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Research article

Equivalence of novel IH-implicit fixed point algorithms for a general class of contractive maps

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Abstract: In this paper, a novel implicit IH-multistep fixed point algorithm and convergence result for a general class of contractive maps is introduced without any imposition of the "sum conditions" on the countably finite family of the iteration parameters. Furthermore, it is shown that the convergence of the proposed iteration scheme is equivalent to some other implicit IH-type iterative schemes (e.g., implicit IH-Noor, implicit IH-Ishikawa and implicit IH-Mann) for the same class of maps. Also, some numerical examples are given to illustrate that the equivalence is true. Our results complement, improve and unify several equivalent results recently announced in literature.

Keywords: strong convergence; implicit multistep IH-iterative scheme; real Hilbert space; general contractive operator; normed linear space
Mathematics Subject Classification: 47H09, 47H10, 47J05, 65J15

1. Introduction

The problem of solving a nonlinear equation and that of approximating fixed points of corresponding contractive-type mapping are closely related. In line with this, there is a practical and theoretical interest in finding approximate fixed points of various contractive-type operators. Existing literature is filled with several methods for achieving this.

Let (Z, d) be a complete metric space and $\Gamma : Z \longrightarrow Z$ a self-map of Z. Assume that $F(\Gamma) = \{q \in Z : \Gamma q = q\}$ is the set of fixed points of Γ.

In recent years, implicit iterative schemes for approximating fixed points of nonlinear mappings have attracted the attention of different researchers all over the world. Regarding this direction, some authors have explored implicit iterations in terms of their qualitative features with regard to convergence, stability and equivalence of convergence in various spaces (see [1–8], and the references therein). Implicit iterations are indispensable from a numerical point of view due to the fact that they give an accurate approximation as compared to explicit iterations. Using computer-oriented programs, it has been observed that approximation of a fixed point via implicit iterative schemes has the potential to reduce the computational cost of the fixed-point problem (see [4] for more details). Other areas in which iteration techniques have found practical values are in solving the root-finding problems (see [9, 10]) and in generating fractal patterns (for details see [11]). In the area of the convergence of implicit and explicit iterations in different spaces, numerous research papers have been published (see [12–39] and the references contained in them).

In computational mathematics, it is of paramount importance (theoretically and practically) to check for the equivalence of iterations so as to avoid duplication of results. For recent works in this direction, see [3, 40] and the references contained therein. Among the works relating to the Kirk-type iteration scheme and equivalence of convergence results, the results in [37] caught our attention for the obvious reason that the sum conditions imposed on the countably finite family of the iteration parameters are too strong and could constitute a computational hazard for the effective implementation of the iterative scheme in applications. For instance, considering the explicit iterative method (1.1) as described in [37], the iterative scheme of the sequence $\{y_n\}_{n=0}^{\infty}$ is defined by

$$
\begin{aligned}
y_{n+1} &= \gamma_{n,0} z_n^1 + \sum_{r=1}^{\ell_1} \gamma_{n,k} \Gamma^r z_n^1, \sum_{r=0}^{\ell_1} \alpha_{n,r} = 1; \\
z_n^t &= \alpha_{n,0}^t z_n^{t+1} + \sum_{s=1}^{\ell_{t+1}} \alpha_{n,s}^t \Gamma^s z_n^{t+1}, \sum_{s=0}^{\ell_{t+1}} \alpha_{n,s}^t = 1, t = 1, 2, \cdots, u-2; \\
z_n^{u-1} &= \sum_{s=0}^{\ell_u} \alpha_{n,t}^{u-1} \Gamma^s y_n, \sum_{s=0}^{\ell_u} \alpha_{n,t}^{u-1} = 1, u \geq 2, n \geq 0,
\end{aligned}
\tag{1.1}
$$

where y_0 is an arbitrary point in X, $\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq \ell_u$ for each u, $\alpha_{n,s}^t, \gamma_{n,t} \geq 0$ and $\gamma_{n,0}, \alpha_{n,0}, \neq 0$ for each i, $\alpha_{n,s}^i, \gamma_{n,t} \in [0, 1], t = 1, 2, \cdots, u-2, s = 0, 1, 2, \cdots, \ell_u$ and ℓ_1 and ℓ_u are fixed integers for each u (which is a generalization of different explicit iterations), which introduces some inherent challenges with the topmost being that of the necessary and sufficient condition for the convergence of (1.1) to the fixed point of the contractive-type mapping Γ. This condition requires that the sum of the countably finite family of the iteration parameters be at unity (i.e., $\sum_{r=0}^{\ell_1} \alpha_{n,r} = 1; \sum_{s=0}^{\ell_{t+1}} \alpha_{n,s}^t = 1$ and $\sum_{s=0}^{\ell_u} \alpha_{n,t}^{u-1} = 1$) which, as explained above, is not only complex and time consuming but also mandates a huge computational cost.

In view of the aforementioned challenges, it becomes pertinent to ask the following question:

Question 1.1. *Can it be possible to construct a more effective implicit multistep iterative scheme that will address the challenges mentioned above and still maintain the results in [3]?*

To address these challenges, Agwu and Igbokwe [41,42] introduced the following explicit iterative schemes:

Let H be a Hilbert space and let $\Gamma : H \longrightarrow H$ be a self-map of X. For an arbitrary $x_0 \in H$ define the sequence $\{x_n\}_{n=0}^{\infty}$ iteratively, for $s = 1, 2, \cdots, k-2$, as follows:

$$\begin{cases} x_{n+1} = \delta_{n,1}x_n + \sum_{j=2}^{\ell_1} \delta_{n,j} \prod_{i=1}^{j-1}(1-\delta_{n,i})\Gamma^{j-1}y_n^1 + \prod_{i=1}^{\ell_1}(1-\delta_{n,i})\Gamma^{\ell_1}y_n^1; \\ y_n^s = \alpha_{n,1}^s x_n + \sum_{j=2}^{\ell_{s+1}} \alpha_{n,j}^s \prod_{i=1}^{j-1}(1-\alpha_{n,i}^s)\Gamma^{j-1}y_n^{s+1} + \prod_{i=1}^{\ell_{s+1}}(1-\alpha_{n,i}^s)\Gamma^{\ell_1}y_n^{s+1}; \\ y_n^{k-1} = \sum_{j=1}^{\ell_k} \alpha_{n,j}^{k-1} \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{k-1})\Gamma^{j-1}x_n + \prod_{i=1}^{\ell_k}(1-\alpha_{n,i}^{k-1})\Gamma^{\ell_k}x_n, k \geq 2, n \geq 1, \end{cases} \tag{1.2}$$

where $\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq \ell_k$, for each j, $\{\{\delta_{n,j}\}_{n=0}^{\infty}\}_{j=1}^{\ell_k}, \{\{\alpha_{n,j}\}_{n=0}^{\infty}\}_{j=1}^{\ell_k} \in [0,1]$ for each k and $\ell_1, \ell_2, \cdots, \ell_k$ are fixed integers (for each k). The iteration scheme defined by (1.2) is called the multistep IH-iteration scheme.

Again, for any $x_0 \in X$, the sequence $\{x_n\}_{n=0}^{\infty}$ is defined recursively, for $s = 1, 2, \cdots, k-2$, by

$$\begin{cases} x_{n+1} = \delta_{n,1}y_n^1 + \sum_{j=2}^{\ell_1} \delta_{n,j} \prod_{i=1}^{j-1}(1-\delta_{n,i})\Gamma^{j-1}y_n^1 + \prod_{i=1}^{\ell_1}(1-\delta_{n,i})\Gamma^{\ell_1}y_n^1; \\ y_n^s = \alpha_{n,1}^s y_n^{s+1} + \sum_{j=2}^{\ell_{s+1}} \alpha_{n,j}^s \prod_{i=1}^{j-1}(1-\alpha_{n,i}^s)\Gamma^{j-1}y_n^{s+1} + \prod_{i=1}^{\ell_{s+1}}(1-\alpha_{n,i}^s)\Gamma^{\ell_1}y_n^{s+1}; \\ y_n^{k-1} = \sum_{j=1}^{\ell_k} \alpha_{n,j}^{k-1} \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{k-1})\Gamma^{j-1}x_n + \prod_{i=1}^{\ell_k}(1-\alpha_{n,i}^{k-1})\Gamma^{\ell_k}x_n, k \geq 2, n \geq 1, \end{cases} \tag{1.3}$$

where $\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq \ell_k$, for each j, $\{\{\delta_{n,j}\}_{n=0}^{\infty}\}_{j=1}^{\ell_k}, \{\{\alpha_{n,j}\}_{n=0}^{\infty}\}_{j=1}^{\ell_k} \in [0,1]$ for each k and $\ell_1, \ell_2, \cdots, \ell_k$ are fixed integers (for each k); this is called a multistep DI-iteration scheme. Using (1.2) and (1.3), Agwu and Igbokwe [41,42] achieved strong convergence and stability results without any imposition of the sum conditions on the control sequences.

The above iteration techniques deal with explicit iterations. The case of implicit iterative schemes have not been fully employed to examine the fixed points of nonlinear problems in recent times. Following the results of Chugh et al. [43], in which the authors proved convergence of faster implicit iterative schemes and remarked that this type of scheme has an advantage over the corresponding explicit iterative scheme for nonlinear problems (as they are widely used in many applications when explicit iterative schemes are inefficient), several researchers have concentrated their efforts in this direction.

Most recently, Bosede et al. [3] invented the following implicit multistep iterative scheme: Define the sequence $\{z_n\}_{n=0}^{\infty}$ by

$$\begin{aligned} z_{n+1} &= \gamma_{n,0}z_n^1 + \sum_{r=1}^{\ell_1} \gamma_{n,k}\Gamma^r z_{n+1}, \sum_{r=0}^{\ell_1} \alpha_{n,r} = 1; \\ z_n^t &= \alpha_{n,0}^t z_n^{t+1} + \sum_{s=1}^{\ell_{t+1}} \alpha_{n,s}^t \Gamma^s z_n^t, \sum_{s=0}^{\ell_{t+1}} \alpha_{n,s}^t = 1, t = 1, 2, \cdots, u-2; \\ z_n^{u-1} &= \alpha_{n,t}^{u-1} z_n + \sum_{s=1}^{\ell_u} \alpha_{n,t}^{u-1}\Gamma^s z_n^{u-1}, \sum_{s=0}^{\ell_u} \alpha_{n,t}^{u-1} = 1, u \geq 2, n \geq 0, \end{aligned} \tag{1.4}$$

where z_0 is an arbitrary point in X, $\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq \ell_u$ for each u, $\alpha_{n,s}^t, \gamma_{n,t} \geq 0$ and $\gamma_{n,0}, \alpha_{n,0}, \neq 0$ for each i, $\alpha_{n,s}^i, \gamma_{n,t} \in [0,1], t = 1, 2, \cdots, u-2, s = 0, 1, 2, \cdots, \ell_u$ and ℓ_1 and ℓ_u are fixed integers for each u. Again, (1.4) is a generalization of many implicit iterative schemes (i.e., implicit Kirk-Noor, implicit

Kirk-lshikawa and implicit Kirk-Mann). Despite their usefulness, the same necessary and sufficient conditions (the sum conditions) required for the convergence of (1.1) to the fixed point of a certain contractive-type mapping Γ is evident in (1.4). Consequent to this, the following second question emerges:

Question 1.2. *Is it possible to replicate* (1.2) *for the case of an implicit multistep iterative scheme and still retain the results in [3]?*

Motivated and inspired by the results in [3,41,42] and remark in [4], in this paper, we define a novel iterative scheme for which an affirmative answer is provided for Question 1.2. See [43–45], for more details.

The remaining part of the paper is organized as follows. Section 2 considers some preliminary results required to prove our convergence theorems. Section 3 deals with the strong convergence of the implicit IH-multistep iteration scheme, implicit IH-Noor iteration scheme, implicit IH-Ishikawa iteration scheme and implicit IH-Mann iteration scheme. In Section 4, numerical examples, open problems and the conclusion are considered.

