Results in Physics 46 (2023) 106313

Contents lists available at ScienceDirect

Results in Physics

journal homepage: www.elsevier.com/locate/rinp

Chaotic attractors and fixed point methods in piecewise fractional derivatives and multi-term fractional delay differential equations

Sumati Kumari Panda [a], Thabet Abdeljawad [b], Fahd Jarad [c,d,*]

[a] Department of Mathematics, GMR Institute of Technology, Rajam 532 127, Andhra Pradesh, India
[b] Department of Mathematics and Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia
[c] Department of Mathematics, Cankaya University, Ankara, Turkey
[d] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan

A R T I C L E I N F O

Keywords:
Cyclic mapping
Piecewise Caputo–Fabrizio fractional derivatives
Fractional delay differential equations
Fixed point

A B S T R A C T

Using generalized cyclic contractions, we establish some fixed point results in controlled rectangular metric spaces. Some subsequent outcomes are obtained. Moreover, some necessary conditions to demonstrate the existence of solutions for the multi-term fractional delay differential equations with ωth order and the piecewise equations under the setting of non-singular type derivative are established in this paper. In order to demonstrate the effectiveness of our results, we provided some numerical examples.

Introduction

The area of ''Fractional Differential Equations'' (FDEs) has risen in the past few years due to its applicability in a wide range of real-world perspectives in science, thermodynamics, economics, modelling, and perhaps other disciplines [1,2]. It is well recognized that traditional calculus may be used to explain and simulate important complex behaviour in various disciplines. Nevertheless, fractional differential equations can provide more accurate assessment of many complicated natural systems (see for instance, mathematical modelling of infectious deceases, diffusion for image reconstruction, and interaction between cancer cells and the immune system). Such kind of stated abnormal approaches are unable to characterize actual behaviour of complex dynamics. So it is preferable to use fractional differential equations rather than ordinary differential equations [3]. Consequently, novel scientific findings and techniques are developed explicitly for fractional differential equations. For this reason, a significant number of scientists focus on initial and boundary value problems with different types of derivatives, including Atangana–Baleanu, Caputo–Fabrizio, and Caputo. In recent past, it can be seen a tremendous expansion of the existing research on the topic, with a variety of intriguing and practical outcomes (see [4–14]).

The fixed-point theory first appeared in an article, establishing the existence of solutions to nonlinear equations. Later, this method was enhanced as a sequential approximation method, and in the context of complete normed space, it was illustrated and described as a fixed-point theorem. It provides an approximate method to effectively identify the fixed point. It also guarantees the existence and uniqueness of a fixed point. In the theory of metric spaces, it is a crucial tool. We can guarantee the existence of a solution to the initial problem by using fixed-point theorems, which provide constraints under which a fixed point persists for a particular function. The existence of a solution is equivalent to the existence of a fixed point for an appropriate mapping in a wide range of scientific problems, starting from many disciplines of mathematical problems. Some of those mathematical proofs relating to a conversion of a set's points into points of the same set where it can be demonstrated that at least one point still stands fixed are referred to as fixed-point results. To determine whether an equation has a solution, fixed-point scientific theories are highly helpful.

In this paper, we will introduce different classes of contractive mappings in controlled rectangular metric spaces and prove related fixed point results. Moreover, we establish some necessary conditions to demonstrate the existence of solutions for the multi-term fractional delay differential equations with ωth order and the piecewise equations under the setting of Caputo–Fabrizio derivative.

Karpagam–Zamfirescu type results

In the 21st century, metric fixed point theory has widely used in economics, medical biology, theoretical semantics, space exploration,

* Corresponding author at: Department of Mathematics, Cankaya University, Ankara, Turkey.
E-mail addresses: mumy143143143@gmail.com, sumatikumari.p@gmrit.edu.in (S.K. Panda), tabdeljawad@psu.edu.sa (T. Abdeljawad), fahd@cankaya.edu.tr (F. Jarad).

https://doi.org/10.1016/j.rinp.2023.106313
Received 1 February 2023; Received in revised form 24 February 2023; Accepted 24 February 2023
Available online 1 March 2023

and epidemiological data collection. Metric fixed point theory, in contrast to the majority of either of the computational sub-fields, has really become the subject of independent classical works. Metric fixed point theory is frequently used for the refinement of various metric spaces and to generalize contraction principle. These improvements frequently strive to dissect difficult Banach space geometric features and optimal approximates.

Definition 1 (*[15]*). Given a non-empty set $\mathcal{M}$ and $\varpi : \mathcal{M}\times\mathcal{M} \to [1,+\infty)$. The function $\mathcal{R} : \mathcal{M}\times\mathcal{M} \to [0,+\infty)$ is called a controlled rectangular metric if

- $\mathcal{R}(x,y)=0 \Leftrightarrow x=y$;
- $\mathcal{R}(x,y)=\mathcal{R}(y,x)$;
- $\mathcal{R}(x,y)\le \varpi(x,\alpha)\mathcal{R}(x,\alpha)+\varpi(\alpha,\mu)\mathcal{R}(\alpha,\mu)+\varpi(\mu,y)\mathcal{R}(\mu,y)$

for all $x,y\in\mathcal{M}$ and for all distinct points $\alpha,\mu\in\mathcal{M}$. As in [15], $(\mathcal{M},\mathcal{R})$ referred to a controlled rectangular metric space (shortly, CRMS). Additionally, notations for this space's topological structures like convergence, Cauchy, and completeness are available in [15].

Definition 2 (*[16]*). $\mathscr{T} : \mathscr{O}\cup\mathscr{H} \to \mathscr{O}\cup\mathscr{H}$ is said to be a cyclic map if $\mathscr{T}(\mathscr{O})\subseteq\mathscr{H}$ and $\mathscr{T}(\mathscr{H})\subseteq\mathscr{O}$, where $\mathscr{O}$ and $\mathscr{H}$ be non-empty closed subsets of a complete metric space $(\mathcal{M},\mathcal{R})$.

Contractive mapping definitions.

1. (Karpagam et al. [17]). Let $\mathcal{A}$ and $\mathcal{B}$ be non-empty closed subsets of a complete metric space is denoted by $\mathcal{X}$, with distance function d, $\mathfrak{H}$ be a cyclic mapping there exists some $x\in\mathcal{A}$ and there exists a $k_x\in(0,1)$ such that

   $d(\mathfrak{H}^{2n}x,\mathfrak{H}y)\le k_x d(\mathfrak{H}^{2n-1}x,y).$

2. (Zamfirescu [18]). There exist real numbers $a,b,c,0\le a<1,0\le b,c<\frac{1}{2}$, such that, for each $x,y\in\mathcal{X}$, and $\mathfrak{H}$ a function mapping $\mathcal{X}$ into itself; at least one of the following statements is correct:

   $d(\mathfrak{H}(x),\mathfrak{H}(y))\le ad(x,y);$

   $d(\mathfrak{H}(x),\mathfrak{H}(y))\le b[d(x,\mathfrak{H}(x))+d(y,\mathfrak{H}(y))];$

   $d(\mathfrak{H}(x),\mathfrak{H}(y))\le c[d(x,\mathfrak{H}(y))+d(y,\mathfrak{H}(x))].$

Now we introduce the following definition.

Definition 3. Let $\mathscr{O}$ and $\mathscr{H}$ be non-empty subsets of controlled rectangular metric space $(\mathcal{M},\mathcal{R})$. Suppose $\mathscr{S}$- from $\mathscr{O}\cup\mathscr{H}$ to $\mathscr{O}\cup\mathscr{H}$ be a cyclic mapping such that for some $x\in\mathscr{O}$, there exists $k\in(0,1)$ such that

$$\mathcal{R}(\mathscr{S}^{2w}x,\mathscr{S}y)\le k\mathcal{R}(\mathscr{S}^{2w-1}x,y) \tag{1}$$

for all $w\in\mathbb{N}$ and $y\in\mathscr{O}$. Then $\mathscr{S}$ is so called a *Karpagam type mapping*.

Theorem 1. *Let $\mathscr{O}$ and $\mathscr{H}$ be non-empty subsets of complete controlled rectangular metric space $(\mathcal{M},\mathcal{R})$. Suppose $\mathscr{S} : \mathscr{O}\cup\mathscr{H}\to\mathscr{O}\cup\mathscr{H}$ be a Karpagam type mapping. For $x_0\in\mathscr{O}$, take $x_w=\mathscr{S}^w x_0$. Suppose that,*

$$\lim_{\flat\to\infty}\sup_{m\ge1}\varpi(x_{\flat+1},x_m)\frac{\varpi(x_{\flat+1},x_{\flat+2})+\varpi(x_{\flat+2},x_{\flat+3})}{\varpi(x_\flat,x_{\flat+1})+\varpi(x_{\flat+1},x_{\flat+2})}<1.$$

Assume that, $\lim_{w\to+\infty}\varpi(x_w,x),\lim_{w\to+\infty}\varpi(x,x_w)$ and $\lim_{w,m\to+\infty}\varpi(x_w,x_m)$ exist and are finite for all $w,m\in\mathbb{N},w\ne m$. Then $\mathscr{O}\cap\mathscr{H}$ is non-empty and $\mathscr{S}$ possess a unique fixed point in $\mathscr{O}\cap\mathscr{H}$.

Proof. Let $x=x_0\in\mathscr{O}$ be an arbitrary point. Define the iterative sequence $x_w=\mathscr{S}^w x_0$. Since $x_0\in\mathscr{O}$ and $\mathscr{S}$ is cyclic, we have, $x_{2n}\in\mathscr{O}$ and $x_{2n+1}\in\mathscr{H}$ for all $w\ge0$. By using (1), we get,

$$\mathcal{R}(\mathscr{S}^2x,\mathscr{S}x)\le k\mathcal{R}(\mathscr{S}x,x). \tag{2}$$

Again,

$$\begin{aligned}\mathcal{R}(\mathscr{S}^3x,\mathscr{S}^2x)&=\mathcal{R}(\mathscr{S}^2x,\mathscr{S}^3x)\\&=\mathcal{R}(\mathscr{S}^2x,\mathscr{S}(\mathscr{S}^2x))\\&\le k\mathcal{R}(\mathscr{S}x,\mathscr{S}^2x)\\&\le k^2\mathcal{R}(\mathscr{S}x,x).\end{aligned} \tag{3}$$

Similarly,

$$\begin{aligned}\mathcal{R}(\mathscr{S}^4x,\mathscr{S}^3x)&=\mathcal{R}(\mathscr{S}^2(\mathscr{S}^2x),\mathscr{S}(\mathscr{S}^2x))\\&\le k\mathcal{R}(\mathscr{S}(\mathscr{S}^2x),\mathscr{S}^2x)\\&=k\mathcal{R}(\mathscr{S}^3x,\mathscr{S}^2x)\\&\le k^3\mathcal{R}(\mathscr{S}x,x).\end{aligned}$$

By induction we obtain that

$$\mathcal{R}(x_w,x_{w+1})\le k^w\mathcal{R}(x_0,x_1)\quad\text{for all}\quad w\ge0. \tag{4}$$

Since $k\in(0,1)$, taking the limit of the above inequality as $w\to\infty$, we deduce that

$$\lim_{w\to\infty}\mathcal{R}(x_w,x_{w+1})=0. \tag{5}$$

We shall prove that $\lim_{w\to\infty}\mathcal{R}(x_w,x_{w+2})=0$. We assume that $x_w\ne x_m$ for every $w,m\in\mathbb{N}$. Indeed suppose that $x_w=x_m$ for some $w=m+r$, with $r>0$, so we have $\mathscr{S}x_w=\mathscr{S}x_m$, and

$$\begin{aligned}\mathcal{R}(x_m,x_{m+1})&=\mathcal{R}(x_w,x_{w+1})\\&\le k\mathcal{R}(x_{w-1},x_w).\end{aligned}$$

Since $k\in(0,1)$, therefore $\mathcal{R}(x_m,x_{m+1})=\mathcal{R}(x_w,x_{w+1})<\mathcal{R}(x_{w-1},x_w)$. By continuing this process, we have $\mathcal{R}(x_m,x_{m+1})<\mathcal{R}(x_m,x_{m+1})$, which is a contradiction.

Therefore $\mathcal{R}(x_m,x_w)>0$ for every $w,m\in\mathbb{N},w\ne m$.

To prove $\lim_{w\to\infty}\mathcal{R}(x_w,x_{w+2})=0$, by using (1) we get,

$$\begin{aligned}\mathcal{R}(x_1,x_3)&=\mathcal{R}(x_3,x_1)\\&=\mathcal{R}(\mathscr{S}^2(\mathscr{S}x),\mathscr{S}x)\\&\le\mathcal{R}(\mathscr{S}(\mathscr{S}x),x)\\&=k\mathcal{R}(\mathscr{S}^2x,x)\\&=k\mathcal{R}(x_2,x_0)\\&=k\mathcal{R}(x_0,x_2).\end{aligned}$$

Again we have,

$$\begin{aligned}\mathcal{R}(x_2,x_4)&=\mathcal{R}(x_4,x_2)\\&=\mathcal{R}(\mathscr{S}^2(\mathscr{S}^2x),\mathscr{S}(\mathscr{S}x))\\&\le k\mathcal{R}(\mathscr{S}(\mathscr{S}^2x),\mathscr{S}x)\\&=k\mathcal{R}(\mathscr{S}^3x,\mathscr{S}x)\\&=k\mathcal{R}(x_3,x_1)\\&=k\mathcal{R}(x_1,x_3)\\&\le k^2\mathcal{R}(x_0,x_2).\end{aligned}$$

By induction we obtain that,

$$\mathcal{R}(x_w,x_{w+2})\le k^w\mathcal{R}(x_0,x_2).$$

If we take the limit of the above inequality as $w\to\infty$ we deduce that

$$\lim_{w\to\infty}\mathcal{R}(x_w,x_{w+2})=0. \tag{6}$$

We shall prove that $\{x_w\}$ is a Cauchy sequence in $(\mathcal{M},\mathcal{R})$, i.e.,

$$\lim_{w,m\to\infty}\mathcal{R}(x_w,x_m)=0\quad\text{where}\quad w,m\in\mathbb{N}.$$

For this we will take the following two cases.

Denote $\mathcal{R}_\flat = \mathcal{R}(x_\flat, x_{\flat+1})$ for all $\flat \in \mathbb{N}$.

Case1: Assume that ρ be odd, i.e., $\rho = 2\lambda + 1$, where $\lambda \geq 1$. Then by hypotheses we have $m = w + \rho > w$, we have,

$$\begin{aligned}
\mathcal{R}(x_w, x_m) &= \mathcal{R}(x_w, x_{w+2\lambda+1})\\
&\leq \varpi(x_w, x_{w+1})\mathcal{R}(x_w, x_{w+1}) + \varpi(x_{w+1}, x_{w+2})\mathcal{R}(x_{w+1}, x_{w+2})\\
&\quad + \varpi(x_{w+2}, x_{w+2\lambda+1})\mathcal{R}(x_{w+2}, x_{w+2\lambda+1})\\
&\leq \varpi(x_w, x_{w+1})\mathcal{R}(x_w, x_{w+1}) + \varpi(x_{w+1}, x_{w+2})\mathcal{R}(x_{w+1}, x_{w+2})\\
&\quad + \varpi(x_{w+2}, x_{w+2\lambda+1})\varpi(x_{w+2}, x_{w+3})\mathcal{R}(x_{w+2}, x_{w+3})\\
&\quad + \varpi(x_{w+2}, x_{w+2\lambda+1})\varpi(x_{w+3}, x_{w+4})\mathcal{R}(x_{w+3}, x_{w+4})\\
&\quad + \varpi(x_{w+2}, x_{w+2\lambda+1})\varpi(x_{w+4}, x_{w+2\lambda+1})\mathcal{R}(x_{w+4}, x_{w+2\lambda+1})\\
&\leq \varpi(x_w, x_{w+1})\mathcal{R}(x_w, x_{w+1}) + \varpi(x_{w+1}, x_{w+2})\mathcal{R}(x_{w+1}, x_{w+2})\\
&\quad + \varpi(x_{w+2}, x_{w+2\lambda+1})\varpi(x_{w+2}, x_{w+3})\mathcal{R}(x_{w+2}, x_{w+3})\\
&\quad + \varpi(x_{w+2}, x_{w+2\lambda+1})\varpi(x_{w+3}, x_{w+4})\mathcal{R}(x_{w+3}, x_{w+4})\\
&\quad + \varpi(x_{w+2}, x_{w+2\lambda+1})\varpi(x_{w+4}, x_{w+2\lambda+1})\varpi(x_{w+4}, x_{w+5})\\
&\quad \mathcal{R}(x_{w+4}, x_{w+5})\\
&\quad + \varpi(x_{w+2}, x_{w+2\lambda+1})\varpi(x_{w+4}, x_{w+2\lambda+1})\varpi(x_{w+5}, x_{w+6})\\
&\quad \mathcal{R}(x_{w+5}, x_{w+6})\\
&\quad + \varpi(x_{w+2}, x_{w+2\lambda+1})\varpi(x_{w+4}, x_{w+2\lambda+1})\varpi(x_{w+6}, x_{w+2\lambda+1})\\
&\quad \mathcal{R}(x_{w+6}, x_{w+2\lambda+1})\\
&\leq \varpi(x_w, x_{w+1})\mathcal{R}(x_w, x_{w+1}) + \varpi(x_{w+1}, x_{w+2})\mathcal{R}(x_{w+1}, x_{w+2})\\
&\quad + \varpi(x_{w+2}, x_{w+2\lambda+1})\varpi(x_{w+2}, x_{w+3})\mathcal{R}(x_{w+2}, x_{w+3})\\
&\quad + \varpi(x_{w+2}, x_{w+2\lambda+1})\varpi(x_{w+3}, x_{w+4})\mathcal{R}(x_{w+3}, x_{w+4})\\
&\quad + \varpi(x_{w+2}, x_{w+2\lambda+1})\varpi(x_{w+4}, x_{w+2\lambda+1})\varpi(x_{w+4}, x_{w+5})\\
&\quad \mathcal{R}(x_{w+4}, x_{w+5})\\
&\quad + \varpi(x_{w+2}, x_{w+2\lambda+1})\varpi(x_{w+4}, x_{w+2\lambda+1})\varpi(x_{w+5}, x_{w+6})\\
&\quad \mathcal{R}(x_{w+5}, x_{w+6})\\
&\quad \vdots\\
&\quad + \varpi(x_{w+2}, x_{w+2\lambda+1})\varpi(x_{w+4}, x_{w+2\lambda+1}) \times \cdots\cdots\\
&\quad \times \varpi(x_{w+2\lambda-2}, x_{w+2\lambda+1})\times\\
&\quad [\varpi(x_{w+2\lambda-2}, x_{w+2\lambda+1})\mathcal{R}(x_{w+2\lambda-2}, x_{w+2\lambda+1})\\
&\quad + \varpi(x_{w+2\lambda-1}, x_{w+2\lambda})\mathcal{R}(x_{w+2\lambda-1}, x_{w+2\lambda})]\\
&\quad + \varpi(x_{w+2}, x_{w+2\lambda+1})\varpi(x_{w+4}, x_{w+2\lambda+1}) \times \cdots\cdots\\
&\quad \times \varpi(x_{w+2\lambda-2}, x_{w+2\lambda+1})\varpi(x_{w+2\lambda}, x_{w+2\lambda+1})\mathcal{R}(x_{w+2\lambda}, x_{w+2\lambda+1})\\
&\leq \varpi(x_w, x_{w+1})\mathcal{R}_w + \varpi(x_{w+1}, x_{w+2})\mathcal{R}_{w+1}\\
&\quad + \varpi(x_{w+2}, x_{w+2\lambda+1})\varpi(x_{w+2}, x_{w+3})\mathcal{R}_{w+2}\\
&\quad + \varpi(x_{w+2}, x_{w+2\lambda+1})\varpi(x_{w+3}, x_{w+4})\mathcal{R}_{w+3}\\
&\quad + \varpi(x_{w+2}, x_{w+2\lambda+1})\varpi(x_{w+4}, x_{w+2\lambda+1})\varpi(x_{w+4}, x_{w+5})\mathcal{R}_{w+4}\\
&\quad + \varpi(x_{w+2}, x_{w+2\lambda+1})\varpi(x_{w+4}, x_{w+2\lambda+1})\varpi(x_{w+5}, x_{w+6})\mathcal{R}_{w+5}\\
&\quad \vdots\\
&\quad + \varpi(x_{w+2}, x_{w+2\lambda+1}) \times \cdots\cdots \times \varpi(x_{w+2\lambda-2}, x_{w+2\lambda+1})\times\\
&\quad [\varpi(x_{w+2\lambda-2}, x_{w+2\lambda+1})\mathcal{R}_{w+2\lambda-2} + \varpi(x_{w+2\lambda-1}, x_{w+2\lambda})\mathcal{R}_{w+2\lambda-1}]\\
&\quad + \varpi(x_{w+2}, x_{w+2\lambda+1})\varpi(x_{w+4}, x_{w+2\lambda+1}) \times \cdots\cdots\\
&\quad \times \varpi(x_{w+2\lambda-2}, x_{w+2\lambda+1})\varpi(x_{w+2\lambda}, x_{w+2\lambda+1})\mathcal{R}_{w+2\lambda}.
\end{aligned} \tag{7}$$

From above Eqs. (4) and (7), and Definition 1, we get

$$\begin{aligned}
\mathcal{R}(x_w, x_m) &\leq \varpi(x_w, x_{w+1})\mathcal{R}(x_0, x_1)k^w + \varpi(x_{w+1}, x_{w+2})\mathcal{R}(x_0, x_1)k^{w+1}\\
&\quad + \Sigma_{\flat=w+2}^{\flat=w+2\lambda} \Pi_{\eta=w+2}^{\eta=\flat} \varpi(x_\eta, x_{w+2\lambda+1})\\
&\quad \left[\varpi(x_\flat, x_{\flat+1})k^\flat + \varpi(x_{\flat+1}, x_{\flat+2})k^{\flat+1}\right]\mathcal{R}(x_0, x_1).
\end{aligned} \tag{8}$$

There just, we use that $\varpi(x, y) \geq 1$.

