}{\alpha}(\mathrm{x}-\mathfrak{r})}\mathscr{D}(\mathfrak{r}+\mathfrak{s}-\mathrm{x})d\mathrm{x} \\
&= \frac{1}{\alpha}\int_{\mathfrak{r}}^{\frac{\mathfrak{r}+\mathfrak{s}}{2}} e^{-\frac{1-\alpha}{\alpha}(\mathrm{x}-\mathfrak{r})}\mathscr{D}(\mathrm{x})d\mathrm{x} \\
&= \mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}-}\mathscr{D}(\mathfrak{r}).
\end{aligned}
$$

$$
\implies \mathrm{I}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}+}\mathscr{D}(\mathfrak{s}) = \mathrm{I}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}-}\mathscr{D}(\mathfrak{r}) = \frac{1}{2}\left[\mathrm{I}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}-}\mathscr{D}(\mathfrak{r}) + \mathrm{I}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}+}\mathscr{D}(\mathfrak{s})\right].
$$

This led us to the desired equality. □

First, we prove both the first and second kind Fejér type inequalities in a different approach. Then, we also prove the Hermite-Hadamard inequality using symmetric convex functions.

Theorem 2.1. *Let there be a convex function $\mathscr{G} : [\mathfrak{r}, \mathfrak{s}] \subseteq \mathcal{R} \setminus \{0\} \to \mathcal{R}$ with $\mathfrak{r} < \mathfrak{s}$. If $\mathscr{D} : [\mathfrak{r}, \mathfrak{s}] \subseteq \mathcal{R} \setminus \{0\} \to \mathcal{R}$ be a convex symmetric and integrable function with respect to $\frac{\mathfrak{r}+\mathfrak{s}}{2}$. Then for $\alpha > 0$, the following inequality holds true:*

$$
\mathscr{G}\left(\frac{\mathfrak{r}+\mathfrak{s}}{2}\right)\left[\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}-}\mathscr{D}(\mathfrak{r}) + \mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}+}\mathscr{D}(\mathfrak{s})\right] \le \left[\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}-}(\mathscr{G}\mathscr{D})(\mathfrak{r}) + \mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}+}(\mathscr{G}\mathscr{D})(\mathfrak{s})\right].
$$

Proof. Using the convexity of $\mathscr{G}$ on $[\mathfrak{r}, \mathfrak{s}]$, we have

$$
2\mathscr{G}\left(\frac{\mathfrak{r}+\mathfrak{s}}{2}\right) \le \mathscr{G}\left(\Phi\mathfrak{r} + (1-\Phi)\mathfrak{s}\right) + \mathscr{G}\left(\Phi\mathfrak{s} + (1-\Phi)\mathfrak{r}\right). \tag{2.1}
$$

Upon multiplication of both sides of the inequality (2.1) by $e^{-\frac{1-\alpha}{\alpha}(\mathfrak{s}-\mathfrak{r})\Phi}\mathscr{D}\left(\Phi\mathfrak{s} + (1-\Phi)\mathfrak{r}\right)$ and then integrating the resultant over $\left[0, \frac{1}{2}\right]$, we obtain

$$
\begin{aligned}
&2\mathscr{G}\left(\frac{\mathfrak{r}+\mathfrak{s}}{2}\right)\int_0^{\frac{1}{2}} e^{-\frac{1-\alpha}{\alpha}(\mathfrak{s}-\mathfrak{r})\Phi}\mathscr{D}\left(\Phi\mathfrak{s} + (1-\Phi)\mathfrak{r}\right)d\Phi \\
&\le \int_0^{\frac{1}{2}} e^{-\frac{1-\alpha}{\alpha}(\mathfrak{s}-\mathfrak{r})\Phi}\mathscr{G}\left(\Phi\mathfrak{r} + (1-\Phi)\mathfrak{s}\right)\mathscr{D}\left(\Phi\mathfrak{s} + (1-\Phi)\mathfrak{r}\right)d\Phi \\
&+ \int_0^{\frac{1}{2}} e^{-\frac{1-\alpha}{\alpha}(\mathfrak{s}-\mathfrak{r})\Phi}\mathscr{G}\left(\Phi\mathfrak{s} + (1-\Phi)\mathfrak{r}\right)\mathscr{D}\left(\Phi\mathfrak{s} + (1-\Phi)\mathfrak{r}\right)d\Phi.
\end{aligned} \tag{2.2}
$$

Since $\mathscr{D}$ is symmetric with respect to $\frac{\mathfrak{r}+\mathfrak{s}}{2}$, we have $\mathscr{D}(\mathrm{x}) = \mathscr{D}(\mathfrak{r}+\mathfrak{s}-\mathrm{x})$.
Moreover, setting $\mathrm{x} = \Phi\mathfrak{s} + (1-\Phi)\mathfrak{r}$ and $d\mathrm{x} = (\mathfrak{s}-\mathfrak{r})d\Phi$ in (2.2), we have

$$
2\mathscr{G}\left(\frac{\mathfrak{r}+\mathfrak{s}}{2}\right)\frac{1}{\mathfrak{s}-\mathfrak{r}}\int_{\mathfrak{r}}^{\frac{\mathfrak{r}+\mathfrak{s}}{2}} e^{-\frac{1-\alpha}{\alpha}(\mathrm{x}-\mathfrak{r})}\mathscr{D}(\mathrm{x})d\mathrm{x} = 2\mathscr{G}\left(\frac{\mathfrak{r}+\mathfrak{s}}{2}\right)\frac{\alpha}{\mathfrak{s}-\mathfrak{r}}\left[\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}-}\mathscr{D}(\mathfrak{r})\right]
$$

$$\leq \frac{1}{\mathfrak{s}-\mathfrak{r}}\int_{\mathfrak{r}}^{\frac{\mathfrak{r}+\mathfrak{s}}{2}} e^{-\frac{1-\alpha}{\alpha}(\mathrm{x}-\mathfrak{r})}\mathscr{G}(\mathfrak{r}+\mathfrak{s}-\mathrm{x})\mathscr{D}(\mathrm{x})d\mathrm{x} + \frac{1}{\mathfrak{s}-\mathfrak{r}}\int_{\mathfrak{r}}^{\frac{\mathfrak{r}+\mathfrak{s}}{2}} e^{-\frac{1-\alpha}{\alpha}(\mathrm{x}-\mathfrak{r})}\mathscr{G}(\mathrm{x})\mathscr{D}(\mathrm{x})d\mathrm{x}$$
$$= \frac{1}{\mathfrak{s}-\mathfrak{r}}\int_{\frac{\mathfrak{r}+\mathfrak{s}}{2}}^{\mathfrak{s}} e^{-\frac{1-\alpha}{\alpha}(\mathfrak{s}-\mathrm{x})}\mathscr{G}(\mathrm{x})\mathscr{D}(\mathfrak{r}+\mathfrak{s}-\mathrm{x})d\mathrm{x} + \frac{1}{\mathfrak{s}-\mathfrak{r}}\int_{\mathfrak{r}}^{\frac{\mathfrak{r}+\mathfrak{s}}{2}} e^{-\frac{1-\alpha}{\alpha}(\mathrm{x}-\mathfrak{r})}\mathscr{G}(\mathrm{x})\mathscr{D}(\mathrm{x})d\mathrm{x}$$
$$= \frac{\alpha}{\mathfrak{s}-\mathfrak{r}}\left[\frac{1}{\alpha}\int_{\frac{\mathfrak{r}+\mathfrak{s}}{2}}^{\mathfrak{s}} e^{-\frac{1-\alpha}{\alpha}(\mathfrak{s}-\mathrm{x})}\mathscr{G}(\mathrm{x})\mathscr{D}(\mathrm{x})d\mathrm{x} + \frac{1}{\alpha}\int_{\mathfrak{r}}^{\frac{\mathfrak{r}+\mathfrak{s}}{2}} e^{-\frac{1-\alpha}{\alpha}(\mathrm{x}-\mathfrak{r})}\mathscr{G}(\mathrm{x})\mathscr{D}(\mathrm{x})d\mathrm{x}\right].$$

It follows from the above developments and Lemma 2.1 that,

$$\frac{\alpha}{\mathfrak{s}-\mathfrak{r}}\mathscr{G}\left(\frac{\mathfrak{r}+\mathfrak{s}}{2}\right)\left[\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}-}\mathscr{D}(\mathfrak{r}) + \mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}+}\mathscr{D}(\mathfrak{s})\right] \leq \frac{\alpha}{\mathfrak{s}-\mathfrak{r}}\left[\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}-}(\mathscr{G}\mathscr{D})(\mathfrak{r}) + \mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}+}(\mathscr{G}\mathscr{D})(\mathfrak{s})\right].$$

This concludes the proof of the required result. □

Example 2.1. *Let* $\mathscr{G}(\mathfrak{m}) = e^{\mathfrak{m}}$, $\mathfrak{m} \in [1,9]$ *and* $\mathscr{D}(\mathfrak{m}) = (5-\mathfrak{m})^2$, *is non-negative symmetric about* $\mathfrak{m} = 5$. *Let* $0 < \alpha < 1$, *then*

$$\mathscr{G}\left(\frac{\mathfrak{r}+\mathfrak{s}}{2}\right)\left[\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}-}\mathscr{D}(\mathfrak{r}) + \mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}+}\mathscr{D}(\mathfrak{s})\right]$$
$$= e^5\left[\mathrm{I}^{\alpha}_{5-}\mathscr{D}(1) + \mathrm{I}^{\alpha}_{5+}\mathscr{D}(9)\right]$$
$$= e^5\left[\frac{1}{\alpha}\int_1^5 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{m}-1)}\ (5-\mathfrak{m})^2\, d\mathfrak{m} + \frac{1}{\alpha}\int_5^9 e^{-\frac{1-\alpha}{\alpha}(9-\mathfrak{m})}\ (5-\mathfrak{m})^2\, d\mathfrak{m}\right].$$

and

$$\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}-}(\mathscr{G}\mathscr{D})(\mathfrak{r}) + \mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}+}(\mathscr{G}\mathscr{D})(\mathfrak{s})$$
$$= \mathrm{I}^{\alpha}_{5-}(\mathscr{G}\mathscr{D})(1) + \mathrm{I}^{\alpha}_{5+}(\mathscr{G}\mathscr{D})(9)$$
$$= \frac{1}{\alpha}\int_1^5 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{m}-1)}\ e^{\mathfrak{m}}\,(5-\mathfrak{m})^2\, d\mathfrak{m} + \frac{1}{\alpha}\int_5^9 e^{-\frac{1-\alpha}{\alpha}(9-\mathfrak{m})}\ e^{\mathfrak{m}}\,(5-\mathfrak{m})^2\, d\mathfrak{m}.$$

The graphical representation of Theorem 2.1 is shown in the graph below (see Figure 1) for $0 < \alpha < 1$*:*

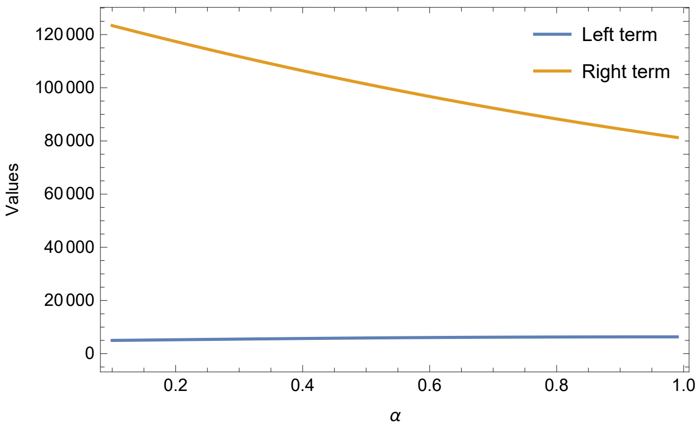

Figure 1. The graphical representation of Theorem 2.1 for $0 < \alpha < 1$.