2. Preliminary

Throughout the remaining sections, $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+, \mathbb{R}^+, \mathbb{N}$ and H will denote a monotone increasing subadditive function, the set of positive real numbers, the set of natural numbers and a real Hilbert space, respectively. Also, the following definition, lemmas and propositions will be needed in order to establish our main results.

Definition 2.1. *([24]) Suppose Y is a metric space and let $\Gamma : Y \longrightarrow Y$ be a self-map of Y. Let $\{x_n\}_{n=0}^{\infty} \subseteq Y$ be a sequence generated by the iteration scheme*

$$x_{n+1} = g(\Gamma, x_n), \tag{2.1}$$

where $x_0 \in Y$ is the initial approximation and g is some function. Suppose $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point q of Γ. Let $\{t_n\}_{n=0}^{\infty} \subseteq Y$ be an arbitrary sequence and set $\epsilon_n = d(t_n, g(\Gamma, t_n)), n = 1, 2, \cdots$ Then, (2.1) *is said to be Γ-stable if and only if* $\lim_{n\to\infty} \epsilon_n = 0$ *implies* $\lim_{n\to\infty} x_n = q$.

Note that in practice, the sequence $\{t_n\}_{n=0}^{\infty}$ could be obtained the using the following approach: Let $x_0 \in Y$. Set $x_{n+1} = g(\Gamma, x_n)$ and let $t_0 = x_0$. Since, $x_1 = g(\Gamma, x_0)$, following the rounding in the function Γ, the value t_1 (which is estimated to be equal to x_1) could be calculated to give t_2, an approximate value of $g(\Gamma, t_1)$. The procedure is continued to yield the sequence $\{t_n\}_{n=0}^{\infty}$, which is approximately the same as the sequence $\{x_n\}_{n=0}^{\infty}$.

Lemma 2.1. *(See, e.g., [37]) Let $\{\tau_n\}_{n=0}^{\infty} \in \mathbb{R}^+ : \tau_n \to 0$ as $n \to \infty$. For $0 \le \delta < 1$, let $\{w_n\}_{n=0}^{\infty}$ be a sequence of positive numbers satisfying $w_{n+1} \le \delta w_n + \tau_n, n = 0, 1, 2, \cdots$. Then, $w_n \to 0$ as $n \to \infty$.*

Lemma 2.2. *([28]) Let $(Y, \|.\|)$ be a normed space and $\Gamma : Y \longrightarrow Y$ a selfmap of Y. Let $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be monotonic increasing subadditive function such that $\phi(0) = 0$ and $\phi(Mr) = M\phi(r)$ for all $0 \le \rho < 1, M \ge 0$ and $r \in \mathbb{R}^+$. Then, $\forall i \in \mathbb{N}$ and $\forall s, t \in Y$; we have*

$$\|\Gamma^j s - \Gamma^j t\| \le \rho^j \|s - t\| + \sum_{i=0}^{j} \binom{j}{i} \rho^{j-1} \phi(\|s - \Gamma s\|). \tag{2.2}$$

Proposition 2.1. *([38]) Let $\{\alpha_i\}_{i=k}^{N} \subseteq \mathbb{R}$ be a countable subset of the set of real numbers $\mathbb{R}$, where k is a fixed nonnegative integer and $N \in \mathbb{N}$ is any integer with $k+1 \leq N$. Then, the following identity holds:*

$$\alpha_k + \sum_{i=k+1}^{N} \alpha_i \prod_{j=k}^{i-1}(1-\alpha_j) + \prod_{j=k}^{N}(1-\alpha_j) = 1. \tag{2.3}$$

Proposition 2.2. *([38]) Let k be a fixed nonnegative integer, $t, u, v \in H$ and $N \in \mathbb{N}$ with $k+1 \leq N$. Let $\{v_i\}_{i=1}^{N-1} \subseteq H$ and $\{\alpha_i\}_{i=1}^{N} \subseteq [0,1]$. Define*

$$y = \alpha_k t + \sum_{i=k+1}^{N} \alpha_i \prod_{j=k}^{i-1}(1-\alpha_j)v_{i-1} + \prod_{j=k}^{N}(1-\alpha_j)v.$$

Then,

$$\begin{aligned}
\|y-u\|^2 \quad = \quad & \alpha_k\|t-u\|^2 + \sum_{i=k+1}^{N} \alpha_i \prod_{j=k}^{i-1}(1-\alpha_j)\|v_{i-1}-u\|^2 + \prod_{j=k}^{N}(1-\alpha_j)\|v-u\|^2 \\
& -\alpha_k\Big[\sum_{i=k+1}^{N} \alpha_i \prod_{j=k}^{i-1}(1-\alpha_j)\|t-v_{i-1}\|^2 + \prod_{j=k}^{i-1}(1-\alpha_j)\|t-v\|^2\Big] \\
& -(1-\alpha_k)\Big[\sum_{i=k+1}^{N} \alpha_i \prod_{j=k}^{i-1}(1-\alpha_j)\|v_{i-1}-(\alpha_{i+1}+w_{i+1})\|^2 \\
& +\alpha_N \prod_{j=k}^{i-1}(1-\alpha_j)\|v-v_{N-1}\|^2\Big],
\end{aligned}$$

where $w_k = \sum_{i=k+1}^{N} \alpha_i \prod_{j=k}^{i-1}(1-\alpha_j)v_{i-1} + \prod_{j=k}^{i-1}(1-\alpha_j)v, k = 1, 2, \cdots, N$ and $w_n = (1-c_n)v$.

3. Main results

In this section, we introduce the following implicit IH-type iterative schemes.

Let $(Z, |.\|)$ be a normed linear space, E a nonempty closed convex subset of Z and $\Gamma : E \longrightarrow E$ a self-map of E. For an arbitrary $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ is defined iteratively, for $j = 1, 2, \cdots, r-2$, by

$$\begin{cases}
x_{n+1} = \delta_{n,1}x_n^{(1)} + \sum_{i=2}^{\ell_1} \delta_{n,i} \prod_{t=1}^{i-1}(1-\delta_{n,t})\Gamma^{i-1}x_{n+1} + \prod_{t=1}^{\ell_1}(1-\delta_{n,t})\Gamma^{\ell_1}x_{n+1}; \\
x_n^{(j)} = \gamma_{n,1}^{j}x_n^{(j+1)} + \sum_{i=2}^{\ell_{j+1}} \gamma_{n,i}^{j} \prod_{t=1}^{i-1}(1-\gamma_{n,t}^{j})\Gamma^{i-1}x_n^{(j)} + \prod_{t=1}^{\ell_{j+1}}(1-\gamma_{n,t}^{j})\Gamma^{\ell_{j+1}}x_n^{(j)}; \\
x_n^{(r-1)} = \gamma^{r-1}x_n + \sum_{i=2}^{\ell_r} \gamma_{n,i}^{r-1} \prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-1})\Gamma^{i-1}x_n^{(r-1)} + \prod_{t=1}^{\ell_r}(1-\gamma_{n,t}^{r-1})\Gamma^{\ell_r}x_n^{(r-1)}, r \geq 2,
\end{cases} \tag{3.1}$$

where $n \geq 1, \ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq \ell_q$ for each j, $\delta_{n,i} \geq 0, \delta_{n,1} \neq 0, \gamma_{n,i}^{j} \geq 0,$ and $\gamma_{n,1}^{j} \neq 0$ for each j, $\{\{\delta_{n,i}\}_{n=0}^{\infty}\}_{i=1}^{\ell_r}, \{\{\gamma_{n,i}^{j}\}_{n=0}^{\infty}\}_{i=1}^{\ell_q} \in [0,1]$ for each j and $\ell_1, \ell_2, \cdots, \ell_r$ are fixed integers (for each j); this is called an implicit IH-multistep iteration.

Equation (3.1) represents a general iteration scheme for getting other implicit IH-type iterations. Indeed, if $r = 3$ in (3.1), we get a three-step implicit IH-Noor iteration, as follows:

$$\begin{cases} x_{n+1} = \delta_{n,1}x_n^{(1)} + \sum_{i=2}^{\ell_1}\delta_{n,i}\prod_{t=1}^{i-1}(1-\delta_{n,t})\Gamma^{i-1}x_{n+1} + \prod_{t=1}^{\ell_1}(1-\delta_{n,t})\Gamma^{\ell_1}x_{n+1}; \\ x_n^{(1)} = \gamma_{n,1}^1 x_n^{(2)} + \sum_{i=2}^{\ell_2}\gamma_{n,i}^1\prod_{t=1}^{i-1}(1-\gamma_{n,t}^1)\Gamma^{i-1}x_n^{(1)} + \prod_{t=1}^{\ell_2}(1-\gamma_{n,t}^1)\Gamma^{\ell_2}x_n^{(1)}; \\ x_n^{(2)} = \gamma_{n,1}^2 x_n + \sum_{i=2}^{\ell_3}\gamma_{n,i}^2\prod_{t=1}^{i-1}(1-\gamma_{n,t}^2)\Gamma^{i-1}x_n^{(2)} + \prod_{t=1}^{\ell_3}(1-\gamma_{n,t}^2)\Gamma^{\ell_3}x_n^{(2)}, n \geq 1, \end{cases} \tag{3.2}$$

where $\ell_1 \geq \ell_2 \geq \ell_3$, $\delta_{n,i} \geq 0, \delta_{n,1} \neq 0, \gamma_{n,1}^1, \gamma_{n,1}^2 \geq 0, \gamma_{n,1}^1 \neq 0, \gamma_{n,1}^2 \neq 0, \{\{\delta_{n,i}\}_{n=0}^{\infty}\}_{i=1}^{\ell_3}, \{\{\gamma_{n,i}^j\}_{n=0}^{\infty}\}_{i=1}^{\ell_3} \in [0,1]$ and ℓ_1, ℓ_2 and ℓ_3 are fixed positive integers. Again, if $r = 2$ in (3.1), we get a two-step implicit IH-lshikawa iteration as shown below:

$$\begin{cases} x_{n+1} = \delta_{n,1}x_n^{(1)} + \sum_{i=2}^{\ell_1}\delta_{n,i}\prod_{t=1}^{i-1}(1-\delta_{n,t})\Gamma^{i-1}x_{n+1} + \prod_{t=1}^{\ell_1}(1-\delta_{n,t})\Gamma^{\ell_1}x_{n+1}; \\ x_n^{(1)} = \gamma_{n,1}^1 x_n + \sum_{i=2}^{\ell_2}\gamma_{n,i}^1\prod_{t=1}^{i-1}(1-\gamma_{n,t}^1)\Gamma^{i-1}x_n^1 + \prod_{t=1}^{\ell_2}(1-\gamma_{n,t}^1)\Gamma^{\ell_2}x_n^1, \end{cases} \tag{3.3}$$

where $\ell_1 \geq \ell_2$, $\delta_{n,i} \geq 0, \delta_{n,1} \neq 0, \gamma_{n,1}^1 \geq 0, \gamma_{n,1}^1 \neq 0, \{\{\delta_{n,i}\}_{n=0}^{\infty}\}_{i=1}^{\ell_2}, \{\{\gamma_{n,i}^j\}_{n=0}^{\infty}\}_{i=1}^{\ell_2} \in [0,1]$ and ℓ_1 and ℓ_2 are fixed positive integers. Lastly, if $r = 2$ *and* $\ell_2 = 0$ in (3.1), we have a one-step implicit IH-Mann iterative scheme as follows:

$$x_{n+1} = \delta_{n,1}x_n^{(1)} + \sum_{i=2}^{\ell_1}\delta_{n,i}\prod_{t=1}^{i-1}(1-\delta_{n,t})\Gamma^{i-1}x_{n+1} + \prod_{t=1}^{\ell_1}(1-\delta_{n,t})\Gamma^{\ell_1}x_{n+1}, \tag{3.4}$$

where $\delta_{n,i} \geq 0, \delta_{n,1} \neq 0, \{\{\delta_{n,i}\}_{n=0}^{\infty}\}_{i=1}^{\ell_1} \in [0,1]$ and ℓ_1 is a fixed positive integer.

