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Research article

Distance and similarity measures of intuitionistic fuzzy hypersoft sets with application: Evaluation of air pollution in cities based on air quality index

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Abstract: Decision-making in a vague, undetermined and imprecise environment has been a great issue in real-life problems. Many mathematical theories like fuzzy, intuitionistic and neutrosophic sets have been proposed to handle such kinds of environments. Intuitionistic fuzzy sets (IFSS) were formulated by Atanassov in 1986 and analyze the truth membership, which assists in evidence, along with the fictitious membership. This article describes a composition of the intuitionistic fuzzy set (IFS) with the hypersoft set, which assists in coping with multi-attributive decision-making issues. Similarity measures are the tools to determine the similarity index, which evaluates how similar two objects are. In this study, we develop some distance and similarity measures for IFHSS with the help of aggregate operators. Also, we prove some new results, theorems and axioms to check the validity of the proposed study and discuss a real-life problem. The air quality index (AQI) is one of the major factors of the environment which is affected by air pollution. Air pollution is one of the extensive worldwide problems, and now it is well acknowledged to be deleterious to human health. A decision-maker determines þ = region (different geographical areas) and the factors $\{\mathfrak{z} = human\ activiteis, Ч = humidity\ level, \zeta = air\ pollution\}$ which enhance the AQI by applying decision-making techniques. This analysis can be used to determine whether a geographical area has a good, moderate or hazardous AQI. The suggested technique may also be applied to a large number of the existing hypersoft sets. For a remarkable environment, alleviating techniques must be undertaken.

Keywords: fuzzy set; intuitionistic fuzzy set; soft set; hypersoft set; intuitionistic fuzzy hypersoft set; similarity measures; distance measures; air quality index; air pollution
Mathematics Subject Classification: 15B15, 90B50, 03B52, 03E72, 03E75

1. Introduction

Making decisions and problem-solving are the most complicated states in our life. So, we must identify the best of multiple choices to tackle these. In this account, multi-attribute decision making helps us to make a selection. However, it is possible to accumulate unreliable facts during decision-making. In different stages of life, decisions involve factors such as uncertainty, vagueness and unreliability in data, which are the most crucial components in tackling the complications. To pursue these issues, various mathematical theories have been introduced, like probability theory, fuzzy sets, soft sets, fuzzy soft sets, intuitionistic soft sets, etc. Fuzzy set theory (FS) was proposed by Zadeh [1] in 1965. After that, the interval value fuzzy set (IVFS) and others were compared by Lee et al. in [2]. It is a structural method for dealing with issues involving inconsistency, ambiguity and inaccuracy of assessments. In correspondence to probability theory, the fuzzy set theory suggested a unique transformation for analyzing available evidence and preferences in group decision-making. Moreover, Pappis [3] presented the applications of fuzzy set theory. In 1983, Atanassov [4] put forward the theory of the intuitionistic fuzzy set (IFS), which is an extension of FS. It was an alternative approach to sort out these uncertainties, vagueness and fuzziness and described the degree of satisfiability and non-satisfiability. However, it is worth noting that single membership and non-membership degrees did not deal precisely with these situations. Then, vague set (VS) theory was introduced by Liu et al. [5]. This theory was like the IFS, which is a generalization of FS. In a VS, interval-based membership is used instead of point-base membership, and it is more effective in capturing the vagueness of the data. IFS and VS are considered equivalent in the literature. In this way, IFS is isomorphic to VS. After that, Smarandache [6] proposed the concept of a neutrosophic set (NS), which is formed by adding indeterminacy in intuitionistic sets. A NS is used to rank the possibilities and helps us to select an appropriate alternative. It involves truthfulness, indeterminacy and falseness, which give us direction to overcome these imprecisions. Molodtsov, a Russian analyst, was the first to suggest soft set (SS) theory [7] in 1999. It is a parameterized subset of a universal set and a wide mathematical tool for managing uncertainty and ill-defined things which is hassle-free from the above complexities. He effectively applied soft set theory in a variety of areas involving smoothness of functions, game theory etc. In the past few years, there has been strong interest in the algebraic structure of soft set theory. Aktas and Cagman [8] presented the idea of soft matrices, which are the characterization of soft sets. They also linked soft sets with the theory of fuzzy and rough sets by illustrating the difference. Soft semi-rings were originated by Ali et al. [9] by applying soft set theory and discussing their properties. Zou and Xiao [10] demonstrated the data interpretation using a soft set under incomplete information. In 2001, Maji et al. [11] suggested the idea of the fuzzy soft set (FSS), which gave a more generalized conclusion and a combination of fuzzy set and soft set. Moreover, they discussed the applications of soft sets in analyzing problems and making the best decisions. Majumdar et al. [12] generalized the FSS by associating a degree with the parameters of FS. It was the most suitable concept, as it includes the uncertainty corresponding to each value of the parameter. The concept of an intuitionistic fuzzy soft set (IFSS) [13] is modified and reformulated as a composition of intuitionistic fuzzy and soft sets. It involves the parameters which reflect the validity of the data which is provided and helps us to choose the best among them. Additionally, using an intuitionistic set, many theories have been proposed, including similarity measure, distance measure and entropy measure; and their applications have been presented in medical diagnosis, HR selection and pattern recognition. Liang et al. [14] proposed the similarities measure by using IFS. De et al. [15] used the intuitionistic set for medical diagnosis. Ejegwa et al. [16] suggested career determination by using an intuitionistic set. Li, Deng-Feng [17] suggested the similarities measure and pattern recognition. Szmidt et al. [18] suggested group decision making using IFS. Wei et al. [19] gave an approach to entropy similarity measure using IFS. Jafar et al. [20] discussed the comprehensive study of the application of IFSM. Mitchell [21] discussed the similarity measure and its application to pattern recognition.

Then, Smarandache [22] enhanced the concept of a soft set and proposed a hypersoft set theory in 2018. It is a more generalized theory than the soft sets, tackles vagueness and assists us in making the best decision. Zulqarnain et al. [23] gave an inclusive study on the applications of intuitionistic hypersoft sets. Yolcu and Ozturk [24] presented the fuzzy hypersoft sets and their application for decision-making. Debnath [25] presented the fuzzy hypersoft sets and their weightage operator for decision making. Yolcu et al. [26] proposed intuitionistic fuzzy hypersoft sets (IFHSS). Zulqarnain et al. [27,28] proposed the aggregate operators of IFHSS and interval-valued intuitionistic fuzzy hypersoft sets (IVIFHSS) with application to multi-criteria decisoin making (MCDM) problems. Some more definitions and operators on the set structures, like picture fuzzy, interval-valued picture fuzzy, FP-intuitionistic multi fuzzy N-soft sets.raphs on interval-valued Fermatean neutrosophic graphs, single-valued pentapartitioned neutrosophic graphs, with applications have been proposed by [29–33].

One of the most significant environmental parameters is the air quality index (AQI), which evaluates the quality of the air in any particular region. A variety of air quality indices have been established to evaluate the health effects of air pollution due to the continuously rising levels of air pollution in the majority of the world's areas. Air pollution is caused by a variety of factors, including industrial and transportation emissions, brick kiln smoke, agricultural waste and biomass burning and construction site dust. Another source of air pollution includes large-scale tree-cutting to create room for new roads and structures. All these air pollutants (i.e., CO, SO2, O3, and NO2), benzene, toluene, ethyl benzene, xylene, and 1, 3-butadiene were included in the suggested index because of their severe effects on human health. Experts have given the criteria that described which AQI level is best for human health. Many researchers used different techniques to analyze the air quality index. For example, Sowlat et al. [34] discussed the fuzzy-based air quality index, and Kumar and Goyal [35] presented the forecasting of daily AQI. Zhan et al. [36] suggested the driving factors of AQI. Saqlain et al. [37] gave similarity measures for NHSSs, and Jafar et al. [38] proposed trigonometric similarity measures for NHSSs with application to renewable energy source selection. Linear Diophantine fuzzy sets [39] and spherical linear Diophantine fuzzy sets [40] are new fuzzy extensions for modeling uncertainties in real-life circumstances. The idea of cubic bipolar fuzzy-VIKOR method using new distance and entropy measures and Einstein averaging aggregation operators with application to renewable energy was presented in [41].

1.1. Motivation

Intuitionistic hypersoft set theory is highly beneficial in solving decision-making issues, but it only deals with attributes of alternatives about characteristics, and thus direct comparison of two sets of variables is not easy. If a DM wants to analyze the comparison between two sets, it can be done with the help of similarity measures and distance measures, for which [26] introduced the intuitionistic hypersoft set. Using the definition, we have proposed the similarity measures and distance measures under the intuitionistic hypersoft set environment.

The intuitionistic soft set theory is restricted with membership and non-membership grades in selecting the optimal alternative in a decision-making problem. To deal with a decision analysis problem that possesses some attributes which can be further categorized, the idea of intuitionistic hypersoft set theory is more effective and reliable. The advantages of the proposed theory are the following:

(1) The proposed method is a new approach for any multi-attribute decision making (MADM) problem, particularly with a large number of attributes, along with a simple computing approach.
(2) The proposed operators are more consistent and accurate when compared to existing approaches for MADM problems in an intuitionistic context, demonstrating their applicability.
(3) New distance and similarity measures for IFHSS are developed with the help of aggregate operators. Proposed information measures are designed to cover certain drawbacks of extension techniques.
(4) The suggested method also analyzes the interrelationship of qualities in practical application, while existing approaches cannot.

1.2. Layout of the paper

The paper is organized as follows. In Section 2, we review some basic definitions to understand the rest of the article, i.e., intuitionistic set, soft set, hypersoft set and intuitionistic fuzzy hypersoft set (IFHSS), and necessary results. In Section 3, distance measures of IFHSS with theorems and propositions and their desirable properties are established. In Section 4, similarity measures of IFHSS are developed. In Section 5, by using these distance and similarity measures, a decision-making problem (application/case study) is presented. In Section 6, results, discussion and comparison are given. Finally, the conclusion and future directions are presented in the last section.

2. Preliminary section

In this section, we discuss the definitions of intuitionistic set, soft set, hypersoft set and intuitionistic hypersoft set.