Let

$$\mathcal{S}_{\mathcal{Z}} = \Sigma_{\flat=0}^{\flat=\mathcal{Z}} \Pi_{\eta=0}^{\eta=\flat} \varpi(x_\eta, x_{w+2\lambda+1})\left[\varpi(x_\flat, x_{\flat+1})k^\flat + \varpi(x_{\flat+1}, x_{\flat+2})k^{\flat+1}\right]\mathcal{R}(x_0, x_1).$$

Then Eq. (8) can be written as,

$$\mathcal{R}(x_w, x_m) \leq \mathcal{R}(x_0, x_1)[\varpi(x_w, x_{w+1})k^w + \varpi(x_{w+1}, x_{w+2})k^{w+1} + \mathcal{S}_{m-1} - \mathcal{S}_{w+1}].$$

Now let,

$$a_\flat = \Pi_{\eta=0}^{\flat} \varpi(x_\eta, x_m)\left[\varpi(x_\flat, x_{\flat+1})k^\flat + \varpi(x_{\flat+1}, x_{\flat+2})k^{\flat+1}\right].$$

Now consider,

$$\begin{aligned}
&\lim_{\flat\to\infty} \sup_{m\geq 1} \frac{a_{\flat+1}}{a_\flat}\\
&= \lim_{\flat\to\infty} \sup_{m\geq 1} \frac{\Pi_{\eta=0}^{\flat+1} \varpi(x_\eta, x_m)\left[\varpi(x_{\flat+1}, x_{\flat+2})k^{\flat+1} + \varpi(x_{\flat+2}, x_{\flat+3})k^{\flat+2}\right]}{\Pi_{\eta=0}^{\flat} \varpi(x_\eta, x_m)\left[\varpi(x_\flat, x_{\flat+1})k^\flat + \varpi(x_{\flat+1}, x_{\flat+2})k^{\flat+1}\right]}\\
&= \lim_{\flat\to\infty} \sup_{m\geq 1} \varpi(x_{\flat+1}, x_m)\frac{\varpi(x_{\flat+1}, x_{\flat+2})k + \varpi(x_{\flat+2}, x_{\flat+3})k^2}{\varpi(x_\flat, x_{\flat+1}) + \varpi(x_{\flat+1}, x_{\flat+2})k}\\
&\leq \lim_{\flat\to\infty} \sup_{m\geq 1} \varpi(x_{\flat+1}, x_m)\frac{\varpi(x_{\flat+1}, x_{\flat+2}) + \varpi(x_{\flat+2}, x_{\flat+3})}{\varpi(x_\flat, x_{\flat+1}) + \varpi(x_{\flat+1}, x_{\flat+2})}\\
&< 1.
\end{aligned}$$

Thus the series

$$\Sigma_{\flat=n+2}^{\flat=\infty} \Pi_{\eta=n+2}^{\eta=\flat} \varpi(x_\eta, x_{w+2\lambda+1})\left[\varpi(x_\flat, x_{\flat+1})k^\flat + \varpi(x_{\flat+1}, x_{\flat+2})k^{\flat+1}\right]$$

$\mathcal{R}(x_0, x_1)$ is converges.

On the other-side,

$$\lim_{w\to\infty} \varpi(x_w, x_{w+1})\mathcal{R}(x_0, x_1)k^w = \lim_{w\to\infty} \varpi(x_{w+1}, x_{w+2})\mathcal{R}(x_0, x_1)k^{w+1} = 0.$$

From this, we can conclude that

$$\lim_{w,m\to\infty} \mathcal{R}(x_w, x_m) = 0.$$

Case2: Let ρ be even, i.e., $\rho = 2\lambda$, where $\lambda \geq 1$ with similarly to case1, we have,

$$\begin{aligned}
\mathcal{R}(x_w, x_{w+2\lambda}) &\leq \varpi(x_w, x_{w+2})\mathcal{R}(x_w, x_{w+2}) + \varpi(x_{w+2}, x_{w+3})\mathcal{R}(x_{w+2}, x_{w+3})\\
&\quad + \varpi(x_{w+3}, x_{w+2\lambda})\mathcal{R}(x_{w+3}, x_{w+2\lambda})\\
&\leq \varpi(x_w, x_{w+2})\mathcal{R}(x_w, x_{w+2}) + \varpi(x_{w+2}, x_{w+3})\mathcal{R}(x_{w+2}, x_{w+3})\\
&\quad + \varpi(x_{w+3}, x_{w+2\lambda})[\varpi(x_{w+3}, x_{w+4})\mathcal{R}(x_{w+3}, x_{w+4})\\
&\quad + \varpi(x_{w+4}, x_{w+5})\mathcal{R}(x_{w+4}, x_{w+5}) + \varpi(x_{w+5}, x_{w+2\lambda})\\
&\quad \mathcal{R}(x_{w+5}, x_{w+2\lambda})]\\
&= \varpi(x_w, x_{w+2})\mathcal{R}(x_w, x_{w+2}) + \varpi(x_{w+2}, x_{w+3})\mathcal{R}(x_{w+2}, x_{w+3})\\
&\quad + \varpi(x_{w+3}, x_{w+2\lambda})\varpi(x_{w+3}, x_{w+4})\mathcal{R}(x_{w+3}, x_{w+4})
\end{aligned}$$

$$
\begin{aligned}
&+ \varpi(x_{w+3}, x_{w+2\lambda})\varpi(x_{w+4}, x_{w+5})\mathcal{R}(x_{w+4}, x_{w+5}) \\
&+ \varpi(x_{w+3}, x_{w+2\lambda})\varpi(x_{w+5}, x_{w+2\lambda})\mathcal{R}(x_{w+5}, x_{w+2\lambda}) \\
\leq\ & \varpi(x_w, x_{w+2})\mathcal{R}(x_w, x_{w+2}) + \varpi(x_{w+2}, x_{w+3})\mathcal{R}(x_{w+2}, x_{w+3}) \\
&+ \varpi(x_{w+3}, x_{w+2\lambda})[\varpi(x_{w+3}, x_{w+4})\mathcal{R}(x_{w+3}, x_{w+4}) \\
&+ \varpi(x_{w+4}, x_{w+5})\mathcal{R}(x_{w+4}, x_{w+5}) + \varpi(x_{w+5}, x_{w+2\lambda}) \\
&\mathcal{R}(x_{w+5}, x_{w+2\lambda})] \\
\leq\ & \varpi(x_w, x_{w+2})\mathcal{R}(x_w, x_{w+2}) + \varpi(x_{w+2}, x_{w+3})\mathcal{R}(x_{w+2}, x_{w+3}) \\
&+ \varpi(x_{w+3}, x_{w+2\lambda})\varpi(x_{w+3}, x_{w+4})\mathcal{R}(x_{w+3}, x_{w+4}) \\
&+ \varpi(x_{w+3}, x_{w+2\lambda})\varpi(x_{w+4}, x_{w+5})\mathcal{R}(x_{w+4}, x_{w+5}) \\
&+ \varpi(x_{w+3}, x_{w+2\lambda})\varpi(x_{w+5}, x_{w+2\lambda})[\varpi(x_{w+5}, x_{w+6}) \\
&\mathcal{R}(x_{w+5}, x_{w+6}) \\
&+ \varpi(x_{w+6}, x_{w+7})\mathcal{R}(x_{w+6}, x_{w+7}) + \varpi(x_{w+7}, x_{w+2\lambda}) \\
&\mathcal{R}(x_{w+7}, x_{w+2\lambda})] \\
\leq\ & \varpi(x_w, x_{w+2})\mathcal{R}(x_w, x_{w+2}) + \varpi(x_{w+2}, x_{w+3})\mathcal{R}(x_{w+2}, x_{w+3}) \\
&+ \varpi(x_{w+3}, x_{w+2\lambda})\varpi(x_{w+3}, x_{w+4})\mathcal{R}(x_{w+3}, x_{w+4}) \\
&+ \varpi(x_{w+3}, x_{w+2\lambda})\varpi(x_{w+4}, x_{w+5})\mathcal{R}(x_{w+4}, x_{w+5}) \\
&+ \varpi(x_{w+3}, x_{w+2\lambda})\varpi(x_{w+5}, x_{w+2\lambda})\varpi(x_{w+5}, x_{w+6}) \\
&\mathcal{R}(x_{w+5}, x_{w+6}) \\
&+ \varpi(x_{w+3}, x_{w+2\lambda})\varpi(x_{w+5}, x_{w+2\lambda})\varpi(x_{w+6}, x_{w+7}) \\
&\mathcal{R}(x_{w+6}, x_{w+7}) \\
&+ \varpi(x_{w+3}, x_{w+2\lambda})\varpi(x_{w+5}, x_{w+2\lambda})\varpi(x_{w+7}, x_{w+2\lambda}) \\
&\mathcal{R}(x_{w+7}, x_{w+2\lambda}) \\
\leq\ & \varpi(x_w, x_{w+2})\mathcal{R}(x_w, x_{w+2}) + \varpi(x_{w+2}, x_{w+3})\mathcal{R}(x_{w+2}, x_{w+3}) \\
&+ \varpi(x_{w+3}, x_{w+2\lambda})[\varpi(x_{w+3}, x_{w+4})\mathcal{R}(x_{w+3}, x_{w+4}) \\
&+ \varpi(x_{w+4}, x_{w+5})\mathcal{R}(x_{w+4}, x_{w+5})] \\
&+ \varpi(x_{w+3}, x_{w+2\lambda})\varpi(x_{w+5}, x_{w+2\lambda})[\varpi(x_{w+5}, x_{w+6}) \\
&\mathcal{R}(x_{w+5}, x_{w+6}) \\
&+ \varpi(x_{w+6}, x_{w+7})\mathcal{R}(x_{w+6}, x_{w+7})] \\
&+ \varpi(x_{w+3}, x_{w+2\lambda})\varpi(x_{w+5}, x_{w+2\lambda})\varpi(x_{w+7}, x_{w+2\lambda}) \\
&\mathcal{R}(x_{w+7}, x_{w+2\lambda}).
\end{aligned}
$$

Upon carrying out this methodology repeatedly and applying the CRMS triangle inequality, we obtain,

$$
\begin{aligned}
\mathcal{R}(x_w, x_{w+2\lambda}) \leq\ & \varpi(x_w, x_{w+2})\mathcal{R}(x_w, x_{w+2}) + \varpi(x_{w+2}, x_{w+3})\mathcal{R}(x_{w+2}, x_{w+3}) \\
&+ \varpi(x_{w+3}, x_{w+2\lambda})[\varpi(x_{w+3}, x_{w+4})\mathcal{R}(x_{w+3}, x_{w+4}) \\
&+ \varpi(x_{w+4}, x_{w+5})\mathcal{R}(x_{w+4}, x_{w+5})] \\
&+ \varpi(x_{w+3}, x_{w+2\lambda})\varpi(x_{w+5}, x_{w+2\lambda})[\varpi(x_{w+5}, x_{w+6}) \\
&\mathcal{R}(x_{w+5}, x_{w+6}) \\
&+ \varpi(x_{w+6}, x_{w+7})\mathcal{R}(x_{w+6}, x_{w+7})] + \cdots\cdot \\
&+ \varpi(x_{w+3}, x_{w+2\lambda})\varpi(x_{w+5}, x_{w+2\lambda})\varpi(x_{w+7}, x_{w+2\lambda})\ldots \\
&\varpi(x_{w+2\lambda-3}, x_{w+2\lambda}) \\
&\times [\varpi(x_{w+2\lambda-3}, x_{w+2\lambda-2})\mathcal{R}(x_{w+2\lambda-3}, x_{w+2\lambda-2}) \\
&+ \varpi(x_{w+2\lambda-2}, x_{w+2\lambda-1})\mathcal{R}(x_{w+2\lambda-2}, x_{w+2\lambda-1})] \\
&+ \varpi(x_{w+3}, x_{w+2\lambda})\varpi(x_{w+5}, x_{w+2\lambda})\varpi(x_{w+7}, x_{w+2\lambda})\ldots \\
&\varpi(x_{w+2\lambda-3}, x_{w+2\lambda}) \\
&\times \varpi(x_{w+2\lambda-1}, x_{w+2\lambda})[\varpi(x_{w+2\lambda-1}, x_{w+2\lambda}) \\
&\mathcal{R}(x_{w+2\lambda-1}, x_{w+2\lambda}) \\
&\qquad + \varpi(x_{w+2\lambda}, x_{w+2\lambda+1})\mathcal{R}(x_{w+2\lambda}, x_{w+2\lambda+1})] \\
\leq\ & \varpi(x_w, x_{w+2})k^w\mathcal{R}(x_0, x_2) + \varpi(x_{w+2}, x_{w+3})k^{w+2}\mathcal{R}(x_0, x_1) \\
&+ \cdots + \varpi(x_{w+3}, x_{w+2\lambda})\varpi(x_{w+5}, x_{w+2\lambda}) \times \cdots \\
&\times \varpi(x_{w+2\lambda-3}, x_{w+2\lambda}) \\
&[\varpi(x_{w+2\lambda-3}, x_{w+2\lambda-2})k^{w+2\lambda-3}\mathcal{R}(x_0, x_1) \\
&+ \varpi(x_{w+2\lambda-2}, x_{w+2\lambda-1})k^{w+2\lambda-2}\mathcal{R}(x_0, x_1)] \\
&+ \varpi(x_{w+3}, x_{w+2\lambda})\varpi(x_{w+5}, x_{w+2\lambda}) \times \cdots \\
&\times \varpi(x_{w+2\lambda-3}, x_{w+2\lambda})\varpi(x_{w+2\lambda-1}, x_{w+2\lambda}) \\
&[\varpi(x_{w+2\lambda-1}, x_{w+2\lambda})k^{w+2\lambda-1}\mathcal{R}(x_0, x_1) \\
&+ \varpi(x_{w+2\lambda}, x_{w+2\lambda+1})k^{w+2\lambda}\mathcal{R}(x_0, x_1)].
\end{aligned} \tag{9}
$$

Thus we conclude,

$$
\begin{aligned}
\mathcal{R}(x_w, x_m) \leq\ & \varpi(x_w, x_{w+2})\mathcal{R}(x_0, x_2)k^w + \varpi(x_{w+2}, x_{w+3})\mathcal{R}(x_0, x_1)k^{w+2} \\
&+ \Sigma_{\flat=w+3}^{\flat=w+2\lambda-1} \Pi_{\eta=w+3}^{\eta=\flat}\varpi(x_\eta, x_{w+2\lambda}) \\
&\left[\varpi(x_\flat, x_{\flat+1})k^\flat + \varpi(x_{\flat+1}, x_{\flat+2})k^{\flat+1}\right]\mathcal{R}(x_0, x_1).
\end{aligned} \tag{10}
$$

There just, we use that $\varpi(x, y) \geq 1$.

Let

$$\mathcal{S}_{\mathcal{Z}} = \Sigma_{\flat=0}^{\flat=\mathcal{Z}} \Pi_{\eta=0}^{\eta=\flat}\varpi(x_\eta, x_{w+2\lambda})\left[\varpi(x_\flat, x_{\flat+1})k^\flat + \varpi(x_{\flat+1}, x_{\flat+2})k^{\flat+1}\right]\mathcal{R}(x_0, x_1).$$

Then we have,

$$
\begin{aligned}
\mathcal{R}(x_w, x_m) \leq\ & \mathcal{R}(x_0, x_2)\varpi(x_w, x_{w+2})k^w + \mathcal{R}(x_0, x_1)[\varpi(x_{w+2}, x_{w+3})k^{w+2} \\
&+ \mathcal{S}_{m-1} - \mathcal{S}_{w+2}].
\end{aligned}
$$

Now let,

$$a_\flat = \Pi_{\eta=0}^{\flat}\varpi(x_\eta, x_m)\left[\varpi(x_\flat, x_{\flat+1})k^\flat + \varpi(x_{\flat+1}, x_{\flat+2})k^{\flat+1}\right].$$

Now consider, $\lim_{\flat\to\infty}\sup_{m\geq1}\frac{a_{\flat+1}}{a_\flat}$

$$
\begin{aligned}
&= \lim_{\flat\to\infty}\sup_{m\geq1}\varpi(x_{\flat+1}, x_m)\frac{\varpi(x_{\flat+1}, x_{\flat+2})k + \varpi(x_{\flat+2}, x_{\flat+3})k^2}{\varpi(x_\flat, x_{\flat+1}) + \varpi(x_{\flat+1}, x_{\flat+2})k} \\
&\leq \lim_{\flat\to\infty}\sup_{m\geq1}\varpi(x_{\flat+1}, x_m)\frac{\varpi(x_{\flat+1}, x_{\flat+2}) + \varpi(x_{\flat+2}, x_{\flat+3})}{\varpi(x_\flat, x_{\flat+1}) + \varpi(x_{\flat+1}, x_{\flat+2})} \\
&< 1.
\end{aligned}
$$

By using the Ratio Test, we conclude that the series

$$\Sigma_{\flat=0}^{\flat=\infty}\Pi_{\eta=0}^{\eta=\flat}\varpi(x_\eta, x_{w+2\lambda})\left[\varpi(x_\flat, x_{\flat+1})k^\flat + \varpi(x_{\flat+1}, x_{\flat+1})k^{\flat+1}\right]$$
$\mathcal{R}(x_0, x_1)$ *converges*.

Hence $\mathcal{R}(x_w, x_m)$ is converges as w, m go towards ∞. Thus by case1 and case2, we have

$$\lim_{w,m\to\infty}\mathcal{R}(x_w, x_m) = 0.$$

We conclude that the $\{\mathscr{S}^w x\}$ is a Cauchy sequence. As a result there exist a $\sigma \in \mathscr{O} \cup \mathscr{H}$ such that $\mathscr{S}^w x \to \sigma$. Note that $\{\mathscr{S}^{2w}x\}$ is a sequence in $\mathscr{O}$ and $\{\mathscr{S}^{2w-1}x\}$ is a sequence in $\mathscr{H}$ such that both sequences tends to same limit σ. As $\mathscr{O}$ and $\mathscr{H}$ are closed, $\sigma \in \mathscr{O} \cap \mathscr{H}$. Hence $\mathscr{O} \cap \mathscr{H} \neq \emptyset$.