For the sake of convenience, especially in our attempt to prove our proposed equivalence, (3.2)–(3.4) shall be rewritten in the manner shown below: Let $(Z, |.||)$ be a normed linear space, E a nonempty closed convex subset of Z and $\Gamma : E \longrightarrow E$ a self-map of E. For an arbitrary $w_0 \in E$, the sequence $\{w_n\}_{n=0}^{\infty}$ is defined iteratively by

$$\begin{cases} w_{n+1} = \delta_{n,1}w_n^{(1)} + \sum_{i=2}^{\ell_1}\delta_{n,i}\prod_{t=1}^{i-1}(1-\delta_{n,t})\Gamma^{i-1}w_{n+1} + \prod_{t=1}^{\ell_1}(1-\delta_{n,t})\Gamma^{\ell_1}w_{n+1}; \\ w_n^{(1)} = \gamma_{n,1}^1 w_n^{(2)} + \sum_{i=2}^{\ell_2}\gamma_{n,i}^1\prod_{t=1}^{i-1}(1-\gamma_{n,t}^1)\Gamma^{i-1}w_n^{(1)} + \prod_{t=1}^{\ell_2}(1-\gamma_{n,t}^1)\Gamma^{\ell_2}w_n^{(1)}; \\ w_n^{(2)} = \gamma_{n,1}^2 w_n + \sum_{i=2}^{\ell_3}\gamma_{n,i}^2\prod_{t=1}^{i-1}(1-\gamma_{n,t}^2)\Gamma^{i-1}w_n^{(2)} + \prod_{t=1}^{\ell_3}(1-\gamma_{n,t}^2)\Gamma^{\ell_3}w_n^{(2)}, n \geq 1, \end{cases} \tag{3.5}$$

where $\ell_1 \geq \ell_2 \geq \ell_3$, $\delta_{n,i} \geq 0, \delta_{n,1} \neq 0, \gamma_{n,1}^1, \gamma_{n,1}^2 \geq 0, \gamma_{n,1}^1 \neq 0, \gamma_{n,1}^2 \neq 0, \{\{\delta_{n,i}\}_{n=0}^{\infty}\}_{i=1}^{\ell_3}, \{\{\gamma_{n,i}^j\}_{n=0}^{\infty}\}_{i=1}^{\ell_3} \in [0,1]$ and ℓ_1, ℓ_2 and ℓ_3 are fixed positive integers; this is called an implicit HI-Noor iterative scheme. Further, for an arbitrary $z_0 \in E$, a two-step implicit IH-lshikawa iteration will be defined as follows:

$$\begin{cases} z_{n+1} = \delta_{n,1}z_n^{(1)} + \sum_{i=2}^{\ell_1}\delta_{n,i}\prod_{t=1}^{i-1}(1-\delta_{n,t})\Gamma^{i-1}z_{n+1} + \prod_{t=1}^{\ell_1}(1-\delta_{n,t})\Gamma^{\ell_1}z_{n+1}; \\ z_n^{(1)} = \gamma_{n,1}^1 z_n + \sum_{i=2}^{\ell_2}\gamma_{n,i}^1\prod_{t=1}^{i-1}(1-\gamma_{n,t}^1)\Gamma^{i-1}z_n^1 + \prod_{t=1}^{\ell_2}(1-\gamma_{n,t}^1)\Gamma^{\ell_2}z_n^1, \end{cases} \tag{3.6}$$

where $\ell_1 \geq \ell_2$, $\delta_{n,i} \geq 0, \delta_{n,1} \neq 0, \gamma_{n,1}^1 \geq 0, \gamma_{n,1}^1 \neq 0, \{\{\delta_{n,i}\}_{n=0}^{\infty}\}_{i=1}^{\ell_2}, \{\{\gamma_{n,i}^j\}_{n=0}^{\infty}\}_{i=1}^{\ell_2} \in [0,1]$ and ℓ_1 *and* ℓ_2 are fixed positive integers. Also, for an arbitrary $u_0 \in E$, a one-step implicit IH-Mann iterative scheme will be defined as follows:

$$u_{n+1} = \delta_{n,1}u_n^{(1)} + \sum_{i=2}^{\ell_1}\delta_{n,i}\prod_{t=1}^{i-1}(1-\delta_{n,t})\Gamma^{i-1}u_{n+1} + \prod_{t=1}^{\ell_1}(1-\delta_{n,t})\Gamma^{\ell_1}u_{n+1}, \tag{3.7}$$

where $\delta_{n,i} \geq 0, \delta_{n,1} \neq 0, \{\{\delta_{n,i}\}_{n=0}^{\infty}\}_{i=1}^{\ell_1} \in [0,1]$ and ℓ_1 is a fixed positive integer.

Remark 3.1. *If* $\ell_1 = \ell_2 = \ell_3 = 2$, $\delta_{n,1} = \delta_n, \delta_{n,2} = \beta_n, \gamma^1_{n,1} = \gamma_n, \gamma^1_{n,2} = \alpha_n, \gamma^2_{n,1} = \sigma_n, \gamma^2_{n,2} = \tau_n$ *and* $\Gamma^2 = \Gamma$, *then we obtain the following iteration method from* (3.5)*:*

$$\begin{cases} w_{n+1} = \delta_n w_n^{(1)} + (1-\delta_n)[\beta_n w_{n+1} + (1-\beta_n)\Gamma w_{n+1}], \\ w_n^{(1)} = \gamma_n w_n^{(2)} + (1-\gamma_n)[\alpha_n w_n^{(1)} + (1-\alpha_n)\Gamma w_n^{(1)}], \\ w_n^{(2)} = \sigma_n w_n + (1-\sigma_n)[\tau_n w_n^{(1)} + (1-\tau_n)\Gamma w_n^{(2)}]. \end{cases} \tag{3.8}$$

Also, if $\beta_n = \alpha_n = \tau_n = 0$ *and* $\sigma_n = 0$, *then we obtain the following well known iterative methods:*

(i) *If* $\beta_n = \alpha_n = \tau_n = 0$, *then we have*

$$\begin{cases} w_{n+1} = \delta_n w_n^{(1)} + (1-\delta_n)\Gamma w_{n+1}, \\ w_n^{(1)} = \gamma_n w_n^{(2)} + (1-\gamma_n)\Gamma w_n^{(1)}, \\ w_n^{(2)} = \sigma_n w_n + (1-\sigma_n)\Gamma w_n^{(2)}, \end{cases} \tag{3.9}$$

which is called the implicit Noor iteration method.

(ii) *If* $\sigma_n = 0$ *in* (3.9)*, then we have*

$$\begin{cases} w_{n+1} = \delta_n w_n^{(1)} + (1-\delta_n)\Gamma w_{n+1}, \\ w_n^{(1)} = \gamma_n w_n^{(2)} + (1-\gamma_n)\Gamma w_n^{(1)}, \end{cases} \tag{3.10}$$

which is called the implicit Ishikawa iteration method.

(ii) *If* $\gamma_n = 0$ *in* (3.10)*, then we have*

$$w_{n+1} = \delta_n w_n^{(1)} + (1-\delta_n)\Gamma w_{n+1}, \tag{3.11}$$

which is called the implicit Mann iteration method.

Now, we present our convergence theorems.

Theorem 3.1. *Let* H *be a real Hilbert space,* D *a nonempty closed and convex subset of* H *and* $\Gamma : D \longrightarrow D$ *a self-map of* H *satisfying the contractive condition*

$$\|\Gamma^i x - \Gamma^i y\| \le \rho\|x - y\|, \tag{3.12}$$

where $x, y \in H$ *and* $0 \le \rho^j < 1$. *For an arbitrary* $x_0 \in H$, *let* $\{x_n\}_{n=0}^{\infty}$ *be the implicit IH-multistep iteration scheme defined by* (3.1) *with* $\sum_{n=1}^{\infty}(1-\delta_{n,1}) = \infty$. *Then,*

(I) *the fixed point* q *of* Γ *satisfying condition* (3.12) *is unique;*

(II) *the implicit IH-multistep iteration scheme converges strongly to the unique fixed point* q *of* Γ.

Proof. First, we establish that the mapping Γ satisfying (3.12) has a unique fixed point. Assume there exist $q_1, q_2 \in F(\Gamma)$ and $q_1 \neq q_2$, *with* $0 < \|q_1 - q_2\|$. Then,

$$0 < \|q_1 - q_2\| = \|\Gamma q_1 - \Gamma q_2\| \le \rho^i\|q_1 - q_2\|. \tag{3.13}$$

The Eq (3.13) implies that $(1-\rho^i)\|q_1 - q_2\| \le 0$. Since $\rho \in [0, 1)$, it follows that $0 < 1 - \rho^i$ and $\|q_1 - q_2\| \le 0$. Also, since the norm is nonnegative, we get $\|q_1 - q_2\| = 0$. That is, $q_1 = q_2 = q($ *say* $)$. Therefore, q is the unique fixed point of Γ.

(III) Now, we prove that the sequence defined by (3.1) converges strongly to q. From (3.1), (3.12) and Proposition 2.4 with $x_{n+1} = y, q = u, x^{(1)} = t, k = 1, \Gamma^{i-1}x_{n+1} = v_{j-1}$ and $\Gamma^{\ell_1} = v$, we get

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|\delta_{n,1}x_n^{(1)} + \sum_{i=2}^{\ell_1}\delta_{n,i}\prod_{t=1}^{i-1}(1-\delta_{n,t})\Gamma^{i-1}x_{n+1} + \prod_{t=1}^{\ell_1}(1-\delta_{n,t})\Gamma^{\ell_1}x_{n+1} - q\|^2 \\
&\leq \delta_{n,1}\|x_n^{(1)} - q\|^2 + \sum_{i=2}^{\ell_1}\delta_{n,i}\prod_{t=1}^{i-1}(1-\delta_{n,t})\|\Gamma^{i-1}x_{n+1} - q\|^2 \\
&\quad + \prod_{t=1}^{\ell_1}(1-\delta_{n,t})\|\Gamma^{\ell_1}x_{n+1} - q\|^2 \\
&\leq \delta_{n,1}\|x_n^{(1)} - q\|^2 + \Big(\sum_{i=2}^{\ell_1}\delta_{n,i}\prod_{t=1}^{i-1}(1-\delta_{n,t})(\rho^i)^2 + \prod_{t=1}^{\ell_1}(1-\delta_{n,t})(\rho^i)^2\Big)\|x_{n+1} - q\|^2 \\
&\leq \left(\frac{\delta_{n,1}}{1 - \Big[\Big(\sum_{i=2}^{\ell_1}\delta_{n,i}\prod_{t=1}^{i-1}(1-\delta_{n,t})(\rho^i)^2 + \prod_{t=1}^{\ell_1}(1-\delta_{n,t})(\rho^i)^2\Big)\Big]}\right) \\
&\quad \times\|x_n^{(1)} - q\|^2. \qquad (3.14)
\end{aligned}$$