Theorem 2.2. *Let there be a convex function $\mathscr{G} : [\mathfrak{r}, \mathfrak{s}] \subseteq \mathcal{R} \setminus \{0\} \rightarrow \mathcal{R}$ with $\mathfrak{r} < \mathfrak{s}$. If $\mathscr{D} : [\mathfrak{r}, \mathfrak{s}] \subseteq \mathcal{R} \setminus \{0\} \rightarrow \mathcal{R}$ be a convex symmetric and integrable function with respect to $\frac{\mathfrak{r}+\mathfrak{s}}{2}$. Then for $\alpha > 0$, the following inequality holds true:*

$$\left[\mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}-}(\mathscr{G}\mathscr{D})(\mathfrak{r}) + \mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}+}(\mathscr{G}\mathscr{D})(\mathfrak{s})\right] \leq \frac{\mathscr{G}(\mathfrak{r}) + \mathscr{G}(\mathfrak{s})}{2}\left[\mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}-}(\mathscr{D})(\mathfrak{r}) + \mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}+}(\mathscr{D})(\mathfrak{s})\right].$$

Proof. Since $\mathscr{G}$ is convex function, we have

$$\mathscr{G}(\Phi\mathfrak{r} + (1-\Phi)\mathfrak{s}) + \mathscr{G}(\Phi\mathfrak{s} + (1-\Phi)\mathfrak{r}) \leq \mathscr{G}(\mathfrak{r}) + \mathscr{G}(\mathfrak{s}). \tag{2.3}$$

Multiplying both side of the above inequality (2.3) by $e^{-\frac{1-\alpha}{\alpha}(\mathfrak{s}-\mathfrak{r})\Phi}\mathscr{D}(\Phi\mathfrak{s} + (1-\Phi)\mathfrak{r})$ and upon integration of the obtained result over $\left[0, \frac{1}{2}\right]$, one has

$$\begin{aligned}
&\int_0^{\frac{1}{2}} e^{-\frac{1-\alpha}{\alpha}(\mathfrak{s}-\mathfrak{r})\Phi}\mathscr{G}(\Phi\mathfrak{r} + (1-\Phi)\mathfrak{s})\,\mathscr{D}(\Phi\mathfrak{s} + (1-\Phi)\mathfrak{r})\,d\Phi\\
&+ \int_0^{\frac{1}{2}} e^{-\frac{1-\alpha}{\alpha}(\mathfrak{s}-\mathfrak{r})\Phi}\mathscr{G}(\Phi\mathfrak{s} + (1-\Phi)\mathfrak{r})\,\mathscr{D}(\Phi\mathfrak{s} + (1-\Phi)\mathfrak{r})\,d\Phi\\
&\leq [\mathscr{G}(\mathfrak{r}) + \mathscr{G}(\mathfrak{s})]\int_0^{\frac{1}{2}} e^{-\frac{1-\alpha}{\alpha}(\mathfrak{s}-\mathfrak{r})\Phi}\mathscr{D}(\Phi\mathfrak{s} + (1-\Phi)\mathfrak{r})\,d\Phi.
\end{aligned}$$

It follows

$$\frac{\alpha}{\mathfrak{s}-\mathfrak{r}}\left[\mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}-}(\mathscr{G}\mathscr{D})(\mathfrak{r}) + \mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}+}(\mathscr{G}\mathscr{D})(\mathfrak{s})\right] \leq \alpha\frac{[\mathscr{G}(\mathfrak{r}) + \mathscr{G}(\mathfrak{s})]}{\mathfrak{s}-\mathfrak{r}}\left[\mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}-}(\mathscr{D})(\mathfrak{r})\right].$$

Furthermore, using the Lemma 2.1, we obtain

$$\frac{\alpha}{\mathfrak{s}-\mathfrak{r}}\left[\mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}-}(\mathscr{G}\mathscr{D})(\mathfrak{r}) + \mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}+}(\mathscr{G}\mathscr{D})(\mathfrak{s})\right] \leq \frac{\mathscr{G}(\mathfrak{r}) + \mathscr{G}(\mathfrak{s})}{2}\frac{\alpha}{\mathfrak{s}-\mathfrak{r}}\left[\mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}-}(\mathscr{D})(\mathfrak{r}) + \mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}+}(\mathscr{D})(\mathfrak{s})\right].$$

This concludes the proof of the desired result. □

Example 2.2. *Let $\mathscr{G}(\mathfrak{m}) = e^{\mathfrak{m}}$, $\mathfrak{m} \in [1, 9]$ and $\mathscr{D}(\mathfrak{m}) = (5-\mathfrak{m})^2$, is non-negative symmetric about $\mathfrak{m} = 5$. Let $0 < \alpha < 1$, then*

$$\begin{aligned}
&\frac{e+e^9}{2}\left[\mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}-}(\mathscr{D})(\mathfrak{r}) + \mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}+}(\mathscr{D})(\mathfrak{s})\right]\\
&= \frac{e+e^9}{2}\left[\mathtt{I}^{\alpha}_{5-}\mathscr{D}(1) + \mathtt{I}^{\alpha}_{5+}\mathscr{D}(9)\right]\\
&= \frac{e+e^9}{2}\left[\frac{1}{\alpha}\int_1^5 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{m}-1)}\ (5-\mathfrak{m})^2\,d\mathfrak{m} + \frac{1}{\alpha}\int_5^9 e^{-\frac{1-\alpha}{\alpha}(9-\mathfrak{m})}\ (5-\mathfrak{m})^2\,d\mathfrak{m}\right].
\end{aligned}$$

And

$$\begin{aligned}
&\mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}-}(\mathscr{G}\mathscr{D})(\mathfrak{r}) + \mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}+}(\mathscr{G}\mathscr{D})(\mathfrak{s})\\
&= \mathtt{I}^{\alpha}_{5-}(\mathscr{G}\mathscr{D})(1) + \mathtt{I}^{\alpha}_{5+}(\mathscr{G}\mathscr{D})(9)\\
&= \frac{1}{\alpha}\int_1^5 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{m}-1)}\ e^{\mathfrak{m}}(5-\mathfrak{m})^2\,d\mathfrak{m} + \frac{1}{\alpha}\int_5^9 e^{-\frac{1-\alpha}{\alpha}(9-\mathfrak{m})}\ e^{\mathfrak{m}}(5-\mathfrak{m})^2\,d\mathfrak{m}.
\end{aligned}$$

The graphical representation of Theorem 2.2 is shown in the graph below (see Figure 2) for $0 < \alpha < 1$*:*

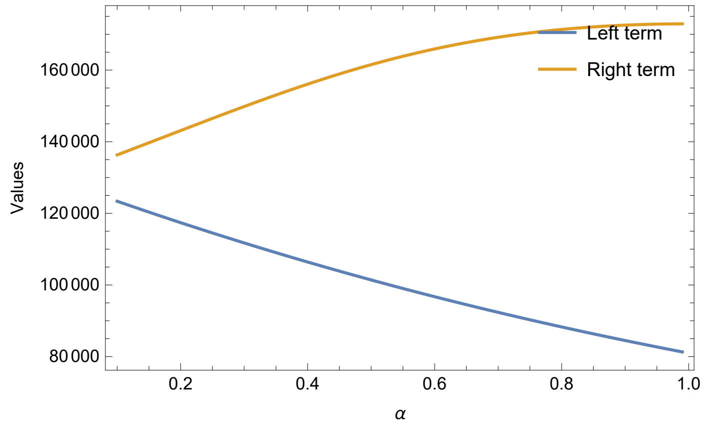

Figure 2. The graphical representation of Theorem 2.2 for $0 < \alpha < 1$.

Theorem 2.3. *Let there be a convex function* $\mathscr{G} : [\mathfrak{r}, \mathfrak{s}] \subseteq \mathcal{R} \setminus \{0\} \to \mathcal{R}$ *with* $\mathfrak{r} < \mathfrak{s}$*. Then for* $\alpha > 0$*, the following fractional integral inequality holds true:*

$$\mathscr{G}\left(\frac{\mathfrak{r}+\mathfrak{s}}{2}\right) \leq \frac{1-\alpha}{2(1-e^{-\frac{\rho c}{2}})}\left[\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}-}\mathscr{G}(\mathfrak{r}) + \mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}+}\mathscr{G}(\mathfrak{s})\right] \leq \frac{\mathscr{G}(\mathfrak{r})+\mathscr{G}(\mathfrak{s})}{2}. \tag{2.4}$$

Proof. By the hypothesis of convexity, we have

$$\begin{aligned}\mathscr{G}\left(\frac{\mathfrak{r}+\mathfrak{s}}{2}\right) &= \mathscr{G}\left(\frac{\Phi\mathfrak{r}+(1-\Phi)\mathfrak{s}+\Phi\mathfrak{s}+(1-\Phi)\mathfrak{r}}{2}\right)\\ &\leq \frac{\mathscr{G}\left(\Phi\mathfrak{r}+(1-\Phi)\mathfrak{s}\right)+\mathscr{G}\left(\Phi\mathfrak{s}+(1-\Phi)\mathfrak{r}\right)}{2}.\end{aligned} \tag{2.5}$$

Upon multiplication of both sides of the inequality (2.5) by $2e^{-\frac{1-\alpha}{\alpha}(\mathfrak{s}-\mathfrak{r})\Phi}$ and then integrating the obtained result over $\left[0, \frac{1}{2}\right]$, we have

$$\begin{aligned}&2\mathscr{G}\left(\frac{\mathfrak{r}+\mathfrak{s}}{2}\right)\int_0^{\frac{1}{2}} e^{-\frac{1-\alpha}{\alpha}(\mathfrak{s}-\mathfrak{r})\Phi}d\Phi\\ &\leq \int_0^{\frac{1}{2}} e^{-\frac{1-\alpha}{\alpha}(\mathfrak{s}-\mathfrak{r})\Phi}\mathscr{G}\left(\Phi\mathfrak{r}+(1-\Phi)\mathfrak{s}\right)d\Phi + \int_0^{\frac{1}{2}} e^{-\frac{1-\alpha}{\alpha}(\mathfrak{s}-\mathfrak{r})\Phi}\mathscr{G}\left(\Phi\mathfrak{s}+(1-\Phi)\mathfrak{r}\right)d\Phi.\end{aligned} \tag{2.6}$$