Definition 2.1. [4] Intuitionistic set theory was proposed by Atanassove in 1983 and shows the degree of belongingness and non-belongingness. Let Û be the universe of discourse and Υ^{int} be the intuitionistic set, defined as

$$\Upsilon^{int}\colon \hat{\text{U}} \rightarrow [0,1]^2.$$

Definition 2.2. [7] In 1999, the term soft set was introduced by Molodtsov to make decisions in parametric family of alternatives. Let $\xi = \{\xi_1, \xi_2, \xi_3, \ldots . \xi_n\}$ be the set of alternatives and Ꞧ be the set of attributes. Let $P(\xi)$ denote the power set of ξ and Ꭓ ⊂ Ꞧ; then, a pair (η, Ꭓ) is called a soft set over ξ as it follows the following mapping.

$$\eta : \text{Ꭓ} \rightarrow P\,(\xi).$$

Definition 2.3. [22] The term hypersoft set was introduced by Smarandache in 2018 to deal with sub-attributions. Let $\xi = \{\xi_1, \xi_2, \xi_3, \ldots . \xi_n\}$ be the set of alternatives, $\text{Ꞧ} = \{\text{Ꞧ}_1, \text{Ꞧ}_2, \ldots\ldots, \text{Ꞧ}_n\}$ be the set of attributes and $\{\varkappa_1, \varkappa_2, \varkappa_3, \ldots, \varkappa_n\} \subset \text{Ꞧ}$ be the set of attributive values. Then, a pair $(\eta, \text{Ꞧ}_1 \times \text{Ꞧ}_2 \times \ldots . . \times \text{Ꞧ}_n)$ is called a hypersoft set over;

$$\eta\colon \text{Ꞧ}_1 \times \text{Ꞧ}_2 \times \ldots . . \times \text{Ꞧ}_n \rightarrow P(\xi).$$

Definition 2.4. [26] Let $\xi = \{\xi_1, \xi_2, \xi_3, \ldots . \xi_n\}$ be the finite set of alternatives, and $P(\xi)$ denotes the power set of ξ. Let $\text{Ꞧ} = \{\text{Ꞧ}_1, \text{Ꞧ}_2, \ldots\ldots, \text{Ꞧ}_n\}$ be the well-defined attributions, whose corresponding attributive values Ꭓ are the set of $\{\varkappa_1, \varkappa_2, \varkappa_3, \ldots, \varkappa_n\}$ having distinct elements. The mapping can be defined as $\eta\colon \mathfrak{P} \rightarrow P(\xi)$. Then, the pair $(\eta, \mathfrak{P})$ is called IFHSS over ξ, such that

$\eta(\mathfrak{P}) = \{\langle \xi\,, (ƚ_{ϐ}(\eta(\mathfrak{P})), ғ_{ϐ}(\eta(\mathfrak{P}))\rangle\}$, where $ƚ, ғ$ are the truthfulness and falseness deals with the membership and non-membership value, respectively.

Definition 2.5. [26] Let ϐ and $\ddot{\Upsilon}$ be the two IFHSS where $ϐ = \{\langle \xi\,, ƚ_{ϐ}(\eta(\mathfrak{P})), ғ_{ϐ}(\eta(\mathfrak{P}))\rangle\}$ and $\ddot{\Upsilon} = \{\langle \xi\,, ƚ_{\ddot{\Upsilon}}(\eta(\mathfrak{P})), ғ_{\ddot{\Upsilon}}(\eta(\mathfrak{P}))\rangle\}$, and then following operations are defined.

Addition:

$$ϐ + \ddot{\Upsilon} = \{\langle v, ƚ_{ϐ}(\eta(\mathfrak{P})) + ƚ_{\ddot{\Upsilon}}(\eta(\mathfrak{P})) - ƚ_{ϐ}(\eta(\mathfrak{P})).ƚ_{\ddot{\Upsilon}}(\eta(\mathfrak{P})), ғ_{ϐ}(\eta(\mathfrak{P}))ғ_{\ddot{\Upsilon}}(\eta(\mathfrak{P}))\rangle\}.$$

Multiplication:

$$ϐ \times \ddot{\Upsilon} = \{\langle v, ƚ_{ϐ}(\eta(\mathfrak{P}))ƚ_{\ddot{\Upsilon}}(\eta(\mathfrak{P})), ғ_{ϐ}(\eta(\mathfrak{P})) + ғ_{\ddot{\Upsilon}}(\eta(\mathfrak{P})) - ғ_{ϐ}(\eta(\mathfrak{P}))ғ_{\ddot{\Upsilon}}(\eta(\mathfrak{P}))\rangle\}.$$

Subtraction:

$$ϐ - \ddot{\Upsilon} = \left\{\langle v, \frac{ƚ_{ϐ}(\eta(\mathfrak{P})) - ƚ_{\ddot{\Upsilon}}(\eta(\mathfrak{P}))}{1 - ƚ_{\ddot{\Upsilon}}(\eta(\mathfrak{P}))}, \frac{ғ_{ϐ}(\eta(\mathfrak{P}))}{ғ_{\ddot{\Upsilon}}(\eta(\mathfrak{P}))}\rangle\right\}$$

holds only when $ϐ \geq \ddot{\Upsilon}\,, ƚ_{\ddot{\Upsilon}}(\eta(\mathfrak{P})) \neq 1, ғ_{\ddot{\Upsilon}}(\eta(\mathfrak{P})) \neq 0.$

Division:

$$ϐ\,/\,\ddot{\Upsilon} = \left\{\langle v, \frac{ƚ_{ϐ}(\eta(\mathfrak{P}))}{ƚ_{\ddot{\Upsilon}}(\eta(\mathfrak{P}))}, \frac{ғ_{ϐ}(\eta(\mathfrak{P})) - ғ_{\ddot{\Upsilon}}(\eta(\mathfrak{P}))}{1 - ғ_{\ddot{\Upsilon}}(\eta(\mathfrak{P}))}\rangle\right\}$$

holds only when $ϐ \leq \ddot{\Upsilon}, ƚ_{\ddot{\Upsilon}}\,(P(\xi)) \neq 0, ғ_{\ddot{\Upsilon}}\,(P(\xi)) \neq 1.$

Definition 2.6. [26] Consider ϐ and $\ddot{\Upsilon}$ to be the two IFHSS where

$$ϐ = \{\langle \xi\,, ƚ_{ϐ}(\eta(\mathfrak{P})), ғ_{ϐ}(\eta(\mathfrak{P}))\rangle\} \text{ and } \ddot{\Upsilon} = \{\langle \xi\,, ƚ_{\ddot{\Upsilon}}(\eta(\mathfrak{P})), ғ_{\ddot{\Upsilon}}(\eta(\mathfrak{P}))\rangle\}$$

and then the following operations are defined.

Complement:

$$ϐ^{c} = \{\langle v, ғ_{ϐ}(\eta(\mathfrak{P})), ƚ_{ϐ}(\eta(\mathfrak{P}))\rangle\}.$$

It is based on the dependency intuitionistic theory, all the truthfulness and falseness are dependent, and $ϐ^{c} = \{\langle v, 1 - ƚ_{ϐ}(\eta(\mathfrak{P})), 1 - ғ_{ϐ}(\eta(\mathfrak{P}))\rangle\}$.

This case is based on the independency intuitionistic theory, and all the truthfulness and falseness are dependent.

Inclusion: $ϐ \subseteq \ddot{\Upsilon}$ iff $ƚ_{ϐ}(\eta(\mathfrak{P})) \leq ƚ_{\ddot{\Upsilon}}(\eta(\mathfrak{P})), ғ_{ϐ}(\eta(\mathfrak{P})) \geq ғ_{\ddot{\Upsilon}}(\eta(\mathfrak{P}))$.

Equality: $ϐ = \ddot{\Upsilon}$ iff $ϐ \subseteq \ddot{\Upsilon}$ and $\ddot{\Upsilon} \subseteq ϐ$.

Union: If ϐ and $\ddot{\Upsilon}$ are the two IFHSS, then the union of ϐ and $\ddot{\Upsilon}$ is

$$\{\langle v, ƚ_{ϐ}(\eta(\mathfrak{P})) \vee ƚ_{\ddot{\Upsilon}}(\eta(\mathfrak{P})), ғ_{ϐ}(\eta(\mathfrak{P})) \wedge ғ_{\ddot{\Upsilon}}(\eta(\mathfrak{P}))\rangle\}.$$

Intersection: If ϐ and $\ddot{\Upsilon}$ are the two IFHSS, then intersection of ϐ and $\ddot{\Upsilon}$ is

$$\{\langle v, ƚ_{ϐ}(\eta(\mathfrak{P})) \wedge ƚ_{\ddot{\Upsilon}}(\eta(\mathfrak{P})), ғ_{ϐ}(\eta(\mathfrak{P})) \vee ғ_{\ddot{\Upsilon}}(\eta(\mathfrak{P}))\rangle\}.$$

Definition 2.7. [26] Let Û be the universe discourse and ϐ be an IFHSS. Then, ϐ is said to be an absolute IFHSS if

$$ƚ_{ϐ}(\eta(\mathfrak{P})) = 1 \; and \; ғ_{ϐ}(\eta(\mathfrak{P})) = 0.$$

Let $ß$ be the single value IFHSS. Then, $ß$ is said to be empty IFHSS if

$$ŧ_{ß}(\eta(\mathfrak{P})) = 0 \; and \; ғ_{ß}(\eta(\mathfrak{P})) = 1.$$

3. Distance measure of IFHSS

In this section, we propose distance measures of IFHSS with theorems, propositions and their desirable properties.

The similarity measure determines how similar two items are. The similarity measure is based on the direct operation of membership, non-membership, hesitation and upper bound of membership function. Similarities measure is used to broaden the theories and suggests many real-life applications, including medical diagnosis, physics education, pattern recognition, defect detection and multi-attribute decision making.

Definition 3.1. Let a mapping $\mathfrak{D}$ defined as $\mathfrak{D}$: È (ȁ) × È (ȁ) → [0,1] be called a distance measure if $\mathfrak{D}$ satisfies the following axioms for $ß$, Ϋ (two IFHSS) and ƀ ∈ Ɽ ⊆ È (ȁ).