We prove that $\mathscr{S}\sigma = \sigma$. Consider,

$$
\begin{aligned}
\mathcal{R}(\mathscr{S}\sigma, \sigma) &= \lim_{w\to\infty}\mathcal{R}(\mathscr{S}\sigma, \mathscr{S}^{2w}x) \\
&= \lim_{w\to\infty}\mathcal{R}(\mathscr{S}^{2w}x, \mathscr{S}\sigma) \\
&\leq \lim_{w\to\infty}k\mathcal{R}(\mathscr{S}^{2w-1}x, \sigma) \\
&= k\mathcal{R}(\sigma, \sigma) \\
&= 0,
\end{aligned} \tag{11}
$$

which yields $\mathcal{R}(\mathscr{S}\sigma, \sigma) = 0$. Therefore, $\mathscr{S}\sigma = \sigma$.

We can easily prove uniqueness. For this, Suppose that the sequence $\{x_w\}$ has two limit points σ_1, σ_2 in $\mathscr{O} \cap \mathscr{H}$.

i.e., $\lim_{w\to\infty} x_w = \sigma_1$ and $\lim_{w\to\infty} x_w = \sigma_2$. Here $\{x_w\}$ is a Cauchy sequence for $x_w \neq x_m$ for $w \neq m$. Hence from hypotheses, we have $\mathcal{R}(\sigma_1, \sigma_2) \leq \varpi(\sigma_1, x_w)\mathcal{R}(\sigma_1, x_w) + \varpi(x_w, x_m)\mathcal{R}(x_w, x_m) + \varpi(x_m, \sigma_2)\mathcal{R}(x_m,$

σ_2). Letting $w, m \to \infty$ in the above inequality, $\mathcal{R}(\sigma_1, \sigma_2) = 0 \Rightarrow \sigma_1 = \sigma_2$. Hence the sequence $\{\mathscr{S} x_w\}$ has unique limit point in $\mathscr{O} \cap \mathscr{H}$. Thus $\mathscr{S}$ has a unique fixed point in $\mathscr{O} \cap \mathscr{H}$.

Theorem 2. *Let $\{\mathscr{A}_\flat\}_{\flat=1}^{p}$ be non-empty closed subsets of a complete CRMS and suppose $\mathscr{S} : \bigcup_{\flat=1}^{p} \mathscr{A}_\flat \to \bigcup_{\flat=1}^{p} \mathscr{A}_\flat$ is a cyclical operator and there exists real numbers $r \in [0, \frac{1}{2})$, $s \in [0, \frac{1}{2})$ and $t \in [0, \frac{1}{2})$ such that for each pair $(x, y) \in \mathscr{A}_\flat \times \mathscr{A}_{\flat+1}$, for $1 \le \flat \le p$, at-least one of the following is true.*

($\mathcal{Z}^\star$). $\mathcal{R}(\mathscr{S}^{2w} x, \mathscr{S} y) \le r \mathcal{R}(\mathscr{S}^{2w-1} x, y)$;

($\mathcal{Z}^*$). $\mathcal{R}(\mathscr{S}^{2w} x, \mathscr{S} y) \le s[\mathcal{R}(\mathscr{S}^{2w-1} x, \mathscr{S}^{2w} x) + \mathcal{R}(y, \mathscr{S} y)]$;

($\mathcal{Z}^\bullet$). $\mathcal{R}(\mathscr{S}^{2w} x, \mathscr{S} y) \le t[\mathcal{R}(\mathscr{S}^{2w-1} x, \mathscr{S} y) + \mathcal{R}(\mathscr{S}^{2w} x, y)]$.

Let us suppose following conditions:

1. *For $x_0 \in \mathscr{A}_\flat$, take $x_w = \mathscr{S}^w x_0$. Suppose that,*
$$\lim_{\flat\to\infty} \sup_{m\ge 1} \varpi(x_{\flat+1}, x_m) \frac{\varpi(x_{\flat+1}, x_{\flat+2}) + \varpi(x_{\flat+2}, x_{\flat+3})}{\varpi(x_\flat, x_{\flat+1}) + \varpi(x_{\flat+1}, x_{\flat+2})} < 1.$$
Assume that, $\lim_{w\to+\infty} \varpi(x_w, x), \lim_{w\to+\infty} \varpi(x, x_w)$ and $\lim_{w,m\to+\infty} \varpi(x_w, x_m)$ exist and are finite for all $w, m \in \mathbb{N}, w \ne m$;
2. $0 \le \max\{r, \frac{s+\varpi(x_{2w-1},x_{2w})}{1-s\varpi(x_{2w},\mathscr{S} y)}, \frac{s\varpi(y,x_{2w-1})}{1-s\varpi(x_{2w},\mathscr{S} y)}, \frac{t+\varpi(\mathscr{S} y,x_{2w-1})}{1-t\varpi(x_{2w},\mathscr{S} y)}, \frac{t\varpi(x_{2w-1},y)}{1-t\varpi(x_{2w},\mathscr{S} y)}\} < \frac{1}{2}$.

Then $\mathscr{S}$ has a unique fixed point γ in $\bigcap_{\flat=1}^{p} \mathscr{A}_\flat$. Moreover, the Picard iteration x_w given by $\mathscr{S} x_w = x_{w+1}$, $w \ge 0$ converges to γ for any starting point $x_0 \in \bigcup_{\flat=1}^{p} \mathscr{A}_\flat$.

Proof. Suppose there exists $x = x_0 \in \mathscr{A}_\flat$. Define the iterative sequence $x_w = \mathscr{S}^w x_0$. Since $x_0 \in \mathscr{A}_\flat$ and $\mathscr{S}$ is a cyclical operator, we have $x_{2w} \in \mathscr{A}_\flat$ and $x_{2w+1} \in \mathscr{A}_{\flat+1}$ for all $w \ge 0$.

Using the CRMS conditions, we will prove that each of the three relations ($\mathcal{Z}^\star$), ($\mathcal{Z}^*$) and ($\mathcal{Z}^\bullet$) can be written in the following equivalent manner.

$$\mathcal{R}(\mathscr{S}^{2w} x, \mathscr{S} y) \le \psi \mathcal{R}(\mathscr{S}^{2w-1} x, y) + \psi \mathcal{R}(\mathscr{S}^{2w-1} x, \mathscr{S}^{2w} x)$$

and

$$\mathcal{R}(\mathscr{S}^{2w} x, \mathscr{S} y) \le \psi \mathcal{R}(\mathscr{S}^{2w-1} x, \mathscr{S} y) + \psi \mathcal{R}(\mathscr{S}^{2w-1} x, y),$$

where $\psi = \max\{r, \frac{s+\varpi(x_{2w-1},x_{2w})}{1-s\varpi(x_{2w},\mathscr{S} y)}, \frac{s\varpi(y,x_{2w-1})}{1-s\varpi(x_{2w},\mathscr{S} y)}, \frac{t+\varpi(\mathscr{S} y,x_{2w-1})}{1-t\varpi(x_{2w},\mathscr{S} y)}, \frac{t\varpi(x_{2w-1},y)}{1-t\varpi(x_{2w},\mathscr{S} y)}\}$.

For proving the above two inequalities let $\flat \in 1, 2, 3, \dots, p$ and two points $x \in \mathscr{A}_\flat, y \in \mathscr{A}_{\flat+1}$. At-least one of the ($\mathcal{Z}^\star$), ($\mathcal{Z}^*$) or ($\mathcal{Z}^\bullet$) is true.

If ($\mathcal{Z}^*$) holds, then we have,

$$\begin{aligned}\mathcal{R}(\mathscr{S}^{2w} x, \mathscr{S} y) &\le s[\mathcal{R}(\mathscr{S}^{2w} x, \mathscr{S}^{2w-1} x) + \mathcal{R}(y, \mathscr{S} y)] \\ &\le s[\mathcal{R}(\mathscr{S}^{2w} x, \mathscr{S}^{2w-1} x) + \{\varpi(y, \mathscr{S}^{2w-1} x)\mathcal{R}(y, \mathscr{S}^{2w-1} x) \\ &+ \varpi(\mathscr{S}^{2w-1} x, \mathscr{S}^{2w} x)\mathcal{R}(\mathscr{S}^{2w-1} x, \mathscr{S}^{2w} x) \\ &+ \varpi(\mathscr{S}^{2w} x, \mathscr{S} y)\mathcal{R}(\mathscr{S}^{2w} x, \mathscr{S} y)\}]\end{aligned}$$

which implies,

$$\begin{aligned}[1 - s\varpi(\mathscr{S}^{2w} x, \mathscr{S} y)]\mathcal{R}(\mathscr{S}^{2w} x, \mathscr{S} y) \le{}& [s + \varpi(\mathscr{S}^{2w-1} x, \mathscr{S}^{2w} x)] \\ &\times \mathcal{R}(\mathscr{S}^{2w} x, \mathscr{S}^{2w-1} x) \\ &+ s\varpi(y, \mathscr{S}^{2w-1} x)\mathcal{R}(y, \mathscr{S}^{2w-1} x)\end{aligned}$$

this can be deduced as,

$$\begin{aligned}\mathcal{R}(\mathscr{S}^{2w} x, \mathscr{S} y) \le{}& \frac{s + \varpi(\mathscr{S}^{2w-1} x, \mathscr{S}^{2w} x)}{1 - s\varpi(\mathscr{S}^{2w} x, \mathscr{S} y)}\mathcal{R}(\mathscr{S}^{2w} x, \mathscr{S}^{2w-1} x) \\ &+ \frac{s\varpi(y, \mathscr{S}^{2w-1} x)}{1 - s\varpi(\mathscr{S}^{2w} x, \mathscr{S} y)}\mathcal{R}(y, \mathscr{S}^{2w-1} x).\end{aligned} \tag{12}$$

If ($\mathcal{Z}^\bullet$) holds, then we have,

$$\begin{aligned}\mathcal{R}(\mathscr{S}^{2w} x, \mathscr{S} y) &\le t[\mathcal{R}(\mathscr{S}^{2w-1} x, \mathscr{S} y) + \mathcal{R}(\mathscr{S}^{2w} x, y)] \\ &\le t[\mathcal{R}(\mathscr{S}^{2w-1} x, \mathscr{S} y) + \{\varpi(\mathscr{S}^{2w} x, \mathscr{S} y)\mathcal{R}(\mathscr{S}^{2w} x, \mathscr{S} y) \\ &+ \varpi(\mathscr{S} y, \mathscr{S}^{2w-1} x)\mathcal{R}(\mathscr{S} y, \mathscr{S}^{2w-1} x) \\ &+ \varpi(\mathscr{S}^{2w-1} x, y)\mathcal{R}(\mathscr{S}^{2w-1} x, y)\}]\end{aligned}$$

which implies,

$$\begin{aligned}[1 - t\varpi(\mathscr{S}^{2w} x, \mathscr{S} y)]\mathcal{R}(\mathscr{S}^{2w} x, \mathscr{S} y) \le{}& [t + \varpi(\mathscr{S} y, \mathscr{S}^{2w-1} x)] \\ &\times \mathcal{R}(\mathscr{S}^{2w-1} x, \mathscr{S} y) \\ &+ t\varpi(\mathscr{S}^{2w-1} x, y)\mathcal{R}(\mathscr{S}^{2w-1} x, y),\end{aligned}$$

this can be deduced as,

$$\begin{aligned}\mathcal{R}(\mathscr{S}^{2w} x, \mathscr{S} y) \le{}& \frac{t + \varpi(\mathscr{S} y, \mathscr{S}^{2w-1} x)}{1 - t\varpi(\mathscr{S}^{2w} x, \mathscr{S} y)}\mathcal{R}(\mathscr{S}^{2w-1} x, \mathscr{S} y) \\ &+ \frac{t\varpi(\mathscr{S}^{2w-1} x, y)}{1 - t\varpi(\mathscr{S}^{2w} x, \mathscr{S} y)}\mathcal{R}(\mathscr{S}^{2w-1} x, y).\end{aligned} \tag{13}$$

Therefore by denoting,

$$\psi = \max\left\{ r, \frac{s + \varpi(x_{2w-1}, x_{2w})}{1 - s\varpi(x_{2w}, \mathscr{S} y)}, \frac{s\varpi(y, x_{2w-1})}{1 - s\varpi(x_{2w}, \mathscr{S} y)}, \frac{t + \varpi(\mathscr{S} y, x_{2w-1})}{1 - t\varpi(x_{2w}, \mathscr{S} y)}, \frac{t\varpi(x_{2w-1}, y)}{1 - t\varpi(x_{2w}, \mathscr{S} y)} \right\}.$$

We have $0 < \psi < \frac{1}{2}$, then Eqs. (12) and (13) reduces to

$$\mathcal{R}(\mathscr{S}^{2w} x, \mathscr{S} y) \le \psi \mathcal{R}(\mathscr{S}^{2w} x, \mathscr{S}^{2w-1} x) + \psi \mathcal{R}(y, \mathscr{S}^{2w-1} x) \tag{14}$$

and

$$\mathcal{R}(\mathscr{S}^{2w} x, \mathscr{S} y) \le \psi \mathcal{R}(\mathscr{S}^{2w-1} x, \mathscr{S} y) + \psi \mathcal{R}(\mathscr{S}^{2w-1} x, y). \tag{15}$$

Let $x_0 \in \bigcup_{\flat=1}^{p} \mathscr{A}_\flat$ and let $x_w = \mathscr{S}^w x_0$, $w = 1, 2, \dots$ It follows that there exists $\flat \in \{1, 2, \dots..p\}$ such that $x_0 \in \mathscr{A}_\flat$ and $x_1 = \mathscr{S} x_0 \in \mathscr{A}_\flat$, due to $\mathscr{S}(\mathscr{A}_\flat) \subseteq \mathscr{A}_{\flat+1}$ for all $\flat \in \{1, 2, 3, \dots, p\}$.

In addition, from Eq. (14), we get,

$$\begin{aligned}&\mathcal{R}(\mathscr{S}^2 x, \mathscr{S} x) \le \psi \mathcal{R}(\mathscr{S}^2 x, \mathscr{S} x) + \psi \mathcal{R}(x, \mathscr{S} x) \\ &\Rightarrow (1 - \psi)\mathcal{R}(\mathscr{S}^2 x, \mathscr{S} x) \le \psi \mathcal{R}(\mathscr{S} x, x) \\ &\Rightarrow \mathcal{R}(\mathscr{S}^2 x, \mathscr{S} x) \le \frac{\psi}{1 - \psi}\mathcal{R}(\mathscr{S} x, x) \\ &\Rightarrow \mathcal{R}(x_2, x_1) \le \Lambda \mathcal{R}(x_1, x_0),\end{aligned}$$

where $\Lambda = \frac{\psi}{1-\psi}$, which lies between 0 and 1. Similarly,

$$\begin{aligned}\mathcal{R}(\mathscr{S}^3 x, \mathscr{S}^2 x) &= \mathcal{R}(\mathscr{S}^2(\mathscr{S} x), \mathscr{S}(\mathscr{S} x)) \\ &\le \psi \mathcal{R}(\mathscr{S}^2(\mathscr{S} x), \mathscr{S}(\mathscr{S} x)) + \psi \mathcal{R}(\mathscr{S} x, \mathscr{S}(\mathscr{S} x)) \\ &= \psi \mathcal{R}(\mathscr{S}^3 x, \mathscr{S}^2 x) + \psi \mathcal{R}(\mathscr{S}^2 x, \mathscr{S} x)\end{aligned}$$

$$\Rightarrow (1 - \psi)\mathcal{R}(\mathscr{S}^3 x, \mathscr{S}^2 x) \le \psi \mathcal{R}(\mathscr{S}^2 x, \mathscr{S} x)$$

which implies,

$$\begin{aligned}\mathcal{R}(\mathscr{S}^3 x, \mathscr{S}^2 x) &\le \frac{\psi}{1 - \psi}\mathcal{R}(\mathscr{S}^2 x, \mathscr{S} x) \\ &= \Lambda \mathcal{R}(x_2, x_1) \\ &\le \Lambda^2 \mathcal{R}(x_1, x_0)\end{aligned}$$

which gives,

$$\mathcal{R}(x_3, x_2) \le \Lambda^2 \mathcal{R}(x_1, x_0).$$

Which can be generalized by induction, such as,

$$\mathcal{R}(x_w, x_{w+1}) \le \Lambda^w \mathcal{R}(x_0, x_1), \quad w \ge 0. \tag{16}$$

Since $0 < \Lambda < 1$, taking $\lim_{w\to\infty}$ both sides of Eq. (16), then we get,

$$\lim_{w\to\infty} \mathcal{R}(x_w, x_{w+1}) = 0. \tag{17}$$

We shall prove that $\lim_{w\to\infty} \mathcal{R}(x_w, x_{w+2}) = 0$. We assume that $x_w \ne x_m$ for every $w, m \in \mathbb{N}$. Indeed suppose that $x_w = x_m$ for some $w = m + r$, with $r > 0$, so we have $\mathscr{S} x_w = \mathscr{S} x_m$, and

$$\begin{aligned}\mathcal{R}(x_m, x_{m+1}) &= \mathcal{R}(x_w, x_{w+1}) \\ &\le k\mathcal{R}(x_{w-1}, x_w).\end{aligned}$$

Since $k \in (0, 1)$, therefore $\mathcal{R}(x_m, x_{m+1}) = \mathcal{R}(x_w, x_{w+1}) < \mathcal{R}(x_{w-1}, x_w)$. By continuing this process, we have $\mathcal{R}(x_m, x_{m+1}) < \mathcal{R}(x_m, x_{m+1})$, which is a contradiction.

Therefore $\mathcal{R}(x_m, x_w) > 0$ for every $w, m \in \mathbb{N}, w \neq m$.

To prove $\lim_{w\to\infty} \mathcal{R}(x_w, x_{w+2}) = 0$, by using Eq. (14), we get,

$$\begin{aligned}\mathcal{R}(x_1, x_3) &= \mathcal{R}(x_3, x_1)\\ &= \mathcal{R}(\mathcal{S}^2(\mathcal{S}x), \mathcal{S}x)\\ &\leq \psi\mathcal{R}(\mathcal{S}^3x, \mathcal{S}^2x) + \psi\mathcal{R}(x, \mathcal{S}^2x)\\ &= \psi\mathcal{R}(x_3, x_2) + \psi\mathcal{R}(x_0, x_2)\\ &\leq \psi\Lambda^2\mathcal{R}(x_0, x_1) + \psi\mathcal{R}(x_0, x_2).\end{aligned} \tag{18}$$

Now consider,

$$\begin{aligned}\mathcal{R}(x_2, x_4) &= \mathcal{R}(x_4, x_2)\\ &= \mathcal{R}(\mathcal{S}^2(\mathcal{S}^2x), \mathcal{S}(\mathcal{S}x))\\ &\leq \psi\mathcal{R}(\mathcal{S}^4x, \mathcal{S}^3x) + \psi\mathcal{R}(\mathcal{S}x, \mathcal{S}^3x)\\ &= \psi\mathcal{R}(x_4, x_3) + \psi\mathcal{R}(x_1, x_3)\\ &\leq \psi\Lambda^3\mathcal{R}(x_0, x_1) + \psi[\psi\Lambda^2\mathcal{R}(x_0, x_1) + \psi\mathcal{R}(x_0, x_2)]\\ &= (\psi\Lambda^3 + \psi^2\Lambda^2)\mathcal{R}(x_0, x_1) + \psi^2\mathcal{R}(x_0, x_2).\end{aligned} \tag{19}$$

Similarly by using Eqs. (14) and (19)

$$\begin{aligned}\mathcal{R}(x_2, x_4) &\leq \psi\mathcal{R}(x_5, x_4) + \psi\mathcal{R}(x_2, x_4)\\ &\leq \psi\Lambda^4\mathcal{R}(x_0, x_1) + \psi[(\psi\Lambda^3 + \psi^2\Lambda^2)\mathcal{R}(x_0, x_1) + \psi^2\mathcal{R}(x_0, x_2)]\\ &= (\psi\Lambda^3 + \psi^2\Lambda^3 + \psi^3\Lambda^2)\mathcal{R}(x_0, x_1) + \psi^3\mathcal{R}(x_0, x_2).\end{aligned} \tag{20}$$

By induction, we can easily prove that

$$\begin{aligned}\mathcal{R}(x_w, x_{w+2}) &\leq (\psi\Lambda^{w+1} + \psi^2\Lambda^w + \cdots. + \psi^w\Lambda^2)\mathcal{R}(x_0, x_1) + \psi^w\mathcal{R}(x_0, x_2)\\ &= \left(\frac{\psi}{\Lambda}\Lambda^{w+2} + \frac{\psi^2}{\Lambda^2}\Lambda^{w+2} + \cdots. + \frac{\psi^w}{\Lambda^w}\Lambda^{w+2}\right)\mathcal{R}(x_0, x_1)\\ &+ \psi^w\mathcal{R}(x_0, x_2)\\ &= \left(\Lambda^{w+2}\Sigma_{\flat=0}^{w-1}\left(\frac{\psi}{\Lambda}\right)^{\flat+1}\right)\mathcal{R}(x_0, x_1) + \psi^w\mathcal{R}(x_0, x_2).\end{aligned}$$

Letting $w \to \infty$ on both sides, we get,

$$\lim_{w\to\infty} \mathcal{R}(x_w, x_{w+2}) = 0. \tag{21}$$

Similarly to the proof of Theorem 1, we can easily prove that $\mathcal{S}^w x$ is a Cauchy sequence in $(\mathcal{M}, \mathcal{R})$.

i.e., $\lim_{w,m\to\infty} \mathcal{R}(x_w, x_m) = 0, \ \forall w, m \in \mathbb{N}$,

for each $x_0 \in \bigcup_{\flat=1}^p \mathcal{A}_\flat$ and hence a convergent sequence too. Let γ be its limit.

i.e., $\lim_{w\to\infty} \mathcal{R}(\mathcal{S}^w x, \gamma) = 0$.