Furthermore, let $\mathfrak{m} = \Phi\mathfrak{s} + (1-\Phi)\mathfrak{r} \implies d\Phi = \frac{d\mathfrak{m}}{\mathfrak{s}-\mathfrak{r}}$. Then inequality (2.6) gives

$$\begin{aligned}&\frac{2(1-e^{-\frac{\rho c}{2}})}{\rho}\mathscr{G}\left(\frac{\mathfrak{r}+\mathfrak{s}}{2}\right)\\ &\leq \left[\frac{1}{\mathfrak{s}-\mathfrak{r}}\int_{\mathfrak{r}}^{\frac{\mathfrak{r}+\mathfrak{s}}{2}} e^{-\frac{1-\alpha}{\alpha}(\mathfrak{s}-\mathfrak{r})\frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{s}-\mathfrak{r}}}\mathscr{G}\left(\mathfrak{r}+\mathfrak{s}-\mathfrak{m}\right)d\mathfrak{m} + \frac{1}{\mathfrak{s}-\mathfrak{r}}\int_{\mathfrak{r}}^{\frac{\mathfrak{r}+\mathfrak{s}}{2}} e^{-\frac{1-\alpha}{\alpha}(\mathfrak{s}-\mathfrak{r})\frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{s}-\mathfrak{r}}}\mathscr{G}\left(\mathfrak{m}\right)d\mathfrak{m}\right]\\ &= \frac{\alpha}{\mathfrak{s}-\mathfrak{r}}\left[\frac{1}{\alpha}\int_{\frac{\mathfrak{r}+\mathfrak{s}}{2}}^{\mathfrak{s}} e^{-\frac{1-\alpha}{\alpha}(\mathfrak{s}-\mathfrak{m})}\mathscr{G}\left(\mathfrak{m}\right)d\mathfrak{m} + \frac{1}{\alpha}\int_{\mathfrak{r}}^{\frac{\mathfrak{r}+\mathfrak{s}}{2}} e^{-\frac{1-\alpha}{\alpha}(\mathfrak{m}-\mathfrak{r})}\mathscr{G}\left(\mathfrak{m}\right)d\mathfrak{m}\right]\\ &= \frac{\alpha}{\mathfrak{s}-\mathfrak{r}}\left[\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}-}\mathscr{G}(\mathfrak{r}) + \mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}+}\mathscr{G}(\mathfrak{s})\right].\end{aligned} \tag{2.7}$$

This concludes the proof of the first part of the inequality (2.4). To prve the next part of inequality, under the given hypothesis, we have

$$\mathscr{G}(\Phi\mathfrak{r} + (1-\Phi)\mathfrak{s}) + \mathscr{G}(\Phi\mathfrak{s} + (1-\Phi)\mathfrak{r}) \leq \mathscr{G}(\mathfrak{r}) + \mathscr{G}(\mathfrak{s}). \tag{2.8}$$

Upon multiplication of both sides of the inequality (2.8) by $e^{-\frac{1-\alpha}{\alpha}(\mathfrak{s}-\mathfrak{r})\Phi}$ and integrating over $\left[0, \frac{1}{2}\right]$, we have

$$\int_0^{\frac{1}{2}} e^{-\frac{1-\alpha}{\alpha}(\mathfrak{s}-\mathfrak{r})\Phi}\mathscr{G}(\Phi\mathfrak{r} + (1-\Phi)\mathfrak{s})\, d\Phi + \int_0^{\frac{1}{2}} e^{-\frac{1-\alpha}{\alpha}(\mathfrak{s}-\mathfrak{r})\Phi}\mathscr{G}(\Phi\mathfrak{s} + (1-\Phi)\mathfrak{r})\, d\Phi$$
$$\leq [\mathscr{G}(\mathfrak{r}) + \mathscr{G}(\mathfrak{s})] \int_0^{\frac{1}{2}} e^{-\frac{1-\alpha}{\alpha}(\mathfrak{s}-\mathfrak{r})\Phi} d\Phi.$$

Consequently, we obtain

$$\frac{\alpha}{\mathfrak{s}-\mathfrak{r}}\left[\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}-}\mathscr{G}(\mathfrak{r}) + \mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}+}\mathscr{G}(\mathfrak{s})\right] \leq \frac{\mathscr{G}(\mathfrak{r}) + \mathscr{G}(\mathfrak{s})}{2}\frac{2(1-e^{-\frac{\rho_c}{2}})}{\rho_c}. \tag{2.9}$$

Consequently, it follows from the above developments (2.7) and (2.9) that

$$\mathscr{G}\left(\frac{\mathfrak{r}+\mathfrak{s}}{2}\right) \leq \frac{1-\alpha}{2(1-e^{-\frac{\rho_c}{2}})}\left[\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}-}\mathscr{G}(\mathfrak{r}) + \mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}+}\mathscr{G}(\mathfrak{s})\right] \leq \frac{\mathscr{G}(\mathfrak{r}) + \mathscr{G}(\mathfrak{s})}{2}.$$

This concludes the proof of the required result. □

Remark 2.1. If one chooses $\alpha \to 1$ i.e., $\frac{\rho_c}{2} \to 0$, then

$$\lim_{\alpha\to 1}\frac{1-\alpha}{2(1-e^{-\frac{\rho_c}{2}})} = \frac{1}{\mathfrak{s}-\mathfrak{r}}$$

and hence Theorem 2.3 retrieves the classical Hermite-Hadamard inequality (1.1).

Example 2.3. *Let* $\mathscr{G}(\mathfrak{m}) = e^{\mathfrak{m}}$, $\mathfrak{m} \in [1, 9]$ *and* $0 < \alpha < 1$, *then*

$$\frac{\mathscr{G}(\mathfrak{r}) + \mathscr{G}(\mathfrak{s})}{2} = \frac{e + e^9}{2},$$
$$\mathscr{G}\left(\frac{\mathfrak{r}+\mathfrak{s}}{2}\right) = e^5$$

and

$$\frac{1-\alpha}{2(1-e^{-\frac{\rho_c}{2}})}\left[\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}-}\mathscr{G}(\mathfrak{r}) + \mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}+}\mathscr{G}(\mathfrak{s})\right]$$
$$= \frac{1-\alpha}{2\left(1-e^{-\frac{4(1-\alpha)}{\alpha}}\right)}\left[\mathrm{I}^{\alpha}_{5-}(\mathscr{G})(1) + \mathrm{I}^{\alpha}_{5+}(\mathscr{G})(9)\right]$$
$$= \frac{1-\alpha}{2\left(1-e^{-\frac{4(1-\alpha)}{\alpha}}\right)}\left[\frac{1}{\alpha}\int_1^5 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{m}-1)}\, e^{\mathfrak{m}} d\mathfrak{m} + \frac{1}{\alpha}\int_5^9 e^{-\frac{1-\alpha}{\alpha}(9-\mathfrak{m})}\, e^{\mathfrak{m}} d\mathfrak{m}\right].$$

The graphical representation of Theorem 2.3 is shown in the graph below (see Figure 3) for $0 < \alpha < 1$*:*

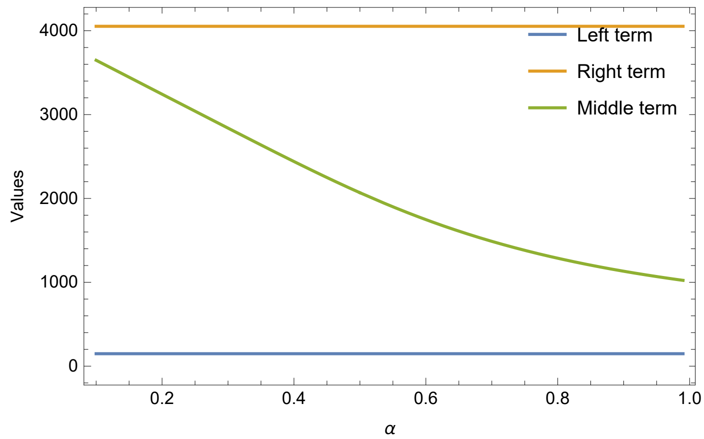

Figure 3. The graphical representation of Theorem 2.3 for $0 < \alpha < 1$.

3. Improved inclusions via harmonically convex functions

The family of Lebesgue measurable functions is represented here by $\mathcal{L}[\mathfrak{r}, \mathfrak{s}]$. In this section, for brevity we use, $\rho_h = \frac{1-\alpha}{\alpha}\frac{\mathfrak{s}-\mathfrak{r}}{\mathfrak{r}\mathfrak{s}}$ wherever needed. If $\alpha \to 1$, then $\rho_h = \frac{1-\alpha}{\alpha}\frac{\mathfrak{s}-\mathfrak{r}}{\mathfrak{r}\mathfrak{s}} \to 0$.

Lemma 3.1. *Let $\mathscr{D} : [\mathfrak{r}, \mathfrak{s}] \subseteq \mathcal{R} \setminus \{0\} \to \mathcal{R}$ be a harmonically symmetric and integrable function with respect to $\frac{2\mathfrak{r}\mathfrak{s}}{\mathfrak{r}+\mathfrak{s}}$. Then for $\alpha > 0$, the following equality holds true:*

$$\mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}+}\mathscr{D} \circ \mathtt{K}\left(\frac{1}{\mathfrak{r}}\right) = \mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}-}\mathscr{D} \circ \mathtt{K}\left(\frac{1}{\mathfrak{s}}\right) = \frac{1}{2}\left[\mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}+}\mathscr{D} \circ \mathtt{K}\left(\frac{1}{\mathfrak{r}}\right) + \mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}-}\mathscr{D} \circ \mathtt{K}\left(\frac{1}{\mathfrak{s}}\right)\right],$$

where $\mathtt{K}(\mathtt{x}) = \frac{1}{\mathtt{x}}, \quad \mathtt{x} \in \left[\frac{1}{\mathfrak{s}}, \frac{1}{\mathfrak{r}}\right]$.