- $\mathfrak{D}_1$: $0 \le \mathfrak{D}(ß, Ϋ) \le 1$
- $\mathfrak{D}_2$: $\mathfrak{D}(ß, Ϋ) = 0$ iff $ß = Ϋ$
- $\mathfrak{D}_3$: $\mathfrak{D}(ß, Ϋ) = \mathfrak{D}(Ϋ, ß)$
- $\mathfrak{D}_4$: $ß \subseteq Ϋ \subseteq Ɽ$

$$\mathfrak{D}(ß, Ϋ) = \frac{1}{n}\sum_{ἰ=1}^{n} max\left\{\left|ƫ_{ß}(ƴ(ƀ))_{ἰ} - ƫ_{Ϋ}(ƴ(ƀ))_{ἰ}\right|, \left|ƒ_{ß}(ƴ(ƀ))_{ἰ} - ƒ_{Ϋ}(ƴ(ƀ))_{ἰ}\right|\right\}$$

$$ƫ_{ß}(ƴ(ƀ))_{ἰ} \le ƫ_{Ϋ}(ƴ(ƀ))_{ἰ} \le ƫ_{Ɽ}(ƴ(ƀ))_{ἰ}$$

$$ƒ_{ß}(ƴ(ƀ))_{ἰ} \ge ƒ_{Ϋ}(ƴ(ƀ))_{ἰ} \ge ƒ_{Ɽ}(ƴ(ƀ))_{ἰ}, \tag{A}$$

where ƫ and ƒ represent the truthfulness and falsity degree, and $ἰ = \{1,2,3,\dots,n\}$ represent the sub-attributes.

Remark: In Definition 3.1, the axiom shows that these mappings will define a metric space, and further, this can be extended to the topic of topology on it.

We have to prove that $\mathfrak{D}(ß, Ɽ) \ge \mathfrak{D}(ß, Ϋ)$. For this, we will discuss two cases.

Case 1. Consider $\mathfrak{D}(ß, Ɽ)$:

$$\left|ƫ_{ß}(ƴ(ƀ))_{ἰ} - ƫ_{Ɽ}(ƴ(ƀ))_{ἰ}\right| \ge \left|ƒ_{ß}(ƴ(ƀ))_{ἰ} - ƒ_{Ɽ}(ƴ(ƀ))_{ἰ}\right|,$$

$$\mathfrak{D}(ß, Ɽ) = \left|ƫ_{ß}(ƴ(ƀ))_{ἰ} - ƫ_{Ɽ}(ƴ(ƀ))_{ἰ}\right| \; \forall ἰ,$$

$$\left|ƒ_{ß}(ƴ(ƀ))_{ἰ} - ƒ_{Ϋ}(ƴ(ƀ))_{ἰ}\right| \le \left|ƒ_{ß}(ƴ(ƀ))_{ἰ} - ƒ_{Ɽ}(ƴ(ƀ))_{ἰ}\right| \le \left|ƫ_{ß}(ƴ(ƀ))_{ἰ} - ƫ_{Ɽ}(ƴ(ƀ))_{ἰ}\right|. \tag{1}$$

$$\left|ƒ_{Ϋ}(ƴ(ƀ))_{ἰ} - ƒ_{Ɽ}(ƴ(ƀ))_{ἰ}\right| \le \left|ƒ_{ß}(ƴ(ƀ))_{ἰ} - ƒ_{Ɽ}(ƴ(ƀ))_{ἰ}\right| \le \left|ƫ_{ß}(ƴ(ƀ))_{ἰ} - ƫ_{Ɽ}(ƴ(ƀ))_{ἰ}\right|. \tag{2}$$

On the other hand,

$$\left|ƫ_{ß}(ƴ(ƀ))_{ἰ} - ƫ_{Ϋ}(ƴ(ƀ))_{ἰ}\right| \le \left|ƫ_{ß}(ƴ(ƀ))_{ἰ} - ƫ_{Ɽ}(ƴ(ƀ))_{ἰ}\right|$$

and

$$\left|ȶ_{Ÿ}\left(ƴ(ƅ)\right)_{ɨ}-ȶ_{Ʀ}\left(ƴ(ƅ)\right)_{ɨ}\right| \le \left|ȶ_{ϐ}\left(ƴ(ƅ)\right)_{ɨ}-ȶ_{Ʀ}\left(ƴ(ƅ)\right)_{ɨ}\right|$$

Combining (1) and (2), we get

$$\frac{1}{n}\sum_{ɨ=1}^{n} max\left\{\left|ȶ_{ϐ}\left(ƴ(ƅ)\right)_{ɨ}-ȶ_{Ÿ}\left(ƴ(ƅ)\right)_{ɨ}\right|,\left|ƒ_{ϐ}\left(ƴ(ƅ)\right)_{ɨ}-ƒ_{Ÿ}\left(ƴ(ƅ)\right)_{ɨ}\right|\right\}$$
$$\le\frac{1}{n}\sum_{ɨ=1}^{n} max\left\{\left|ȶ_{ϐ}\left(ƴ(ƅ)\right)_{ɨ}-ȶ_{Ʀ}\left(ƴ(ƅ)\right)_{ɨ}\right|,\left|ƒ_{ϐ}\left(ƴ(ƅ)\right)_{ɨ}-ƒ_{Ʀ}\left(ƴ(ƅ)\right)_{ɨ}\right|\right\}$$

and

$$\frac{1}{n}\sum_{ɨ=1}^{n} max\left\{\left|ȶ_{Ÿ}\left(ƴ(ƅ)\right)_{ɨ}-ȶ_{Ʀ}\left(ƴ(ƅ)\right)_{ɨ}\right|,\left|ƒ_{Ÿ}\left(ƴ(ƅ)\right)_{ɨ}-ƒ_{Ʀ}\left(ƴ(ƅ)\right)_{ɨ}\right|\right\}\le\frac{1}{n}\sum_{ɨ=1}^{n} max\left\{\left|ȶ_{ϐ}\left(ƴ(ƅ)\right)_{ɨ}-ȶ_{Ʀ}\left(ƴ(ƅ)\right)_{ɨ}\right|,\left|ƒ_{ϐ}\left(ƴ(ƅ)\right)_{ɨ}-ƒ_{Ʀ}\left(ƴ(ƅ)\right)_{ɨ}\right|\right\}. \tag{B}$$

So, we conclude that

$$\mathfrak{D}(ϐ,Ʀ)\ge\mathfrak{D}(ϐ,Ÿ) \text{ and } \mathfrak{D}(ϐ,Ʀ)\ge\mathfrak{D}(Ÿ,Ʀ).$$

Case 2. Consider $\mathfrak{D}(ϐ,Ʀ)$:

$$\left|ȶ_{ϐ}\left(ƴ(ƅ)\right)_{ɨ}-ȶ_{Ʀ}\left(ƴ(ƅ)\right)_{ɨ}\right|\le\left|ƒ_{ϐ}\left(ƴ(ƅ)\right)_{ɨ}-ƒ_{Ʀ}\left(ƴ(ƅ)\right)_{ɨ}\right|,$$

$$\mathfrak{D}(ϐ\,,Ʀ)=\left|ƒ_{ϐ}\left(ƴ(ƅ)\right)_{ɨ}-ƒ_{Ʀ}\left(ƴ(ƅ)\right)_{ɨ}\right|.$$

However, $\forall$ ɨ,

$$\left|ȶ_{ϐ}\left(ƴ(ƅ)\right)_{ɨ}-ȶ_{Ÿ}\left(ƴ(ƅ)\right)_{ɨ}\right|\le\left|ȶ_{ϐ}\left(ƴ(ƅ)\right)_{ɨ}-ȶ_{Ʀ}\left(ƴ(ƅ)\right)_{ɨ}\right|\le\left|ƒ_{ϐ}\left(ƴ(ƅ)\right)_{ɨ}-ƒ_{Ʀ}\left(ƴ(ƅ)\right)_{ɨ}\right|. \tag{1'}$$

$$\left|ȶ_{Ÿ}\left(ƴ(ƅ)\right)_{ɨ}-ȶ_{Ʀ}\left(P(v)\right)_{ɨ}\right|\le\left|ȶ_{ϐ}\left(ƴ(ƅ)\right)_{ɨ}-ȶ_{Ʀ}\left(ƴ(ƅ)\right)_{ɨ}\right|\le\left|ƒ_{ϐ}\left(P(v)\right)_{ɨ}-ƒ_{Ʀ}\left(ƴ(ƅ)\right)_{ɨ}\right|. \tag{2'}$$

Combining (1') and (2'), we get

$$\frac{1}{n}\sum_{ɨ=1}^{n} max\left\{\left|ȶ_{ϐ}\left(ƴ(ƅ)\right)_{ɨ}-ȶ_{Ÿ}\left(ƴ(ƅ)\right)_{ɨ}\right|,\left|ƒ_{ϐ}\left(ƴ(ƅ)\right)_{ɨ}-ƒ_{Ÿ}\left(ƴ(ƅ)\right)_{ɨ}\right|\right\}$$
$$\le\frac{1}{n}\sum_{ɨ=1}^{n} max\left\{\left|ȶ_{ϐ}\left(ƴ(ƅ)\right)_{ɨ}-ȶ_{Ʀ}\left(ƴ(ƅ)\right)_{ɨ}\right|,\left|ƒ_{ϐ}\left(ƴ(ƅ)\right)_{ɨ}-ƒ_{Ʀ}\left(ƴ(ƅ)\right)_{ɨ}\right|\right\}$$

and

$$\frac{1}{n}\sum_{ɨ=1}^{n} max\left\{\left|ȶ_{Ÿ}\left(ƴ(ƅ)\right)_{ɨ}-ȶ_{Ʀ}\left(ƴ(ƅ)\right)_{ɨ}\right|,\left|ƒ_{Ÿ}\left(ƴ(ƅ)\right)_{ɨ}-ƒ_{Ʀ}\left(ƴ(ƅ)\right)_{ɨ}\right|\right\}\le\frac{1}{n}\sum_{ɨ=1}^{n} max\left\{\left|ȶ_{ϐ}\left(ƴ(ƅ)\right)_{ɨ}-ȶ_{Ʀ}\left(ƴ(ƅ)\right)_{ɨ}\right|,\left|ƒ_{ϐ}\left(ƴ(ƅ)\right)_{ɨ}-ƒ_{Ʀ}\left(ƴ(ƅ)\right)_{ɨ}\right|\right\}.$$