Again, from (3.1), (3.12) and Proposition 2.4 with $x_n^{(j)} = y, q = u, x_n^{j+1} = t, k = 1, \Gamma^{i-1}x_n^{(j)}v_{j-1}$ and $\Gamma_{\ell_2} = v$, we have the following estimates (for $j = 1$):

$$\begin{aligned}
\|x_n^{(1)} - q\|^2 &= \|\gamma_{n,1}^1 x_n^{(2)} + \sum_{i=2}^{\ell_2}\gamma_{n,i}^1\prod_{t=1}^{i-1}(1-\gamma_{n,t}^1)\Gamma^{i-1}x_n^{(1)} + \prod_{t=1}^{\ell_2}(1-\gamma_{n,t}^1)\Gamma^{\ell_2}x_n^{(1)} - q\|^2 \\
&\leq \gamma_{n,1}^1\|x_n^{(2)} - q\|^2 + \sum_{i=2}^{\ell_2}\gamma_{n,i}^1\prod_{t=1}^{i-1}(1-\gamma_{n,t}^1)\|\Gamma^{i-1}x_n^{(1)} - q\|^2 + \prod_{t=1}^{\ell_2}(1-\gamma_{n,t}^1)\|\Gamma^{\ell_2}x_n^{(1)} - q\|^2 \\
&\leq \gamma_{n,1}^1\|x_n^{(2)} - q\|^2 + \sum_{i=2}^{\ell_2}\gamma_{n,i}^1\prod_{t=1}^{i-1}(1-\gamma_{n,t}^1)(\rho^i)^2\|x_n^{(1)} - q\|^2 + \prod_{t=1}^{\ell_2}(1-\gamma_{n,t}^1(\rho^i)^2\|x_n^{(1)} - q\|^2 \\
&= \gamma_{n,1}^1\|x_n^{(2)} - q\|^2 + \Big(\sum_{i=2}^{\ell_2}\gamma_{n,i}^1\prod_{t=1}^{i-1}(1-\gamma_{n,t}^1)(\rho^i)^2 + \prod_{t=1}^{\ell_2}(1-\gamma_{n,t}^1(\rho^i)^2\Big)\|x_n^{(1)} - q\|^2 \\
&\leq \frac{\gamma_{n,1}^1}{1 - \Big[\Big(\sum_{i=2}^{\ell_2}\gamma_{n,i}^1\prod_{t=1}^{i-1}(1-\gamma_{n,t}^1)(\rho^i)^2 + \prod_{t=1}^{\ell_2}(1-\gamma_{n,t}^1(\rho^i)^2)\Big)\Big]}\|x_n^{(2)} - q\|^2. \qquad (3.15)
\end{aligned}$$

Furthermore, using (3.1), (3.12) and Proposition 2.4 with

$$x_n^{(j)} = y, q = u, x_n^{j+1} = t, k = 1, \Gamma^{i-1}x_n^{(j)}v_{j-1} \text{ and } \Gamma_{\ell_3} = v,$$

we have the following estimates (for $j = 2$):

$$\begin{aligned}
\|x_n^{(2)} - q\|^2 &= \|\gamma_{n,1}^2 x_n^{(3)} + \sum_{i=2}^{\ell_3}\gamma_{n,i}^2\prod_{t=1}^{i-1}(1-\gamma_{n,t}^2)\Gamma^{i-1}x_n^{(2)} + \prod_{t=1}^{\ell_3}(1-\gamma_{n,t}^2)\Gamma^{\ell_3}x_n^{(2)} - q\|^2 \\
&\leq \gamma_{n,1}^2\|x_n^{(3)} - q\|^2 + \sum_{i=2}^{\ell_3}\gamma_{n,i}^2\prod_{t=1}^{i-1}(1-\gamma_{n,t}^2)\|\Gamma^{i-1}x_n^{(2)} - q\|^2 + \prod_{t=1}^{\ell_3}(1-\gamma_{n,t}^2)\|\Gamma^{\ell_3}x_n^{(2)} - q\|^2
\end{aligned}$$

$$
\begin{aligned}
&\le \gamma_{n,1}^{2}\|x_n^{(3)}-q\|^2+\sum_{i=2}^{\ell_3}\gamma_{n,i}^{2}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{2})(\rho^i)^2\|x_n^{(2)}-q\|^2+\prod_{t=1}^{\ell_3}(1-\gamma_{n,t}^{2}(\rho^i)^2\|x_n^{(2)}-q\|^2\\
&= \gamma_{n,1}^{2}\|x_n^{(3)}-q\|^2+\Big(\sum_{i=2}^{\ell_3}\gamma_{n,i}^{2}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{2})(\rho)^2+\prod_{t=1}^{\ell_3}(1-\gamma_{n,t}^{2}(\rho^i)^2\Big)\|x_n^{(2)}-q\|^2\\
&\le \frac{\gamma_{n,1}^{2}}{1-\Big[\Big(\sum_{i=2}^{\ell_2}\gamma_{n,i}^{2}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{2})(\rho^i)^2+\prod_{t=1}^{\ell_3}(1-\gamma_{n,t}^{2}(\rho^i)^2\Big)\Big]}\|x_n^{(3)}-q\|^2. \qquad (3.16)
\end{aligned}
$$

Continuing in this manner, using (3.1), (3.12) and Proposition 2.4 with $x_n^{(j)}=y, q=u, x_n^{j+1}=t, k=1, \Gamma^{i-1}x_n^{(j)}v_{j-1}$ and $\Gamma_{\ell_{r-2}}=v$, we have the following estimates (with $j=r-2$ and $j=r-1$) for $\|x_n^{(r-1)}-q\|^2$ and $\|x_n^{(r-1)}-q\|^2$:

$$
\begin{aligned}
\|x_n^{(r-2)}-q\|^2 &= \|\gamma_{n,1}^{r-2}x_n^{(r-1)}+\sum_{i=2}^{\ell_{r-2}}\gamma_{n,i}^{r-2}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})\Gamma^{i-1}x_n^{(r-2)}+\prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2})\Gamma^{\ell_{r-1}}x_n^{(r-2)}-q\|^2\\
&\le \gamma_{n,1}^{r-2}\|x_n^{(r-1)}-q\|^2+\sum_{i=2}^{\ell_{r-1}}\gamma_{n,i}^{r-2}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})\|\Gamma^{i-1}x_n^{(r-2)}-q\|^2\\
&\quad+\prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2})\|\Gamma^{\ell_{r-1}}x_n^{(r-2)}-q\|^2\\
&\le \gamma_{n,1}^{r-2}\|x_n^{(r-1)}-q\|^2+\sum_{i=2}^{\ell_{r-1}}\gamma_{n,i}^{r-2}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})(\rho^i)^{r-2}\|x_n^{(r-2)}-q\|^2\\
&\quad+\prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2}(\rho^i)^2\|x_n^{(r-2)}-q\|^2\\
&= \gamma_{n,1}^{r-2}\|x_n^{(r-1)}-q\|^2+\Big(\sum_{i=2}^{\ell_{r-1}}\gamma_{n,i}^{r-2}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})(\rho^i)^2\\
&\quad+\prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2}(\rho^i)^2\Big)\|x_n^{(r-2)}-q\|^2\\
&\le \left(\frac{\gamma_{n,1}^{r-2}}{1-\Big[\Big(\sum_{i=2}^{\ell_{r-2}}\gamma_{n,i}^{r-2}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})(\rho^i)^2+\prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2}(\rho^i)^2\Big)\Big]}\right)\\
&\quad\times\|x_n^{(r-2)}-q\|^2 \qquad (3.17)
\end{aligned}
$$

and

$$
\begin{aligned}
\|x_n^{(r-1)}-q\|^2 &= \|\gamma_{n,1}^{r-1}x_n^{(r)}+\sum_{i=2}^{\ell_{r-1}}\gamma_{n,i}^{r-1}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-1})\Gamma^{i-1}x_n^{(r-1)}+\prod_{t=1}^{\ell_r}(1-\gamma_{n,t}^{r-1})\Gamma^{\ell_r}x_n^{(r-1)}-q\|^2\\
&\le \gamma_{n,1}^{r-1}\|x_n^{(r)}-q\|^2+\sum_{i=2}^{\ell_r}\gamma_{n,i}^{r-1}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-1})\|\Gamma^{i-1}x_n^{(r-1)}-q\|^2\\
&\quad+\prod_{t=1}^{\ell_r}(1-\gamma_{n,t}^{r-1})\|\Gamma^{\ell_r}x_n^{(r-2)}-q\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq \gamma_{n,1}^{r-1}\|x_n^{(r)} - q\|^2 + \sum_{i=2}^{\ell_r} \gamma_{n,i}^{r-1} \prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-1})(\rho^i)^{r-1}\|x_n^{(r-1)} - q\|^2 \\
&\quad + \prod_{t=1}^{\ell_r}(1-\gamma_{n,t}^{r-1}(\rho^i)^2\|x_n^{(r-1)} - q\|^2 \\
&= \gamma_{n,1}^{r-1}\|x_n^{(r)} - q\|^2 + \Big(\sum_{i=2}^{\ell_r} \gamma_{n,i}^{r-1} \prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-1})(\rho^i)^2 \\
&\quad + \prod_{t=1}^{\ell_r}(1-\gamma_{n,t}^{r-1}(\rho^i)^2\Big)\|x_n^{(r-1)} - q\|^2 \\
&\leq \frac{\gamma_{n,1}^{r-1}}{1-\Big[\Big(\sum_{i=2}^{\ell_{r-1}} \gamma_{n,i}^{r-1} \prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-1})(\rho^i)^2 + \prod_{t=1}^{\ell_r}(1-\gamma_{n,t}^{r-1}(\rho^i)^2\Big)\Big]} \\
&\quad \times\|x_n^{(r)} - q\|^2. \qquad (3.18)
\end{aligned}
$$

Now, from (3.14)–(3.18), we have

$$
\begin{aligned}
\|x_{n+1} - q\|^2 \leq\; & \left(\frac{\delta_{n,1}}{1-\Big[\Big(\sum_{i=2}^{\ell_1} \delta_{n,i} \prod_{t=1}^{i-1}(1-\delta_{n,t})(\rho^i)^2 + \prod_{t=1}^{\ell_1}(1-\delta_{n,t})(\rho^i)^2\Big)\Big]}\right) \\
& \times\left(\frac{\gamma_{n,1}^{1}}{1-\Big[\Big(\sum_{i=2}^{\ell_2} \gamma_{n,i}^{1} \prod_{t=1}^{i-1}(1-\gamma_{n,t}^{1})(\rho^i)^2 + \prod_{t=1}^{\ell_2}(1-\gamma_{n,t}^{1}(\rho^i)^2\Big)\Big]}\right) \\
& \times\left(\frac{\gamma_{n,1}^{2}}{1-\Big[\Big(\sum_{i=2}^{\ell_2} \gamma_{n,i}^{2} \prod_{t=1}^{i-1}(1-\gamma_{n,t}^{2})(\rho^i)^2 + \prod_{t=1}^{\ell_3}(1-\gamma_{n,t}^{2}(\rho^i)^2\Big)\Big]}\right) \\
& \times\left(\frac{\gamma_{n,1}^{r-2}}{1-\Big[\Big(\sum_{i=2}^{\ell_{r-2}} \gamma_{n,i}^{r-2} \prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})(\rho^i)^2 + \prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2}(\rho^i)^2\Big)\Big]}\right) \\
& \times\left(\frac{\gamma_{n,1}^{r-1}}{1-\Big[\Big(\sum_{i=2}^{\ell_{r-1}} \gamma_{n,i}^{r-1} \prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-1})(\rho^i)^2 + \prod_{t=1}^{\ell_r}(1-\gamma_{n,t}^{r-1}(\rho^i)^2\Big)\Big]}\right) \\
& \times\|x_n - q\|^2. \qquad (3.19)
\end{aligned}
$$

Let

$$
D_1 = 1 - \frac{\delta_{n,1}}{1-\Big[\Big(\sum_{i=2}^{\ell_1} \delta_{n,i} \prod_{t=1}^{i-1}(1-\delta_{n,t})(\rho^i)^2 + \prod_{t=1}^{\ell_1}(1-\delta_{n,t})(\rho^i)^2\Big)\Big]}.
$$

Then,

$$
\begin{aligned}
D_1 &= \frac{1-\Big[\Big(\sum_{i=2}^{\ell_1} \delta_{n,i} \prod_{t=1}^{i-1}(1-\delta_{n,t})(\rho^i)^2 + \prod_{t=1}^{\ell_1}(1-\delta_{n,t})(\rho^i)^2\Big)\Big] + \delta_{n,1}}{1-\Big[\Big(\sum_{i=2}^{\ell_1} \delta_{n,i} \prod_{t=1}^{i-1}(1-\delta_{n,t})(\rho^i)^2 + \prod_{t=1}^{\ell_1}(1-\delta_{n,t})(\rho^i)^2\Big)\Big]} \\
&\geq 1-\Big[\Big(\sum_{i=2}^{\ell_1} \delta_{n,i} \prod_{t=1}^{i-1}(1-\delta_{n,t})(\rho^i)^2 + \prod_{t=1}^{\ell_1}(1-\delta_{n,t})(\rho^i)^2\Big)\Big] + \delta_{n,1}.
\end{aligned}
$$