Proof. Let $\mathscr{D}$ be a harmonically symmetric function with respect to $\frac{2\mathfrak{r}\mathfrak{s}}{\mathfrak{r}+\mathfrak{s}}$. Then using the harmonically symmetric property of $\mathscr{D}$, given as $\mathscr{D}\left(\frac{1}{\Phi}\right) = \mathscr{D}\left(\frac{1}{\frac{1}{\mathfrak{r}}+\frac{1}{\mathfrak{s}}-\Phi}\right)$ for $\alpha > 0$.

$$\begin{aligned}
\mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}+}\mathscr{D} \circ \mathtt{K}\left(\frac{1}{\mathfrak{r}}\right) &= \frac{1}{\alpha}\int_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}}^{\frac{1}{\mathfrak{r}}} e^{-\frac{1-\alpha}{\alpha}\left(\frac{1}{\mathfrak{r}}-\Phi\right)}\mathscr{D}\left(\frac{1}{\Phi}\right)d\Phi \\
&= \frac{1}{\alpha}\int_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}}^{\frac{1}{\mathfrak{r}}} e^{-\frac{1-\alpha}{\alpha}\left(\frac{1}{\mathfrak{r}}-\Phi\right)}\mathscr{D}\left(\frac{1}{\frac{1}{\mathfrak{r}}+\frac{1}{\mathfrak{s}}-\Phi}\right)d\Phi \\
&= -\frac{1}{\alpha}\int_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}}^{\frac{1}{\mathfrak{s}}} e^{-\frac{1-\alpha}{\alpha}\left(\mathtt{x}-\frac{1}{\mathfrak{s}}\right)}\mathscr{D}\left(\frac{1}{\mathtt{x}}\right)d\mathtt{x} \\
&= \frac{1}{\alpha}\int_{\frac{1}{\mathfrak{s}}}^{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}} e^{-\frac{1-\alpha}{\alpha}\left(\mathtt{x}-\frac{1}{\mathfrak{s}}\right)}\mathscr{D}\left(\frac{1}{\mathtt{x}}\right)d\mathtt{x} \\
&= \mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}-}\mathscr{D} \circ \mathtt{K}\left(\frac{1}{\mathfrak{s}}\right).
\end{aligned}$$

Consequently, it follows from the above developments that

$$\mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}+}\mathscr{D} \circ \mathtt{K}\left(\frac{1}{\mathfrak{r}}\right) = \mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}-}\mathscr{D} \circ \mathtt{K}\left(\frac{1}{\mathfrak{s}}\right) = \frac{1}{2}\left[\mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}+}\mathscr{D} \circ \mathtt{K}\left(\frac{1}{\mathfrak{r}}\right) + \mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}-}\mathscr{D} \circ \mathtt{K}\left(\frac{1}{\mathfrak{s}}\right)\right],$$

where $\mathtt{K}(x) = \frac{1}{x}, \quad x \in \left[\frac{1}{\mathfrak{s}}, \frac{1}{\mathfrak{r}}\right]$. □

Now, we use the above result to produce new Hadamard-Fejér type inequalities of both first and second kind for harmonically convex functions.

Let us begin with the Hadamard-Fejér type inequality of the first kind.

Theorem 3.1. *Let there be a harmonically convex function $\mathscr{G} : [\mathfrak{r}, \mathfrak{s}] \subseteq \mathcal{R} \setminus \{0\} \to \mathcal{R}$ with $\mathfrak{r} < \mathfrak{s}$. If $\mathscr{D} : [\mathfrak{r}, \mathfrak{s}] \subseteq \mathcal{R} \setminus \{0\} \to \mathcal{R}$ be a harmonically symmetric and integrable function with respect to $\frac{2\mathfrak{rs}}{\mathfrak{r}+\mathfrak{s}}$. Then for $\alpha > 0$, the following inequality holds true:*

$$\mathscr{G}\left(\frac{2\mathfrak{rs}}{\mathfrak{r}+\mathfrak{s}}\right)\left[\mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{rs}}-}\mathscr{D} \circ \mathtt{K}\left(\frac{1}{\mathfrak{s}}\right) + \mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{rs}}+}\mathscr{D} \circ \mathtt{K}\left(\frac{1}{\mathfrak{r}}\right)\right]$$
$$\leq \left[\mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{rs}}-}\mathscr{G}\mathscr{D} \circ \mathtt{K}\left(\frac{1}{\mathfrak{s}}\right) + \mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{rs}}+}\mathscr{G}\mathscr{D} \circ \mathtt{K}\left(\frac{1}{\mathfrak{r}}\right)\right].$$

Proof. Since $\mathscr{G}$ is harmonically convex function on $[\mathfrak{r}, \mathfrak{s}]$, we have

$$\mathscr{G}\left(\frac{2\mathfrak{rs}}{\mathfrak{r}+\mathfrak{s}}\right) \leq \frac{\mathscr{G}\left(\frac{\mathfrak{rs}}{\Phi\mathfrak{r}+(1-\Phi)\mathfrak{s}}\right) + \mathscr{G}\left(\frac{\mathfrak{rs}}{\Phi\mathfrak{s}+(1-\Phi)\mathfrak{r}}\right)}{2}.$$

Multiplying both side by $2e^{-\frac{1-\alpha}{\alpha}\frac{\mathfrak{s}-\mathfrak{r}}{\mathfrak{rs}}\Phi}\mathscr{D}\left(\frac{\mathfrak{rs}}{\Phi\mathfrak{s}+(1-\Phi)\mathfrak{r}}\right)$ and then integrating over $[0, \frac{1}{2}]$, we find

$$2\mathscr{G}\left(\frac{2\mathfrak{rs}}{\mathfrak{r}+\mathfrak{s}}\right)\int_0^{\frac{1}{2}} e^{-\frac{1-\alpha}{\alpha}\frac{\mathfrak{s}-\mathfrak{r}}{\mathfrak{rs}}\Phi}\mathscr{D}\left(\frac{\mathfrak{rs}}{\Phi\mathfrak{s}+(1-\Phi)\mathfrak{r}}\right)d\Phi$$
$$\leq \int_0^{\frac{1}{2}} e^{-\frac{1-\alpha}{\alpha}\frac{\mathfrak{s}-\mathfrak{r}}{\mathfrak{rs}}\Phi}\mathscr{D}\left(\frac{\mathfrak{rs}}{\Phi\mathfrak{s}+(1-\Phi)\mathfrak{r}}\right)\mathscr{G}\left(\frac{\mathfrak{rs}}{\Phi\mathfrak{r}+(1-\Phi)\mathfrak{s}}\right)d\Phi$$
$$+ \int_0^{\frac{1}{2}} e^{-\frac{1-\alpha}{\alpha}\frac{\mathfrak{s}-\mathfrak{r}}{\mathfrak{rs}}\Phi}\mathscr{D}\left(\frac{\mathfrak{rs}}{\Phi\mathfrak{s}+(1-\Phi)\mathfrak{r}}\right)\mathscr{G}\left(\frac{\mathfrak{rs}}{\Phi\mathfrak{s}+(1-\Phi)\mathfrak{r}}\right)d\Phi.$$

Since, $\mathscr{D}$ is harmonically symmetric with respect to $\frac{2\mathfrak{rs}}{\mathfrak{r}+\mathfrak{s}}$ i.e

$$\mathscr{D}\left(\frac{1}{x}\right) = \mathscr{D}\left(\frac{1}{\frac{1}{\mathfrak{r}}+\frac{1}{\mathfrak{s}}-x}\right).$$

Also, setting $x = \frac{\Phi\mathfrak{s}+(1-\Phi)\mathfrak{r}}{\mathfrak{rs}} \implies d\Phi = \frac{\mathfrak{rs}}{\mathfrak{s}-\mathfrak{r}}dx$ the above developments proceed as follows:

$$\alpha\frac{2\mathfrak{rs}}{\mathfrak{s}-\mathfrak{r}}\mathscr{G}\left(\frac{2\mathfrak{rs}}{\mathfrak{r}+\mathfrak{s}}\right)\frac{1}{\alpha}\int_{\frac{1}{\mathfrak{s}}}^{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{rs}}} e^{-\frac{1-\alpha}{\alpha}\left(x-\frac{1}{\mathfrak{s}}\right)}\mathscr{D}\left(\frac{1}{x}\right)dx$$
$$\leq \alpha\frac{\mathfrak{rs}}{\mathfrak{s}-\mathfrak{r}}\left[\frac{1}{\alpha}\int_{\frac{1}{\mathfrak{s}}}^{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{rs}}} e^{-\frac{1-\alpha}{\alpha}\left(x-\frac{1}{\mathfrak{s}}\right)}\mathscr{G}\left(\frac{1}{\frac{1}{\mathfrak{r}}+\frac{1}{\mathfrak{s}}-x}\right)\mathscr{D}\left(\frac{1}{x}\right)dx + \frac{1}{\alpha}\int_{\frac{1}{\mathfrak{s}}}^{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{rs}}} e^{-\frac{1-\alpha}{\alpha}\left(x-\frac{1}{\mathfrak{s}}\right)}\mathscr{G}\left(\frac{1}{x}\right)\mathscr{D}\left(\frac{1}{x}\right)dx\right]$$
$$= \alpha\frac{\mathfrak{rs}}{\mathfrak{s}-\mathfrak{r}}\left[\frac{1}{\alpha}\int_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{rs}}}^{\frac{1}{\mathfrak{r}}} e^{-\frac{1-\alpha}{\alpha}\left(\frac{1}{\mathfrak{r}}-x\right)}\mathscr{G}\left(\frac{1}{x}\right)\mathscr{D}\left(\frac{1}{\frac{1}{\mathfrak{r}}+\frac{1}{\mathfrak{s}}-x}\right)dx + \frac{1}{\alpha}\int_{\frac{1}{\mathfrak{s}}}^{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{rs}}} e^{-\frac{1-\alpha}{\alpha}\left(x-\frac{1}{\mathfrak{s}}\right)}\mathscr{G}\left(\frac{1}{x}\right)\mathscr{D}\left(\frac{1}{x}\right)dx\right]$$
$$= \alpha\frac{\mathfrak{rs}}{\mathfrak{s}-\mathfrak{r}}\left[\frac{1}{\alpha}\int_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{rs}}}^{\frac{1}{\mathfrak{r}}} e^{-\frac{1-\alpha}{\alpha}\left(\frac{1}{\mathfrak{r}}-x\right)}\mathscr{G}\left(\frac{1}{x}\right)\mathscr{D}\left(\frac{1}{x}\right)dx + \frac{1}{\alpha}\int_{\frac{1}{\mathfrak{s}}}^{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{rs}}} e^{-\frac{1-\alpha}{\alpha}\left(x-\frac{1}{\mathfrak{s}}\right)}\mathscr{G}\left(\frac{1}{x}\right)\mathscr{D}\left(\frac{1}{x}\right)dx\right]$$

$$= \alpha\frac{\mathfrak{r}\mathfrak{s}}{\mathfrak{s}-\mathfrak{r}}\left[\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}-}\mathscr{G}\mathscr{D}\circ\mathrm{K}\left(\frac{1}{\mathfrak{s}}\right)+\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}+}\mathscr{G}\mathscr{D}\circ\mathrm{K}\left(\frac{1}{\mathfrak{r}}\right)\right].$$