So, we conclude that

$$\mathfrak{D}(ϐ,Ʀ)\ge\mathfrak{D}(ϐ,Ÿ) \text{ and } \mathfrak{D}(ϐ,Ʀ)\ge\mathfrak{D}(Ÿ,Ʀ).$$

Theorem 3.2. Let ϐ and Ÿ be two IFHSS, and then d^{m}(ϐ, Ÿ) for m=1,2,..,5 is a distance between IFHSS defined as the following:

1) $d^1(\mathrm{Ϭ}, \ddot{\Upsilon})=$
$$\frac{1}{2|ƀ|}\sum_{i}(|t^2{}_{\mathrm{Ϭ}i}(y(ƀ)) - t^2{}_{\ddot{\Upsilon}i}(y(ƀ))| + |f^2{}_{\mathrm{Ϭ}i}(y(ƀ)) - f^2{}_{\ddot{\Upsilon}i}(y(ƀ))|)$$
2) $d^2(\mathrm{Ϭ}, \ddot{\Upsilon}) =$
$$\frac{1}{|ƀ|}\sum_{i}(|t^2{}_{\mathrm{Ϭ}i}(y(ƀ)) - t^2{}_{\ddot{\Upsilon}i}(y(ƀ))| \vee |f^2{}_{\mathrm{Ϭ}i}(y(ƀ)) - f^2{}_{\ddot{\Upsilon}i}(y(ƀ))|)$$
3) $d^3(\mathrm{Ϭ}, \ddot{\Upsilon}) =$
$$\frac{\sum_i(|t^2{}_{\mathrm{Ϭ}i}(y(ƀ)) - t^2{}_{\ddot{\Upsilon}i}(y(ƀ))| \vee |f^2{}_{\mathrm{Ϭ}i}(y(ƀ)) - f^2{}_{\ddot{\Upsilon}i}(y(ƀ))|)}{\sum_i(1 + |t^2{}_{\mathrm{Ϭ}i}(y(ƀ)) - t^2{}_{\ddot{\Upsilon}i}(y(ƀ))| \vee |f^2{}_{\mathrm{Ϭ}i}(y(ƀ)) - f^2{}_{\ddot{\Upsilon}i}(y(ƀ))|)}$$
4) $d^4(\mathrm{Ϭ}, \ddot{\Upsilon}) =$
$$1 - \lambda\frac{\sum_i(|t^2{}_{\mathrm{Ϭ}i}(y(ƀ)) \wedge t^2{}_{\ddot{\Upsilon}i}(y(ƀ))|)}{\sum_i(|t^2{}_{\mathrm{Ϭ}i}(y(ƀ)) \vee t^2{}_{\ddot{\Upsilon}i}(y(ƀ))|)} - \mu\,\frac{\sum_i(|f^2{}_{\mathrm{Ϭ}i}(y(ƀ)) \wedge f^2{}_{\ddot{\Upsilon}i}(y(ƀ))|)}{\sum_i(|f^2{}_{\mathrm{Ϭ}i}(y(ƀ)) \vee f^2{}_{\ddot{\Upsilon}i}(y(ƀ))|)}$$
where $\lambda, \mu \in [0,1]$.

5) $d^5(\mathrm{Ϭ}, \ddot{\Upsilon}) =$
$$1 - \frac{\lambda}{|ƀ|}\frac{\sum_i(|t^2{}_{\mathrm{Ϭ}i}(y(ƀ)) \wedge t^2{}_{\ddot{\Upsilon}i}(y(ƀ))|)}{\sum_i(|t^2{}_{\mathrm{Ϭ}i}(y(ƀ)) \vee t^2{}_{\ddot{\Upsilon}i}(y(ƀ))|)} - \frac{\mu}{|\xi|}\frac{\sum_i(|f^2{}_{\mathrm{Ϭ}i}(y(ƀ)) \wedge f^2{}_{\ddot{\Upsilon}i}(y(ƀ))|)}{\sum_i(|f^2{}_{\mathrm{Ϭ}i}(y(ƀ)) \vee f^2{}_{\ddot{\Upsilon}i}(y(ƀ))|)}$$
where, $\mu \in [0,1]$.

If $d^m(\mathrm{Ϭ}, \ddot{\Upsilon})$ for m = 1,2,..,5 satisfied all the axioms of distance, i.e., D_1–D_4, then they are suitable for validity.

Theorem 3.3. Let Ϭ and $\ddot{\Upsilon}$ be two IFHSS then $d^m(\mathrm{Ϭ}, \ddot{\Upsilon})$ for m = 1,2,..,5 is a distance between IFHSS holds the followings:

a) $d^m(\mathrm{Ϭ}, \ddot{\Upsilon}^c) = d^m(\mathrm{Ϭ}^c, \ddot{\Upsilon})$
b) $d^m(\mathrm{Ϭ}, \ddot{\Upsilon}) = d^m(\mathrm{Ϭ} \cap \ddot{\Upsilon}, \mathrm{Ϭ} \cup \ddot{\Upsilon})$
c) $d^m(\mathrm{Ϭ}, \mathrm{Ϭ} \cap \ddot{\Upsilon}) = d^m(\ddot{\Upsilon}, \mathrm{Ϭ} \cup \ddot{\Upsilon})$
d) $d^m(\mathrm{Ϭ}, \mathrm{Ϭ} \cup \ddot{\Upsilon}) = d^m(\ddot{\Upsilon}, \mathrm{Ϭ} \cap \ddot{\Upsilon})$

Proof:

a. $d^1(\mathrm{Ϭ}, \ddot{\Upsilon}^c) = d^1(\mathrm{Ϭ}^c, \ddot{\Upsilon})$

Let
$$\mathrm{Ϭ} = \{\langle v, t_{\mathrm{Ϭ}}(y(ƀ)), f_{\mathrm{Ϭ}}(y(ƀ))\rangle\};$$
$$\ddot{\Upsilon} = \{\langle v, t_{\ddot{\Upsilon}}(y(ƀ)), f_{\ddot{\Upsilon}}(y(ƀ))\}\rangle\};$$
$$\ddot{\Upsilon}^c = \{\langle v, (f_{\ddot{\Upsilon}}(y(ƀ)),\ t_{\ddot{\Upsilon}}(y(ƀ))\rangle\}.$$

Then, by distance $d^1(\mathrm{Ϭ}, \ddot{\Upsilon})$, we have
$$d^1(\mathrm{Ϭ}, \ddot{\Upsilon}) = \frac{1}{2|ƀ|}\sum_i(|t^2{}_{\mathrm{Ϭ}i}(y(ƀ)) - t^2{}_{\ddot{\Upsilon}i}(y(ƀ))| + |f^2{}_{\mathrm{Ϭ}i}(y(ƀ)) - f^2{}_{\ddot{\Upsilon}i}(y(ƀ))|).$$

Then,
$$d^1(\mathrm{Ϭ}, \ddot{\Upsilon}^c) = \frac{1}{2|ƀ|}\sum_{i}(|t^2{}_{\mathrm{Ϭ}i}(y(ƀ)) - f^2{}_{\ddot{\Upsilon}i}(y(ƀ))| + |f^2{}_{\mathrm{Ϭ}i}(y(ƀ)) - t^2{}_{\ddot{\Upsilon}i}(y(ƀ))|)$$
$$= \frac{1}{2|ƀ|}\sum_i(|f^2{}_{\mathrm{Ϭ}i}(y(ƀ)) - t^2{}_{\ddot{\Upsilon}i}(y(ƀ))| + |t^2{}_{\mathrm{Ϭ}i}(y(ƀ)) - f^2{}_{\ddot{\Upsilon}i}(y(ƀ))|)$$
$$= d^1(\mathrm{Ϭ}^c, \ddot{\Upsilon})$$

b. $d^1(Ϭ,Ÿ) = d^1(Ϭ \cap Ÿ, Ϭ \cup Ÿ)$

$$= \frac{1}{2|ḅ|}\sum_{i}(|(\min(ƫ_{Ϭi}(ƴ(ḅ)), ƫ_{Ÿi}(ƴ(ḅ)))^2 - \left(\max\left(ƫ_{Ϭi}(ƴ(ḅ)), ƫ_{Ÿi}(ƴ(ḅ))\right)\right)^2\right|$$
$$+ |(\max(ƒ_{Ϭi}(ƴ(ḅ)), ƒ_{Ÿi}(ƴ(ḅ)))^2 - (\min(ƒ_{Ϭi}(ƴ(ḅ)), ƒ_{Ÿi}(ƴ(ḅ)))^2|)$$
$$=\frac{1}{2|ḅ|}\sum_{i}(|ƫ^2{}_{Ϭi}(ƴ(ḅ)) - ƫ^2{}_{Ÿi}(ƴ(ḅ))|+|ƒ^2{}_{Ϭi}(ƴ(ḅ)) - ƒ^2{}_{Ÿi}(ƴ(ḅ))|) = d^1(Ϭ,Ÿ).$$

4. Similarity measures of IFHSS

In this Section 4, similarity measures of IFHSS have been developed.

The distance measure determines how attributes are closely related to each other. The distance measure is based on the direct operation of membership, and non-membership.