By using definition of cyclical operator of an infinite number of terms of this sequence lie in each $\mathcal{A}_\flat$, for all $\flat = 1, 2, 3, \ldots, p$. Therefore,

$$\gamma \in \bigcap_{\flat=1}^{p} \mathcal{A}_\flat \neq \emptyset.$$

Note that $\mathcal{S}^{2w}x$ is a sequence in $\mathcal{A}_\flat$ and $\mathcal{S}^{2w-1}x$ is a sequence in $\mathcal{A}_{\flat+1}$ such that both sequences tend to same limit γ.

To prove that γ is a fixed point of $\mathcal{S}$, we will use Eq. (14),

$$\begin{aligned}\mathcal{R}(\mathcal{S}\gamma, \gamma) &= \lim_{w\to\infty} \mathcal{R}(\mathcal{S}\gamma, \mathcal{S}^{2w}\gamma)\\ &= \lim_{w\to\infty} \mathcal{R}(\mathcal{S}^{2w}\gamma, \mathcal{S}\gamma)\\ &\leq \psi \lim_{w\to\infty} \mathcal{R}(\mathcal{S}^{2w}\gamma, \mathcal{S}^{2w-1}\gamma) + \psi \lim_{w\to\infty} \mathcal{R}(\gamma, \mathcal{S}^{2w-1}\gamma)\\ &= \psi\mathcal{R}(\gamma, \gamma) + \psi\mathcal{R}(\gamma, \gamma),\end{aligned}$$

which gives $\mathcal{R}(\mathcal{S}\gamma, \gamma) = 0$. Therefore, $\mathcal{S}\gamma = \gamma$.

To prove uniqueness, let us suppose $\mathcal{S}$ has another fixed point $\gamma^\star \in \bigcap_{\flat=1}^p \mathcal{A}_\flat$, $\gamma \neq \gamma^\star$. Thus, $\mathcal{S}\gamma^\star = \gamma^\star$. By using Eq. (14), we obtain,

$$\begin{aligned}\mathcal{R}(\gamma, \gamma^\star) &= \lim_{w\to\infty} \mathcal{R}(\mathcal{S}^{2w}\gamma, \mathcal{S}\gamma^\star)\\ &\leq \psi \lim_{w\to\infty} \mathcal{R}(\mathcal{S}^{2w}\gamma, \mathcal{S}^{2w-1}\gamma) + \psi \lim_{w\to\infty} \mathcal{R}(\gamma^\star, \mathcal{S}^{2w-1}\gamma)\\ &= \psi\mathcal{R}(\gamma, \gamma) + \psi\mathcal{R}(\gamma^\star, \gamma),\end{aligned}$$

which gives,

$$(1-\psi)\mathcal{R}(\gamma^\star, \gamma) \leq 0.$$

$$\Rightarrow \mathcal{R}(\gamma^\star, \gamma) = 0, \quad \text{since } 0 < \psi < \frac{1}{2}.$$

Hence $\gamma = \gamma^\star$. This completes the proof.

Example 1. Let $\mathcal{O} = \mathcal{H} = \mathcal{M} = [0, 1]$. Define $\mathcal{R} : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ by $\mathcal{M}(x, y) = |x - y|^2$ and $\varpi : \mathcal{M} \times \mathcal{M} \to [1, \infty)$ by $\varpi(x, y) = x + y + 3$ for all $x, y \in \mathcal{M}$.

Define $\mathcal{S} : \mathcal{M} \to \mathcal{M}$ by

$$\mathcal{S}x = \begin{cases} 0, & \text{if } \ x \in [0, \frac{1}{2}] \\ \frac{1}{4}, & \text{if } \ x \in [\frac{1}{2}, 1]. \end{cases}$$

Fix any $x \in [0, \frac{1}{2})$.

Case1: If $y \in [0, \frac{1}{2})$, then $\mathcal{S}y = 0$ and since $x \in [0, \frac{1}{2})$,

$$\mathcal{S}x = 0; \mathcal{S}^2x = 0; \ldots.\mathcal{S}^w x = 0, \quad \text{for all } \ w.$$

Then,

$$\begin{aligned}\mathcal{R}(\mathcal{S}^{2w}x, \mathcal{S}y) &= 0\\ &\leq k\mathcal{R}(\mathcal{S}^{2w-1}x, y) \quad \text{for } \ k \in (0, 1).\end{aligned}$$

Case2: If $y \in [\frac{1}{2}, 1)$ then $\mathcal{S}y = \frac{1}{4}$ and $\mathcal{S}^w x = 0$, for all w. Thus,

$$\begin{aligned}\mathcal{R}(\mathcal{S}^{2w}x, \mathcal{S}y) &= |0 - \frac{1}{4}|^2\\ &= \frac{1}{16}\\ &\leq k|0 - y|^2\\ &= k\mathcal{R}(\mathcal{S}^{2w-1}x, y) \quad \text{for } \ k \in (0, 1).\end{aligned}$$

Thus all the conditions of Theorem 1 satisfied and 0 is the unique fixed point of $\mathcal{S}$.

Theorem 3. *Let $\mathcal{S}$ be a self mapping on $\mathcal{M}$, and let $(\mathcal{M}, \mathcal{R})$ be a complete controlled rectangular metric space. Assume that the axioms listed below are true.*

1. For all $x, y \in \mathcal{M}$, we have $\mathcal{R}(\mathcal{S}x, \mathcal{S}y) \leq k\mathcal{R}(x, y)$, $\ k \in [0, 1)$;
2. $\mathcal{S}$ is continuous.
3. For $x_0 \in \mathcal{M}$, take $x_w = \mathcal{S}^w x_0$. Suppose that
$$\lim_{\flat\to\infty} \sup_{m\geq 1} \varpi(x_{\flat+1}, x_m)\frac{\varpi(x_{\flat+1}, x_{\flat+2}) + \varpi(x_{\flat+2}, x_{\flat+3})}{\varpi(x_\flat, x_{\flat+1}) + \varpi(x_{\flat+1}, x_{\flat+2})} < 1.$$
We assume that, for $x \in \mathcal{M}$, we have $\lim_{w\to+\infty} \varpi(x_w, x)$, $\lim_{w\to+\infty} \varpi(x, x_w)$ and $\lim_{w,m\to+\infty} \varpi(x_w, x_m)$ exist and are finite $\forall w, m \in \mathbb{N}$ and $w \neq m$.

Then $\mathcal{S}$ has a unique fixed point in $\mathcal{M}$.

Theorem 4. *Let $\mathcal{O}$ and $\mathcal{H}$ be non-empty closed subset of a complete CRMS and $\mathcal{S} : \mathcal{O} \cup \mathcal{H} \to \mathcal{O} \cup \mathcal{H}$ be a cyclic map. If there exists $k \in (0, 1)$ such that*

$$\mathcal{R}(\mathcal{S}x, \mathcal{S}y) \leq k\mathcal{R}(x, y) \tag{22}$$

for all $x \in \mathcal{O}$ and $y \in \mathcal{H}$. For $x_0 \in \mathcal{O}$, take $x_w = \mathcal{S}^w x_0$. Suppose that

$$\lim_{\flat\to\infty} \sup_{m\geq 1} \varpi(x_{\flat+1}, x_m)\frac{\varpi(x_{\flat+1}, x_{\flat+2}) + \varpi(x_{\flat+2}, x_{\flat+3})}{\varpi(x_\flat, x_{\flat+1}) + \varpi(x_{\flat+1}, x_{\flat+2})} < 1.$$

We assume that, for $x \in \mathcal{M}$, we have $\lim_{w\to+\infty}\varpi(x_w,x)$, $\lim_{w\to+\infty}\varpi(x,x_w)$ and $\lim_{w,m\to+\infty}\varpi(x_w,x_m)$ exist and are finite $\forall w,m\in\mathbb{N}$ and $w\neq m$. Then $\mathscr{S}$ has a unique fixed point in $\mathscr{O}\cap\mathscr{H}$.

Proof. Fix, x(say) x_0 in $\mathscr{O}\cup\mathscr{H}$. Define the iterative sequence $x_n = \mathscr{S}^w x_0$. Thus for $x_0\in\mathscr{O}\cap\mathscr{H}$, by hypotheses we have, $\mathcal{R}(x_1,x_2)\leq k\mathcal{R}(x_0,x_1)$.

By induction we can obtain that $\mathcal{R}(x_w,x_{w+1})\leq k^n\mathcal{R}(x_0,x_1)$ for all $w\geq 0$. Then the above equation yields that $\{\mathscr{S}^w(x)\}$ is a Cauchy sequence. Consequently $\{\mathscr{S}^w(x)\}$ converges to some point $v\in\mathcal{R}$. However in view of Eq. (22) an infinite number of terms of the sequence $\{\mathscr{S}^w(x)\}$ lie in $\mathscr{O}$ and an infinite number of terms lie in $\mathscr{H}$. Therefore $v\in\mathscr{O}\cap\mathscr{H}$, so, $\mathscr{O}\cap\mathscr{H}\neq\emptyset$.

Since $\mathscr{S}$ is cyclic, $\mathscr{S}(\mathscr{O})\subseteq\mathscr{H}$ and $\mathscr{S}(\mathscr{H})\subseteq\mathscr{O}$ leads to $\mathscr{S}:\mathscr{O}\cap\mathscr{H}\to\mathscr{O}\cap\mathscr{H}$ and Eq. (22) implies that $\mathscr{S}$ restricted to $\mathscr{O}\cap\mathscr{H}$ is a contraction mapping. Since Banach contraction mapping principle applies to $\mathscr{S}$ on $\mathscr{O}\cap\mathscr{H}$. By following same pattern in Theorem 4, we can easily prove that $\mathscr{S}$ has a unique fixed point in $\mathscr{O}\cap\mathscr{H}$.

Remark 1. Theorem 4 has the fascinating quality that continuity of $\mathscr{S}$ is no longer required.

To demonstrate that discontinuous maps can meet every requirement of Theorem 4, we used the following example.

Example 2. Let $\mathcal{M}=\mathbb{N}\cup\{\infty\}$ and let $\mathcal{R}:\mathcal{M}\times\mathcal{M}\to[0,\infty)$ be defined by $\mathcal{R}(x,y)=\left|\frac{1}{x}-\frac{1}{y}\right|^2$ and $\varpi:\mathcal{M}\times\mathcal{M}\to[1,\infty)$ by $\varpi(x,y)=|x|+|y|+3$. Then $(\mathcal{M},\mathcal{R})$ is a CRMS.

Let $\mathscr{S}:\mathcal{M}\to\mathcal{M}$ defined by

$$\mathscr{S}x=\begin{cases}4x, & \text{if } x\in\mathbb{N}\\ \infty, & \text{if } x=\infty.\end{cases}$$

Let $\mathscr{O}=\mathbb{N}\cup\{\infty\}$ and $\mathscr{H}=\{4x: x\in\mathbb{N}\}\cup\{\infty\}$ then $\mathcal{M}=\mathscr{O}\cup\mathscr{H}$ and $\mathscr{S}(\mathscr{O})\subseteq\mathscr{H}$ and $\mathscr{S}(\mathscr{H})\subseteq\mathscr{O}$. One can easily prove that $\mathcal{R}(\mathscr{S}x,\mathscr{S}y)\leq k\mathcal{R}(x,y)$ for all $x\in\mathscr{O}$ and $y\in\mathscr{H}$.

Thus the discontinuous mapping $\mathscr{S}$ satisfies all the conditions of Theorem 4.

Connecting fixed point methods to piecewise Caputo–Fabrizio derivative

Several classifications of differential equations have been seen in recent years. Analytical and numerical approaches have been developed by many authors to address complex problems. These problems have been successfully used to model a number of real-world concerns. We noted that the fractional differential operators are utilized to address a huge variety of real-world challenges better progressively.

During recent years, a variety of approaches have been proposed to illustrate the behaviours of some complicated global issues that have emerged in a range of scientific domains. The idea of piecewise equations (PEs) of under fractional order derivative was indeed newly proposed by the authors of [19] with the purpose of modelling nonlinear behaviour of real-world issues.

Thus, under the conceptual theory of piecewise equations with Caputo fractional derivative (shortly, $\mathscr{CFD}$) as,

$$ {}^{\mathscr{PCF}}\mathfrak{D}_\omega^\chi\Theta(\omega)=\Theta(\omega,\Theta(\omega),{}^{\mathscr{PCF}}\mathfrak{D}_\omega^\chi\Theta(\omega)),\quad \omega\in[0,\eta],\ \text{where}\quad \eta>0 \tag{23}$$

$$\Theta(0)=\Theta_0+\alpha(\omega),\Theta_0\in\mathbb{R},$$

such that $\chi\in(0,1]$, $\alpha\in\mathbb{C}([0,\eta])$ and $\Theta:[0,\eta]\times\mathbb{R}\times\mathbb{R}\to\times\mathbb{R}$. Here we prove the existence and unique solution for the above-stated general Cauchy nonlocal implicit problems while keeping in mind the importance and usefulness. The term $\mathscr{PCF}$ refers to piecewise $\mathscr{CFD}$, which uses a non-singular exponential kind kernel to depict the power law singular kernel. Piecewise versions of fractional Calculus is further described in [20]–[21], which we recommend for readers.

Definition 4. The piecewise integral of fractional order $\chi\in(0,1]$ is formulated as having if ϱ is a continuous function:

$$ {}^{\mathscr{PCF}}\mathfrak{I}_\omega^\chi\varrho(\omega)=\begin{cases}\int_0^{\omega_1}\varrho(p)dp, & \text{if}\quad \omega\in[0,\omega_1]\\ \frac{1-\chi}{\mathfrak{CF}(\chi)}\varrho(\omega)+\frac{\chi}{\mathfrak{CF}(\chi)}\int_{\omega_1}^{\omega}\varrho(p)dp, & \text{if}\quad \omega\in[\omega_1,\eta],\end{cases}$$

where the normalizing function is $\mathfrak{CF}(\chi)$.

Definition 5. Considering that ϱ is a continuous function, the piecewise derivative with a classical and exponential decay kernel and fractional order $\varrho\in(0,1]$ is expressed in terms:

$$ {}^{\mathscr{PCF}}\mathfrak{D}_\omega^\chi\varrho(\omega)=\begin{cases}\frac{d\varrho}{d\omega}, & \text{if}\quad \omega\in[0,\omega_1]\\ {}^{\mathscr{CF}}\mathfrak{D}_\omega^\chi\varrho(\omega), & \text{if}\quad \omega\in[\omega_1,\eta],\end{cases}$$

here, ${}^{\mathscr{CF}}\mathfrak{D}_\omega^\chi$ exhibit $\mathscr{CFD}$, for $\omega\in[\omega_1,\eta]$ that has always been described like,

$$ {}^{\mathscr{CF}}\mathfrak{D}_\omega^\chi\varrho(\omega)=\frac{\mathscr{CF}(\chi)}{1-\chi}\int_0^\omega\exp\left(\frac{-\chi(\omega-p)}{1-\chi}\right)\varrho'(p)dp,\quad \omega\geq 0.$$

Lemma 1. *If $\mho$ is indeed a continuous function, then the piecewise equation-based $\mathscr{CFD}$ solution to the specific problem will follow:*

$$ {}^{\mathscr{PCF}}\mathfrak{D}_\omega^\chi\varrho(\omega)=\mho(\omega),\quad \chi\in(0,1],$$

provided by,

$$\varrho(\omega)=\begin{cases}\varrho(0)+\int_0^{\omega_1}\mho(p)dp, & \text{if}\quad \omega\in[0,\omega_1]\\ \varrho(\omega_1)+\frac{1-\chi}{\mathfrak{CF}(\chi)}\mho(\omega)+\frac{\chi}{\mathfrak{CF}(\chi)}\int_{\omega_1}^{\omega}\mho(p)dp, & \text{if}\quad \omega\in[\omega_1,\eta].\end{cases}$$

Then the CRMS is defined as, $\mathcal{M}=\{\sigma:[0,\eta]\to\mathbb{R}/\sigma\in\mathbb{C}([0,\omega_1]\cup[\omega_1,\eta])\}$ endowed with a norm:

$$\mathcal{R}(\sigma_1(\omega),\sigma_2(\omega))=\|\sigma_1(\omega)-\sigma_2(\omega)\|^2 = \sup_{\omega_1\in[0,\eta]}|\sigma_1(\omega)-\sigma_2(\omega)|^2.$$

Remark 2. By utilizing above Lemma, the solution of the problem with piecewise linear equation

$$ {}^{\mathscr{PCF}}\mathfrak{D}_\omega^\chi\sigma(\omega)=\mho(\omega)\quad \chi\in[0,1], \tag{24}$$
$$\sigma(0)=\sigma_0+\alpha(\sigma).$$

which can be deduced as,

$$\sigma(\omega)=\begin{cases}\sigma_0+\alpha(\sigma)+\int_0^{\omega_1}\mho(p)dp, & \text{where}\quad \omega\in[0,\omega_1]\\ \sigma(\omega_1)+\frac{1-\chi}{\mathfrak{CF}(\chi)}\mho(\omega)+\frac{\chi}{\mathfrak{CF}(\chi)}\int_{\omega_1}^{\omega}\mho(p)dp, & \text{where}\quad \omega\in[\omega_1,\eta].\end{cases}$$

Proof. When both sides of Eq. (24) are subjected to the piecewise integral, we have

$$\sigma(\omega)=\begin{cases}\sigma(0)+\int_0^{\omega_1}\mho(p)dp, & \text{where}\quad \omega\in[0,\omega_1]\\ \sigma(\omega_1)+\frac{1-\chi}{\mathfrak{CF}(\chi)}\mho(\omega)+\frac{\chi}{\mathfrak{CF}(\chi)}\int_{\omega_1}^{\omega}\mho(p)dp, & \text{where}\quad \omega\in[\omega_1,\eta].\end{cases} \tag{25}$$

By taking $\sigma(0)=\sigma_0+\alpha(\sigma)$ in Eq. (24), we have

$$\sigma(\omega)=\begin{cases}\sigma_0+\alpha(\sigma)+\int_0^{\omega_1}\mho(p)dp, & \text{where}\quad \omega\in[0,\omega_1]\\ \sigma(\omega_1)+\frac{1-\chi}{\mathfrak{CF}(\chi)}\mho(\omega)+\frac{\chi}{\mathfrak{CF}(\chi)}\int_{\omega_1}^{\omega}\mho(p)dp, & \text{where}\quad \omega\in[\omega_1,\eta].\end{cases}$$

Corollary 1. *The solution to the desired problem Eq. (23) is provided by Remark 2.*

$$\sigma(\omega)=\begin{cases}\sigma_0+\alpha(\sigma)+\int_0^{\sigma_1}\Theta(p,\sigma(p),{}^{\mathscr{PCF}}\mathfrak{D}_p^\chi\sigma(p))dp, & \omega\in[0,\omega_1];\\ \sigma(\omega_1)+\frac{1-\chi}{\mathfrak{CF}(\chi)}\Theta(\omega,\sigma(\omega),{}^{\mathscr{PCF}}\mathfrak{D}_\omega^\chi\sigma(\omega)) & \\ +\frac{\chi}{\mathfrak{CF}(\chi)}\int_{\omega_1}^{\omega}\Theta(p,\sigma(p),{}^{\mathscr{PCF}}\mathfrak{D}_p^\chi\sigma(p))dp, & \omega\in[\omega_1,\eta].\end{cases}$$

Theorem 5. *Let $\mathcal{M} = \{\sigma : [0,\eta] \to \mathbb{R}/\sigma \in \mathbb{C}([0,\omega_1] \cup [\omega_1,\eta])\}$ and $\mathbb{R} : \mathcal{M}\times\mathcal{M} \to [0,\infty)$ be given by $\mathbb{R}(\sigma_1(\omega),\sigma_2(\omega)) = \|(\sigma_1(\omega)-\sigma_2(\omega))^2\| = \sup_{\sigma\in[0,\eta]}|\sigma_1(\omega)-\sigma_2(\omega)|^2$ with $\varpi(\sigma_1(\omega),\sigma_2(\omega)) = |\sigma_1(\omega)| + |\sigma_2(\omega)| + 3$, where $\varpi : \mathcal{M}\times\mathcal{M} \to [1,\infty)$. Clearly $(\mathcal{M},\mathcal{R})$ is a complete CRMS.*

$\mathscr{O} = \mathscr{H} = \mathcal{M}$. It is obvious that $\mathscr{O}$ and $\mathscr{H}$ are closed subsets of $(\mathcal{M},\mathcal{R})$. Define $\mathscr{S} : \mathscr{O}\cup\mathscr{H} \to \mathscr{O}\cup\mathscr{H}$ by

$$\mathscr{S}\sigma(\omega) = \begin{cases} \sigma_0 + \alpha(\sigma) + \int_0^{\sigma_1} \Theta(p,\sigma(p), {}^{\mathscr{PCF}}\mathfrak{D}_p^\chi\sigma(p))dp, & \omega\in[0,\omega_1];\\ \sigma(\omega_1) + \frac{1-\chi}{\mathfrak{CF}(\chi)}\Theta(\omega,\sigma(\omega),{}^{\mathscr{PCF}}\mathfrak{D}_\omega^\chi\sigma(\omega)) \\ \quad + \frac{\chi}{\mathfrak{CF}(\chi)}\int_{\omega_1}^{\omega}\Theta(p,\sigma(p),{}^{\mathscr{PCF}}\mathfrak{D}_p^\chi\sigma(p))dp, & \omega\in[\omega_1,\eta]. \end{cases} \tag{26}$$

Clearly, $\mathfrak{S}(\mathscr{O}) \subset \mathscr{H}$ and $\mathfrak{S}(\mathscr{H}) \subset \mathscr{O}$. Thus $\mathfrak{S}$ is a cyclic map on $\mathscr{O}\cup\mathscr{H}$.