From the above developments and Lemma 3.1, we have

$$\alpha\frac{\mathfrak{r}\mathfrak{s}}{\mathfrak{s}-\mathfrak{r}}\mathscr{G}\left(\frac{2\mathfrak{r}\mathfrak{s}}{\mathfrak{r}+\mathfrak{s}}\right)\left[\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}-}\mathscr{D}\circ\mathrm{K}\left(\frac{1}{\mathfrak{s}}\right)+\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}+}\mathscr{D}\circ\mathrm{K}\left(\frac{1}{\mathfrak{r}}\right)\right]$$
$$\leq \alpha\frac{\mathfrak{r}\mathfrak{s}}{\mathfrak{s}-\mathfrak{r}}\left[\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}-}\mathscr{G}\mathscr{D}\circ\mathrm{K}\left(\frac{1}{\mathfrak{s}}\right)+\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}+}\mathscr{G}\mathscr{D}\circ\mathrm{K}\left(\frac{1}{\mathfrak{r}}\right)\right].$$

This concludes the proof of the required result. □

Example 3.1. *Let* $\mathscr{G}(\mathfrak{m}) = \mathfrak{m}^2$, $\mathfrak{m} \in [1,4]$, $\mathscr{D}(\mathfrak{m}) = \left(\frac{5\mathfrak{m}-8}{8\mathfrak{m}}\right)^2$ *and* $0 < \alpha < 1$, *then*

$$\mathscr{G}\left(\frac{2\mathfrak{r}\mathfrak{s}}{\mathfrak{r}+\mathfrak{s}}\right)\left[\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}-}\mathscr{D}\circ\mathrm{K}\left(\frac{1}{\mathfrak{s}}\right)+\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}+}\mathscr{D}\circ\mathrm{K}\left(\frac{1}{\mathfrak{r}}\right)\right]$$
$$= \frac{64}{25}\left[\frac{1}{\alpha}\int_{\frac{1}{4}}^{\frac{5}{8}} e^{-\frac{1-\alpha}{\alpha}\left(\mathfrak{m}-\frac{1}{4}\right)}\left(\frac{5-8\mathfrak{m}}{8}\right)^2 d\mathfrak{m} + \frac{1}{\alpha}\int_{\frac{5}{8}}^{1} e^{-\frac{1-\alpha}{\alpha}(1-\mathfrak{m})}\left(\frac{5-8\mathfrak{m}}{8}\right)^2 d\mathfrak{m}\right]$$

and

$$\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}-}\mathscr{G}\mathscr{D}\circ\mathrm{K}\left(\frac{1}{\mathfrak{s}}\right)+\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}+}\mathscr{G}\mathscr{D}\circ\mathrm{K}\left(\frac{1}{\mathfrak{r}}\right)$$
$$= \mathrm{I}^{\alpha}_{\frac{5}{8}-}\mathscr{G}\mathscr{D}\circ\mathrm{K}\left(\frac{1}{4}\right)+\mathrm{I}^{\alpha}_{\frac{5}{8}+}\mathscr{G}\mathscr{D}\circ\mathrm{K}(1)$$
$$= \frac{1}{\alpha}\int_{\frac{1}{4}}^{\frac{5}{8}} e^{-\frac{1-\alpha}{\alpha}\left(\mathfrak{m}-\frac{1}{4}\right)}\left(\frac{1}{\mathfrak{m}}\right)^2\left(\frac{5-8\mathfrak{m}}{8}\right)^2 d\mathfrak{m} + \frac{1}{\alpha}\int_{\frac{5}{8}}^{1} e^{-\frac{1-\alpha}{\alpha}(1-\mathfrak{m})}\left(\frac{1}{\mathfrak{m}}\right)^2\left(\frac{5-8\mathfrak{m}}{8}\right)^2 d\mathfrak{m}.$$

The graphical representation of Theorem 3.1 is shown in the graph below (see Figure 4) for $0 < \alpha < 1$*:*

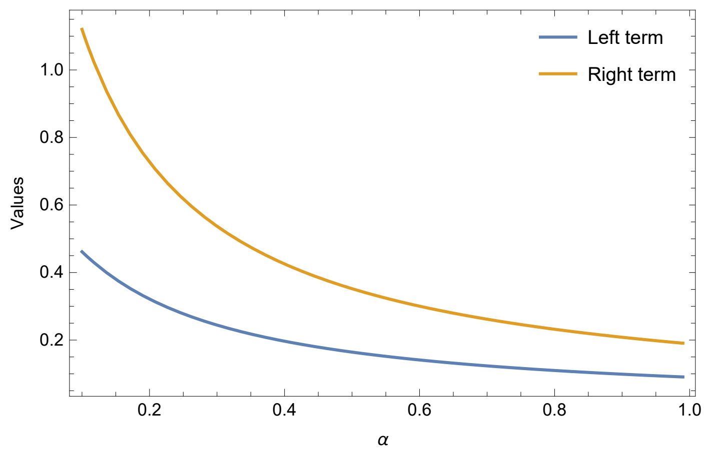

Figure 4. The graphical representation of Theorem 3.1 for $0 < \alpha < 1$.

Now, we will establish the Fejér type inequality of the second kind.

Theorem 3.2. *Let there be a harmonically convex function $\mathscr{G} : [\mathfrak{r}, \mathfrak{s}] \subseteq \mathcal{R} \setminus \{0\} \to \mathcal{R}$ with $\mathfrak{r} < \mathfrak{s}$. If $\mathscr{D} : [\mathfrak{r}, \mathfrak{s}] \subseteq \mathcal{R} \setminus \{0\} \to \mathcal{R}$ be a harmonically symmetric and integrable function with respect to $\frac{2\mathfrak{r}\mathfrak{s}}{\mathfrak{r}+\mathfrak{s}}$. Then for $\alpha > 0$, the following inequality holds true:*

$$\begin{aligned}
&\left[\mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}-}\mathscr{G}\mathscr{D}\circ\mathtt{K}\left(\frac{1}{\mathfrak{s}}\right)+\mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}+}\mathscr{G}\mathscr{D}\circ\mathtt{K}\left(\frac{1}{\mathfrak{r}}\right)\right]\\
&\leq\frac{\mathscr{G}(\mathfrak{r})+\mathscr{G}(\mathfrak{s})}{2}\left[\mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}-}\mathscr{D}\circ\mathtt{K}\left(\frac{1}{\mathfrak{s}}\right)+\mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}+}\mathscr{D}\circ\mathtt{K}\left(\frac{1}{\mathfrak{r}}\right)\right].
\end{aligned}$$

Proof. Since $\mathscr{G}$ is harmonically convex function

$$\mathscr{G}\left(\frac{\mathfrak{r}\mathfrak{s}}{\Phi\mathfrak{r}+(1-\Phi)\mathfrak{s}}\right)+\mathscr{G}\left(\frac{\mathfrak{r}\mathfrak{s}}{\Phi\mathfrak{s}+(1-\Phi)\mathfrak{r}}\right)\leq\mathscr{G}(\mathfrak{r})+\mathscr{G}(\mathfrak{s}).$$

Multiplying both side by $e^{-\frac{1-\alpha}{\alpha}\frac{\mathfrak{s}-\mathfrak{r}}{\mathfrak{r}\mathfrak{s}}\Phi}\mathscr{D}\left(\frac{\mathfrak{r}\mathfrak{s}}{\Phi\mathfrak{s}+(1-\Phi)\mathfrak{r}}\right)$ and then integrating the resultant over $\left[0,\frac{1}{2}\right]$, we find

$$\begin{aligned}
&\int_0^{\frac{1}{2}}e^{-\frac{1-\alpha}{\alpha}\frac{\mathfrak{s}-\mathfrak{r}}{\mathfrak{r}\mathfrak{s}}\Phi}\mathscr{G}\left(\frac{\mathfrak{r}\mathfrak{s}}{\Phi\mathfrak{r}+(1-\Phi)\mathfrak{s}}\right)\mathscr{D}\left(\frac{\mathfrak{r}\mathfrak{s}}{\Phi\mathfrak{s}+(1-\Phi)\mathfrak{r}}\right)d\Phi\\
&+\int_0^{\frac{1}{2}}e^{-\frac{1-\alpha}{\alpha}\frac{\mathfrak{s}-\mathfrak{r}}{\mathfrak{r}\mathfrak{s}}\Phi}\mathscr{G}\left(\frac{\mathfrak{r}\mathfrak{s}}{\Phi\mathfrak{s}+(1-\Phi)\mathfrak{r}}\right)\mathscr{D}\left(\frac{\mathfrak{r}\mathfrak{s}}{\Phi\mathfrak{s}+(1-\Phi)\mathfrak{r}}\right)d\Phi\\
&\leq[\mathscr{G}(\mathfrak{r})+\mathscr{G}(\mathfrak{s})]\int_0^{\frac{1}{2}}e^{-\frac{1-\alpha}{\alpha}\frac{\mathfrak{s}-\mathfrak{r}}{\mathfrak{r}\mathfrak{s}}\Phi}\mathscr{D}\left(\frac{\mathfrak{r}\mathfrak{s}}{\Phi\mathfrak{s}+(1-\Phi)\mathfrak{r}}\right)d\Phi. \qquad (3.1)
\end{aligned}$$

Setting $\mathrm{x}=\frac{\Phi\mathfrak{s}+(1-\Phi)\mathfrak{r}}{\mathfrak{r}\mathfrak{s}}$ and $\mathscr{D}\left(\frac{1}{\mathrm{x}}\right)=\mathscr{D}\left(\frac{1}{\frac{1}{\mathfrak{r}}+\frac{1}{\mathfrak{s}}-\mathrm{x}}\right)$ in (3.1), we have

$$\begin{aligned}
&\frac{\mathfrak{r}\mathfrak{s}}{\mathfrak{s}-\mathfrak{r}}\left[\int_{\frac{1}{\mathfrak{s}}}^{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}}e^{-\frac{1-\alpha}{\alpha}\left(\mathrm{x}-\frac{1}{\mathfrak{s}}\right)}\mathscr{G}\left(\frac{1}{\frac{1}{\mathfrak{r}}+\frac{1}{\mathfrak{s}}-\mathrm{x}}\right)\mathscr{D}\left(\frac{1}{\mathrm{x}}\right)d\mathrm{x}\int_{\frac{1}{\mathfrak{s}}}^{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}}e^{-\frac{1-\alpha}{\alpha}\left(\mathrm{x}-\frac{1}{\mathfrak{s}}\right)}\mathscr{G}\left(\frac{1}{\mathrm{x}}\right)\mathscr{D}\left(\frac{1}{\mathrm{x}}\right)d\mathrm{x}\right]\\
&=\alpha\frac{\mathfrak{r}\mathfrak{s}}{\mathfrak{s}-\mathfrak{r}}\left[\frac{1}{\alpha}\int_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}}^{\frac{1}{\mathfrak{r}}}e^{-\frac{1-\alpha}{\alpha}\left(\frac{1}{\mathfrak{r}}-\mathrm{x}\right)}\mathscr{G}\left(\frac{1}{\mathrm{x}}\right)\mathscr{D}\left(\frac{1}{\mathrm{x}}\right)d\mathrm{x}+\frac{1}{\alpha}\int_{\frac{1}{\mathfrak{s}}}^{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}}e^{-\frac{1-\alpha}{\alpha}\left(\mathrm{x}-\frac{1}{\mathfrak{s}}\right)}\mathscr{G}\left(\frac{1}{\mathrm{x}}\right)\mathscr{D}\left(\frac{1}{\mathrm{x}}\right)d\mathrm{x}\right]\\
&=\alpha\frac{\mathfrak{r}\mathfrak{s}}{\mathfrak{s}-\mathfrak{r}}\left[\mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}-}\mathscr{G}\mathscr{D}\circ\mathtt{K}\left(\frac{1}{\mathfrak{s}}\right)+\mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}+}\mathscr{G}\mathscr{D}\circ\mathtt{K}\left(\frac{1}{\mathfrak{r}}\right)\right]. \qquad (3.2)
\end{aligned}$$