Definition 4.1. Let Ϭ and Ÿ be two IFHSS. A mapping Ƴ defined as Ƴ: È (å) ×È (å) → [0,1] is called a similarity measure between Ϭ and Ÿ if Ƴ holds these axioms:

1) $Ƴ_1$: $0 \leq Ƴ(Ϭ, Ÿ) \leq 1$
2) $Ƴ_2$: $Ƴ(Ϭ, Ÿ) = 0$ iff $Ϭ = Ÿ$
3) $Ƴ_3$: $Ƴ(Ϭ, Ÿ) = Ƴ(Ÿ, Ϭ)$
4) $Ƴ_4$: $Ϭ \subseteq Ÿ \subseteq Ʀ$

$$Ƴ(Ϭ, Ʀ) \leq Ƴ(Ϭ, Ÿ) \text{ and } Ƴ(Ϭ, Ʀ) \leq Ƴ(Ÿ, Ʀ).$$

Theorem 4.2. Let Ϭ and Ÿ be two IFHSS. Then, $Ƴ^m(Ϭ, Ÿ)$ for m= 1,2,...,5 are the similarity measure between IFHSS holds the followings:

1) $Ƴ^1(Ϭ,Ÿ) = 1 - \frac{1}{2|ḅ|}\sum_i(|ƫ^2{}_{Ϭi}(ƴ(ḅ)) - ƫ^2{}_{Ÿi}(ƴ(ḅ))| + |ƒ^2{}_{Ϭi}(ƴ(ḅ)) - ƒ^2{}_{Ÿi}(ƴ(ḅ))|)$;
2) $Ƴ^2(Ϭ,Ÿ) = 1 - \frac{1}{|ḅ|}\sum_i(|ƫ^2{}_{Ϭi}(ƴ(ḅ)) - ƫ^2{}_{Ÿi}(ƴ(ḅ))| \vee |ƒ^2{}_{Ϭi}(ƴ(ḅ)) - ƒ^2{}_{Ÿi}(ƴ(ḅ))|)$;
3) $Ƴ^3(Ϭ,Ÿ) = \frac{\sum_i(1-|ƫ^2{}_{Ϭi}(ƴ(ḅ))-ƫ^2{}_{Ÿi}(ƴ(ḅ))|\vee|ƒ^2{}_{Ϭi}(ƴ(ḅ))-ƒ^2{}_{Ÿi}(ƴ(ḅ))|)}{\sum_i(1+|ƫ^2{}_{Ϭi}(ƴ(ḅ))-ƫ^2{}_{Ÿi}(ƴ(ḅ))|\vee|ƒ^2{}_{Ϭi}(ƴ(ḅ))-ƒ^2{}_{Ÿi}(ƴ(ḅ))|)}$;
4) $Ƴ^4(Ϭ,Ÿ) = \lambda\frac{\sum_i(|ƫ^2{}_{Ϭi}(ƴ(ḅ))\wedge ƫ^2{}_{Ÿi}(ƴ(ḅ))|)}{\sum_i(|ƫ^2{}_{Ϭi}(ƴ(ḅ))\vee ƫ^2{}_{Ÿi}(ƴ(ḅ))|)} + \mu\frac{\sum_i(|ƒ^2{}_{Ϭi}(ƴ(ḅ))\wedge ƒ^2{}_{Ÿi}(ƴ(ḅ))|)}{\sum_i(|ƒ^2{}_{Ϭi}(ƴ(ḅ))\vee ƒ^2{}_{Ÿi}(ƴ(ḅ))|)}$,
   where $\lambda, \mu \in [0,1]$;
5) $Ƴ^5(Ϭ,Ÿ) = \frac{\lambda}{|ḅ|}\frac{\sum_i(|ƫ^2{}_{Ϭi}(ƴ(ḅ))\wedge ƫ^2{}_{Ÿi}(ƴ(ḅ))|)}{\sum_i(|ƫ^2{}_{Ϭi}(ƴ(ḅ))\vee ƫ^2{}_{Ÿi}(ƴ(ḅ))|)} + \frac{\mu}{|ḅ|}\frac{\sum_i(|ƒ^2{}_{Ϭi}(ƴ(ḅ))\wedge ƒ^2{}_{Ÿi}(ƴ(ḅ))|)}{\sum_i(|ƒ^2{}_{Ϭi}(Ƥ(υ))\vee ƒ^2{}_{Ÿi}(ƴ(ḅ))|)}$,
   where $\lambda, \mu \in [0,1]$.

Now, we verify the $Ƴ_1$–$Ƴ_4$ axioms.

$$Ƴ^1(Ϭ,Ÿ) = 1 - \frac{1}{2|ḅ|}\sum_i\left(\left||ƫ^2{}_{Ϭi}(ƴ(ḅ)) - ƫ^2{}_{Ÿi}(ƴ(ḅ))| + |ƒ^2{}_{Ϭi}(ƴ(ḅ)) - ƒ^2{}_{Ÿi}(ƴ(ḅ))|\right|\right)$$

$$Ƴ^1(Ϭ,Ÿ) = 1 \text{ iff } Ϭ = Ÿ.$$

$$1 - \frac{1}{2|ḅ|}\sum_i(|ƫ^2{}_{Ϭi}(ƴ(ḅ)) - ƫ^2{}_{Ÿi}(ƴ(ḅ))| + |ƒ^2{}_{Ϭi}(ƴ(ḅ)) - ƒ^2{}_{Ÿi}(ƴ(ḅ))|) = 1$$
$$\Rightarrow \sum_i(|ƫ^2{}_{Ϭi}(ƴ(ḅ)) - ƫ^2{}_{Ÿi}(ƴ(ḅ))| + |ƒ^2{}_{Ϭi}(ƴ(ḅ)) - ƒ^2{}_{Ÿi}(ƴ(ḅ))|) = 0.$$

This is possible when

$$(|ƫ^2{}_{Ϭi}(ƴ(ḅ)) - ƫ^2{}_{Ÿi}(ƴ(ḅ))| + |ƒ^2{}_{Ϭi}(ƴ(ḅ)) - ƒ^2{}_{Ÿi}(ƴ(ḅ))|) = 0$$

$$\Rightarrow \ |\, t^2_{ϐi}(ƴ(ƀ)) - t^2_{\ddot{Y}i}(ƴ(ƀ))| = 0; |f^2_{ϐi}(ƴ(ƀ)) - f^2_{\ddot{Y}i}(ƴ(ƀ))| = 0$$

$$\Rightarrow t^2_{ϐi}(ƴ(ƀ)) = t^2_{\ddot{Y}i}(ƴ(ƀ));\ f^2_{ϐi}(ƴ(ƀ)) = f^2_{\ddot{Y}i}(ƴ(ƀ))$$

$$\Rightarrow t_{ϐ}(ƴ(ƀ)) = t_{\ddot{Y}}(ƴ(ƀ)), f_{ϐ}(ƴ(ƀ)) = f_{\ddot{Y}}(ƴ(ƀ))$$

$$\Rightarrow ϐ = \ddot{Y}.$$

Conversely, we have to prove that $\Upsilon^1(ϐ, \ddot{Y}) = 1$.

Since $ϐ = \ddot{Y}$,

$\Rightarrow t_{ϐi}(ƴ(ƀ)) = t_{\ddot{Y}i}(ƴ(ƀ)), f_{ϐi}(ƴ(ƀ)) = f_{\ddot{Y}i}(ƴ(ƀ))$

$\Rightarrow t^2_{ϐi}(ƴ(ƀ)) = t^2_{\ddot{Y}i}(ƴ(ƀ)), f^2_{ϐi}(ƴ(ƀ)) = f^2_{\ddot{Y}i}(ƴ(ƀ))$

$\Rightarrow t^2_{ϐi}(ƴ(ƀ)) - t^2_{\ddot{Y}i}(ƴ(ƀ)) = 0, f^2_{ϐi}(ƴ(ƀ)) - f^2_{\ddot{Y}i}(ƴ(ƀ)) = 0$

$\Rightarrow \left(|t^2_{ϐi}(ƴ(ƀ)) - t^2_{\ddot{Y}i}(ƴ(ƀ))| = 0, |f^2_{ϐi}(ƴ(ƀ)) - f^2_{\ddot{Y}i}(ƴ(ƀ))|\right) = 0$

$\Rightarrow \dfrac{1}{2|ƀ|}\sum_i \left(|t^2_{ϐi}(ƴ(ƀ)) - t^2_{\ddot{Y}i}(ƴ(ƀ))| + |f^2_{ϐi}(ƴ(ƀ)) - f^2_{\ddot{Y}i}(ƴ(ƀ))|\right) = 0$

$\Rightarrow 1 - \dfrac{1}{2|ƀ|}\sum_i \left(|t^2_{ϐi}(ƴ(ƀ)) - t^2_{\ddot{Y}i}(ƴ(ƀ))| + |f^2_{ϐi}(ƴ(ƀ)) - f^2_{\ddot{Y}i}(ƴ(ƀ))|\right) = 1 - 0$

$\Rightarrow \Upsilon^1(ϐ, \ddot{Y}) = 1.$

Theorem 4.3. Let $ϐ$ and $\ddot{Y}$ be two IFHSS. Then, $\Upsilon^m(ϐ, \ddot{Y})$ for $m = 1,2,..,5$ are the similarity measures in between IFHSS $ϐ$ and $\ddot{Y}$, we have:

a) $\Upsilon^m(ϐ, \ddot{Y}^c) = \Upsilon^m(ϐ^c, \ddot{Y})$
b) $\Upsilon^m(ϐ, \ddot{Y}) = \Upsilon^m(ϐ \cap \ddot{Y}, ϐ \cup \ddot{Y})$
c) $\Upsilon^m(ϐ, ϐ \cap \ddot{Y}) = \Upsilon^m(\ddot{Y}, ϐ \cup \ddot{Y})$
d) $\Upsilon^m(ϐ, ϐ \cup \ddot{Y}) = \Upsilon^m(\ddot{Y}, ϐ \cap \ddot{Y})$

Proof:

a. $\Upsilon^1(ϐ, \ddot{Y}^c) = \Upsilon^1(ϐ^c, \ddot{Y})$.