The accompanying premise must exist for this investigation of being valid:

($\mathcal{A}_1$) For every $\sigma,\xi\in\mathcal{M}$ and element $\mathscr{C}_\alpha > 0$, we have

$$|\alpha(\sigma)-\alpha(\xi)| \le \mathscr{C}_\alpha|\sigma-\xi|;$$

($\mathcal{A}_2$) For every $\sigma,\xi,\overline{\sigma},\overline{\xi}\in\mathcal{M}$ and the elements $\mathscr{L}_\Theta > 0$ and $0 < \mathscr{M}_\Theta < 1$,

$$|\Theta(\omega,\sigma,\xi) - \Theta(\omega,\overline{\sigma},\overline{\xi})| \le \mathscr{L}_\Theta|\sigma-\overline{\sigma}| + \mathscr{M}_\Theta|\xi-\overline{\xi}\ \|;$$

($\mathcal{A}_3$) For $\sigma_0\in\mathcal{M}$, take $\sigma_w = \mathscr{S}^w\sigma_0$. Suppose that

$$\lim_{\flat\to\infty}\sup_{m\ge1}\varpi(\sigma_{\flat+1},\sigma_m)\frac{\varpi(\sigma_{\flat+1},\sigma_{\flat+2})+\varpi(\sigma_{\flat+2},\sigma_{\flat+3})}{\varpi(\sigma_\flat,\sigma_{\flat+1})+\varpi(\sigma_{\flat+1},\sigma_{\flat+2})} < 1.$$

We assume that, for $\sigma\in\mathcal{M}$, we have $\lim_{w\to+\infty}\varpi(\sigma_w,\sigma), \lim_{w\to+\infty}\varpi(\sigma,\sigma_w)$ and $\lim_{w,m\to+\infty}\varpi(\sigma_w,\sigma_m)$ exist and are finite $\forall w,m\in\mathbb{N}$ and $w\ne m$.

($\mathcal{A}_4$) $\mathcal{K} = \max\left\{\left(\mathscr{C}_\alpha + \frac{\omega_1\mathscr{L}_\Theta}{1-\mathscr{M}_\Theta}\right)^2, \left(\frac{(1-\chi)\mathscr{L}_\Theta}{\mathfrak{CF}(\chi)(1-\mathscr{M}_\Theta)} + \frac{\chi\mathscr{L}_\Theta(\omega-\omega_1)}{\mathfrak{CF}(\chi)(1-\mathscr{M}_\Theta)}\right)^2\right\} < 1.$

In view of hypotheses, $\mathcal{A}_1 - \mathcal{A}_4$, our problem Eq. (23) has a unique solution.

Proof. Consider $\sigma,\overline{\sigma}\in\mathcal{M}$, then we have,

$$\begin{aligned} &|{}^{\mathscr{PCF}}\mathfrak{D}_\omega^\chi(\sigma(\omega)) - {}^{\mathscr{PCF}}\mathfrak{D}_\omega^\chi(\overline{\sigma}(\omega))|\\ &= |\Theta(\omega,\sigma(\omega),{}^{\mathscr{PCF}}\mathfrak{D}_\omega^\chi(\sigma(\omega))) - \Theta(\omega,\overline{\sigma}(\omega),{}^{\mathscr{PCF}}\mathfrak{D}_\omega^\chi(\overline{\sigma}(\omega)))|\\ &\le \mathscr{L}_\Theta|\sigma(\omega)-\overline{\sigma}(\omega)| + \mathscr{M}_\Theta|{}^{\mathscr{PCF}}\mathfrak{D}_\omega^\chi(\sigma(\omega)) - {}^{\mathscr{PCF}}\mathfrak{D}_\omega^\chi(\overline{\sigma}(\omega))| \end{aligned} \tag{27}$$

in light of this, we have Eq. (27)

$$|{}^{\mathscr{PCF}}\mathfrak{D}_\omega^\chi(\sigma(\omega)) - {}^{\mathscr{PCF}}\mathfrak{D}_\omega^\chi(\overline{\sigma}(\omega))| \le \frac{\mathscr{L}_\Theta}{1-\mathscr{M}_\Theta}|\sigma(\omega)-\overline{\sigma}(\omega)|. \tag{28}$$

As a result, we suppose $\sigma,\overline{\sigma}\in\mathcal{M}$, and using (28),

$$|\mathscr{S}\sigma - \mathscr{S}\overline{\sigma}| = \begin{cases} |\alpha(\sigma)-\alpha(\overline{\sigma})| + \int_0^{\sigma_1}|\Theta(p,\sigma(p),{}^{\mathscr{PCF}}\mathfrak{D}_p^\chi\sigma(p)) \\ \quad - \Theta(p,\overline{\sigma}(p),{}^{\mathscr{PCF}}\mathfrak{D}_p^\chi\overline{\sigma}(p))|dp, \quad \omega\in[0,\omega_1];\\ \frac{1-\chi}{\mathfrak{CF}(\chi)}|\Theta(\omega,\sigma(\omega),{}^{\mathscr{PCF}}\mathfrak{D}_\omega^\chi\sigma(\omega)) - \Theta(\omega,\overline{\sigma}(\omega),{}^{\mathscr{PCF}}\mathfrak{D}_\omega^\chi\overline{\sigma}(\omega))|\\ +\frac{\chi}{\mathfrak{CF}(\chi)}\int_{\omega_1}^{\omega}|\Theta(p,\sigma(p),{}^{\mathscr{PCF}}\mathfrak{D}_p^\chi\sigma(p)) - \Theta(p,\overline{\sigma}(p),\\ {}^{\mathscr{PCF}}\mathfrak{D}_p^\chi\overline{\sigma}(p))|dp, \quad \omega\in[\omega_1,\eta]. \end{cases} \tag{29}$$

Thus, Eq. (29) leads to,

$$|\mathscr{S}\sigma - \mathscr{S}\overline{\sigma}| = \begin{cases} \mathscr{C}_\alpha|\sigma-\overline{\sigma}| + \int_0^{\omega_1}\left(\mathscr{L}_\Theta|\sigma(p)-\overline{\sigma}(p)| + \mathscr{M}_\Theta|{}^{\mathscr{PCF}}\mathfrak{D}_p^\chi\sigma(p)\right.\\ \quad \left.- {}^{\mathscr{PCF}}\mathfrak{D}_p^\chi\overline{\sigma}(p)|\right)dp, \quad \omega\in[0,\omega_1];\\ \frac{1-\chi}{\mathfrak{CF}(\chi)}\left(\mathscr{L}_\Theta|\sigma(\omega)-\overline{\sigma}(\omega)| + \mathscr{M}_\Theta|{}^{\mathscr{PCF}}\mathfrak{D}_\omega^\chi\sigma(\omega) - {}^{\mathscr{PCF}}\mathfrak{D}_\omega^\chi\overline{\sigma}(\omega)|\right)\\ +\frac{\chi}{\mathfrak{CF}(\chi)}\int_{\omega_1}^{\omega}\left(\mathscr{L}_\Theta|\sigma(p)-\overline{\sigma}(p)| + \mathscr{M}_\Theta|{}^{\mathscr{PCF}}\mathfrak{D}_p^\chi\sigma(p)\right.\\ \quad\left. - {}^{\mathscr{PCF}}\mathfrak{D}_p^\chi\overline{\sigma}(p)|\right)dp, \quad \omega\in[\omega_1,\eta]. \end{cases} \tag{30}$$

If we simplify further, Eq. (30) provides

$$|\mathscr{S}\sigma - \mathscr{S}\overline{\sigma}| = \begin{cases} \mathscr{C}_\alpha|\sigma-\overline{\sigma}| + \int_0^{\omega_1}\left(\frac{\mathscr{L}_\Theta}{1-\mathscr{M}_\Theta}\right)|\sigma(p)-\overline{\sigma}(p)|dp, \quad \omega\in[0,\omega_1];\\ \frac{1-\chi}{\mathfrak{CF}(\chi)}\frac{\mathscr{L}_\Theta}{1-\mathscr{M}_\Theta}|\sigma(\omega)-\overline{\sigma}(\omega)| + \frac{\chi}{\mathfrak{CF}(\chi)}\frac{\mathscr{L}_\Theta}{1-\mathscr{M}_\Theta}\int_{\omega_1}^{\omega}|\sigma(p)-\overline{\sigma}(p)|dp,\\ \omega\in[\omega_1,\eta]. \end{cases} \tag{31}$$

$$\begin{aligned} &|\mathscr{S}\sigma(\omega) - \mathscr{S}\overline{\sigma}(\omega)|^2\\ &\le \begin{cases} (\mathscr{C}_\alpha)^2|\sigma-\overline{\sigma}|^2 + \left(\int_0^{\omega_1}\left(\frac{\mathscr{L}_\Theta}{1-\mathscr{M}_\Theta}\right)|\sigma(p)-\overline{\sigma}(p)|dp\right)^2\\ +2\mathscr{C}_\alpha\left(\frac{\mathscr{L}_\Theta}{1-\mathscr{M}_\Theta}\right)|\sigma(p)-\overline{\sigma}(p)|^2\int_0^{\omega_1}dp, \omega\in[0,\omega_1];\\ \left(\frac{1-\chi}{\mathfrak{CF}(\chi)}\frac{\mathscr{L}_\Theta}{1-\mathscr{M}_\Theta}\right)^2|\sigma-\overline{\sigma}|^2 + \left(\frac{\chi}{\mathfrak{CF}(\chi)}\frac{\mathscr{L}_\Theta}{1-\mathscr{M}_\Theta}\right)^2\\ \quad\left(\int_{\omega_1}^{\omega}|\sigma(p)-\overline{\sigma}(p)|dp\right)^2\\ +2\frac{\chi(1-\chi)}{(\mathfrak{CF}(\chi))^2}\left(\frac{\mathscr{L}_\Theta}{1-\mathscr{M}_\Theta}\right)^2|\sigma(p)-\overline{\sigma}(p)|^2\int_{\omega_1}^{\omega}dp, \quad \omega\in[\omega_1,\eta]. \end{cases}\\ &= \begin{cases} (\mathscr{C}_\alpha)^2|\sigma-\overline{\sigma}|^2 + \left(\frac{\mathscr{L}_\Theta}{1-\mathscr{M}_\Theta}\right)^2|\sigma(p)-\overline{\sigma}(p)|^2\\ +2\mathscr{C}_\alpha\left(\frac{\mathscr{L}_\Theta}{1-\mathscr{M}_\Theta}\right)|\sigma(p)-\overline{\sigma}(p)|^2\omega_1, \quad \omega\in[0,\omega_1];\\ \left(\frac{1-\chi}{\mathfrak{CF}(\chi)}\frac{\mathscr{L}_\Theta}{1-\mathscr{M}_\Theta}\right)^2|\sigma-\overline{\sigma}|^2 + \left(\frac{\chi}{\mathfrak{CF}(\chi)}\frac{\mathscr{L}_\Theta}{1-\mathscr{M}_\Theta}\right)^2\\ |\sigma(p)-\overline{\sigma}(p)|^2(\omega-\omega_1)^2\\ +2\frac{\chi(1-\chi)}{(\mathfrak{CF}(\chi))^2}\left(\frac{\mathscr{L}_\Theta}{1-\mathscr{M}_\Theta}\right)^2|\sigma(p)-\overline{\sigma}(p)|^2(\omega-\omega_1), \quad \omega\in[\omega_1,\eta]. \end{cases}\\ &= \begin{cases} \left((\mathscr{C}_\alpha)^2 + \frac{\mathscr{L}_\Theta^2\omega_1^2}{(1-\mathscr{M}_\Theta)^2} + \frac{2\mathscr{C}_\alpha\mathscr{L}_\Theta\omega_1}{1-\mathscr{M}_\Theta}\right)|\sigma(p)-\overline{\sigma}(p)|^2, \quad \omega\in[0,\omega_1];\\ \left(\frac{(1-\chi)^2}{(\mathfrak{CF}(\chi))^2}\frac{\mathscr{L}_\Theta^2}{(1-\mathscr{M}_\Theta)^2} + \frac{\chi^2\mathscr{L}_\Theta^2(\omega-\omega_1)^2}{(\mathfrak{CF}(\chi))^2(1-\mathscr{M}_\Theta)^2} + \frac{2\chi(1-\chi)\mathscr{L}_\Theta^2(\omega-\omega_1)}{(\mathfrak{CF}(\chi))^2(1-\mathscr{M}_\Theta)^2}\right)\\ \times|\sigma(p)-\overline{\sigma}(p)|^2, \ \omega\in[\omega_1,\eta]. \end{cases}\\ &= \begin{cases} \left(\mathscr{C}_\alpha + \frac{\mathscr{L}_\Theta\omega_1}{(1-\mathscr{M}_\Theta)}\right)^2|\sigma(p)-\overline{\sigma}(p)|^2, \quad \omega\in[0,\omega_1];\\ \left(\frac{(1-\chi)\mathscr{L}_\Theta}{(\mathfrak{CF}(\chi))(1-\mathscr{M}_\Theta)} + \frac{\chi\mathscr{L}_\Theta(\omega-\omega_1)}{\mathfrak{CF}(\chi)(1-\mathscr{M}_\Theta)}\right)^2|\sigma(p)-\overline{\sigma}(p)|^2, \ \omega\in[\omega_1,\eta]. \end{cases} \end{aligned}$$

Thus, by taking supremum on both sides,

$$\sup|\mathscr{S}\sigma(\omega) - \mathscr{S}\overline{\sigma}(\omega)|^2 \le \sup\begin{cases} \left(\mathscr{C}_\alpha + \frac{\mathscr{L}_\Theta\omega_1}{(1-\mathscr{M}_\Theta)}\right)^2|\sigma(p)-\overline{\sigma}(p)|^2, \quad \omega\in[0,\omega_1];\\ \left(\frac{(1-\chi)\mathscr{L}_\Theta}{(\mathfrak{CF}(\chi))(1-\mathscr{M}_\Theta)} + \frac{\chi\mathscr{L}_\Theta(\omega-\omega_1)}{\mathfrak{CF}(\chi)(1-\mathscr{M}_\Theta)}\right)^2|\sigma(p)-\overline{\sigma}(p)|^2, \quad \omega\in[\omega_1,\eta]. \end{cases}$$

Which implies,

$$\|\mathscr{S}\sigma(\omega) - \mathscr{S}\overline{\sigma}(\omega)\| \le \begin{cases} \left(\mathscr{C}_\alpha + \frac{\mathscr{L}_\Theta\omega_1}{(1-\mathscr{M}_\Theta)}\right)^2\|\sigma(p)-\overline{\sigma}(p)\|, \quad \omega\in[0,\omega_1];\\ \left(\frac{(1-\chi)\mathscr{L}_\Theta}{(\mathfrak{CF}(\chi))(1-\mathscr{M}_\Theta)} + \frac{\chi\mathscr{L}_\Theta(\omega-\omega_1)}{\mathfrak{CF}(\chi)(1-\mathscr{M}_\Theta)}\right)^2\|\sigma(p)-\overline{\sigma}(p)\|, \quad \omega\in[\omega_1,\eta]. \end{cases}$$

Hence the inequality can be written as,

$$\|\mathscr{S}\sigma(\omega) - \mathscr{S}\overline{\sigma}(\omega)\| \le \mathcal{K}\|\sigma(\omega)-\overline{\sigma}(\omega)\|.$$

Thus, all of Theorem 3 requirements have been met. Thus, $\mathscr{S}$ possesses a unique fixed point, indicating that the solution to the Eq. (23) is unique.

Connecting fixed point results to wth order multi-term fractional delay differential equation

Differential equations of any order can be used to model the memory and genetic characteristics of diverse materials and processes. In comparison to the conventional integer-order structures, fractional-order models appear to be more acceptable and practical. Hypothetical advancements in regularity, catastrophic dynamics, and mathematical analysis for fractional equations have increased significantly. Numerous real-world problems have computational methods that use multi-term fractional differential equations, which contain multiple fractional order differential operators. For the most significant characteristics of this category of mathematical analysis, the readers may refer to [22]–[23]. Additionally, the authors of [24] have demonstrated the existence and uniqueness of the solution and examined the findings in relation to four distinct categories of functional stability.

A model of an wth order multi-term fractional delay differential equation was indeed proposed by G.U. Rahaman et al. [25] as below:

$$\begin{cases} \Sigma_{\flat=1}^{w}\xi_\flat\ {}^{\mathfrak{C}}\mathfrak{D}^{\vartheta_\flat}\sigma(a) = \theta(a,\sigma(a),\sigma(\mu a)), \\ \vartheta_1 \in (w-1,w], \quad 0<\vartheta_\flat \le 1, \quad \flat = 2,3\ldots,w, \mu\in(0,1), \xi_\flat\in\mathbb{R}, \\ a\in\mathcal{G}=[0,1]; \\ \sigma(0)=0, \quad \frac{d^\ell\sigma(0)}{d\sigma^\ell}=0, \quad \sigma(1)=\Sigma_{\ell=1}^{w-2}\gamma_\ell\sigma(\tau_\ell), \quad \gamma_\ell\in\mathbb{R}, \text{ and} \\ \tau_\ell\in(0,1) \text{ here } \ell=1,2,3,\ldots,w-2. \end{cases} \tag{32}$$

In addition to presenting existing and unique solutions to the wth order multi-term fractional delay differential equation, the authors of [25] also focused on findings pertaining to different kinds of functional stability. For more related results see, [26] and [27].