Also,

$$\begin{aligned}
&[\mathscr{G}(\mathfrak{r})+\mathscr{G}(\mathfrak{s})]\int_0^{\frac{1}{2}}e^{-\frac{1-\alpha}{\alpha}\frac{\mathfrak{s}-\mathfrak{r}}{\mathfrak{r}\mathfrak{s}}\Phi}\mathscr{D}\left(\frac{\mathfrak{r}\mathfrak{s}}{\Phi\mathfrak{s}+(1-\Phi)\mathfrak{r}}\right)d\Phi\\
&=\alpha\frac{\mathfrak{r}\mathfrak{s}}{\mathfrak{s}-\mathfrak{r}}[\mathscr{G}(\mathfrak{r})+\mathscr{G}(\mathfrak{s})]\frac{1}{\alpha}\int_{\frac{1}{\mathfrak{s}}}^{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}}e^{-\frac{1-\alpha}{\alpha}\left(\mathrm{x}-\frac{1}{\mathfrak{s}}\right)}\mathscr{D}\left(\frac{1}{\mathrm{x}}\right)d\mathrm{x}\\
&=\alpha\frac{\mathfrak{r}\mathfrak{s}}{\mathfrak{s}-\mathfrak{r}}[\mathscr{G}(\mathfrak{r})+\mathscr{G}(\mathfrak{s})]\left[\mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}-}\mathscr{D}\circ\mathtt{K}\left(\frac{1}{\mathfrak{s}}\right)\right]. \qquad (3.3)
\end{aligned}$$

From the above developments (3.2), (3.3) and Lemma 3.1, we have

$$\alpha\frac{\mathfrak{r}\mathfrak{s}}{\mathfrak{s}-\mathfrak{r}}\left[\mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}-}\mathscr{G}\mathscr{D}\circ\mathtt{K}\left(\frac{1}{\mathfrak{s}}\right)+\mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}+}\mathscr{G}\mathscr{D}\circ\mathtt{K}\left(\frac{1}{\mathfrak{r}}\right)\right]$$

$$\leq \alpha \frac{\mathfrak{r}\mathfrak{s}}{\mathfrak{s}-\mathfrak{r}} \frac{\mathscr{G}(\mathfrak{r})+\mathscr{G}(\mathfrak{s})}{2}\left[\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}-}\mathscr{D}\circ\mathrm{K}\left(\frac{1}{\mathfrak{s}}\right)+\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}+}\mathscr{D}\circ\mathrm{K}\left(\frac{1}{\mathfrak{r}}\right)\right].$$

This concludes the proof of the required result. □

Example 3.2. *Let* $\mathscr{G}(\mathfrak{m})=\mathfrak{m}^2$, $\mathfrak{m}\in[1,4]$, $\mathscr{D}(\mathfrak{m})=\left(\frac{5\mathfrak{m}-8}{8\mathfrak{m}}\right)^2$ *and* $0<\alpha<1$, *then*

$$\frac{17}{2}\left[\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}-}\mathscr{D}\circ\mathrm{K}\left(\frac{1}{\mathfrak{s}}\right)+\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}+}\mathscr{D}\circ\mathrm{K}\left(\frac{1}{\mathfrak{r}}\right)\right]$$
$$=\frac{17}{2}\left[\frac{1}{\alpha}\int_{\frac{1}{4}}^{\frac{5}{8}} e^{-\frac{1-\alpha}{\alpha}\left(\mathfrak{m}-\frac{1}{4}\right)}\left(\frac{5-8\mathfrak{m}}{8}\right)^2 d\mathfrak{m}+\frac{1}{\alpha}\int_{\frac{5}{8}}^{1} e^{-\frac{1-\alpha}{\alpha}(1-\mathfrak{m})}\left(\frac{5-8\mathfrak{m}}{8}\right)^2 d\mathfrak{m}\right]$$

and

$$\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}-}\mathscr{G}\mathscr{D}\circ\mathrm{K}\left(\frac{1}{\mathfrak{s}}\right)+\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}+}\mathscr{G}\mathscr{D}\circ\mathrm{K}\left(\frac{1}{\mathfrak{r}}\right)$$
$$=\mathrm{I}^{\alpha}_{\frac{5}{8}-}\mathscr{G}\mathscr{D}\circ\mathrm{K}\left(\frac{1}{4}\right)+\mathrm{I}^{\alpha}_{\frac{5}{8}+}\mathscr{G}\mathscr{D}\circ\mathrm{K}(1)$$
$$=\frac{1}{\alpha}\int_{\frac{1}{4}}^{\frac{5}{8}} e^{-\frac{1-\alpha}{\alpha}\left(\mathfrak{m}-\frac{1}{4}\right)}\left(\frac{5-8\mathfrak{m}}{8}\right)^4 d\mathfrak{m}+\frac{1}{\alpha}\int_{\frac{5}{8}}^{1} e^{-\frac{1-\alpha}{\alpha}(1-\mathfrak{m})}\left(\frac{5-8\mathfrak{m}}{8}\right)^4 d\mathfrak{m}.$$

The graphical representation of Theorem 3.2 is shown in the graph below (see Figure 5) for $0<\alpha<1$*:*

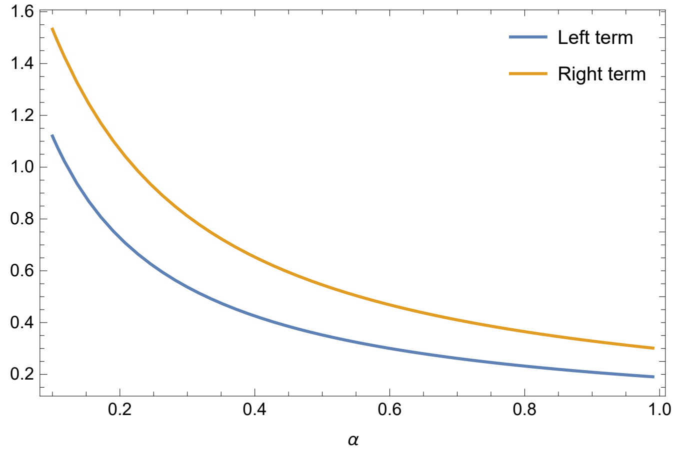

Figure 5. The graphical representation of Theorem 3.2 for $0<\alpha<1$.

Theorem 3.3. *Let there be a harmonically convex function* $\mathscr{G}:[\mathfrak{r},\mathfrak{s}]\subseteq\mathcal{R}\setminus\{0\}\to\mathcal{R}$ *with* $\mathfrak{r}<\mathfrak{s}$. *Then for* $\alpha>0$,

$$\mathscr{G}\left(\frac{2\mathfrak{r}\mathfrak{s}}{\mathfrak{r}+\mathfrak{s}}\right)\leq\frac{(1-\alpha)}{2\left(1-e^{\frac{\rho_h}{2}}\right)}\left[\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}+}\mathscr{G}\circ\mathscr{K}\left(\frac{1}{\mathfrak{r}}\right)+\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}-}\mathscr{G}\circ\mathscr{K}\left(\frac{1}{\mathfrak{s}}\right)\right]\leq\frac{\mathscr{G}(\mathfrak{r})+\mathscr{G}(\mathfrak{s})}{2},\tag{3.4}$$

holds true.

Proof. Since $\mathscr{G}$ is harmonically convex function on $[\mathfrak{r}, \mathfrak{s}]$, we have

$$\mathscr{G}\left(\frac{2\mathfrak{r}\mathfrak{s}}{\mathfrak{r}+\mathfrak{s}}\right)=\mathscr{G}\left(\frac{2\left(\frac{\mathfrak{r}\mathfrak{s}}{\Phi\mathfrak{r}+(1-\Phi)\mathfrak{s}}\right)\left(\frac{\mathfrak{r}\mathfrak{s}}{\Phi\mathfrak{s}+(1-\Phi)\mathfrak{r}}\right)}{\left(\frac{\mathfrak{r}\mathfrak{s}}{\Phi\mathfrak{r}+(1-\Phi)\mathfrak{s}}\right)+\left(\frac{\mathfrak{r}\mathfrak{s}}{\Phi\mathfrak{s}+(1-\Phi)\mathfrak{r}}\right)}\right)\leq\frac{\mathscr{G}\left(\frac{\mathfrak{r}\mathfrak{s}}{\Phi\mathfrak{r}+(1-\Phi)\mathfrak{s}}\right)+\mathscr{G}\left(\frac{\mathfrak{r}\mathfrak{s}}{\Phi\mathfrak{s}+(1-\Phi)\mathfrak{r}}\right)}{2}. \tag{3.5}$$

Multiplying both side of the inequality (3.5) by $2e^{-\frac{1-\alpha}{\alpha}\frac{\mathfrak{s}-\mathfrak{r}}{\mathfrak{r}\mathfrak{s}}\Phi}$ and integrating over $[0, \frac{1}{2}]$, we obtain

$$\begin{aligned}2\mathscr{G}\left(\frac{2\mathfrak{r}\mathfrak{s}}{\mathfrak{r}+\mathfrak{s}}\right)\int_0^{1/2}e^{-\frac{1-\alpha}{\alpha}\frac{\mathfrak{s}-\mathfrak{r}}{\mathfrak{r}\mathfrak{s}}\Phi}d\Phi\leq&\int_0^{1/2}e^{-\frac{1-\alpha}{\alpha}\frac{\mathfrak{s}-\mathfrak{r}}{\mathfrak{r}\mathfrak{s}}\Phi}\mathscr{G}\left(\frac{\mathfrak{r}\mathfrak{s}}{\Phi\mathfrak{r}+(1-\Phi)\mathfrak{s}}\right)d\Phi\\&+\int_0^{1/2}e^{-\frac{1-\alpha}{\alpha}\frac{\mathfrak{s}-\mathfrak{r}}{\mathfrak{r}\mathfrak{s}}\Phi}\mathscr{G}\left(\frac{\mathfrak{r}\mathfrak{s}}{\Phi\mathfrak{s}+(1-\Phi)\mathfrak{r}}\right)d\Phi.\end{aligned}$$