Consider the following:

$$ϐ = \{\langle v, t_{ϐ}(ƴ(ƀ)), f_{ϐ}(ƴ(ƀ))\rangle\};$$

$$\ddot{Y} = \{\langle v, t_{\ddot{Y}}(ƴ(ƀ)), f_{\ddot{Y}}(ƴ(ƀ))\rangle\};$$

$$\ddot{Y}^c = \{\langle v, f_{\ddot{Y}}(ƴ(ƀ)), t_{\ddot{Y}}(ƴ(ƀ))\rangle\}.$$

By definition,

$$\Upsilon^1(ϐ, \ddot{Y}) = 1 - \tfrac{1}{2|ƀ|}\textstyle\sum_i \left(|t^2_{ϐi}(ƴ(ƀ)) - t^2_{\ddot{Y}i}(ƴ(ƀ))| + |f^2_{ϐi}(ƴ(ƀ)) - f^2_{\ddot{Y}i}(ƴ(ƀ))|\right).$$

Then,

$$\Upsilon^1(ϐ, \ddot{Y}^c) = 1 - \frac{1}{2|ƀ|}\sum_i \left(|t^2_{ϐi}(ƴ(ƀ)) - f^2_{\ddot{Y}i}(ƴ(ƀ))| + |f^2_{ϐi}(ƴ(ƀ)) - t^2_{\ddot{Y}i}(ƴ(ƀ))|\right)$$

$$= 1 - \frac{1}{2|ƀ|}\sum_i \left(|f^2_{ϐi}(ƴ(ƀ)) - t^2_{\ddot{Y}i}(ƴ(ƀ))| + |t^2_{ϐi}(ƴ(ƀ)) - f^2_{\ddot{Y}i}(ƴ(ƀ))|\right)$$

$$= \Upsilon^1(ϐ^c, \ddot{Y}).$$

b. $\text{Ƴ}^1(\text{Ɓ},\text{Ÿ}) = \text{Ƴ}^1(\text{Ɓ} \cap \text{Ÿ}, \text{Ɓ} \cup \text{Ÿ})$

$$= 1 - \frac{1}{2|\text{ƀ}|}\sum_i (|(\min(t_{\text{Ɓ}i}(y(\text{ƀ})), t_{\text{Ÿ}i}(y(\text{ƀ}))|)^2 - \left(\max\left(t_{\text{Ɓ}i}(y(\text{ƀ})), t_{\text{Ÿ}i}(y(\text{ƀ}))\right)\right)^2\Big|$$
$$+ |(\max(f_{\text{Ɓ}i}(y(\text{ƀ})), f_{\text{Ÿ}i}(y(\text{ƀ})))^2 - (\min(f_{\text{Ɓ}i}(y(\text{ƀ})), f_{\text{Ÿ}i}(y(\text{ƀ})))^2|)$$
$$= 1 - \frac{1}{2|\text{ƀ}|}\sum_i (|t^2_{\text{Ɓ}i}(y(\text{ƀ})) - t^2_{\text{Ÿ}i}(y(\text{ƀ}))| + |f^2_{\text{Ɓ}i}(y(\text{ƀ})) - f^2_{\text{Ÿ}i}(y(\text{ƀ}))|) = \text{Ƴ}^1(\text{Ɓ},\text{Ÿ}).$$

c. $\text{Ƴ}^1(\text{Ɓ}, \text{Ɓ} \cap \text{Ÿ}) = \text{Ƴ}^1(\text{Ÿ}, \text{Ɓ} \cup \text{Ÿ})$

$$= 1 - \frac{1}{2|\text{ƀ}|}\sum_i (|t^2_{\text{Ɓ}i}(y(\text{ƀ})) - (\min\left(t_{\text{Ɓ}i}(y(\text{ƀ})), t_{\text{Ÿ}i}(y(\text{ƀ}))\right))^2| + |f^2_{\text{Ɓ}i}(y(\text{ƀ}))$$
$$- (\max\left(f_{\text{Ɓ}i}(y(\text{ƀ})), f_{\text{Ÿ}i}(y(\text{ƀ}))\right))^2|)$$
$$= \frac{1}{2|\text{ƀ}|}\sum_i ((|t^2_{\text{Ÿ}i}(y(\text{ƀ})) - (\max\left(t_{\text{Ɓ}i}(y(\text{ƀ})), t_{\text{Ÿ}i}(y(\text{ƀ}))\right))^2| + |f^2_{\text{Ÿ}i}(y(\text{ƀ})) -$$
$$(\min\left(f_{\text{Ɓ}i}(y(\text{ƀ})), f_{\text{Ÿ}i}(y(\text{ƀ}))\right))^2|) = \text{Ƴ}^1(\text{Ÿ}, \text{Ɓ} \cup \text{Ÿ}).$$

5. Numerical application

5.1. Case study 1

Let þ = $\{þ^1, þ^2, þ^3, \ldots, þ^n\}$ represent the regions (different geographical areas), ʒ=$\{ʒ^1, ʒ^2, ʒ^3, \ldots, ʒ^n\}$ represents human activities, and Ч =$\{Ч^1, Ч^2, Ч^3, \ldots, Ч^n\}$ and $\zeta = \{\zeta^1, \zeta^2, \zeta^3, \ldots, \zeta^n\}$ show the humidity level and air pollution at that particular area, respectively. A decision-maker determines þ region and the factors ʒ, Ч, ζ which enhance the AQI by applying a decision-making technique. This analysis can be used to determine whether a geographical area has good, moderate or hazardous AQI.

Let þ = $\{þ^1, þ^2, þ^3 \ldots þ^n\}$ and

$$\text{Ŭ} = \begin{Bmatrix} \text{Ŭ}^1\ (air\ pollutants) \\ \text{Ŭ}^2(\ human\ activities) \\ \text{Ŭ}^3(humidity\ level) \end{Bmatrix}$$

$$\text{Ŭ}^1 = \begin{Bmatrix} (sulfur\ dioxide)SO_{2,} (ozone)O_3, (nitrogen)NO_2, (carbon\ monoxide)CO, \\ (\ particulate\ matter\ )PM_{2.5} \end{Bmatrix}$$

$$\text{Ŭ}^2 = \{smoking\ , transport\ exhaust\ , airbone\ dust\}$$

$$\text{Ŭ}^3 = \{temprature\ , pressure\}$$

Now, we computed the IFHSS by using the attributive values through the following mapping:
Ϭ :Ŭ→ þ and Ÿ : Ŭ→ þ.

In this example, we have calculated the AQI of the different cities based on the data set having different pollutant factors and presented in Tables 1 and 2. Using the proposed distance and similarity measure, we calculated the values of the attributes and alternatives, as shown in Tables 3–7, and Table 8 represents the mean of the affective factors to geographical region which deteriorate the quality of air. The air quality index of each region has been calculated in Table 9, and its graphical order is represented in Figure 1, which shows that þ^{1} region has good air quality as compared to þ^{2} and n^{3}. The AQI of geographical region þ^{3} is very high and can indicate adverse effects on human health. We found that our technique for calculating the air quality index of various regions is helpful for selection. The AQI of various regions can easily be calculated by using this mathematical technique.

Table 1. Decision-making matrix from affective factors to geographical region.

Regions (þ)	Air pollutant (SO_2)	Human activities (transport exhaust)	Humidity (level)
þ^{1}	(0.99,0.50)	(1.00,0.10)	(0.06,0.10)
þ^{2}	(0.60,0.40)	(0.60,0.23)	(0.10,0.20)
þ^{3}	(0.20,0.70)	(0.40, 0.50)	(0.20,0.10)

Table 2. Decision-making matrix from ideal affective factors to geographical region.

Regions (þ)	Air pollutant (SO_2)	Human activities (transport exhaust)	Humidity (level)
þ^{1}	(0.06,0.02)	(0.40,0.00)	(0.26,0.20)
þ^{2}	(0.04,0.01)	(0.30,0.10)	(0.28,0.30)
þ^{3}	(0.01,0.00)	(0.10,0.00)	(0.30,0.10)

Table 3. Distance similarity measures using $\Upsilon^{1}(\beta, \ddot{\Upsilon})$.

Similarity measures	Regions (þ)	Air pollutant (SO_2)	Human activities (transport exhaust)	Humidity (level)
	þ^{1}	0.8915	0.8853	0.8841
$\Upsilon^{1}(\beta, \ddot{\Upsilon})$	þ^{2}	0.9586	0.9525	0.9483
	þ^{3}	0.9583	0.9521	0.9510

Table 4. Distance similarity measures using $\Upsilon^{2}(\beta, \ddot{\Upsilon})$.

Similarity measures	Regions (þ)	Air pollutant (SO_2)	Human activities (transport exhaust)	Humidity (level)
	þ^{1}	0.8119	0.8031	0.7943
$\Upsilon^{2}(\beta, \ddot{\Upsilon})$	þ^{2}	0.9386	0.9303	0.9210
	þ^{3}	0.9230	0.9190	0.9210

Table 5. Distance similarity measures using $\Upsilon^{3}(\beta, \ddot{\Upsilon})$.

Similarity measures	Regions (þ)	Air pollutant (SO_2)	Human activities (transport exhaust)	Humidity (level)
	þ^{1}	0.2293	0.2076	0.1866
$\Upsilon^{3}(\beta, \ddot{\Upsilon})$	þ^{2}	0.6602	0.6230	0.5831
	þ^{3}	0.5916	0.5748	0.5831

Table 6. Distance similarity measures using $ϒ^{4}(ϐ, Ÿ)$.

Similarity measures	Regions (þ)	Air pollutant (SO_2)	Human activities (transport exhaust)	Humidity (level)
$ϒ^{4}(ϐ, Ÿ)$	$þ^1$	0.0675	0.0496	0.0157
	$þ^2$	0.2022	0.1387	0.0292
	$þ^3$	0.5365	0.3381	0.1249

Table 7. Distance similarity measures using $ϒ^{5}(ϐ, Ÿ)$.

Similarity measures	Regions (þ)	Air pollutant (SO_2)	Human activities (transport exhaust)	Humidity (temperature)
$ϒ^{5}(ϐ, Ÿ)$	$þ^1$	0.0067	0.0049	0.0016
	$þ^2$	0.0202	0.0138	0.0029
	$þ^3$	0.0536	0.0338	0.0124

Table 8. Mean of distance similarity measures.

Regions (þ)	Air pollutant (SO_2)	Human activities (transport exhaust)	Humidity (temperature)
$þ^1$	0.4014	0.3901	0.3765
$þ^2$	0.5559	0.5316	0.4969
$þ^3$	0.6126	0.5635	0.5185

Table 9. Ranking of regions according to AQI.

Regions (þ)	AQI	Ranking
$þ^1$	0.3893	1
$þ^2$	0.5281	2
$þ^3$	0.5649	3

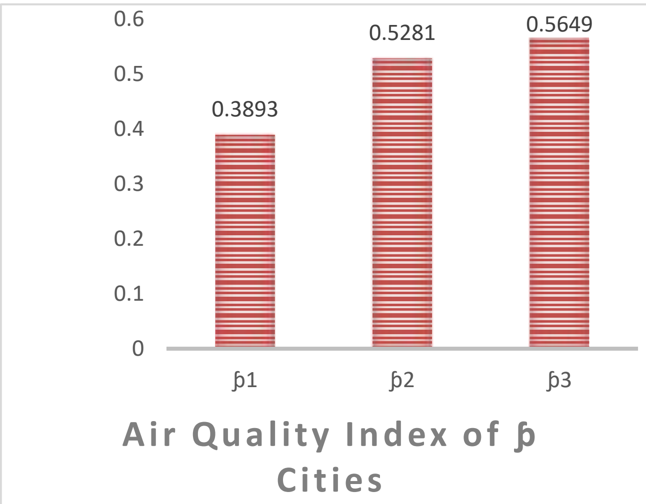

Figure 1. Ranking of AQI for different geographical locations (cities).