Theorem 6. *The CRMS of all continuous functions defined on* $[0,1]$ *which is denoted by* $C([0,1],\mathbb{R})$ *and endowed with* $\mathcal{R}(\sigma_1(a),\sigma_2(a)) = \|(\sigma_1(a)-\sigma_2(a))^2\| = \sup\{|\sigma_1(a)-\sigma_2(a)|,\ a\in[0,1]\}$. *Let* $\varpi : C([0,1],\mathbb{R})\times C([0,1],\mathbb{R})\to[1,\infty)$ *defined by* $\varpi(\sigma_1(a),\sigma_2(a)) = |\sigma_1(a)+\sigma_2(a)+3|$, *where* $C([0,1],\mathbb{R}) = \{\sigma/\sigma : [0,1]\to\mathbb{R}\}$ *then* $(C([0,1],\mathbb{R}),\mathcal{R})$ *is a complete CRMS.*

Let $\mathscr{O}=\mathscr{H}=C([0,1],\mathbb{R})$. *It is clear that* $\mathscr{O}\&\mathscr{H}$ *are closed subsets of* $(C([0,1],\mathbb{R}),\mathcal{R})$.

Define $\mathscr{S} : \mathscr{O}\cup\mathscr{H}\to\mathscr{O}\cup\mathscr{H}$ *by*

$$\begin{aligned}\mathscr{S}\sigma(a) = \frac{a^{w-1}}{\varrho_1}\Bigg[& \Sigma_{\ell=1}^{w-2}\gamma_\ell\frac{1}{\xi_1\Gamma(\vartheta_1)}\int_0^{\tau_\ell}(\tau_\ell-r)^{\vartheta_1-1}\theta(r,\sigma(r),\sigma(\mu r))dr \\ & - \Sigma_{\ell=1}^{w-2}\Sigma_{\flat=2}^{w}\frac{\xi_\flat}{\xi_1}\gamma_\ell\frac{1}{\Gamma(\vartheta_1-\vartheta_i)}\int_0^{\tau_\ell}(\tau_\ell-r)^{\vartheta_1-\vartheta_\flat-1}\sigma(r)dr \\ & - \frac{1}{\xi_1\Gamma(\vartheta_1)}\int_0^1(1-r)^{\vartheta_1-1}\theta(r,\sigma(r),\sigma(\mu r))dr \\ & + \Sigma_{\flat=2}^{w}\frac{\xi_\flat}{\xi_1}\frac{1}{\Gamma(\vartheta_1-\vartheta_i)}\int_0^1(1-r)^{\vartheta_1-\vartheta_\flat-1}\sigma(r)dr\Bigg] \\ & + \frac{1}{\xi_1\Gamma(\vartheta_1)}\int_0^a(a-r)^{\vartheta_1-1}\theta(r,\sigma(r),\sigma(\mu r))dr \\ & + \Sigma_{\flat=2}^{w}\frac{\xi_\flat}{\xi_1}\frac{1}{\Gamma(\vartheta_1-\vartheta_i)}\int_0^a(a-r)^{\vartheta_1-\vartheta_\flat-1}\sigma(r)dr.\end{aligned} \tag{33}$$

Clearly, $\mathscr{S}(\mathscr{O})\subset\mathscr{H}$ *and* $\mathscr{S}(\mathscr{H})\subset\mathscr{O}$. *Thus* $\mathscr{S}$ *is a cyclic map on* $\mathscr{O}\cup\mathscr{H}$. *We need the following presumptions in order to demonstrate the existence and uniqueness outcomes for the problem (32):*

$(\mathcal{A}_1)$. *The function* $\theta : \mathcal{G}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *is continuous;*

$(\mathcal{A}_2)$. $|\theta(a,\sigma_1,\sigma_2)-\theta(a,\beta_1,\beta_2)| \le \Lambda_1|\sigma_1-\beta_1|+\Lambda_2|\sigma_2-\beta_2|$ *for each* $a\in\mathcal{G}$ *and* $\sigma_1,\sigma_2,\beta_1,\beta_2\in\mathbb{R}$ *where* $\Lambda_1,\Lambda_2>0$.

$(\mathcal{A}_3)$.

$$\begin{aligned}\Lambda = \Bigg[& \left(\Sigma_{\ell=1}^{w-2}(|\gamma_\ell|\tau_\ell^{\vartheta_1}+1+\varrho_1)\frac{1}{|\xi_1\|\varrho_1|\Gamma(\vartheta_1+1)}\Lambda_3\right)^2 \\ & + \left(\Sigma_{\ell=1}^{w-2}\Sigma_{\flat=2}^{w}\frac{|\xi_\flat|}{|\xi_1\|\varrho_1|\Gamma(\vartheta_1-\vartheta_\flat+1)}(|\gamma_\ell|\tau_\ell^{\vartheta_1-\vartheta_\flat}+1+|\varrho_1|)\right)^2 \\ & + 2\Bigg(\Sigma_{\flat=2}^{w}\Sigma_{\ell=1}^{w-2}\left(|\gamma_\ell|\tau_\ell^{\vartheta_1-\vartheta_\flat}\left((1+\varrho_1)(\tau_\ell^{\vartheta_\flat}+1)+|\gamma_\ell|\tau_\ell^{\vartheta_1}\right)+(\varrho_1+1)^2\right) \\ & \times\frac{|\xi_\flat\Lambda_3|}{|\xi_1|^2|\varrho_1|.^2\Gamma(\vartheta_1-\vartheta_\flat+1)\Gamma(\vartheta_1+1)}\Bigg)^2\Bigg] < 1.\end{aligned}$$

Then the w*th order multi-term fractional delay differential problem (32) has unique solution.*

Proof. $\sigma_1\in\mathcal{O}$ and $\sigma_2\in\mathcal{H}$, *i.e.,* $\sigma_1,\sigma_2\in C([0,1],\mathbb{R})$.

Consider, $|\mathscr{S}\sigma_1(a)-\mathscr{S}\sigma_2(a)|$

$$\begin{aligned}= \frac{|a^{w-1}|}{|\varrho_1|}\Bigg[& \Sigma_{\ell=1}^{w-2}|\gamma_\ell|\frac{1}{|\xi_1|\Gamma(\vartheta_1)}\int_0^{\tau_\ell}(\tau_\ell-r)^{\vartheta_1-1}|\theta(r,\sigma_1(r),\sigma_1(\mu r)) \\ & - \theta(r,\sigma_2(r),\sigma_2(\mu r))|dr \\ & + \Sigma_{\ell=1}^{w-2}\Sigma_{\flat=2}^{w}\frac{|\xi_\flat|}{|\xi_1|}|\gamma_\ell|\frac{1}{\Gamma(\vartheta_1-\vartheta_i)}\int_0^{\tau_\ell}(\tau_\ell-r)^{\vartheta_1-\vartheta_\flat-1}|\sigma_1(r)-\sigma_2(r)|dr \\ & + \frac{1}{|\xi_1|\Gamma(\vartheta_1)}\int_0^1(1-r)^{\vartheta_1-1}|\theta(r,\sigma_1(r),\sigma_1(\mu r))-\theta(r,\sigma_2(r),\sigma_2(\mu r))|dr \\ & + \Sigma_{\flat=2}^{w}\frac{|\xi_\flat|}{|\xi_1|}\frac{1}{\Gamma(\vartheta_1-\vartheta_i)}\int_0^1(1-r)^{\vartheta_1-\vartheta_\flat-1}|\sigma_1(r)-\sigma_2(r)|dr\Bigg] \\ & + \frac{1}{|\xi_1|\Gamma(\vartheta_1)}\int_0^a(a-r)^{\vartheta_1-1}|\theta(r,\sigma_1(r),\sigma_1(\mu r))-\theta(r,\sigma_2(r),\sigma_2(\mu r))|dr \\ & + \Sigma_{\flat=2}^{w}\frac{|\xi_\flat|}{|\xi_1|}\frac{1}{\Gamma(\vartheta_1-\vartheta_i)}\int_0^a(a-r)^{\vartheta_1-\vartheta_\flat-1}|\sigma_1(r)-\sigma_2(r)|dr\end{aligned} \tag{34}$$

$$\begin{aligned}= \frac{|a^{w-1}|}{|\varrho_1|}\Bigg[& \Sigma_{\ell=1}^{w-2}|\gamma_\ell|\frac{1}{|\xi_1|\Gamma(\vartheta_1)}\int_0^{\tau_\ell}(\tau_\ell-r)^{\vartheta_1-1}(\Lambda_1|\sigma_1-\sigma_2| \\ & + \Lambda_2|\sigma_1-\sigma_2|)dr \\ & + \Sigma_{\ell=1}^{w-2}\Sigma_{\flat=2}^{w}\frac{|\xi_\flat|}{|\xi_1|}|\gamma_\ell|\frac{1}{\Gamma(\vartheta_1-\vartheta_i)}\int_0^{\tau_\ell}(\tau_\ell-r)^{\vartheta_1-\vartheta_\flat-1}|\sigma_1-\sigma_2|dr \\ & + \frac{1}{|\xi_1|\Gamma(\vartheta_1)}\int_0^1(1-r)^{\vartheta_1-1}(\Lambda_1|\sigma_1-\sigma_2|+\Lambda_2|\sigma_1-\sigma_2|)dr \\ & + \Sigma_{\flat=2}^{w}\frac{|\xi_\flat|}{|\xi_1|}\frac{1}{\Gamma(\vartheta_1-\vartheta_i)}\int_0^1(1-r)^{\vartheta_1-\vartheta_\flat-1}|\sigma_1-\sigma_2|dr\Bigg] \\ & + \frac{1}{|\xi_1|\Gamma(\vartheta_1)}\int_0^a(a-r)^{\vartheta_1-1}(\Lambda_1|\sigma_1-\sigma_2|+\Lambda_2|\sigma_1-\sigma_2|)dr \\ & + \Sigma_{\flat=2}^{w}\frac{|\xi_\flat|}{|\xi_1|}\frac{1}{\Gamma(\vartheta_1-\vartheta_i)}\int_0^a(a-r)^{\vartheta_1-\vartheta_\flat-1}|\sigma_1-\sigma_2|dr.\end{aligned} \tag{35}$$

Since $\Lambda_3=\Lambda_1+\Lambda_2$, we get

$$\begin{aligned}|\mathscr{S}\sigma_1(a)-\mathscr{S}\sigma_2(a)| \le |\sigma_1-\sigma_2|\Bigg(\frac{1}{|\varrho_1|}\Bigg[& \Lambda_3\Sigma_{\ell=1}^{w-2}|\gamma_\ell|\frac{1}{|\xi_1|\Gamma(\vartheta_1)} \\ & \int_0^{\tau_\ell}(\tau_\ell-r)^{\vartheta_1-1}dr \\ & + \Sigma_{\ell=1}^{w-2}\Sigma_{\flat=2}^{w}\frac{|\xi_\flat|}{|\xi_1|}|\gamma_\ell|\frac{1}{\Gamma(\vartheta_1-\vartheta_i)}\int_0^{\tau_\ell}(\tau_\ell-r)^{\vartheta_1-\vartheta_\flat-1}dr \\ & + \frac{\Lambda_3}{|\xi_1|\Gamma(\vartheta_1)}\int_0^1(1-r)^{\vartheta_1-1}dr \\ & + \Sigma_{\flat=2}^{w}\frac{|\xi_\flat|}{|\xi_1|}\frac{1}{\Gamma(\vartheta_1-\vartheta_i)}\int_0^1(1-r)^{\vartheta_1-\vartheta_\flat-1}dr\Bigg] \\ & + \frac{\Lambda_3}{|\xi_1|\Gamma(\vartheta_1)}\int_0^a(a-r)^{\vartheta_1-1}dr \\ & + \Sigma_{\flat=2}^{w}\frac{|\xi_\flat|}{|\xi_1|}\frac{1}{\Gamma(\vartheta_1-\vartheta_i)}\int_0^a(a-r)^{\vartheta_1-\vartheta_\flat-1}dr\Bigg).\end{aligned} \tag{36}$$

The integral in Eq. (36) is evaluated, and the result is,

$$\begin{aligned}|\mathscr{S}\sigma_1(a)-\mathscr{S}\sigma_2(a)| \le |\sigma_1-\sigma_2|\Bigg[& \Sigma_{\ell=1}^{w-2}(|\gamma_\ell|\tau_\ell^{\vartheta_1}+1+\varrho_1) \\ & \frac{1}{|\xi_1\|\varrho_1|\Gamma(\vartheta_1+1)}\Lambda_3 \\ & + \Sigma_{\ell=1}^{w-2}\Sigma_{\flat=2}^{w}\frac{|\xi_\flat|}{|\xi_1\|\varrho_1|\Gamma(\vartheta_1-\vartheta_\flat+1)} \\ & (|\gamma_\ell|\tau_\ell^{\vartheta_1-\vartheta_\flat}+1+|\varrho_1|)\Bigg].\end{aligned} \tag{37}$$

Now, we get

$$
\begin{aligned}
&|\mathscr{S}\sigma_1(a)-\mathscr{S}\sigma_2(a)|^2\\
&\le |\sigma_1-\sigma_2|^2\Bigg[\left(\Sigma_{\ell=1}^{w-2}(|\gamma_\ell|\tau_\ell^{\vartheta_1}+1+\varrho_1)\frac{1}{|\xi_1\|\varrho_1|\Gamma(\vartheta_1+1)}\Lambda_3\right)^2\\
&+\left(\Sigma_{\ell=1}^{w-2}\Sigma_{\flat=2}^{w}\frac{|\xi_\flat|}{|\xi_1\|\varrho_1|\Gamma(\vartheta_1-\vartheta_\flat+1)}(|\gamma_\ell|\tau_\ell^{\vartheta_1-\vartheta_\flat}+1+|\varrho_1|)\right)^2\\
&+2\Sigma_{\ell=1}^{w-2}\Sigma_{\ell=1}^{w-2}\Sigma_{\flat=2}^{w}(|\gamma_\ell|\tau_\ell^{\vartheta_1}+1+\varrho_1)(|\gamma_\ell|\tau_\ell^{\vartheta_1-\vartheta_\flat}+1+|\varrho_1|)\\
&\times\frac{|\xi_\flat\Lambda_3|}{|\xi_1|^2|\varrho_1|^2\Gamma(\vartheta_1-\vartheta_\flat+1)\Gamma(\vartheta_1+1)}\Bigg]\\
&\le |\sigma_1-\sigma_2|^2\Bigg[\left(\Sigma_{\ell=1}^{w-2}(|\gamma_\ell|\tau_\ell^{\vartheta_1}+1+\varrho_1)\frac{1}{|\xi_1\|\varrho_1|\Gamma(\vartheta_1+1)}\Lambda_3\right)^2\\
&+\left(\Sigma_{\ell=1}^{w-2}\Sigma_{\flat=2}^{w}\frac{|\xi_\flat|}{|\xi_1\|\varrho_1|\Gamma(\vartheta_1-\vartheta_\flat+1)}(|\gamma_\ell|\tau_\ell^{\vartheta_1-\vartheta_\flat}+1+|\varrho_1|)\right)^2\\
&+2\Bigg(\Sigma_{\flat=2}^{w}\Sigma_{\ell=1}^{w-2}(|\gamma_\ell|\tau_\ell^{\vartheta_1}+1+\varrho_1)(|\gamma_\ell|\tau_\ell^{\vartheta_1-\vartheta_\flat}+1+|\varrho_1|)\\
&\times\frac{|\xi_\flat\Lambda_3|}{|\xi_1|^2|\varrho_1|^2\Gamma(\vartheta_1-\vartheta_\flat+1)\Gamma(\vartheta_1+1)}\Bigg)^2\Bigg]\\
&\le |\sigma_1-\sigma_2|^2\Bigg[\left(\Sigma_{\ell=1}^{w-2}(|\gamma_\ell|\tau_\ell^{\vartheta_1}+1+\varrho_1)\frac{1}{|\xi_1\|\varrho_1|\Gamma(\vartheta_1+1)}\Lambda_3\right)^2\\
&+\left(\Sigma_{\ell=1}^{w-2}\Sigma_{\flat=2}^{w}\frac{|\xi_\flat|}{|\xi_1\|\varrho_1|\Gamma(\vartheta_1-\vartheta_\flat+1)}(|\gamma_\ell|\tau_\ell^{\vartheta_1-\vartheta_\flat}+1+|\varrho_1|)\right)^2\\
&+2\Bigg(\Sigma_{\flat=2}^{w}\Sigma_{\ell=1}^{w-2}(|\gamma_\ell|^2\tau_\ell^{2\vartheta_1-\vartheta_\flat}+|\gamma_\ell|\tau_\ell^{\vartheta_1}+|\gamma_\ell|\tau_\ell^{\vartheta_1}|\varrho_1|\\
&+|\gamma_\ell|\tau_\ell^{\vartheta_1-\vartheta_\flat}+1+|\varrho_1|+\varrho_1|\gamma_\ell|\tau_\ell^{\vartheta_1-\vartheta_\flat}+\varrho_1+\varrho_1^2)\\
&\times\frac{|\xi_\flat\Lambda_3|}{|\xi_1|^2|\varrho_1|^2\Gamma(\vartheta_1-\vartheta_\flat+1)\Gamma(\vartheta_1+1)}\Bigg)^2\Bigg]\\
&\le |\sigma_1-\sigma_2|^2\Bigg[\left(\Sigma_{\ell=1}^{w-2}(|\gamma_\ell|\tau_\ell^{\vartheta_1}+1+\varrho_1)\frac{1}{|\xi_1\|\varrho_1|\Gamma(\vartheta_1+1)}\Lambda_3\right)^2\\
&+\left(\Sigma_{\ell=1}^{w-2}\Sigma_{\flat=2}^{w}\frac{|\xi_\flat|}{|\xi_1\|\varrho_1|\Gamma(\vartheta_1-\vartheta_\flat+1)}(|\gamma_\ell|\tau_\ell^{\vartheta_1-\vartheta_\flat}+1+|\varrho_1|)\right)^2\\
&+2\Bigg(\Sigma_{\flat=2}^{w}\Sigma_{\ell=1}^{w-2}\Big(|\gamma_\ell|\tau_\ell^{\vartheta_1-\vartheta_\flat}\Big(|\gamma_\ell|\tau_\ell^{\vartheta_1}+\tau_\ell^{\vartheta_\flat}+\tau_\ell^{\vartheta_\flat}\varrho_1+1+\varrho_1\Big)+(\varrho_1+1)^2\Big)\\
&\times\frac{|\xi_\flat\Lambda_3|}{|\xi_1|^2|\varrho_1|^2\Gamma(\vartheta_1-\vartheta_\flat+1)\Gamma(\vartheta_1+1)}\Bigg)^2\Bigg].
\end{aligned}
\tag{38}
$$

This can be reduced to,

$$
\begin{aligned}
|\mathscr{S}\sigma_1(a)-\mathscr{S}\sigma_2(a)|^2 \le |\sigma_1-\sigma_2|^2\Bigg[\Bigg(&\Sigma_{\ell=1}^{w-2}(|\gamma_\ell|\tau_\ell^{\vartheta_1}+1+\varrho_1)\\
&\frac{1}{|\xi_1\|\varrho_1|\Gamma(\vartheta_1+1)}\Lambda_3\Bigg)^2\\
&+\left(\Sigma_{\ell=1}^{w-2}\Sigma_{\flat=2}^{w}\frac{|\xi_\flat|}{|\xi_1\|\varrho_1|\Gamma(\vartheta_1-\vartheta_\flat+1)}\right.\\
&\left.(|\gamma_\ell|\tau_\ell^{\vartheta_1-\vartheta_\flat}+1+|\varrho_1|)\right)^2\\
&+2\Bigg(\Sigma_{\flat=2}^{w}\Sigma_{\ell=1}^{w-2}\Big(|\gamma_\ell|\tau_\ell^{\vartheta_1-\vartheta_\flat}\Big((1+\varrho_1)(\tau_\ell^{\vartheta_\flat}+1)\\
&+|\gamma_\ell|\tau_\ell^{\vartheta_1}\Big)+(\varrho_1+1)^2\Big)\\
&\times\frac{|\xi_\flat\Lambda_3|}{|\xi_1|^2|\varrho_1|^2\Gamma(\vartheta_1-\vartheta_\flat+1)\Gamma(\vartheta_1+1)}\Bigg)^2\Bigg]\\
&\le \Lambda|\sigma_1-\sigma_2|^2.
\end{aligned}
\tag{39}
$$

By taking supremum on both sides, we get

$$\|(\mathscr{S}\sigma_1(a)-\mathscr{S}\sigma_2(a))^2\| \le \Lambda\|(\sigma_1-\sigma_2)^2\|.$$

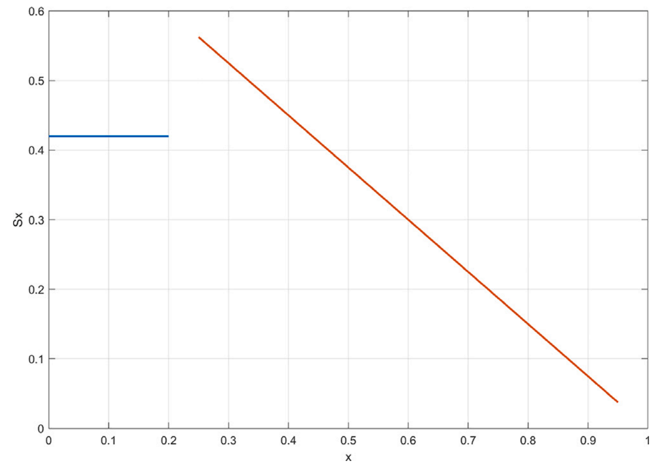

Fig. 1. Graphical representation of given discontinuous function.