Let $\mathfrak{m}=\frac{\Phi\mathfrak{s}+(1-\Phi)\mathfrak{r}}{\mathfrak{r}\mathfrak{s}}$, then $d\mathfrak{m}=\frac{\mathfrak{s}-\mathfrak{r}}{\mathfrak{r}\mathfrak{s}}d\Phi$

$$\begin{aligned}&\frac{2(1-e^{-\frac{\rho_h}{2}})}{\rho_h}\mathscr{G}\left(\frac{2\mathfrak{r}\mathfrak{s}}{\mathfrak{r}+\mathfrak{s}}\right)\\&\leq\int_{\frac{1}{\mathfrak{s}}}^{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}}e^{-\frac{1-\alpha}{\alpha}\frac{\mathfrak{s}-\mathfrak{r}}{\mathfrak{r}\mathfrak{s}}\frac{\mathfrak{r}\mathfrak{s}}{\mathfrak{s}-\mathfrak{r}}(\mathfrak{m}-\frac{1}{\mathfrak{s}})}\left(\frac{\mathfrak{r}\mathfrak{s}}{\mathfrak{s}-\mathfrak{r}}\right)\mathscr{G}\left(\frac{1}{\frac{1}{\mathfrak{r}}+\frac{1}{\mathfrak{s}}-\mathfrak{m}}\right)d\mathfrak{m}+\int_{\frac{1}{\mathfrak{s}}}^{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}}e^{-\frac{1-\alpha}{\alpha}\frac{\mathfrak{s}-\mathfrak{r}}{\mathfrak{r}\mathfrak{s}}\frac{\mathfrak{r}\mathfrak{s}}{\mathfrak{s}-\mathfrak{r}}(\mathfrak{m}-\frac{1}{\mathfrak{s}})}\left(\frac{\mathfrak{r}\mathfrak{s}}{\mathfrak{s}-\mathfrak{r}}\right)\mathscr{G}\left(\frac{1}{\mathfrak{m}}\right)d\mathfrak{m}\\&=\frac{\mathfrak{r}\mathfrak{s}}{\mathfrak{s}-\mathfrak{r}}\left[\int_{\frac{1}{\mathfrak{s}}}^{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}}e^{-\frac{1-\alpha}{\alpha}(\mathfrak{m}-\frac{1}{\mathfrak{s}})}\mathscr{G}\left(\frac{1}{\frac{1}{\mathfrak{r}}+\frac{1}{\mathfrak{s}}-\mathfrak{m}}\right)d\mathfrak{m}+\int_{\frac{1}{\mathfrak{s}}}^{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}}e^{-\frac{1-\alpha}{\alpha}(\mathfrak{m}-\frac{1}{\mathfrak{s}})}\mathscr{G}\left(\frac{1}{\mathfrak{m}}\right)d\mathfrak{m}\right]\\&=\frac{\mathfrak{r}\mathfrak{s}}{\mathfrak{s}-\mathfrak{r}}\left[\int_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}}^{\frac{1}{\mathfrak{r}}}e^{-\frac{1-\alpha}{\alpha}(\frac{1}{\mathfrak{r}}-\mathfrak{m})}\mathscr{G}\left(\frac{1}{\mathfrak{m}}\right)d\mathfrak{m}+\int_{\frac{1}{\mathfrak{s}}}^{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}}e^{-\frac{1-\alpha}{\alpha}(\mathfrak{m}-\frac{1}{\mathfrak{s}})}\mathscr{G}\left(\frac{1}{\mathfrak{m}}\right)d\mathfrak{m}\right]\\&=\alpha\frac{\mathfrak{r}\mathfrak{s}}{\mathfrak{s}-\mathfrak{r}}\left[\mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}+}\mathscr{G}\circ\mathscr{K}\left(\frac{1}{\mathfrak{r}}\right)+\mathtt{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}-}\mathscr{G}\circ\mathscr{K}\left(\frac{1}{\mathfrak{s}}\right)\right].\end{aligned} \tag{3.6}$$

This gives us the first part of the inequality (3.4). Now, for the next part, we use the hypotheses of harmonically convex function i.e.

$$\mathscr{G}\left(\frac{\mathfrak{r}\mathfrak{s}}{\Phi\mathfrak{r}+(1-\Phi)\mathfrak{s}}\right)+\mathscr{G}\left(\frac{\mathfrak{r}\mathfrak{s}}{\Phi\mathfrak{s}+(1-\Phi)\mathfrak{r}}\right)\leq\mathscr{G}(\mathfrak{r})+\mathscr{G}(\mathfrak{s}). \tag{3.7}$$

Multiplying both side of the above inequality (3.7) by $e^{-\frac{1-\alpha}{\alpha}\frac{\mathfrak{s}-\mathfrak{r}}{\mathfrak{r}\mathfrak{s}}\Phi}$ and then integrating the resultant over $[0, 1]$, we obtain

$$\begin{aligned}&\int_0^{\frac{1}{2}}e^{-\frac{1-\alpha}{\alpha}\frac{\mathfrak{s}-\mathfrak{r}}{\mathfrak{r}\mathfrak{s}}\Phi}\mathscr{G}\left(\frac{\mathfrak{r}\mathfrak{s}}{\Phi\mathfrak{r}+(1-\Phi)\mathfrak{s}}\right)d\Phi+\int_0^{\frac{1}{2}}e^{-\frac{1-\alpha}{\alpha}\frac{\mathfrak{s}-\mathfrak{r}}{\mathfrak{r}\mathfrak{s}}\Phi}\mathscr{G}\left(\frac{\mathfrak{r}\mathfrak{s}}{\Phi\mathfrak{s}+(1-\Phi)\mathfrak{r}}\right)d\Phi\\&\leq[\mathscr{G}(\mathfrak{r})+\mathscr{G}(\mathfrak{s})]\int_0^{\frac{1}{2}}e^{-\frac{1-\alpha}{\alpha}\frac{\mathfrak{s}-\mathfrak{r}}{\mathfrak{r}\mathfrak{s}}\Phi}d\Phi\\&=\frac{2(1-e^{-\frac{\rho_h}{2}})}{\rho_h}\frac{\mathscr{G}(\mathfrak{r})+\mathscr{G}(\mathfrak{s})}{2}.\end{aligned}$$

Consequently from the first inequality (3.6), we have

$$\int_0^{\frac{1}{2}}e^{-\frac{1-\alpha}{\alpha}\frac{\mathfrak{s}-\mathfrak{r}}{\mathfrak{r}\mathfrak{s}}\Phi}\mathscr{G}\left(\frac{\mathfrak{r}\mathfrak{s}}{\Phi\mathfrak{r}+(1-\Phi)\mathfrak{s}}\right)d\Phi+\int_0^{\frac{1}{2}}e^{-\frac{1-\alpha}{\alpha}\frac{\mathfrak{s}-\mathfrak{r}}{\mathfrak{r}\mathfrak{s}}\Phi}\mathscr{G}\left(\frac{\mathfrak{r}\mathfrak{s}}{\Phi\mathfrak{s}+(1-\Phi)\mathfrak{r}}\right)d\Phi$$

$$= \alpha \frac{\mathfrak{r}\mathfrak{s}}{\mathfrak{s}-\mathfrak{r}}\left[\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}+}\mathcal{G}\circ\mathcal{K}\left(\frac{1}{\mathfrak{r}}\right)+\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}-}\mathcal{G}\circ\mathcal{K}\left(\frac{1}{\mathfrak{s}}\right)\right] \leq \frac{2(1-e^{-\frac{\rho_h}{2}})}{\rho_h}\frac{\mathcal{G}(\mathfrak{r})+\mathcal{G}(\mathfrak{s})}{2}. \tag{3.8}$$

From the above developments (3.6) and (3.8), it follows

$$\mathcal{G}\left(\frac{2\mathfrak{r}\mathfrak{s}}{\mathfrak{r}+\mathfrak{s}}\right) \leq \frac{(1-\alpha)}{2\left(1-e^{\frac{\rho_h}{2}}\right)}\left[\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}+}\mathcal{G}\circ\mathcal{K}\left(\frac{1}{\mathfrak{r}}\right)+\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}-}\mathcal{G}\circ\mathcal{K}\left(\frac{1}{\mathfrak{s}}\right)\right] \leq \frac{\mathcal{G}(\mathfrak{r})+\mathcal{G}(\mathfrak{s})}{2}.$$

This concludes the proof of the required result. □

Remark 3.1. If one chooses $\alpha \to 1$ i.e., $\frac{\rho_h}{2} \to 0$, then

$$\lim_{\alpha\to 1}\frac{1-\alpha}{2(1-e^{-\frac{\rho_h}{2}})} = \frac{\mathfrak{r}\mathfrak{s}}{\mathfrak{s}-\mathfrak{r}}$$

and hence Theorem 2.3 retrieves the classical Hermite-Hadamard inequality (1.3) for harmonically convex function.