5.2. Case study 2

A company is recruiting a new candidate who works in HR. The job is to maintain the policies, understand the needs of the organization and make sure these needs are fulfilled on time. They have published the advertisement in the newspaper, and many candidates apply for it. Assume that there is a set of $ɳ = \{ɳ^1, ɳ^2, ɳ^3 \ldots ɳ^n\}$ candidates (alternative) chosen for an interview. To find the best HR manager for the organization, the group of decision-makers has been assigned tasks to select the candidates $\{ɳ^1, ɳ^2, ɳ^3 \ldots ɳ^n\}$ based on $ɱ$ different criteria $\{ɱ^1, ɱ^2, ɱ^3, \ldots, ɱ^n\}$ involving managing employee attitude $\{a^1, a^2, \ldots, a^n\}$, production booster $\{ß^1, ß^2, \ldots, ß^n\}$ and traditional personnel HRM $\{\hbar^1, \hbar^2, \ldots, \hbar^n\}$. Consider $ɳ = \{ɳ^1, ɳ^2, ɳ^3 \ldots ɳ^n\}$ and

$$ɱ = \begin{Bmatrix} ɱ^1(managing\ employee\ attitute) \\ ɱ^2(production\ booster) \\ ɱ^3(traditional\ personnel\ HRM) \end{Bmatrix}$$

where

$$ɱ^1 = \{resolve\ conflicts, management\ survey\}$$
$$ɱ^2 = \{problem\ solving\ groups, information\ sharing, organization\ development\}$$
$$ɱ^3 = \{hiring, promotion, change\ managment\}$$

Now, we construct the IFHSS by using the attributive values with the following mappings: $ƙ: ɱ \to P(ɳ)$ and $Ƴ: ɱ \to P(ɳ)$.

In this example, by using the proposed distance similarity measure, we calculated the similarity measures $Ƴ(ƙ, Ƴ)$ shown in Tables 10–17, and Table 18 represents the means of the respective attributes corresponding to each candidate. The attributive values corresponding to each candidate are presented in Table 18, and ranking of the candidates is made in descending order of the similarity values, which shows that $ɳ^2$ is the best candidate who is eligible for HR manager.

Table 10. Decision-making matrix from criteria to alternatives.

Candidates ($ɳ$)	Managing Employee Attitude ($resolve\ conflicts$)	Production Booster ($problem\ solving\ groups$)	Traditional Personnel HRM ($hiring$)
$ɳ^1$	(0.70,0.10)	(0.99,0.10)	(0.36,0.10)
$ɳ^2$	(0.60,0.20)	(0.45,0.27)	(0.40,0.20)
$ɳ^3$	(0.55,0.35)	(0.65, 0.20)	(0.30,0.10)

Table 11. Decision-making matrix from ideal affective criteria to alternatives.

Candidates ($ɳ$)	Managing Employee Attitude ($resolve\ conflicts$)	Production Booster ($problem\ solving\ groups$)	Traditional Personnel HRM ($hiring$)
$ɳ^1$	(0.80,0.02)	(0.30,0.25)	(0.40,0.20)
$ɳ^2$	(0.60,0.25)	(0.40,0.15)	(0.30,0.28)
$ɳ^3$	(0.50,0.30)	(0.50,0.00)	(0.20,0.10)

Table 12. Distance similarity measures using Υ^{1}(ƙ, Υ).

Similarity measures	Candidates (ɳ)	Managing Employee Attitude (*resolve conflicts*)	Production Booster (*problem solving groups*)	Traditional Personnel HRM (*hiring*)
Υ^{1}(ƙ , Υ)	ɳ ^{1}	0.9408	0.9523	0.9510
	ɳ ^{2}	0.9778	0.9888	0.9784
	ɳ ^{3}	0.9542	0.9767	0.9826

Table 13. Distance similarity measures using Υ^{2}(ƙ, Υ).

Similarity measures	Candidates (ɳ)	Managing Employee Attitude (*resolve conflicts*)	Production Booster (*problem solving groups*)	Traditional Personnel HRM (*hiring*)
Υ^{2}(ƙ , Υ)	ɳ ^{1}	0.8909	0.9151	0.9110
	ɳ ^{2}	0.9607	0.9857	0.9697
	ɳ ^{3}	0.9260	0.9609	0.9725

Table 14. Distance similarity measures using Υ^{3}(ƙ, Υ).

Similarity measures	Candidates (ɳ)	Managing Employee Attitude (*resolve conflicts*)	Production Booster (*problem solving groups*)	Traditional Personnel HRM (*hiring*)
Υ^{3}(ƙ, Υ)	ɳ ^{1}	0.4668	0.5590	0.5425
	ɳ ^{2}	0.7686	0.9090	0.8165
	ɳ ^{3}	0.6042	0.7694	0.8320

Table 15. Distance similarity measures using Υ^{4}(ƙ, Υ).

Similarity measures	Candidates (ɳ)	Managing Employee Attitude (*resolve conflicts*)	Production Booster (*problem solving groups*)	Traditional Personnel HRM (*hiring*)
Υ^{4}(ƙ, Υ)	ɳ ^{1}	0.3303	0.3537	0.3189
	ɳ ^{2}	0.6278	0.7348	0.5216
	ɳ ^{3}	0.3434	0.5615	0.6377

Table 16. Distance similarity measures using $\Upsilon^5(\Bbbk, \Upsilon)$.

Similarity measures	Candidates (ꞑ)	Managing Employee Attitude $(resolve\ conflicts)$	Production Booster $(problem\ solving\ groups)$	Traditional Personnel HRM $(hiring)$
$\Upsilon^5(\Bbbk, \Upsilon)$	ꞑ ^{1}	0.0330	0.0353	0.0319
	ꞑ ^{2}	0.0627	0.0734	0.0521
	ꞑ ^{3}	0.0343	0.0561	0.0637

Table 17. Means of distance similarity measures.

Candidates (ꞑ)	Managing Employee Attitude $(resolve\ conflicts)$	Production Booster $(problem\ solving\ groups)$	Traditional Personnel HRM $(hiring)$
ꞑ ^{1}	0.5323	0.5630	0.5510
ꞑ ^{2}	0.6795	0.7383	0.6676
ꞑ ^{3}	0.5724	0.6649	0.6977

Table 18. Ranking of candidates by distance similarity measures.

Candidates (ꞑ)	Mean similarity value	Ranking
ꞑ ^{1}	0.5487	3
ꞑ ^{2}	0.6951	1
ꞑ ^{3}	0.6450	2

6. Result discussion and comparison

Intuitionistic hypersoft set theory is highly beneficial in solving decision-making issues, but it only deals with attributes of alternatives about characteristics. Thus, direct comparison of two sets of variables is not easy, and if a DM wants to analyze the comparison between two sets, then it can be done with the help of similarity measures and distance measures. In this regard, [26] introduces the intuitionistic hypersoft set. Using the definition, we have proposed the similarity measures and distance measures under the intuitionistic hypersoft set environment. To discuss the effectiveness and applicability of the proposed study, two case studies have been considered. Comparison with the existing techniques is presented in Table 19. The result shows that proposed distance and similarity measures are helpful for selection when an attribute is further sub-divided. Furthermore, if the DM increases the number of the parameters and sub-divided sets, this technique can be employed in the same manner easily. The suggested method analyzes the interrelationships of qualities in practical application, while existing approaches cannot.

The superiority of the proposed technique is presented below:

(1) To obtain accuracy and precision in an uncertain environment, we cannot ignore the sub-attributive values, and the hypersoft set structure is the only set in which sub-attributes are considered to account.

(2) IFHSS is practical for sorting out the uncertainty and fuzziness within the problems and describing the degrees of membership and non-membership values.
(3) Comparison of two sets is difficult when the DM has to consider the importance of attributive values. Then, similarity and distance measures can be used.

Table 19. Comparison of the current and prior studies.

Researcher	*Set Structure*	*Membership value*	*Non-member ship value*	*Based on Distance, Similarity measure*	*Weighted measurement of distance*	*Computation of max-min distance measure*
Yolcu, A. et al. and Debnath, S. [24,25]	FHS	yes	no	no	no	no
Yolcu, A. et al. [26]	IFHS	yes	yes	no	no	no
Saqlain M. et al. and Jafar et al. [37,38]	NHSS	yes	yes	yes	yes	no
Musa S. Y., & Asaad B. A. [42]	BPHS	yes	yes	no	no	no
Smarandache, F. [43]	Indeterm hypersoft set	yes	yes	no	no	no
Saqlain, M. et al. (proposed)	IFHSS	yes	yes	yes	yes	yes

A comparison has been made in Table 19, which presents the hybrids of hypersoft set structures with distance measure, weighted distance measure and distance based on min-max. The results of the Table 19 show that the distance measures based on the min-max approach are new and practical.