Thus, all of Theorem 3 requirements have been met. Thus, $\mathscr{S}$ possesses a unique fixed point, indicating that the solution to the problem (32) is unique.

Numerical study

Example 3. Let $\mathcal{M} = [0,1]$. Define $\mathcal{R} : \mathcal{M}\times\mathcal{M} \to [0,\infty)$ by $\mathcal{R}(x,y) = |x-y|^2$ and $\varpi : \mathcal{M}\times\mathcal{M} \to [1,\infty)$ by $\varpi(x,y) = x+y+3$ for all $x,y\in\mathcal{M}$. Then $(\mathcal{M},\mathcal{R})$ is a complete CRMS.

Let $\mathscr{O} = [0,0.56]$ and $\mathscr{H} = [0.25,1]$. Clearly, $\mathscr{O}$ and $\mathscr{H}$ are closed subsets of $\mathcal{M} = [0,1]$.

Define $\mathscr{S} : \mathscr{O}\cup\mathscr{H} \to \mathscr{O}\cup\mathscr{H}$ by

$$\mathscr{S}x = \begin{cases} 0.42, & \text{if } 0 \le x \le 0.25\\ 0.75(1-x), & \text{if } 0.25 < x \le 1,\end{cases}$$

which is discontinuous as shown in Fig. 1.

First we will prove that $\mathscr{S}$ is a cyclic map. For this we need to prove $\mathscr{S}(\mathscr{O}) \subseteq \mathscr{H}$.

So, if $x = 0 \in \mathscr{O}$, then $\mathscr{S}x = \mathscr{S}0 = 0.42 \in \mathscr{H}$.

If $x = 0.56 \in \mathscr{O}$, then $\mathscr{S}x = \mathscr{S}0.56 = 0.75(1-0.56) = 0.33 \in \mathscr{H}$.

If $0 < x < 0.56$, suppose $0 < x \le 0.25$ then $\mathscr{S}x = 0.42 \in \mathscr{H}$. Suppose $0.25 < x < 0.56$, then

$$
\begin{aligned}
&-0.25 > -x > -0.56\\
&-0.56 < -x < -0.25\\
&\Rightarrow 1-0.56 < 1-x < 1-0.25\\
&\Rightarrow 0.33 < 0.75(1-x) < 0.56\\
&\Rightarrow 0.33 < \mathscr{S}x < 0.56\\
&\Rightarrow \mathscr{S}x \in (0.33,0.56) \in [0.25,1]
\end{aligned}
$$

which implies, $\mathscr{S}x \in \mathscr{H}$, thus $\mathscr{S}(\mathscr{O}) \subseteq \mathscr{H}$.

Now we will prove $\mathscr{S}(\mathscr{H}) \subseteq \mathscr{O}$.

If $x = 0.25$ then $\mathscr{S}x = \mathscr{S}0.25 = 0.42 \in \mathscr{O}$.

If $x = 1$ then $\mathscr{S}x = \mathscr{S}1 = 0 \in \mathscr{O}$.

If $0.25 < x < 1$, then $-0.25 > -x > -1$ which implies

$$
\begin{aligned}
&-1 < -x < -0.25\\
&\Rightarrow 0 < 1-x < 0.75\\
&\Rightarrow 0 < 0.75(1-x) < 0.56\\
&\Rightarrow 0 < \mathscr{S}x < 0.56\\
&\Rightarrow \mathscr{S}x \in (0,0.56) \in [0,0.56]
\end{aligned}
$$

which implies, $\mathscr{S}x \in \mathscr{O}$, thus $\mathscr{S}(\mathscr{H}) \subseteq \mathscr{O}$.

Hence $\mathscr{S}(\mathscr{O}) \subseteq \mathscr{H}$ and $\mathscr{S}(\mathscr{H}) \subseteq \mathscr{O}$. Therefore $\mathscr{S}$ is a cyclic map.

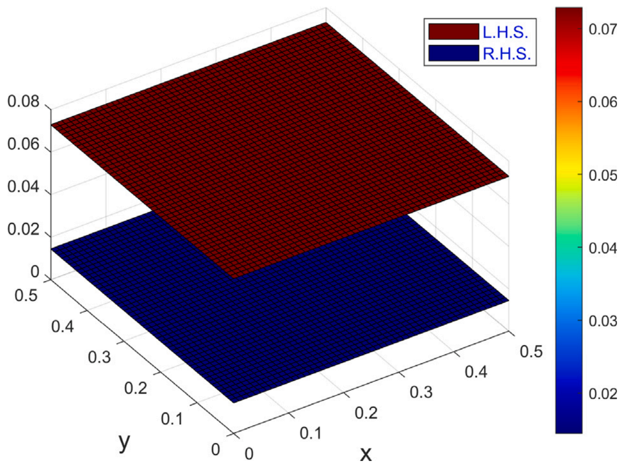

Fig. 2. The value of the comparison of the L.H.S. and the R.H.S. of Eq. (1) in Case-2.

Table 1
Numerical comparisons of L.H.S and R.H.S of Example 3 of subcase-I of Case-3.

x	y	$\mathcal{R}(\mathscr{S}^{2w}x, \mathscr{S}y)$	$k\mathcal{R}(\mathscr{S}^{2w-1}x, y)$
0	0.025	0	0.1404
0	0.075	0	0.1075
0	0.125	0	0.0783
0	0.175	0	0.0540
0	0.225	0	0.0342

Fix any $x \in [0, 0.56]$

Let $x = 0$, then $\mathscr{S}x = \mathscr{S}^2x = \mathscr{S}^3x = \cdots .\mathscr{S}^nx = 0.42$.

Therefore, $\mathscr{S}^{2w}x = \mathscr{S}^{2w-1}x = 0.42$.

Case:1 If $y = 0$, then $\mathscr{S}y = \mathscr{S}0 = 0.42$.

Consider,

$$\begin{aligned}\mathcal{R}(\mathscr{S}^{2w}x, \mathscr{S}y) &= 0\\ &\le k\mathcal{R}(\mathscr{S}^{2w-1}x, y) \quad \text{for} \quad k \in (0,1).\end{aligned}$$

Thus, $\mathcal{R}(\mathscr{S}^{2w}x, \mathscr{S}y) \le k\mathcal{R}(\mathscr{S}^{2w-1}x, y)$.

Case:2 If $y = 0.56$, then $\mathscr{S}y = \mathscr{S}0.56 = 0.75(1 - 0.5) = 0.33$.

Consider,

$$\begin{aligned}\mathcal{R}(\mathscr{S}^{2w}x, \mathscr{S}y) &= \mathcal{R}(0.42, 0.33) = 0.00081\\ &\le k\mathcal{R}(0.42, 0.56)\\ &= k\mathcal{R}(\mathscr{S}^{2w-1}x, y) \quad \text{for} \quad k = 0.9 \in (0,1).\end{aligned}$$

Thus, $\mathcal{R}(\mathscr{S}^{2w}x, \mathscr{S}y) \le k\mathcal{R}(\mathscr{S}^{2w-1}x, y)$ (see Fig. 2).

Case:3 If $0 < y < 0.56$.

Sub-case:I If $0 < y \le 0.25$, then $\mathscr{S}y = 0.42$.

Consider,

$$\begin{aligned}\mathcal{R}(\mathscr{S}^{2w}x, \mathscr{S}y) &= \mathcal{R}(0.42, 0.42) = 0\\ &\le k|0.42 - y|^2\\ &= k\mathcal{R}(\mathscr{S}^{2w-1}x, y) \quad \text{for} \quad k \in (0,1).\end{aligned}$$

Therefore, $\mathcal{R}(\mathscr{S}^{2w}x, \mathscr{S}y) \le k\mathcal{R}(\mathscr{S}^{2w-1}x, y)$ (See Tables 1–3)

Sub-case:II If $0.25 < y < 0.56$, then $\mathscr{S}y = 0.75(1 - y)$.

Table 2
Numerical comparisons of L.H.S and R.H.S of Example 3 of subcase-II of Case-3.

x	y	$\mathcal{R}(\mathscr{S}^{2w}x, \mathscr{S}y)$	$k\mathcal{R}(\mathscr{S}^{2w-1}x, y)$ with $k = 0.9$
0	0.062	0.080372	0.115348
0	0.186	0.03629	0.04928
0	0.31	0.009506	0.01089
0	0.434	0.00002	0.000176
0	0.558	0.007832	0.01714

Table 3
Picard iterations.

x_0	$x_0 = 0.08$	$x_0 = 0.16$	$x_0 = 0.26$	$x_0 = 0.32$
x_0	0.05	0.15	0.35	0.75
x_1	0.42	0.42	0.42	0.42
x_2	0.435	0.435	0.435	0.435
x_3	0.42375	0.42375	0.42375	0.42375
x_4	0.432188	0.432188	0.432188	0.432188
x_5	0.425859	0.425859	0.425859	0.425859
⋮	⋮	⋮	⋮	⋮
x_{42}	0.428571	0.428571	0.428571	0.428571
x_{43}	0.428571	0.428571	0.428571	0.428571
x_{44}	0.428571	0.428571	0.428571	0.428571
x_{45}	0.428571	0.428571	0.428571	0.428571

Consider,

$$\begin{aligned}\mathcal{R}(\mathscr{S}^{2w}x, \mathscr{S}y) &= \mathcal{R}(0.42, 0.75(1-y))\\ &= |0.42 - 0.75(1-y)|^2\\ &= |0.75y - 0.33|^2\\ &= 0.56|y - 0.44|^2\\ &\le 0.9|y - 0.42|^2\\ &= 0.9|0.42 - y|^2\\ &= k\mathcal{R}(\mathscr{S}^{2w-1}x, y) \quad \text{for} \quad k = 0.9 \in (0,1).\end{aligned}$$

Thus all the conditions of Theorem 1 satisfied, and 0.428571 is the unique fixed point of $\mathscr{S}$. In Figs. 6, 3, 4 and 5, we give some 2D graphs and 3D surfaces, which show the comparison of the left hand side (L.H.S.) and the right hand side (R.H.S.) of condition (1) by using MATLAB.

Next, we carry out some numerical and analytical experiments and for approximating the fixed point of $\mathscr{S}$ in Fig. 4. Furthermore, the convergence behaviour of these iterations is shown in Fig. 4.

Analytical version of Convergence behaviour: Let $\mathcal{A}(x) = x - 0.75(1 - x) = 0$.

Here, $x_{w+1} = 0.75(1 - x_w)$; $\quad w = 1, 2, 3, \ldots$.

Let $x_0 = 0.05$; $\quad x_1 = \mathscr{S}x_0 = \mathscr{S}(0.05) = 0.42$.

Then,

$$\begin{aligned}x_{w+1} &= \mathscr{S}(x_w)\\ \mathscr{S}(x) &= 0.75(1 - x)\\ \mathscr{S}'(x) &= -0.75.\end{aligned}$$

Let root of $\mathcal{A}(x)$ is 0.42857 (say r). Then $\mathscr{S}'(r) = \mathscr{S}'(0.42857) = -0.75$. Thus, $|\mathscr{S}'(r)| = |\mathscr{S}'(0.42857)| = |-0.75| = 0.75$, since $|\mathscr{S}'(r)| < 1$.

The iteration $x_{w+1} = \mathscr{S}x_n$ converges and it converges to 0.42857 which is the unique fixed point of $\mathscr{S}$. Moreover, differentiating $\mathscr{S}'(x)$ w.r.t x, thus $\mathscr{S}''(x) = 0$ which implies $\mathscr{S}''(r) = 0$. Hence the function Converges with order 3.

Example 4.

$$\mathfrak{P}\mathfrak{C}\mathfrak{F}\mathfrak{D}_x^{0.3}\sigma(\omega) = \frac{exp(-x)cos|\sigma(\omega)| + sin|^{\mathfrak{C}\mathfrak{F}}\mathfrak{D}_x^{0.3}\sigma(\omega)|}{70 + \omega^9}, \quad \omega \in [0, \eta];$$

$$\sigma(0) = 0.5 + \frac{cos|\sigma|}{100}.$$

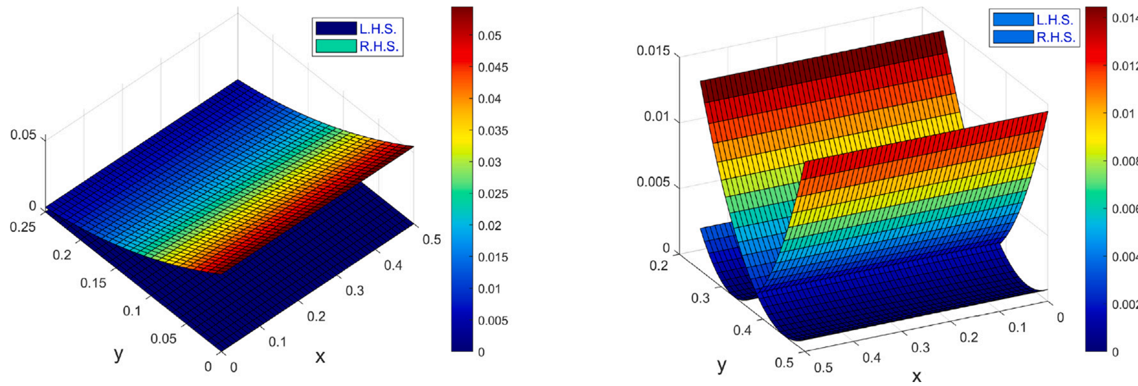

Fig. 3. The comparison of the value of L.H.S. and the R.H.S. of Eq. (1) in Case-3 of subcase-1 and Case-3 of subcase-2.

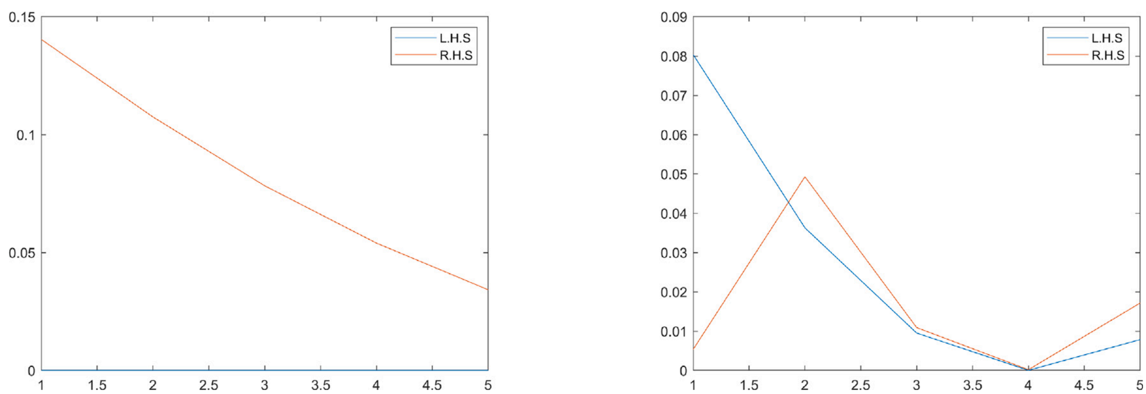

Fig. 4. The comparison of the value of L.H.S. and the R.H.S. of Eq. (1) in Case-3 of subcase-1 and Case-3 of subcase-2.

Solution: Taking $\eta = 1$ and $\omega_1 = 0.5$

$$|\Theta(\omega, \sigma(\omega), {}^{\mathfrak{PCF}}\mathfrak{D}_x^{0.3})\sigma(\omega)| = \left|\frac{|exp(-\omega)cos|\sigma(\omega)| + sin|{}^{\mathfrak{PCF}}\mathfrak{D}_x^{0.3}\sigma(\omega)|}{70+\omega^9}\right|$$
$$\leq \left|\frac{e^{-\omega}cos|\sigma(\omega)|}{70+\omega^9}\right| + \left|\frac{sin|{}^{\mathfrak{PCF}}\mathfrak{D}_x^{0.3}\sigma(\omega)|}{70+\omega^9}\right|$$

since $\omega \in [0,1]$;

$$\leq \frac{|\sigma(\omega)|}{70} + \frac{1}{70}\left|{}^{\mathfrak{PCF}}\mathfrak{D}_x^{0.3}\sigma(\omega)\right|.$$

Therefore, $\mathscr{L}_\Theta = \frac{1}{70}$; $\mathscr{M}_\Theta = \frac{1}{70}$; $\mathscr{C}_\alpha = \frac{1}{100}$.

It is easy, one can prove that by using hypotheses:

$$\|\mathscr{S}\sigma(\omega) - \mathscr{S}\overline{\sigma}(\omega)\| \leq \begin{cases} \left(\mathscr{C}_\alpha + \frac{\mathscr{L}_\Theta\omega_1}{(1-\mathscr{M}_\Theta)}\right)^2 |\sigma(p) - \overline{\sigma}(p)|^2, & \omega \in [0,\omega_1]; \\ \left(\frac{(1-\chi)\mathscr{L}_\Theta}{(\mathfrak{CF}(\chi))(1-\mathscr{M}_\Theta)} + \frac{\chi\mathscr{L}_\Theta(\omega-\omega_1)}{\mathfrak{CF}(\chi)(1-\mathscr{M}_\Theta)}\right)^2 |\sigma(p) - \overline{\sigma}(p)|^2, \\ \omega \in [\omega_1, \eta]. \end{cases}$$

Consider,

$$\mathcal{K} = \max\left\{\left(\mathscr{C}_\alpha + \frac{\omega_1\mathscr{L}_\Theta}{1-\mathscr{M}_\Theta}\right)^2, \left(\frac{(1-\chi)\mathscr{L}_\Theta}{\mathfrak{CF}(\chi)(1-\mathscr{M}_\Theta)} + \frac{\chi\mathscr{L}_\Theta(\omega-\omega_1)}{\mathfrak{CF}(\chi)(1-\mathscr{M}_\Theta)}\right)^2\right\}$$
$$= \max\{0.0002, 0.0001\}$$
$$= 0.0002 < 1. \tag{40}$$

Hence by using Theorem 5, we have unique solution. The unique solution of Theorem 5 is,

$$\sigma(\omega) = \begin{cases} 0.5 + \frac{cos|\sigma|}{100} + \int_0^{0.5} \frac{e^{-p}cos|\sigma(\omega)|+sin|{}^{\mathfrak{PCF}}\mathfrak{D}_p^{0.3}\sigma(p)|}{70+p^9}dp, & \omega \in [0,0.5] \\ \sigma(0.5) + \frac{0.7}{\mathfrak{CF}(0.3)}\frac{e^{-\omega}cos|\sigma(\omega)|+sin|{}^{\mathfrak{PCF}}\mathfrak{D}_\omega^{0.3}\sigma(\omega)|}{70+\omega^9} \\ +\frac{0.3}{\mathfrak{CF}(0.3)}\int_{0.5}^1 \frac{e^{-p}cos|\sigma(\omega)|+sin|{}^{\mathfrak{PCF}}\mathfrak{D}_\omega^{0.3}\sigma(p)|}{70+p^9}dp & \omega \in [0.5,1]. \end{cases}$$

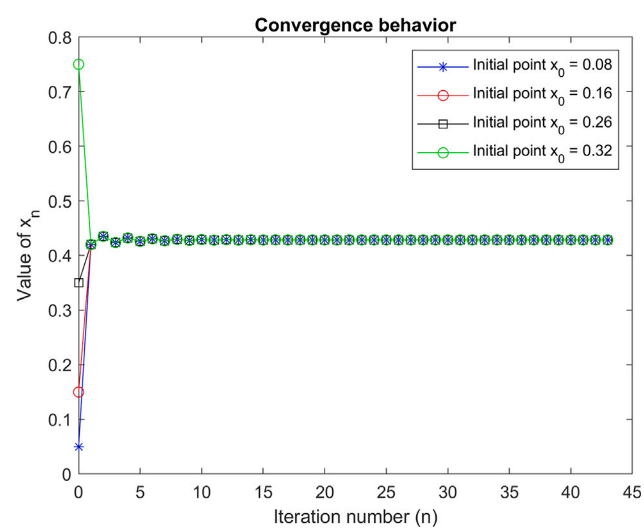

Fig. 5. Convergence behaviour for Example 3.