Example 3.3. *Let* $\mathcal{G}(\mathfrak{m}) = \mathfrak{m}^2$, $\mathfrak{m} \in [1,4]$, $\mathrm{K}(\mathfrak{m}) = \frac{1}{\mathfrak{m}}$ *and* $0 < \alpha < 1$, *then*

$$\mathcal{G}\left(\frac{2\mathfrak{r}\mathfrak{s}}{\mathfrak{r}+\mathfrak{s}}\right) = \frac{64}{25},$$

$$\begin{aligned}&\frac{(1-\alpha)}{2\left(1-e^{\frac{\rho_h}{2}}\right)}\left[\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}+}\mathcal{G}\circ\mathcal{K}\left(\frac{1}{\mathfrak{r}}\right)+\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2\mathfrak{r}\mathfrak{s}}-}\mathcal{G}\circ\mathcal{K}\left(\frac{1}{\mathfrak{s}}\right)\right]\\ &= \frac{(1-\alpha)}{2\left(1-e^{\frac{3(1-\alpha)}{8\alpha}}\right)}\left[\mathrm{I}^{\alpha}_{\frac{5}{8}+}\mathcal{G}\circ\mathcal{K}\left(\frac{1}{\mathfrak{r}}\right)+\mathrm{I}^{\alpha}_{\frac{5}{8}-}\mathcal{G}\circ\mathcal{K}\left(\frac{1}{\mathfrak{s}}\right)\right]\\ &= \frac{(1-\alpha)}{2\left(1-e^{\frac{3(1-\alpha)}{8\alpha}}\right)}\left[\frac{1}{\alpha}\int_{\frac{1}{4}}^{\frac{5}{8}} e^{-\frac{1-\alpha}{\alpha}\left(\mathfrak{m}-\frac{1}{4}\right)}\left(\frac{1}{\mathfrak{m}}\right)^2 d\mathfrak{m} + \frac{1}{\alpha}\int_{\frac{5}{8}}^{1} e^{-\frac{1-\alpha}{\alpha}(1-\mathfrak{m})}\left(\frac{1}{\mathfrak{m}}\right)^2 d\mathfrak{m}\right]\end{aligned}$$

and

$$\frac{\mathcal{G}(\mathfrak{r})+\mathcal{G}(\mathfrak{s})}{2} = \frac{17}{2}.$$

The graphical representation of Theorem 3.3 is shown in the graph below (see Figure 6) for $0 < \alpha < 1$*:*

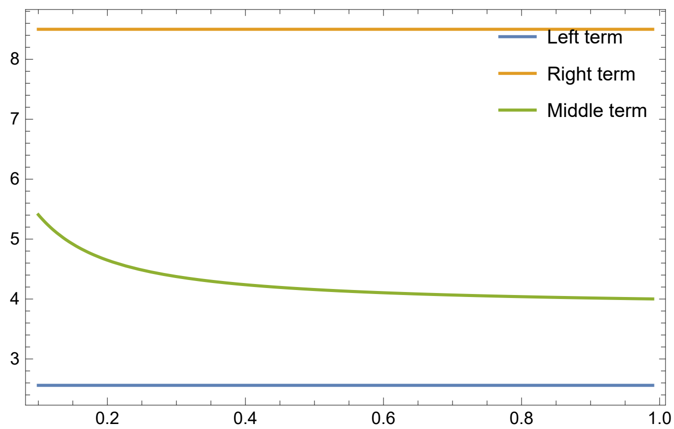

Figure 6. The graphical representation of Theorem 3.3 for $0 < \alpha < 1$.

4. Applications

Example 4.1. *Let $\mathbb{C}^n$ be the set of $n \times n$ complex matrices, $\mathbb{M}_n$ denote the algebra of $n \times n$ complex matrices, and $\mathbb{M}_n^+$ denote the strictly positive matrices in $\mathbb{M}_n$. That is, for any nonzero $u \in \mathbb{C}^n$, $\mathrm{A} \in \mathbb{M}_n^+$ if $\langle \mathrm{A}u, u\rangle > 0$.*

Sababheh [51], proved that $\mathscr{G}(\kappa) = \| \mathrm{A}^\kappa \mathrm{X} \mathrm{B}^{1-\kappa} + \mathrm{A}^{1-\kappa}\mathrm{X}\mathrm{B}^\kappa \|$, $\mathrm{A}, \mathrm{B} \in \mathbb{M}_n^+, \mathrm{X} \in \mathbb{M}_n$ is convex for all $\kappa \in [0, 1]$.

Then, by using Theorem 2.3, we have

$$\begin{aligned}
&\| \mathrm{A}^{\frac{\mathfrak{r}+\mathfrak{s}}{2}} \mathrm{X} \mathrm{B}^{1-(\frac{\mathfrak{r}+\mathfrak{s}}{2})} + \mathrm{A}^{1-(\frac{\mathfrak{r}+\mathfrak{s}}{2})} \mathrm{X} \mathrm{B}^{\frac{\mathfrak{r}+\mathfrak{s}}{2}} \| \\
&\leq \frac{1-\alpha}{2(1-e^{-\frac{\rho_c}{2}})}\Big[\mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}+} \| \mathrm{A}^{\mathfrak{s}}\mathrm{X}\mathrm{B}^{1-\mathfrak{s}} + \mathrm{A}^{1-\mathfrak{s}}\mathrm{X}\mathrm{B}^{\mathfrak{s}} \| + \mathrm{I}^{\alpha}_{\frac{\mathfrak{r}+\mathfrak{s}}{2}-} \| \mathrm{A}^{\mathfrak{r}}\mathrm{X}\mathrm{B}^{1-\mathfrak{r}} + \mathrm{A}^{1-\mathfrak{r}}\mathrm{X}\mathrm{B}^{\mathfrak{r}} \| \Big] \\
&\leq \frac{\| \mathrm{A}^{\mathfrak{r}}\mathrm{X}\mathrm{B}^{1-\mathfrak{r}} + \mathrm{A}^{1-\mathfrak{r}}\mathrm{X}\mathrm{B}^{\mathfrak{r}} \| + \| \mathrm{A}^{\mathfrak{s}}\mathrm{X}\mathrm{B}^{1-\mathfrak{s}} + \mathrm{A}^{1-\mathfrak{s}}\mathrm{X}\mathrm{B}^{\mathfrak{s}} \|}{2}.
\end{aligned}$$

Example 4.2. The q-digamma(psi) function ψ_Φ given as (see [52]):

$$\begin{aligned}
\psi_\Phi(\zeta) &= -\ln(1-\Phi) + \ln\Phi \sum_{k=0}^{\infty} \frac{\Phi^{k+\zeta}}{1-\Phi^{k+\zeta}} \\
&= -\ln(1-\Phi) + \ln\Phi \sum_{k=1}^{\infty} \frac{\Phi^{k\zeta}}{1-\Phi^{k\zeta}}.
\end{aligned}$$

For $\Phi > 1$ and $\zeta > 0$, Φ-digamma function ψ_Φ can be given as:

$$\begin{aligned}
\psi_\Phi(\zeta) &= -\ln(\Phi-1) + \ln\Phi\Big[\zeta - \frac{1}{2} - \sum_{k=0}^{\infty} \frac{\Phi^{-(k+\zeta)}}{1-\Phi^{-(k+\zeta)}}\Big] \\
&= -\ln(\Phi-1) + \ln\Phi\Big[\zeta - \frac{1}{2} - \sum_{k=1}^{\infty} \frac{\Phi^{-k\zeta}}{1-\Phi^{-k\zeta}}\Big].
\end{aligned}$$

If we set $\mathscr{G}(\zeta) = \psi'_\Phi(\zeta)$ in Theorem 2.3, then we have the following inequality.

$$\psi'_\Phi\Big(\frac{\mathfrak{r}+\mathfrak{s}}{2}\Big) \leq \frac{1-\alpha}{2\alpha(1-e^{-\frac{\Phi_c}{2}})}\Big[\int_{\frac{\mathfrak{r}+\mathfrak{s}}{2}}^{\mathfrak{s}} e^{-\frac{1-\alpha}{\alpha}(\mathfrak{s}-\mathfrak{m})}\psi'_\Phi(\mathfrak{m})\,d\mathfrak{m} + \int_{\mathfrak{r}}^{\frac{\mathfrak{r}+\mathfrak{s}}{2}} e^{-\frac{1-\alpha}{\alpha}(\mathfrak{m}-\mathfrak{r})}\psi'_\Phi(\mathfrak{m})\,d\mathfrak{m}\Big] \leq \frac{\psi'_\Phi(\mathfrak{r}) + \psi'_\Phi(\mathfrak{s})}{2}.$$

Modified Bessel functions

Example 4.3. *Let the function $\mathscr{J}_\rho : \mathcal{R} \to [1, \infty)$ be defined [52] as*

$$\mathscr{J}_\rho(\mathfrak{m}) = 2^\rho \Gamma(\rho+1)\mathfrak{m}^{-\rho}\mathrm{I}_\rho(\mathfrak{m}), \quad \mathfrak{m} \in \mathcal{R}.$$

Here, we consider the modified Bessel function of first kind given in

$$\mathscr{J}_\rho(\mathfrak{m}) = \sum_{n=0}^{\infty} \frac{\left(\frac{\mathfrak{m}}{2}\right)^{\rho+2n}}{n!\Gamma(\rho+n+1)}.$$

The first and second order derivative are given as

$$\mathscr{I}_\rho'(\mathfrak{m}) = \frac{\mathfrak{m}}{2(\rho+1)}\mathscr{I}_{\rho+1}(\mathfrak{m}).$$

$$\mathscr{I}_\rho''(\mathfrak{m}) = \frac{1}{4(\rho+1)}\left[\frac{u^2}{(\rho+1)}\mathscr{I}_{\rho+2}(\mathfrak{m}) + 2\mathscr{I}_{\rho+1}(\mathfrak{m})\right].$$

If we use, $\mathscr{G}(\mathfrak{m}) = \mathscr{I}_\rho'(\mathfrak{m})$ *and the above functions in Theorem 2.3, we have*

$$\begin{aligned}\frac{\mathfrak{r}+\mathfrak{s}}{2}\mathscr{I}_{\rho+1}\left(\frac{\mathfrak{r}+\mathfrak{s}}{2}\right) &\le \frac{1-\alpha}{2\alpha(1-e^{-\frac{\rho c}{2}})}\left[\int_{\frac{\mathfrak{r}+\mathfrak{s}}{2}}^{\mathfrak{s}} e^{-\frac{1-\alpha}{\alpha}(\mathfrak{s}-\mathfrak{m})}\mathfrak{m}\mathscr{I}_{\rho+1}(\mathfrak{m})\,d\mathfrak{m} + \int_{\mathfrak{r}}^{\frac{\mathfrak{r}+\mathfrak{s}}{2}} e^{-\frac{1-\alpha}{\alpha}(\mathfrak{m}-\mathfrak{r})}\mathfrak{m}\mathscr{I}_{\rho+1}(\mathfrak{m})\,d\mathfrak{m}\right] \\ &\le \frac{\mathfrak{r}\mathscr{I}_{\rho+1}(\mathfrak{r}) + \mathfrak{s}\mathscr{I}_{\rho+1}(\mathfrak{s})}{2}.\end{aligned}$$

5. Concluding remarks

The use of fractional calculus for finding various integral inequalities via convex functions has skyrocketed in recent years. This paper addresses a novel sort of Fejér type integral inequalities. In order to generalize some H-H-F (Hermite-Hadamard-Fejér) type inequalities, a new fractional integral operator with exponential kernel is employed. New midpoint type inequalities for both convex and harmonically convex functions are studied. Applications related to matrices, q-digamma and modifed Bessel functions are presented as well.

6. Future scopes

We will use our theories and methods to create new inequalities for future research by combining these new weighted generalized fractional integral operators with Chebyshev, Simpson, Jensen-Mercer Markov, Bullen, Newton, and Minkowski type inequalities. Quantum calculus, fuzzy interval-valued analysis, and interval-valued analysis can all be used to establish these kinds of inequalities. The idea of Digamma functions and other special functions will be integrated with this kind of inequality as the major focus. We also aim to find other novel inequalities using finite products of functions. In future, we will employ the concept of cr-order defined by Bhunia and Samanta [53] to present different inequalities for cr-convexity and cr-harmonically convexity [54].

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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