Consequently,

$$D_1 \leq \sum_{i=2}^{\ell_1} \delta_{n,i} \prod_{t=1}^{i-1} (1 - \delta_{n,t})(\rho^i)^2 + \prod_{t=1}^{\ell_1} (1 - \delta_{n,t})(\rho^i)^2 + \delta_{n,1}. \tag{3.20}$$

Since $\rho \in [0, 1)$, it follows that $\rho^i \leq \rho < 1$ for $i \in \mathbb{N}$. Also, since from Proposition 2.3

$$\sum_{i=2}^{\ell_1} \delta_{n,i} \prod_{t=1}^{i-1} (1 - \delta_{n,t})(\rho^i)^2 + \prod_{t=1}^{\ell_1} (1 - \delta_{n,t})(\rho^i)^2 = (1 - \delta_{n,1} - \prod_{t=1}^{\ell_1} (1 - \delta_{n,t}))\rho + \prod_{t=1}^{\ell_1} (1 - \delta_{n,t})\rho,$$

it follows (from (3.20)) that

$$D_1 = (1 - \delta_{n,1}) + \delta_{n,1}. \tag{3.21}$$

Using similar argument as above, we obtain

$$\frac{\gamma^1_{n,1}}{1 - \left[\left(\sum_{i=2}^{\ell_2} \gamma^1_{n,i} \prod_{t=1}^{i-1} (1 - \gamma^1_{n,t})(\rho^i)^2 + \prod_{t=1}^{\ell_3} (1 - \gamma^1_{n,t}(\rho^i)^2\right)\right]} \leq (1 - \gamma^1_{n,1})\rho + \gamma^1_{n,1}, \tag{3.22}$$

$$\frac{\gamma^2_{n,1}}{1 - \left[\left(\sum_{i=2}^{\ell_2} \gamma^2_{n,i} \prod_{t=1}^{i-1} (1 - \gamma^2_{n,t})(\rho^i)^2 + \prod_{t=1}^{\ell_3} (1 - \gamma^2_{n,t}(\rho^i)^2\right)\right]} \leq (1 - \gamma^2_{n,1})\rho + \gamma^2_{n,1}, \tag{3.23}$$

$$\vdots$$

$$\frac{\gamma^{r-2}_{n,1}}{1 - \left[\left(\sum_{i=2}^{\ell_{r-2}} \gamma^{r-2}_{n,i} \prod_{t=1}^{i-1} (1 - \gamma^{r-2}_{n,t})(\rho^i)^2 + \prod_{t=1}^{\ell_{r-2}} (1 - \gamma^{r-2}_{n,t}(\rho^i)^2\right)\right]} \leq (1 - \gamma^{r-2}_{n,1})\rho + \gamma^{r-2}_{n,1}, \tag{3.24}$$

and

$$\frac{\gamma^{r-1}_{n,1}}{1 - \left[\left(\sum_{i=2}^{\ell_{r-1}} \gamma^2_{n,i} \prod_{t=1}^{i-1} (1 - \gamma^{r-1}_{n,t})(\rho^i)^2 + \prod_{t=1}^{\ell_{r-1}} (1 - \gamma^{r-1}_{n,t}(\rho^i)^2\right)\right]} \leq (1 - \gamma^{r-1}_{n,1})\rho + \gamma^{r-1}_{n,1}. \tag{3.25}$$

Now, using (3.19)–(3.25), we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq [(1 - \delta_{n,1}) + \delta_{n,1}][(1 - \gamma^1_{n,1}) + \gamma^1_{n,1}][(1 - \gamma^2_{n,1})\rho + \gamma^2_{n,1}] \\ &\quad \times \cdots \times [(1 - \gamma^{r-2}_{n,1})\rho + \gamma^{r-2}_{n,1}][(1 - \gamma^{r-1}_{n,1})\rho + \gamma^{r-1}_{n,1}]\|x_n - q\|^2 \\ &\leq [1 - (1 - \delta_{n,1})(1 - \rho)]\|x_n - q\|^2. \end{aligned} \tag{3.26}$$

From (3.26) and Lemma 2.2, we have $x_n \to q$ as $n \to \infty$ and this completes the proof.

Since (3.1) includes (3.5)–(3.7), the corollary below follows immediately from Theorem 3.1. □

Corollary 3.1. *Let H be a real Hilbert space, D a nonempty closed and convex subset of H and $\Gamma : H \longrightarrow H$ a self-map of H satisfying the contractive condition*

$$\|\Gamma^i x - \Gamma^i y\| \leq \rho^i \|x - y\|, \tag{3.27}$$

where $x, y \in H$ and $0 \leq \rho^j < 1$. For arbitrary $w_0 = z_0 = u_0 \in H$, let $\{w_n\}_{n=0}^\infty$, $\{z_n\}_{n=0}^\infty$ and $\{u_n\}_{n=0}^\infty$ be the implicit IH-Noor, implicit multistepIH-Ishikawa and implicit IH-Mann iteration scheme defined by (3.5)–(3.7)*, respectively. Then,*

(*I*) *the fixed point q of Γ defined by* (3.26) *is unique;*
(*II*) *the implicit IH-Noor iteration scheme* (3.5) *converges strongly to the unique fixed point q of Γ;*
(*III*) *the implicit IH-Ishikawa iteration scheme* (3.6) *converges strongly to the unique fixed point q of Γ;*
(*IV*) *the implicit IH-Mann iteration scheme* (3.7) *converges strongly to the unique fixed point q of Γ*

Theorem 3.2. *Let H be a real Hilbert space, E a nonempty closed and convex subset of H and $\Gamma : E \longrightarrow E$, with $q \in F(\Gamma)$, satisfying the following condition:*

$$\|\Gamma^i x - q\| \leq \rho^i \|x - y\|, \tag{3.28}$$

where $\rho^i \in [0, 1)$. Let $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be the implicit IH-multistep and implicit IH-Mann iteration schemes defined by (3.1) *and* (3.7)*, respectively with $\sum_{n=1}^{\infty}(1 - \delta_{n,1}) = \infty$. If $x_0 = u_0 \in E$, then* (*a*) *and* (*b*) *below are equivalent.*

(a) *Implicit IH-Mann iterative scheme $\{u_n\}_{n=1}^{\infty}$ defined by* (3.7) *converges to q;*
(b) *Implicit IH-multistep iterative scheme $\{x_n\}_{n=1}^{\infty}$ defined by* (3.1) *converges to q.*

Proof. First, we prove that $(a) \Rightarrow (b)$. Assume $u_n \to q$ as $n \to \infty$. Since, (3.1), (3.7) and (3.28) imply

$$\|u_{n+1} - x_{n+1}\|^2 = \|\delta_{n,1}(u_n - x_n) - \sum_{i=2}^{\ell_1} \delta_{n,i} \prod_{t=1}^{i-1}(1 - \delta_{n,t})\Big[\Gamma^{i-1}x_{n+1} - \Gamma_{i-1}u_{n+1}\Big] - \prod_{t=1}^{\ell_1}(\Gamma x_{n+1} - \Gamma u_{n+1})\|^2,$$

it follows that

$$\begin{aligned}
\|u_{n+1} - x_{n+1}\|^2 &\leq \delta_{n,1}\|u_n - x_n\|^2 + \|\sum_{i=2}^{\ell_1} \delta_{n,i} \prod_{t=1}^{i-1}(1 - \delta_{n,t})\Big[\Gamma^{i-1}x_{n+1} - \Gamma_{i-1}u_{n+1}\Big] \\
&\quad - \prod_{t=1}^{\ell_1}(\Gamma x_{n+1} - \Gamma u_{n+1})\|^2 \\
&\leq \delta_{n,1}\|u_n - x_n\|^2 + \sum_{i=2}^{\ell_1} \delta_{n,i} \prod_{t=1}^{i-1}(1 - \delta_{n,t})\|\Gamma^{i-1}x_{n+1} - \Gamma_{i-1}u_{n+1}\|^2 \\
&\quad + \prod_{t=1}^{\ell_1} \|\Gamma x_{n+1} - \Gamma u_{n+1})\|^2 \\
&\leq \delta_{n,1}\|u_n - x_n\|^2 + \sum_{i=2}^{\ell_1} \delta_{n,i} \prod_{t=1}^{i-1}(1 - \delta_{n,t})(\rho^i)^2\|x_{n+1} - u_{n+1}\|^2 \\
&\quad + \prod_{t=1}^{\ell_1}(\rho^i)^2\|x_{n+1} - u_{n+1})\|^2 \\
&\leq \left(\frac{\delta_{n,1}}{1 - \Big[\sum_{i=2}^{\ell_1} \delta_{n,i} \prod_{t=1}^{i-1}(1 - \delta_{n,t})(\rho^i)^2 + \prod_{t=1}^{\ell_1}(\rho^i)^2\Big]}\right) \times \|u_n - x_n\|^2.
\end{aligned} \tag{3.29}$$

Also, using (3.1), we have

$$\|u_n - x_n^{(1)}\|^2 = \|\gamma_{n,1}(u_n - x_n^{(2)}) - \sum_{i=2}^{\ell_2} \gamma_{n,i}^1 \prod_{t=1}^{i-1}(1 - \gamma_{n,t}^1)\Big[\Gamma^{i-1}u_n - u_n - \Big(\Gamma^{i-1}x_n^{(1)} - \Gamma_{i-1}u_n\Big)\Big]$$