Numerical simulation via Cobra attractor

In this section we shall consider the well-known Cobra attractor:

$$\dot{x}(\omega) = a(y-x) + byz$$
$$\dot{y}(\omega) = cx + dxz^2$$
$$\dot{z}(\omega) = \gamma z + \epsilon|x|$$

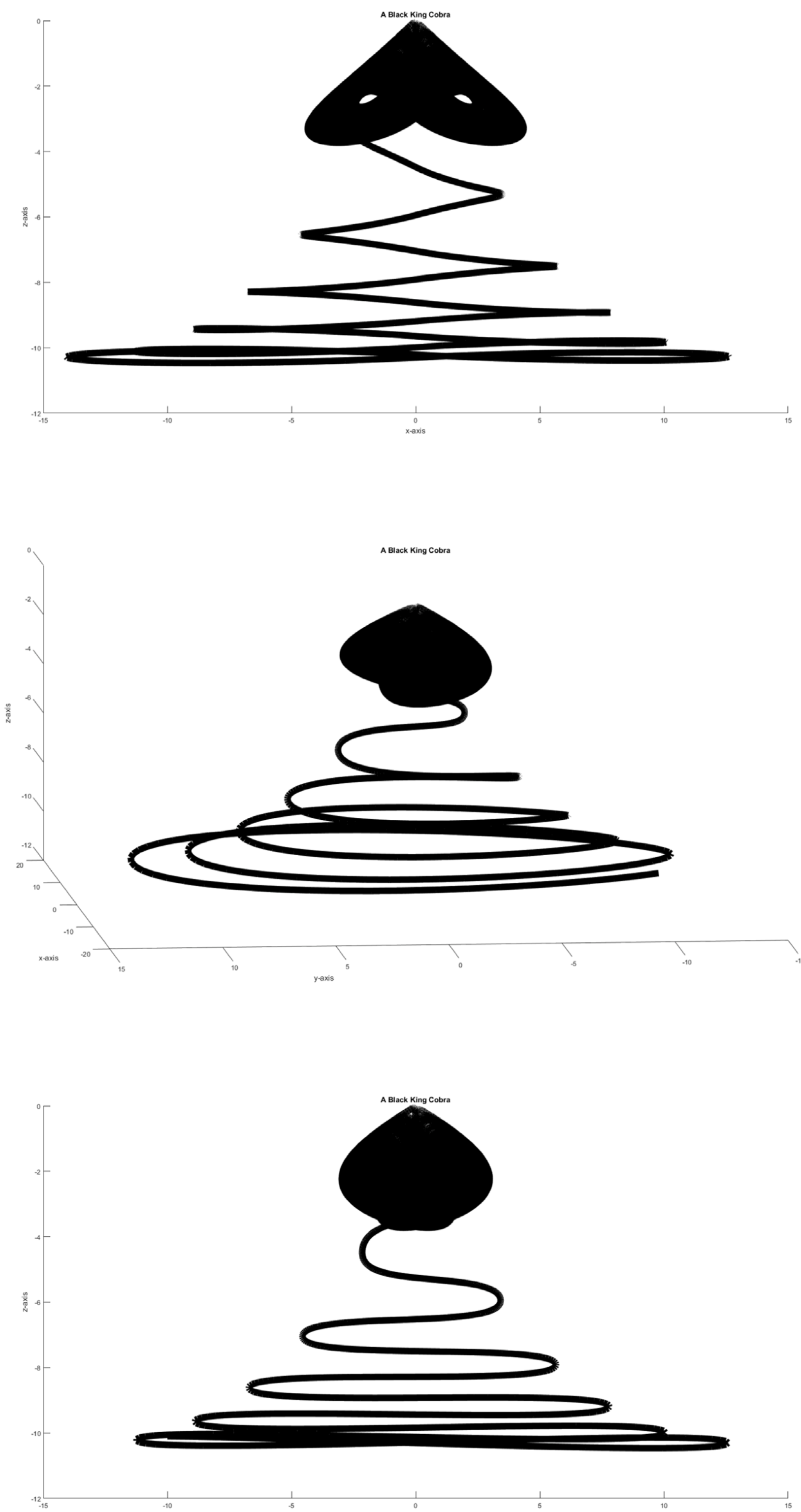

Fig. 6. The above represented King-Cobra attractors are in $x-y, x-z, x-y-z$ planes.

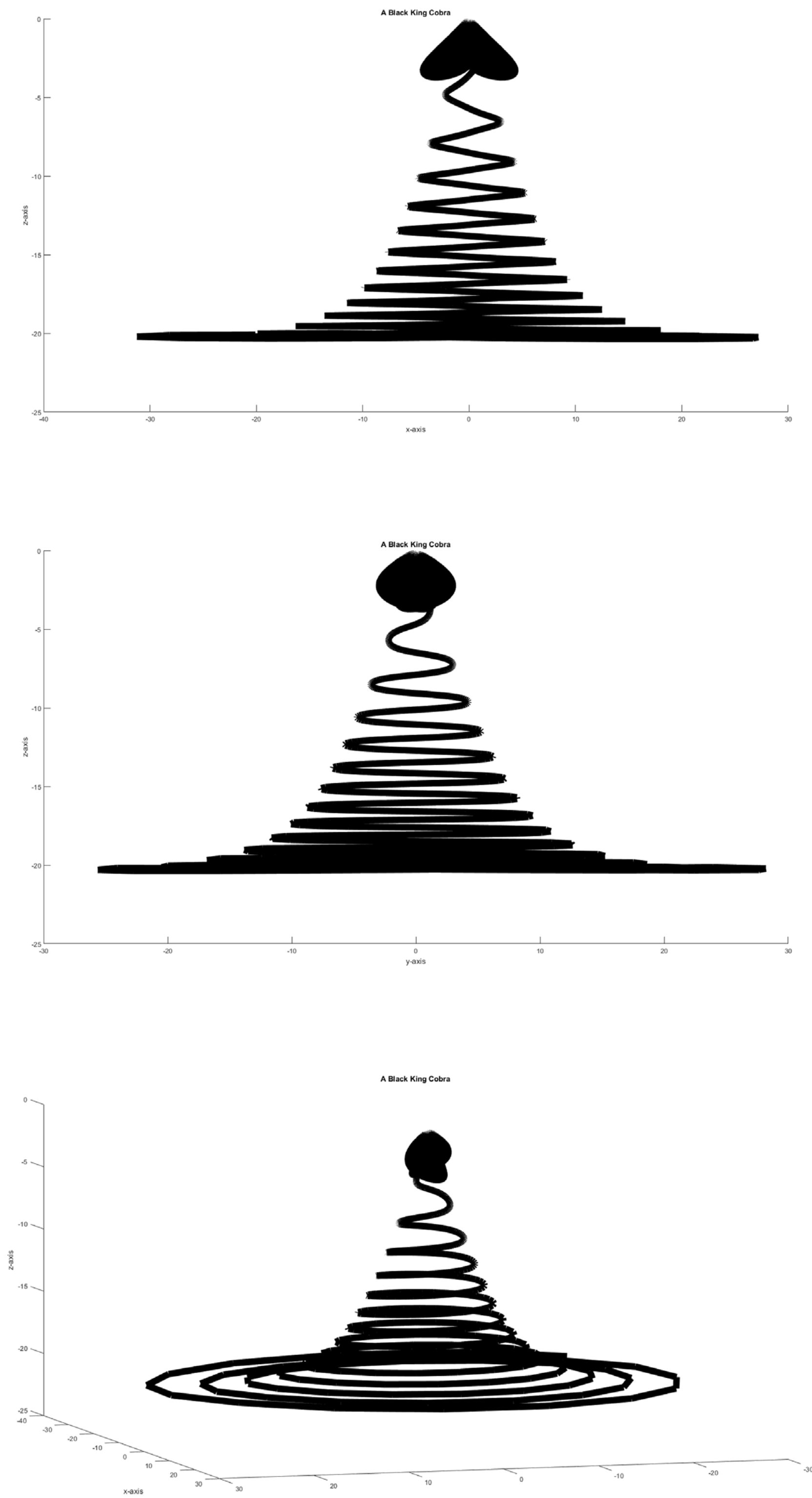

Fig. 6. (*continued*).

We now apply the suggested piecewise derivative:

$$\begin{cases} \dot{x}(\kappa) & = a(y-x)+byz, \quad \text{if } \kappa \in [0,\omega_1] \\ \dot{y}(\kappa) & = cx+dxz^2 \\ \dot{z}(\kappa) & = \gamma z+\epsilon|x| \\ {}_0^{\mathscr{C}\mathscr{F}}\mathfrak{D}_\kappa^\delta x(\kappa) & = a(y-x)+byz \\ {}_0^{\mathscr{C}\mathscr{F}}\mathfrak{D}_\kappa^\delta y(\kappa) & = cx+dxz^2 \quad \text{if } \kappa \in [\omega_1,\xi] \\ {}_0^{\mathscr{C}\mathscr{F}}\mathfrak{D}_\kappa^\delta z(\kappa) & = \gamma z+\epsilon|x|. \end{cases} \tag{41}$$

For simplicity, we let,

$$f_1(x,y,z,\kappa) = a(y-x)+byz$$
$$f_2(x,y,z,\kappa) = cx+dxz^2$$
$$f_3(x,y,z,\kappa) = \gamma z+\epsilon|x|$$

The functions $(f_i)_{i=1,2,3}$ satisfying the conditions under which the existence of the unique solution is achieved. Therefore due to its nonlinearity we shall rely on the numerical solution. using existing method for the classical case we have,

$$\begin{cases} x_{w+1} & = x_w+\frac{3}{2}hf_1(x_w,y_w,z_w,\kappa_w)-\frac{h}{2}f_1(x_{w-1},y_{w-1},z_{w-1},\kappa_{w-1}) \\ y_{w+1} & = y_w+\frac{3}{2}hf_2(x_w,y_w,z_w,\kappa_w)-\frac{h}{2}f_2(x_{w-1},y_{w-1},z_{w-1},\kappa_{w-1}) \\ z_{w+1} & = z_w+\frac{3}{2}hf_3(x_w,y_w,z_w,\kappa_w)-\frac{h}{2}f_3(x_{w-1},y_{w-1},z_{w-1},\kappa_{w-1}) \\ & \text{if } \kappa_w<\omega_1. \end{cases} \tag{42}$$

$$\begin{cases} x_{w+1} & = x_w+(1-\delta)[f_1(x_{w+1}^p,y_{w+1}^p,z_{w+1}^p,\kappa_{w+1})-f_1(x_w,y_w,z_w,\kappa_w)] \\ & +\delta[\frac{3}{2}hf_1(x_w,y_w,z_w,\kappa_w)-\frac{h}{2}f_1(x_{w-1},y_{w-1},z_{w-1},\kappa_{w-1})] \\ y_{w+1} & = y_w+(1-\delta)[f_2(x_{w+1}^p,y_{w+1}^p,z_{w+1}^p,\kappa_{w+1})-f_2(x_w,y_w,z_w,\kappa_w)] \\ & +\delta[\frac{3}{2}hf_2(x_w,y_w,z_w,\kappa_w)-\frac{h}{2}f_2(x_{w-1},y_{w-1},z_{w-1},\kappa_{w-1})] \\ z_{w+1} & = z_w+(1-\delta)[f_3(x_{w+1}^p,y_{w+1}^p,z_{w+1}^p,\kappa_{w+1})-f_3(x_w,y_w,z_w,\kappa_w)] \\ & +\delta[\frac{3}{2}hf_3(x_w,y_w,z_w,\kappa_w)-\frac{h}{2}f_3(x_{w-1},y_{w-1},z_{w-1},\kappa_{w-1})] \\ & \text{if } \kappa_w>T. \end{cases} \tag{43}$$

In the above $x_{w+1}^p, y_{w+1}^p, z_{w+1}^p$ are predictor of x_{w+1}, y_{w+1} and z_{w+1} respectively and are calculated as,

$$\begin{cases} x_{w+1}^p & = x_w+\Delta\kappa f_1(x_w,y_w,z_w,\kappa_w) \\ y_{w+1}^p & = y_w+\Delta\kappa f_2(x_w,y_w,z_w,\kappa_w) \\ z_{w+1}^p & = z_w+\Delta\kappa f_3(x_w,y_w,z_w,\kappa_w). \end{cases} \tag{44}$$

Thus the final scheme as given by,

$$\begin{aligned} x_{w+1} = x_w + (1-\delta)[f_1(x_w & \\ & + hf_1(x_w,y_w,z_w,\kappa_w), y_w+hf_2(x_w,y_w,z_w,\kappa_w), hf_3(x_w,y_w,z_w,\kappa_w)) \\ & - f_1(x_w,y_w,z_w,\kappa_w)] + \delta[\frac{3}{2}hf_1(x_w,y_w,z_w,\kappa_w) \\ & - \frac{h}{2}f_1(x_{w-1},y_{w-1},z_{w-1},\kappa_{w-1})] \end{aligned}$$

We shall present numerical simulation figures as shown below.

To perform this simulation the following parameters were used $c=5, d=-1, \gamma=-5, d=-1, \epsilon=-6$ for the figure with King-Cobra $x-y, x-z, x-y-z$, and the used initial conditions are $x(0)=-10.1, y(0)=-2, z(0)=-7.1$ for the appropriate King-kobra figure we used $x(0)=-20.1, y(0)=-10, z(0)=-20.1$.

Conclusion

The existence results for fractional differential equations has drawn the attention of many researchers. In the current work, we have discussed some new fixed point results for cyclic mapping in controlled rectangular metric spaces with application to the existence of solutions to the multi-term fractional delay differential equations with wth order and the piecewise equations under the setting of non-singular type derivative. Numerical simulations were carried out for various values of fractional orders; a system of nonlinear equations was presented and numerically solved.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

The author T. Abdeljawad would like to thank Prince Sultan University, Saudi Arabia for paying the APC and for the support through the TAS research lab.

References

[1] Kochubei AN, Luchko YF, Tarasov VE, Petras I. Handbook of fractional calculus with applications: Applications in Physics part A. ;: De Gruyter; 2019.
[2] Kochubei AN, Luchko YF, Tarasov VE, Petras I. Handbook of fractional calculus with applications: Applications in physics part B. De Gruyter; 2019.
[3] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and applications of fractional differential equations. Elsevier; 2006.
[4] Atangana A, Baleanu D. New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model. Therm Sci 2016;20(2):763–9. http://dx.doi.org/10.2298/TSCI160111018A.
[5] Caputo M, Fabrizio M. A new definition of fractional derivative without singular kernel. Prog Fract Differ Appl 2015;1(2):1–13.
[6] Panda SK, Ravichandran C, Hazarika B. Results on system of Atangana-Baleanu fractional order Willis aneurysm and nonlinear singularly perturbed boundary value problems. Chaos Solitons Fractals 2021;142:110390.
[7] Panda SK, Abdeljawad T, Ravichandran C. A complex valued approach to the solutions of Riemann–Liouville integral, Atangana-Baleanu integral operator and non-linear telegraph equation via fixed point method. Chaos, Solitons Fractals 2020;130:109439.
[8] Panda Sumati Kumari, Atangana Abdon, Abdeljawad Thabet. Existence results and numerical study on novel coronavirus 2019-Ncov/Sars-Cov-2 model using differential operators based on the generalized Mittag-Leffler kernel and fixed points. Fractals 2022;30(08):2240214.
[9] Das A, Rabbani M, Hazarika B, Panda SK. A fixed point theorem using condensing operators and its applications to Erdelyi–Kober bivariate fractional integral equations. Turkish J Math 2022;46(6):2513–29.
[10] Gao W, Baskonus HM. Deeper investigation of modified epidemiological computer virus model containing the Caputo operator. Chaos Solitons Fractals 2022;158:112050.
[11] Agilan P, Julietraja K, Mlaiki N, Mukheimer A. Intuitionistic fuzzy stability of an Euler–Lagrange symmetry additive functional equation via direct and Fixed Point Technique (FPT). Symmetry 2022;14:2454. http://dx.doi.org/10.3390/sym14112454.
[12] Ma Yong-Ki, Williams W Kavitha, Vijayakumar V, Nisar KS, Shukla A. Results on Atangana-Baleanu fractional semilinear neutral delay integro-differential systems in Banach space. J King Saud Univ-Sci 2022;34(6):102158.
[13] Panda SK. Applying fixed point methods and fractional operators in the modelling of novel coronavirus 2019-nCoV/SARS-CoV-2. Results Phys 2020;19:103433.
[14] Panda SK, Atangana A, Nieto JJ. New insights on novel coronavirus 2019-nCoV/SARS-CoV-2 modelling in the aspect of fractional derivatives and fixed points. Math Biosci Eng 2021;18(6):8683–726.
[15] Zubaira ST, Gopalana K, Abdeljawad T. Controlled b-Branciari metric type spaces and related fixed point theorems with applications. Filomat 2020;34(13):4253–69.
[16] Kirk WA, Srinavasan PS, Veeramani P. Fixed points for mapping satisfying cyclical contractive conditions. Fixed Point Theory 2003;4:79–89.
[17] Karpagam S, Agrawal S. Best proximity points theorems for cyclic Meir-Keeler contraction maps. Nonlinear Anal 2011;74:1040–6.
[18] Zamfirescu T. Fix point theorems in metric spaces. Arch Math (Basel) 1972;23:292–8.

[19] Atangana A, Araz SI. New concept in calculus: Piecewise differential and integral operators. Chaos Solitons Fractals 2021;145:110638. http://dx.doi.org/10.1016/j.chaos.2020.110638.

[20] Zhang T, Xiong L. Periodic motion for impulsive fractional functional differential equations with piecewise Caputo derivative. Appl Math Lett 2020;101:106072.

[21] Shah K, Abdeljawad T, Ali A. Mathematical analysis of the Cauchy type dynamical system under piecewise equations with Caputo fractional derivative. Chaos Solitons Fractals 2022;161:112356.

[22] Ahmad B, Alghamdi N, Alsaedi A, Ntouyas SK. Multi-term fractional differential equations with nonlocal boundary conditions. Open Math 2018;16(1):1519–36.

[23] Pasupathi Agilan, et al. Direct and fixed-point stability-instability of additive functional equation in Banach and quasi-beta normed spaces. Symmetry 2022;14(8):1700.

[24] Ahmad B, Alghamdi N, Alsaedi A, Ntouya SK. A system of coupled multi-term fractional differential equations with three-point coupled boundary conditions. Fract Calc Appl Anal 2019;22(3):601–18.

[25] Rahman GU, Agarwal RP, Ahmad D. Existence and stability analysis of *n*th order multi term fractional delay differential equation. Chaos Solitons Fractals 2022;155:111709.

[26] Sultan B, Azmi FM, Sultan M, Mahmood T, Mlaiki N, Souayah N. Boundedness of fractional integrals on grand weighted Herz-Morrey spaces with variable exponent. Fractal Fract 2022;6(11):660.

[27] Berhail A, Tabouche N, Alzabut J, et al. Using the Hilfer–Katugampola fractional derivative in initial-value Mathieu fractional differential equations with application to a particle in the plane. Adv Cont Discr Mod 2022;2022:44. http://dx.doi.org/10.1186/s13662-022-03716-6.