7. Conclusions

Intuitionistic hypersoft set theory is highly beneficial in solving decision-making issues, but it only deals with attributes of alternatives about characteristics. Thus, direct comparison of two sets of variables is not easy, and if a DM wants to analyze the comparison between two sets, then it can be done with the help of similarity measures and distance measures. To cope with situations involving multiple criteria and multiple-attribute decision-making problems, we suggested several distances and similarity measures for IFHSS by using aggregate operators. Also, we have proved some new results, theorems and axioms to check the validity of the defined study and discussed a real-life problem. The direction of this research work can be extended to the neutrosophic hypersoft set, m-polar hypersoft set, bipolar hypersoft set [42], indetermhypersoft set [43] and other uncertain environments.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. L. A. Zadeh, Fuzzy sets as a basis for a theory of possibility, *Fuzzy Set. Syst.*, **1** (1978), 3–28. https://doi.org/10.1016/0165-0114(78)90029-5
2. K. M. Lee, K. LEE, K. J. CIOS, Comparison of interval-valued fuzzy sets, intuitionistic fuzzy sets, and bipolar-valued fuzzy sets, *Comput. Inform. Technol.*, 2001, 433–439. https://doi.org/10.1142/9789812810885_0055
3. C. P. Pappis, C. I. Siettos, T. K. Dasaklis, *Fuzzy Sets, systems, and applications*, Springer, Boston, MA, 2013. https://doi.org/10.1007/978-1-4419-1153-7_370
4. K. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Set. Syst.*, **20** (1986), 87–96. https://doi.org/10.1016/S0165-0114(86)80034-3
5. Y. Liu, G. Wang, L. Feng, *A general model for transforming vague sets into fuzzy sets*, Springer, Berlin, Heidelberg, 2008. https://doi.org/10.1007/978-3-540-87563-5_8
6. F. Smarandache, Neutrosophic set a generalization of the intuitionistic fuzzy set, *Int. J. Pure Appl. Math.*, **24** (2005), 287–297.
7. D. Molodtsov, Soft set theory first results, *Comput. Math. Appl.*, **37** (1999), 19–31. https://doi.org/10.1016/S0898-1221(99)00056-5
8. H. Aktaş, N. Çağman, Soft sets and soft groups, *Inform. Sci.*, **177** (2007), 2726–2735. https://doi.org/10.1016/j.ins.2006.12.008
9. M. I. Ali, F. Feng, X. Liu, W. K. Min, M. Shabir, On some new operations in soft set theory, *Comput. Math. Appl.*, **57** (2009), 1547–1553. https://doi.org/10.1016/j.camwa.2008.11.009
10. Y. Zou, Z. Xiao, Data analysis approaches of soft sets under incomplete information, *Knowl.-Based Syst.*, **21** (2008), 941–945. https://doi.org/10.1016/j.knosys.2008.04.004
11. P. K. Maji, R. Biswas, A. R. Roy, Fuzzy soft sets, *J. Fuzzy Math.*, **9** (2001), 589–602.
12. P. Majumdar, S. K. Samanta, Generalised fuzzy soft sets, *Comput. Math. Appl.*, **59** (2010), 1425–1432. https://doi.org/10.1016/j.camwa.2009.12.006
13. C. Naim, S. Karataş, Intuitionistic fuzzy soft set theory and its decision making, *J. Intell. Fuzzy Syst.*, **24** (2013), 829–836.
14. Z. Liang, P. Shi, Similarity measures on intuitionistic fuzzy sets, *Pattern Recogn. Lett.*, **24** (2003), 2687–2693. https://doi.org/10.1016/S0167-8655(03)00111-9
15. S. K. De, R. Biswas, A. R. Roy, An application of intuitionistic fuzzy sets in medical diagnosis, *Fuzzy Set. Syst.*, **117** (2001), 209–213. https://doi.org/10.1016/S0165-0114(98)00235-8
16. P. A. Ejegwa, A. J. Akubo, O. M. Joshua, Intuitionistic fuzzy set and its application in career determination via normalized Euclidean distance method, *Eur. Sci. J.*, **10** (2014), 529–536.
17. D. F. Li, Some measures of dissimilarity in intuitionistic fuzzy structures, *J. Comput. Syst. Sci.*, **68** (2004), 115–122. https://doi.org/10.1016/j.jcss.2003.07.006

18. E. Szmidt, J. Kacprzyk, Intuitionistic fuzzy sets in group decision making, *Note. Intuition. Fuzzy Set.*, **2** (1996),15–32. Available from: http://ifigenia.org/wiki/issue:nifs/2/1/15-32.
19. C. P. Wei, P. Wang, Y. Z. Zhang, Entropy, similarity measure of interval-valued intuitionistic fuzzy sets and their applications, *Inform. Sci.*, **181** (2011), 4273–4286. https://doi.org/10.1016/j.ins.2011.06.001
20. M. N. Jafar, A. Saeed, M. Waheed, A. Shafiq, A comprehensive study of intuitionistic fuzzy soft matrices and its applications in selection of laptop by using score function, *Int. J. Comput. Appl.*, **177** (2020), 8–17. https://doi.org/10.5120/ijca2020919844
21. B. H. Mitchell, On the Dengfeng-Chuntian similarity measure and its application to pattern recognition, *Pattern Recogn. Lett.*, **24** (2003), 3101–3104. https://doi.org/10.1016/S0167-8655(03)00169-7
22. F. Smarandache, Extension of soft set to hypersoft set, and then to plithogenic hypersoft set, *Neutrosophic Sets Syst.*, **22** (2018), 168–170. https://doi.org/10.5281/zenodo.2159754
23. R. M. Zulqarnain, X. L. Xin, M. Saeed, Extension of TOPSIS method under intuitionistic fuzzy hypersoft environment based on correlation coefficient and aggregation operators to solve decision making problem, *AIMS Math.*, **6** (2021), 2732–2755. https://doi.org/10.3934/math.2021167
24. A. Yolcu, T. Y. Öztürk, *Fuzzy hypersoft sets and its application to decision-making*, Theory and Application of Hypersoft Set, Belgium, Brussels: Pons Publishing House, 2021, 138–154.
25. S. Debnath, Fuzzy hypersoft sets and its weightage operator for decision making, *J. Fuzzy Ext. Appl.*, **2** (2021), 163–170. https://doi.org/10.22105/jfea.2021.275132.1083
26. A. Yolcu, F. Smarandache, T. Y. Öztürk, Intuitionistic fuzzy hypersoft sets, *Commun. Fac. Sci. Univ.*, **70** (2021), 443–455. https://doi.org/10.31801/cfsuasmas.788329
27. R. M. Zulqarnain, I. Siddique, R. Ali, D. Pamucar, D. Marinkovic, D. Bozanic, Robust aggregation operators for intuitionistic fuzzy hypersoft set with their application to solve mcdm problem, *Entropy*, **23** (2021), 688. https://doi.org/10.3390/e23060688
28. R. M. Zulqarnain, I. Siddique, F. Jarad, H. Karamti, A. Iampan, Aggregation operators for interval-valued intuitionistic fuzzy hypersoft set with their application in material selection, *Math. Probl. Eng.*, 2022, 1–21. https://doi.org/10.1155/2022/8321964
29. A. Ashraf, K. Ullah, A. Hussain, M. Bari, Interval-valued picture fuzzy Maclaurin symmetric mean operator with application in multiple attribute decision-making, *Rep. Mech. Eng.*, **3** (2022), 301–317. https://doi.org/10.31181/rme20020042022a
30. M. Riaz, H. M. A. Farid, Picture fuzzy aggregation approach with application to third-party logistic provider selection process, *Rep. Mech. Eng.*, **3** (2022), 318–327. https://doi.org/10.31181/rme20023062022r
31. S. Broumi, R. Sundareswaran, M. Shanmugapriya, G. Nordo, M. Talea, A. Bakali, et al., Interval-valued fermatean neutrosophic graphs, *Decis. Making Appl. Manag. Eng.*, **5** (2022), 176–200. https://doi.org/10.31181/dmame0311072022b
32. A. K. Das, C. Granados, FP-intuitionistic multi fuzzy N-soft set and its induced FP-Hesitant N soft set in decision-making, *Decis. Making Appl. Manag. Eng.*, **5** (2022), 67–89. https://doi.org/10.31181/dmame181221045d
33. S. Das, R. Das, S. Pramanik, Single valued pentapartitioned neutrosophic graphs, *Neutrosophic Sets Syst.*, **50** (2022), 225–238. https://doi.org/10.5281/zenodo.6774779
34. M. H. Sowlat, H. Gharibi, M. Yunesian, M. T. Mahmoudi, S. Lotfi, A novel, fuzzy-based air quality index (FAQI) for air quality assessment, *Atmos. Environ.*, **45** (2011), 2050–2059. https://doi.org/10.1016/j.atmosenv.2011.01.060

35. A. Kumar, P. Goyal, Forecasting of daily air quality index in Delhi, *Sci. Total Environ.*, **409** (2011), 5517–5523. https://doi.org/10.1016/j.scitotenv.2011.08.069
36. D. Zhan, M. P. Kwan, W. Zhang, X. Yu, B. Meng, Q. Liu, The driving factors of air quality index in China, *J. Clean. Prod.*, **197** (2018), 1342–1351. https://doi.org/10.1016/j.jclepro.2018.06.108
37. M. Saqlain, S. Moin, M. N. Jafar, M. Saeed, S. Broumi, Single and multi-valued neutrosophic hypersoft set and tangent similarity measure of single valued neutrosophic hypersoft sets, *Neutrosophic Sets Syst.*, **32** (2020), 317–329. https://doi.org/10.5281/zenodo.3723165
38. M. N. Jafar, M. Saeed, M. Saqlain, M. S. Yang, Trigonometric similarity measures for neutrosophic hypersoft sets with application to renewable energy source selection, *IEEE Access*, **9** (2021), 129178–129187. https://doi.org/10.1109/ACCESS.2021.3112721
39. M. Riaz, M. R. Hashmi, Linear Diophantine fuzzy set and its applications towards multi-attribute decision-making problems, *J. Intell. Fuzzy Syst.*, **37** (2019), 5417–5439. https://doi.org/10.3233/JIFS-190550
40. M. Riaz, R. M. Hashmi, D. Pamucar, Y. M. Chu, Spherical linear Diophantine fuzzy sets with modeling uncertainties in MCDM, *Comp. Model. Eng. Sci.*, **126** (2021), 1125–1164. https://doi.org/10.32604/cmes.2021.013699
41. M. Riaz, A. Habib, M. Saqlain, M. S. Yang, Cubic bipolar fuzzy-VIKOR method using new distance and entropy measures and Einstein averaging aggregation operators with application to renewable energy, *Int. J. Fuzzy Syst.*, 2022, 1–34. https://doi.org/10.1007/s40815-022-01383-z
42. S. Y. Musa, B. A. Asaad, Bipolar hypersoft sets, *Mathematics*, **9** (2021), 1826. https://doi.org/10.3390/math9151826
43. F. Smarandache, Introduction to the indetermsoft set and indetermhypersoft set, *Neutrosophic Sets Sy.*, **50** (2022), 629–650. https://doi.org/10.5281/zenodo.6774960