$$
\begin{aligned}
& -\prod_{t=1}^{\ell_2}(1-\gamma^1_{n,t})\Big(\Gamma^{\ell_2}x_n^{(1)}-\Gamma^{\ell_2}u_n\Big)\|^2 \\
\leq\ & \gamma_{n,1}\|u_n-x_n^{(2)}\|^2+\|\sum_{i=2}^{\ell_2}\gamma^1_{n,i}\prod_{t=1}^{i-1}(1-\gamma^1_{n,t})\Big[\Gamma^{i-1}u_n-u_n-\Big(\Gamma^{i-1}x_n^{(1)}-\Gamma_{i-1}u_n\Big)\Big] \\
& -\prod_{t=1}^{\ell_2}(1-\gamma^1_{n,t})\Big(\Gamma^{\ell_2}x_n^{(1)}-\Gamma^{\ell_2}u_n\Big)\|^2 \\
=\ & \gamma_{n,1}\|u_n-x_n^{(2)}\|^2+\|\sum_{i=2}^{\ell_2}\gamma^1_{n,i}\prod_{t=1}^{i-1}(1-\gamma^1_{n,t})(\Gamma^{i-1}u_n-u_n) \\
& -\prod_{t=1}^{\ell_2}(1-\gamma^1_{n,t})(\Gamma^{\ell_2}u_n-u_n)-\Big[\sum_{i=2}^{\ell_2}\gamma^1_{n,i}\prod_{t=1}^{i-1}(1-\gamma^1_{n,t})Big(\Gamma^{i-1}x_n^{(1)}-\Gamma_{i-1}u_n\Big) \\
& -\prod_{t=1}^{\ell_2}(1-\gamma^1_{n,t})\Big(\Gamma^{\ell_2}u_n-\Gamma^{\ell_2}x_n^{(1)}\Big)\Big]\|^2 \\
\leq\ & \gamma_{n,1}\|u_n-x_n^{(2)}\|^2+\|\sum_{i=2}^{\ell_2}\gamma^1_{n,i}\prod_{t=1}^{i-1}(1-\gamma^1_{n,t})(\Gamma^{i-1}u_n-u_n) \\
& -\prod_{t=1}^{\ell_2}(1-\gamma^1_{n,t})(\Gamma^{\ell_2}u_n-u_n)-\Big[\sum_{i=2}^{\ell_2}\gamma^1_{n,i}\prod_{t=1}^{i-1}(1-\gamma^1_{n,t})Big(\Gamma^{i-1}x_n^{(1)}-\Gamma_{i-1}u_n\Big) \\
& -\prod_{t=1}^{\ell_2}(1-\gamma^1_{n,t})\Big(\Gamma^{\ell_2}u_n-\Gamma^{\ell_2}x_n^{(1)}\Big)\Big]\|^2 \\
\leq\ & \gamma_{n,1}\|u_n-x_n^{(2)}\|^2+\|\sum_{i=2}^{\ell_2}\gamma^1_{n,i}\prod_{t=1}^{i-1}(1-\gamma^1_{n,t})(\Gamma^{i-1}u_n-u_n) \\
& -\prod_{t=1}^{\ell_2}(1-\gamma^1_{n,t})(\Gamma^{\ell_2}u_n-u_n)\|^2+\|\sum_{i=2}^{\ell_2}\gamma^1_{n,i}\prod_{t=1}^{i-1}(1-\gamma^1_{n,t})\Big(\Gamma^{i-1}x_n^{(1)}-\Gamma_{i-1}u_n\Big) \\
& -\prod_{t=1}^{\ell_2}(1-\gamma^1_{n,t})\Big(\Gamma^{\ell_2}u_n-\Gamma^{\ell_2}x_n^{(1)}\Big)\|^2 \\
\leq\ & \gamma_{n,1}\|u_n-x_n^{(2)}\|^2+\sum_{i=2}^{\ell_2}\gamma^1_{n,i}\prod_{t=1}^{i-1}(1-\gamma^1_{n,t})\|\Gamma^{i-1}u_n-u_n\|^2 \\
& +\prod_{t=1}^{\ell_2}(1-\gamma^1_{n,t})\|\Gamma^{\ell_2}u_n-u_n)\|^2+\sum_{i=2}^{\ell_2}\gamma^1_{n,i}\prod_{t=1}^{i-1}(1-\gamma^1_{n,t})\|\Gamma^{i-1}x_n^{(1)}-\Gamma_{i-1}u_n\|^2 \\
& +\prod_{t=1}^{\ell_2}(1-\gamma^1_{n,t})\|\Gamma^{\ell_2}u_n-\Gamma^{\ell_2}x_n^{(1)}\|^2.
\end{aligned}
\tag{3.30}
$$

Applying condition (3.28) to (3.30), we get

$$\begin{aligned}
\|u_n - x_n^{(1)}\|^2 &\leq \gamma_{n,1}^1 \|u_n - x_n^{(2)}\|^2 + \sum_{i=2}^{\ell_2} \gamma_{n,i}^1 \prod_{t=1}^{i-1} (1-\gamma_{n,t}^1) \|\Gamma^{i-1} u_n - u_n\|^2 \\
&\quad + \prod_{t=1}^{\ell_2} (1-\gamma_{n,t}^1) \|\Gamma^{\ell_2} u_n - u_n)\|^2 + \sum_{i=2}^{\ell_2} \gamma_{n,i}^1 \prod_{t=1}^{i-1} (1-\gamma_{n,t}^1)(\rho^i)^2 \|x_n^{(1)} - u_n\|^2 \\
&\quad + \prod_{t=1}^{\ell_2} (1-\gamma_{n,t}^1)(\rho^i)^2 \|u_n - x_n^{(1)}\|^2 \\
&\leq \frac{\gamma_{n,1}}{1 - \left[\sum_{i=2}^{\ell_2} \gamma_{n,i}^1 \prod_{t=1}^{i-1} (1-\gamma_{n,t}^1)(\rho^i)^2 + \prod_{t=1}^{\ell_2} (1-\gamma_{n,t}^1)(\rho^i)^2 \right]} \|u_n - x_n^{(2)}\|^2 \\
&\quad + \frac{\sum_{i=2}^{\ell_2} \gamma_{n,i}^1 \prod_{t=1}^{i-1} (1-\gamma_{n,t}^1)}{1 - \left[\sum_{i=2}^{\ell_2} \gamma_{n,i}^1 \prod_{t=1}^{i-1} (1-\gamma_{n,t}^1)(\rho^i)^2 + \prod_{t=1}^{\ell_2} (1-\gamma_{n,t}^1)(\rho^i)^2 \right]} \|\Gamma^{i-1} u_n - u_n\|^2 \\
&\quad + \frac{\prod_{t=1}^{\ell_2} (1-\gamma_{n,t}^1)}{1 - \left[\sum_{i=2}^{\ell_2} \gamma_{n,i}^1 \prod_{t=1}^{i-1} (1-\gamma_{n,t}^1)(\rho^i)^2 + \prod_{t=1}^{\ell_2} (1-\gamma_{n,t}^1)(\rho^i)^2 \right]} \\
&\quad \times \|\Gamma^{\ell_2} u_n - u_n)\|^2. \qquad (3.31)
\end{aligned}$$

Observe that

$$\begin{aligned}
\|\Gamma^{i-1} u_n - u_n\|^2 &= \|u_n - q - (\Gamma^{i-1} u_n - \Gamma^{i-1} q)\|^2 \\
&\leq \|u_n - q\|^2 + \|\Gamma^{i-1} u_n - \Gamma^{i-1} q\|^2 \leq (1 + (\rho^i)^2) \|u_n - q\|^2. \qquad (3.32)
\end{aligned}$$

From (3.31) and (3.32), we have

$$\begin{aligned}
\|u_n - x_n^{(1)}\|^2 &\leq \frac{\gamma_{n,1}^1}{1 - \left[\sum_{i=2}^{\ell_2} \gamma_{n,i}^1 \prod_{t=1}^{i-1} (1-\gamma_{n,t}^1)(\rho^i)^2 + \prod_{t=1}^{\ell_2} (1-\gamma_{n,t}^1)(\rho^i)^2 \right]} \|u_n - x_n^{(2)}\|^2 \\
&\quad + \frac{\sum_{i=2}^{\ell_2} \gamma_{n,i}^1 \prod_{t=1}^{i-1} (1-\gamma_{n,t}^1)(1 + (\rho^i)^2)}{1 - \left[\sum_{i=2}^{\ell_2} \gamma_{n,i}^1 \prod_{t=1}^{i-1} (1-\gamma_{n,t}^1)(\rho^i)^2 + \prod_{t=1}^{\ell_2} (1-\gamma_{n,t}^1)(\rho^i)^2 \right]} \|u_n - q\|^2 \\
&\quad + \frac{\prod_{t=1}^{\ell_2} (1-\gamma_{n,t}^1)(1 + (\rho^i)^2)}{1 - \left[\sum_{i=2}^{\ell_2} \gamma_{n,i}^1 \prod_{t=1}^{i-1} (1-\gamma_{n,t}^1)(\rho^i)^2 + \prod_{t=1}^{\ell_2} (1-\gamma_{n,t}^1)(\rho^i)^2 \right]} \\
&\quad \times \|u_n - q\|^2. \qquad (3.33)
\end{aligned}$$

Using a similar argument as above, we obtain

$$\begin{aligned}
\|u_n - x_n^{(2)}\|^2 &\leq \frac{\gamma_{n,1}^2}{1 - \left[\sum_{i=2}^{\ell_3} \gamma_{n,i}^2 \prod_{t=1}^{i-1} (1-\gamma_{n,t}^2)(\rho^i)^2 + \prod_{t=1}^{\ell_3} (1-\gamma_{n,t}^2)(\rho^i)^2 \right]} \|u_n - x_n^{(3)}\|^2 \\
&\quad + \frac{\sum_{i=2}^{\ell_3} \gamma_{n,i}^2 \prod_{t=1}^{i-1} (1-\gamma_{n,t}^2)(1 + (\rho^i)^2)}{1 - \left[\sum_{i=2}^{\ell_3} \gamma_{n,i}^2 \prod_{t=1}^{i-1} (1-\gamma_{n,t}^2)(\rho^i)^2 + \prod_{t=1}^{\ell_3} (1-\gamma_{n,t}^2)(\rho^i)^2 \right]} \|u_n - q\|^2
\end{aligned}$$

$$+\frac{\prod_{t=1}^{\ell_3}(1-\gamma_{n,t}^2)(1+(\rho^i)^2)}{1-\Big[\sum_{i=2}^{\ell_3}\gamma_{n,i}^2\prod_{t=1}^{i-1}(1-\gamma_{n,t}^2)(\rho^i)^2+\prod_{t=1}^{\ell_3}(1-\gamma_{n,t}^2)(\rho^i)^2\Big]}\times\|u_n-q\|^2, \tag{3.34}$$

$$\begin{aligned}\|u_n-x_n^{(3)}\|^2 \leq\ & \frac{\gamma_{n,1}^3}{1-\Big[\sum_{i=2}^{\ell_4}\gamma_{n,i}^3\prod_{t=1}^{i-1}(1-\gamma_{n,t}^3)(\rho^i)^2+\prod_{t=1}^{\ell_4}(1-\gamma_{n,t}^3)(\rho^i)^2\Big]}\|u_n-x_n^{(4)}\|^2\\ &+\frac{\sum_{i=2}^{\ell_4}\gamma_{n,i}^3\prod_{t=1}^{i-1}(1-\gamma_{n,t}^3)(1+(\rho^i)^2)}{1-\Big[\sum_{i=2}^{\ell_2}\gamma_{n,i}^3\prod_{t=1}^{i-1}(1-\gamma_{n,t}^3)(\rho^i)^2+\prod_{t=1}^{\ell_3}(1-\gamma_{n,t}^3)(\rho^i)^2\Big]}\|u_n-q\|^2\\ &+\frac{\prod_{t=1}^{\ell_4}(1-\gamma_{n,t}^3)(1+(\rho^i)^2)}{1-\Big[\sum_{i=2}^{\ell_4}\gamma_{n,i}^3\prod_{t=1}^{i-1}(1-\gamma_{n,t}^3)(\rho^i)^2+\prod_{t=1}^{\ell_4}(1-\gamma_{n,t}^3)(\rho^i)^2\Big]}\\ &\times\|u_n-q\|^2\end{aligned} \tag{3.35}$$

and continuing this process $(r-2)$ and $(r-1)$ times yields

$$\begin{aligned}\|u_n-x_n^{(r-2)}\|^2 \leq\ & \frac{\gamma_{n,1}^{r-2}}{1-\Big[\sum_{i=2}^{\ell_{r-1}}\gamma_{n,i}^{r-2}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})(\rho^i)^2+\prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2})(\rho^i)^2\Big]}\|u_n-x_n^{(r-1)}\|^2\\ &+\frac{\sum_{i=2}^{\ell_{r-1}}\gamma_{n,i}^{r-2}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})(1+(\rho^i)^2)}{1-\Big[\sum_{i=2}^{\ell_{r-1}}\gamma_{n,i}^{r-2}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})(\rho^i)^2+\prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2})(\rho^i)^2\Big]}\|u_n-q\|^2\\ &+\frac{\prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2})(1+(\rho^i)^2)}{1-\Big[\sum_{i=2}^{\ell_{r-1}}\gamma_{n,i}^{r-2}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})(\rho^i)^2+\prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2})(\rho^i)^2\Big]}\\ &\times\|u_n-q\|^2\end{aligned} \tag{3.36}$$

and

$$\begin{aligned}\|u_n-x_n^{(r-1)}\|^2 \leq\ & \frac{\gamma_{n,1}^{r-1}}{1-\Big[\sum_{i=2}^{\ell_r}\gamma_{n,i}^{r-1}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-1})(\rho^i)^2+\prod_{t=1}^{\ell_r}(1-\gamma_{n,t}^{r-1})(\rho^i)^2\Big]}\|u_n-x_n^{(r)}\|^2\\ &+\frac{\sum_{i=2}^{\ell_r}\gamma_{n,i}^{r-1}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-1})(1+(\rho^i)^2)}{1-\Big[\sum_{i=2}^{\ell_r}\gamma_{n,i}^{r-1}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-1})(\rho^i)^2+\prod_{t=1}^{\ell_r}(1-\gamma_{n,t}^{r-1})(\rho^i)^2\Big]}\|u_n-q\|^2\\ &+\frac{\prod_{t=1}^{\ell_r}(1-\gamma_{n,t}^{r-1})(1+(\rho^i)^2)}{1-\Big[\sum_{i=2}^{\ell_r}\gamma_{n,i}^{r-1}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})(\rho^i)^2+\prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-1})(\rho^i)^2\Big]}\\ &\times\|u_n-q\|^2.\end{aligned} \tag{3.37}$$

Now, using (3.29) and (3.33)–(3.37), we have

$$
\begin{aligned}
\|u_{n+1}-x_{n+1}\|^2 \;\le\; & \left(\frac{\delta_{n,1}}{1-\Big[\sum_{i=2}^{\ell_1}\delta_{n,i}\prod_{t=1}^{i-1}(1-\delta_{n,t})(\rho^i)^2+\prod_{t=1}^{\ell_1}(1-\delta_{n,t})(\rho^i)^2\Big]}\right)\\
&\times\left(\frac{\gamma_{n,1}^1}{1-\Big[\sum_{i=2}^{\ell_2}\gamma_{n,i}^1\prod_{t=1}^{i-1}(1-\gamma_{n,t}^1)(\rho^i)^2+\prod_{t=1}^{\ell_2}(1-\gamma_{n,t}^1)(\rho^i)^2\Big]}\right)\\
&\times\left(\frac{\gamma_{n,1}^2}{1-\Big[\sum_{i=2}^{\ell_3}\gamma_{n,i}^2\prod_{t=1}^{i-1}(1-\gamma_{n,t}^2)(\rho^i)^2+\prod_{t=1}^{\ell_3}(1-\gamma_{n,t}^2)(\rho^i)^2\Big]}\right)\\
&\times\left(\frac{\gamma_{n,1}^3}{1-\Big[\sum_{i=2}^{\ell_4}\gamma_{n,i}^3\prod_{t=1}^{i-1}(1-\gamma_{n,t}^3)(\rho^i)^2+\prod_{t=1}^{\ell_4}(1-\gamma_{n,t}^3)(\rho^i)^2\Big]}\right)\\
&\times\cdots\\
&\times\left(\frac{\gamma_{n,1}^{r-2}}{1-\Big[\sum_{i=2}^{\ell_{r-1}}\gamma_{n,i}^{r-2}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})(\rho^i)^2+\prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2})(\rho^i)^2\Big]}\right)\\
&\times\left(\frac{\gamma_{n,1}^{r-1}}{1-\Big[\sum_{i=2}^{\ell_r}\gamma_{n,i}^{r-1}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-1})(\rho^i)^2+\prod_{t=1}^{\ell_r}(1-\gamma_{n,t}^{r-1})(\rho^i)^2\Big]}\right)\\
&\times\|u_n-x_n\|^2\\
&+\Bigg\{\left(\frac{\gamma_{n,1}^1}{1-\Big[\sum_{i=2}^{\ell_2}\gamma_{n,i}^1\prod_{t=1}^{i-1}(1-\gamma_{n,t}^1)(\rho^i)^2+\prod_{t=1}^{\ell_2}(1-\gamma_{n,t}^1)(\rho^i)^2\Big]}\right)\\
&\times\left(\frac{\gamma_{n,1}^2}{1-\Big[\sum_{i=2}^{\ell_3}\gamma_{n,i}^2\prod_{t=1}^{i-1}(1-\gamma_{n,t}^2)(\rho^i)^2+\prod_{t=1}^{\ell_3}(1-\gamma_{n,t}^2)(\rho^i)^2\Big]}\right)\\
&\times\left(\frac{\gamma_{n,1}^3}{1-\Big[\sum_{i=2}^{\ell_4}\gamma_{n,i}^3\prod_{t=1}^{i-1}(1-\gamma_{n,t}^3)(\rho^i)^2+\prod_{t=1}^{\ell_4}(1-\gamma_{n,t}^3)(\rho^i)^2\Big]}\right)\\
&\times\cdots\\
&\times\left(\frac{\gamma_{n,1}^{r-2}}{1-\Big[\sum_{i=2}^{\ell_{r-1}}\gamma_{n,i}^{r-2}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})(\rho^i)^2+\prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2})(\rho^i)^2\Big]}\right)\\
&\times\Bigg[\frac{\sum_{i=2}^{\ell_{r-1}}\gamma_{n,i}^{r-2}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})}{1-\Big[\sum_{i=2}^{\ell_{r-1}}\gamma_{n,i}^{r-2}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})(\rho^i)^2+\prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2})(\rho^i)^2\Big]}\\
&+\frac{\prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2})}{1-\Big[\sum_{i=2}^{\ell_{r-1}}\gamma_{n,i}^{r-2}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})(\rho^i)^2+\prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2})(\rho^i)^2\Big]}\Bigg]\\
&+\left(\frac{\gamma_{n,1}^1}{1-\Big[\sum_{i=2}^{\ell_2}\gamma_{n,i}^1\prod_{t=1}^{i-1}(1-\gamma_{n,t}^1)(\rho^i)^2+\prod_{t=1}^{\ell_2}(1-\gamma_{n,t}^1)(\rho^i)^2\Big]}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(\frac{\gamma_{n,1}^{2}}{1-\left[\sum_{i=2}^{\ell_3}\gamma_{n,i}^{2}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{2})(\rho^i)^2+\prod_{t=1}^{\ell_3}(1-\gamma_{n,t}^{2})(\rho^i)^2\right]}\right)\\
&\times\left(\frac{\gamma_{n,1}^{3}}{1-\left[\sum_{i=2}^{\ell_4}\gamma_{n,i}^{3}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{3})(\rho^i)^2+\prod_{t=1}^{\ell_4}(1-\gamma_{n,t}^{3})(\rho^i)^2\right]}\right)\\
&\times\cdots\\
&\times\left(\frac{\gamma_{n,1}^{r-2}}{1-\left[\sum_{i=2}^{\ell_{r-1}}\gamma_{n,i}^{r-2}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})(\rho^i)^2+\prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2})(\rho^i)^2\right]}\right)\\
&\times\left[\frac{\sum_{i=2}^{\ell_{r-2}}\gamma_{n,i}^{r-3}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-3})}{1-\left[\sum_{i=2}^{\ell_{r-2}}\gamma_{n,i}^{r-3}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-3})(\rho^i)^2+\prod_{t=1}^{\ell_{r-2}}(1-\gamma_{n,t}^{r-3})(\rho^i)^2\right]}\right.\\
&\left.+\frac{\prod_{t=1}^{\ell_{r-2}}(1-\gamma_{n,t}^{r-3})}{1-\left[\sum_{i=2}^{\ell_{r-2}}\gamma_{n,i}^{r-3}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-3})(\rho^i)^2+\prod_{t=1}^{\ell_{r-2}}(1-\gamma_{n,t}^{r-3})(\rho^i)^2\right]}\right]\\
&+\frac{\sum_{i=2}^{\ell_2}\gamma_{n,i}^{1}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{1})}{1-\left[\sum_{i=2}^{\ell_2}\gamma_{n,i}^{1}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{1})(\rho^i)^2+\prod_{t=1}^{\ell_2}(1-\gamma_{n,t}^{1})(\rho^i)^2\right]}\\
&\left.+\frac{\prod_{t=1}^{\ell_2}(1-\gamma_{n,t}^{1})}{1-\left[\sum_{i=2}^{\ell_2}\gamma_{n,i}^{1}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{1})(\rho^i)^2+\prod_{t=1}^{\ell_2}(1-\gamma_{n,t}^{1})(\rho^i)^2\right]}\right\}\\
&\times\left(\frac{\delta_{n,1}}{1-\left[\sum_{i=2}^{\ell_1}\delta_{n,i}\prod_{t=1}^{i-1}(1-\delta_{n,t})(\rho^i)^2+\prod_{t=1}^{\ell_1}(1-\delta_{n,t})(\rho^i)^2\right]}\right)\\
&\times(1+(\rho^i)^2)\|u_n-q\|^2. \qquad (3.38)
\end{aligned}
$$

Substituting (3.21)–(3.25) into (3.38) yields

$$
\begin{aligned}
\|u_{n+1}-x_{n+1}\|^2 \leq\ & ((1-\delta_{n,1})+\delta_{n,1})((1-\gamma_{n,1}^{1})+\gamma_{n,1}^{1})((1-\gamma_{n,1}^{2})+\gamma_{n,1}^{2})((1-\gamma_{n,1}^{3})+\gamma_{n,1}^{3})\\
&\times\cdots\\
&\times((1-\gamma_{n,1}^{r-2})+\gamma_{n,1}^{r-2})((1-\gamma_{n,1}^{r-1})+\gamma_{n,1}^{r-1})\|u_n-x_n\|^2\\
&+\Big\{((1-\gamma_{n,1}^{1})+\gamma_{n,1}^{1})((1-\gamma_{n,1}^{2})+\gamma_{n,1}^{2})((1-\gamma_{n,1}^{3})+\gamma_{n,1}^{3})((1-\gamma_{n,1}^{r-2})+\gamma_{n,1}^{r-2})\\
&\times\left[\frac{\sum_{i=2}^{\ell_{r-1}}\gamma_{n,i}^{r-2}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})}{1-\left[\sum_{i=2}^{\ell_{r-1}}\gamma_{n,i}^{r-2}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})(\rho^i)^2+\prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2})(\rho^i)^2\right]}\right.\\
&\left.+\frac{\prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2})}{1-\left[\sum_{i=2}^{\ell_{r-1}}\gamma_{n,i}^{r-2}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})(\rho^i)^2+\prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2})(\rho^i)^2\right]}\right]\\
&+((1-\gamma_{n,1}^{1}+\gamma_{n,1}^{1}))((1-\gamma_{n,1}^{2})+\gamma_{n,1}^{2})((1-\gamma_{n,1}^{3})+\gamma_{n,1}^{3})\\
&\times\cdots\\
&\times((1-\gamma_{n,1}^{r-2})+\gamma_{n,1}^{r-2})
\end{aligned}
$$

$$
\begin{aligned}
&\times\Bigg[\frac{\sum_{i=2}^{\ell_{r-2}}\gamma_{n,i}^{r-3}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-3})}{1-\Big[\sum_{i=2}^{\ell_{r-2}}\gamma_{n,i}^{r-3}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-3})(\rho^i)^2+\prod_{t=1}^{\ell_{r-2}}(1-\gamma_{n,t}^{r-3})(\rho^i)^2\Big]}\\
&+\frac{\prod_{t=1}^{\ell_{r-2}}(1-\gamma_{n,t}^{r-3})}{1-\Big[\sum_{i=2}^{\ell_{r-2}}\gamma_{n,i}^{r-3}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-3})(\rho^i)^2+\prod_{t=1}^{\ell_{r-2}}(1-\gamma_{n,t}^{r-3})(\rho^i)^2\Big]}\Bigg]\\
&+\frac{\sum_{i=2}^{\ell_2}\gamma_{n,i}^{1}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{1})}{1-\Big[\sum_{i=2}^{\ell_2}\gamma_{n,i}^{1}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{1})(\rho^i)^2+\prod_{t=1}^{\ell_2}(1-\gamma_{n,t}^{1})(\rho^i)^2\Big]}\\
&+\frac{\prod_{t=1}^{\ell_2}(1-\gamma_{n,t}^{1})}{1-\Big[\sum_{i=2}^{\ell_2}\gamma_{n,i}^{1}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{1})(\rho^i)^2+\prod_{t=1}^{\ell_2}(1-\gamma_{n,t}^{1})(\rho^i)^2\Big]}\Bigg\}\\
&\times((1-\delta_{n,1})+\delta_{n,1})(1+(\rho^i)^2)\|u_n-q\|^2\\
&\le[1-(1-\delta_{n,1})(1-\rho)]\|u_n-x_n\|^2\\
&+\Bigg\{((1-\gamma_{n,1}^{1})+\gamma_{n,1}^{1})((1-\gamma_{n,1}^{2})+\gamma_{n,1}^{2})((1-\gamma_{n,1}^{3})+\gamma_{n,1}^{3})((1-\gamma_{n,1}^{r-2})+\gamma_{n,1}^{r-2})\\
&\times\Bigg[\frac{\sum_{i=2}^{\ell_{r-1}}\gamma_{n,i}^{r-2}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})}{1-\Big[\sum_{i=2}^{\ell_{r-1}}\gamma_{n,i}^{r-2}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})(\rho^i)^2+\prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2})(\rho^i)^2\Big]}\\
&+\frac{\prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2})}{1-\Big[\sum_{i=2}^{\ell_{r-1}}\gamma_{n,i}^{r-2}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})(\rho^i)^2+\prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2})(\rho^i)^2\Big]}\Bigg]\\
&+((1-\gamma_{n,1}^{1}+\gamma_{n,1}^{1}))((1-\gamma_{n,1}^{2})+\gamma_{n,1}^{2})((1-\gamma_{n,1}^{3})+\gamma_{n,1}^{3})\\
&\times\cdots\\
&\times((1-\gamma_{n,1}^{r-2})+\gamma_{n,1}^{r-2})\\
&\times\Bigg[\frac{\sum_{i=2}^{\ell_{r-2}}\gamma_{n,i}^{r-3}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-3})}{1-\Big[\sum_{i=2}^{\ell_{r-2}}\gamma_{n,i}^{r-3}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-3})(\rho^i)^2+\prod_{t=1}^{\ell_{r-2}}(1-\gamma_{n,t}^{r-3})(\rho^i)^2\Big]}\\
&+\frac{\prod_{t=1}^{\ell_{r-2}}(1-\gamma_{n,t}^{r-3})}{1-\Big[\sum_{i=2}^{\ell_{r-2}}\gamma_{n,i}^{r-3}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-3})(\rho^i)^2+\prod_{t=1}^{\ell_{r-2}}(1-\gamma_{n,t}^{r-3})(\rho^i)^2\Big]}\Bigg]\\
&+\frac{\sum_{i=2}^{\ell_2}\gamma_{n,i}^{1}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{1})}{1-\Big[\sum_{i=2}^{\ell_2}\gamma_{n,i}^{1}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{1})(\rho^i)^2+\prod_{t=1}^{\ell_2}(1-\gamma_{n,t}^{1})(\rho^i)^2\Big]}\\
&+\frac{\prod_{t=1}^{\ell_2}(1-\gamma_{n,t}^{1})}{1-\Big[\sum_{i=2}^{\ell_2}\gamma_{n,i}^{1}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{1})(\rho^i)^2+\prod_{t=1}^{\ell_2}(1-\gamma_{n,t}^{1})(\rho^i)^2\Big]}\Bigg\}\\
&\times((1-\delta_{n,1})+\delta_{n,1})(1+(\rho^i)^2)\|u_n-q\|^2.
\end{aligned}
\tag{3.39}
$$

Let

$$\sigma_n=(1-\delta_{n,1})(1-\rho) \tag{3.40}$$

and

$$
\begin{aligned}
\tau_n \;=\; & \Big\{((1-\gamma^1_{n,1})+\gamma^1_{n,1})((1-\gamma^2_{n,1})+\gamma^2_{n,1})((1-\gamma^3_{n,1})+\gamma^3_{n,1})((1-\gamma^{r-2}_{n,1})+\gamma^{r-2}_{n,1}) \\
& \times\Bigg[\frac{\sum_{i=2}^{\ell_{r-1}}\gamma^{r-2}_{n,i}\prod_{t=1}^{i-1}(1-\gamma^{r-2}_{n,t})}{1-\Big[\sum_{i=2}^{\ell_{r-1}}\gamma^{r-2}_{n,i}\prod_{t=1}^{i-1}(1-\gamma^{r-2}_{n,t})(\rho^i)^2+\prod_{t=1}^{\ell_{r-1}}(1-\gamma^{r-2}_{n,t})(\rho^i)^2\Big]} \\
& +\frac{\prod_{t=1}^{\ell_{r-1}}(1-\gamma^{r-2}_{n,t})}{1-\Big[\sum_{i=2}^{\ell_{r-1}}\gamma^{r-2}_{n,i}\prod_{t=1}^{i-1}(1-\gamma^{r-2}_{n,t})(\rho^i)^2+\prod_{t=1}^{\ell_{r-1}}(1-\gamma^{r-2}_{n,t})(\rho^i)^2\Big]}\Bigg] \\
& +((1-\gamma^1_{n,1}+\gamma^1_{n,1}))((1-\gamma^2_{n,1})+\gamma^2_{n,1})((1-\gamma^3_{n,1})+\gamma^3_{n,1}) \\
& \times\cdots \\
& \times((1-\gamma^{r-2}_{n,1})+\gamma^{r-2}_{n,1}) \\
& \times\Bigg[\frac{\sum_{i=2}^{\ell_{r-2}}\gamma^{r-3}_{n,i}\prod_{t=1}^{i-1}(1-\gamma^{r-3}_{n,t})}{1-\Big[\sum_{i=2}^{\ell_{r-2}}\gamma^{r-3}_{n,i}\prod_{t=1}^{i-1}(1-\gamma^{r-3}_{n,t})(\rho^i)^2+\prod_{t=1}^{\ell_{r-2}}(1-\gamma^{r-3}_{n,t})(\rho^i)^2\Big]} \\
& +\frac{\prod_{t=1}^{\ell_{r-2}}(1-\gamma^{r-3}_{n,t})}{1-\Big[\sum_{i=2}^{\ell_{r-2}}\gamma^{r-3}_{n,i}\prod_{t=1}^{i-1}(1-\gamma^{r-3}_{n,t})(\rho^i)^2+\prod_{t=1}^{\ell_{r-2}}(1-\gamma^{r-3}_{n,t})(\rho^i)^2\Big]}\Bigg] \\
& +\frac{\sum_{i=2}^{\ell_2}\gamma^1_{n,i}\prod_{t=1}^{i-1}(1-\gamma^1_{n,t})}{1-\Big[\sum_{i=2}^{\ell_2}\gamma^1_{n,i}\prod_{t=1}^{i-1}(1-\gamma^1_{n,t})(\rho^i)^2+\prod_{t=1}^{\ell_2}(1-\gamma^1_{n,t})(\rho^i)^2\Big]} \\
& +\frac{\prod_{t=1}^{\ell_2}(1-\gamma^1_{n,t})}{1-\Big[\sum_{i=2}^{\ell_2}\gamma^1_{n,i}\prod_{t=1}^{i-1}(1-\gamma^1_{n,t})(\rho^i)^2+\prod_{t=1}^{\ell_2}(1-\gamma^1_{n,t})(\rho^i)^2\Big]}\Bigg\} \\
& \times((1-\delta_{n,1})+\delta_{n,1})(1+(\rho^i)^2)\|u_n-q\|^2. \qquad (3.41)
\end{aligned}
$$

Then, we get (from (3.39)) that

$$\|u_{n+1}-x_{n+1}\|^2 \le (1-\sigma_n)\|u_n-x_n\|^2+\tau_n. \qquad (3.42)$$

From Lemma 2.3 and (3.42), we conclude that

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \qquad (3.43)$$

Since $\|x_n-q\|\le\|u_n-x_n\|+\|u_n-q\|$, it follows from the assumption $\lim_{n\to\infty}\|u_n-x_n\|=0$ and (3.43) that $\lim_{n\to\infty}\|x_n-q\|=0$.

Next, we prove that $(b)\Rightarrow(a)$. To do this, assume that x $u_n\to q$ as $n\to\infty$. Furthermore, since, (3.1), (3.7) and (3.56) imply

$$\|x_{n+1}-u_{n+1}\|^2=\|\delta_{n,1}(x_n-u_n^{(1)})-\sum_{i=2}^{\ell_1}\delta_{n,i}\prod_{t=1}^{i-1}(1-\delta_{n,t})\Big[\Gamma^{i-1}u_{n+1}-\Gamma_{i-1}x_{n+1}\Big]-\prod_{t=1}^{\ell_1}(\Gamma u_{n+1}-\Gamma x_{n+1})\|^2,$$

it follows that

$$\begin{aligned}
\|x_{n+1}-u_{n+1}\|^2 &\leq \delta_{n,1}\|x_n^{(1)}-u_n\|^2+\|\sum_{i=2}^{\ell_1}\delta_{n,i}\prod_{t=1}^{i-1}(1-\delta_{n,t})\Big[\Gamma^{i-1}u_{n+1}-\Gamma_{i-1}x_{n+1}\Big] \\
&\quad -\prod_{t=1}^{\ell_1}(\Gamma u_{n+1}-\Gamma x_{n+1})\|^2 \\
&\leq \delta_{n,1}\|u_n-x_n\|^2+\sum_{i=2}^{\ell_1}\delta_{n,i}\prod_{t=1}^{i-1}(1-\delta_{n,t})\|-(\Gamma^{i-1}x_{n+1}-\Gamma_{i-1}u_{n+1})\|^2 \\
&\quad +\prod_{t=1}^{\ell_1}\|-(\Gamma x_{n+1}-\Gamma u_{n+1})\|^2 \\
&= \delta_{n,1}\|u_n-x_n\|^2+\sum_{i=2}^{\ell_1}\delta_{n,i}\prod_{t=1}^{i-1}(1-\delta_{n,t})\|\Gamma^{i-1}x_{n+1}-\Gamma_{i-1}u_{n+1}\|^2 \\
&\quad +\prod_{t=1}^{\ell_1}\|\Gamma x_{n+1}-\Gamma u_{n+1}\|^2 \\
&\leq \delta_{n,1}\|u_n-x_n\|^2+\sum_{i=2}^{\ell_1}\delta_{n,i}\prod_{t=1}^{i-1}(1-\delta_{n,t})(\rho^i)^2\|x_{n+1}-u_{n+1}\|^2 \\
&\quad +\prod_{t=1}^{\ell_1}(\rho^i)^2\|x_{n+1}-u_{n+1})\|^2 \\
&\leq \frac{\delta_{n,1}}{1-\Big[\sum_{i=2}^{\ell_1}\delta_{n,i}\prod_{t=1}^{i-1}(1-\delta_{n,t})(\rho^i)^2+\prod_{t=1}^{\ell_1}(\rho^i)^2\Big]}\|u_n-x_n\|^2. \qquad (3.44)
\end{aligned}$$

Again, from (3.1), (3.7), (3.28) and the fact that $(a-b)^2 \leq a^2+b^2$, we obtain

$$\begin{aligned}
\|x_n^{(1)}-u_n\|^2 &= \|\gamma_{n,1}^1(x_n^{(2)}-u_n)-\sum_{i=2}^{\ell_2}\gamma_{n,i}^1\prod_{t=1}^{i-1}(1-\gamma_{n,t})\Big[\Gamma^{i-1}[u_n-\Gamma^{i-1}x_n^{(1)}]\Big] \\
&\quad -\prod_{t=1}^{\ell_2}(u_n-\Gamma^{\ell_2}x_n^{(1)})\|^2 \\
&\leq \gamma_{n,1}^1\|x_n^{(2)}-u_n\|^2+\|\sum_{i=2}^{\ell_2}\gamma_{n,i}^1\prod_{t=1}^{i-1}(1-\gamma_{n,t}^1)\Big[u_n-\Gamma^{i-1}u_n-(\Gamma^{i-1}x_n^{(1)}-\Gamma^{i-1}u_n)]\Big] \\
&\quad -\prod_{t=1}^{\ell_1}(1-\gamma_{n,t}^1)(u_n-\Gamma^{\ell_2}u_n+\Gamma^{\ell_2}u_n-\Gamma^{\ell_2}x_n^{(1)})\|^2 \\
&= \gamma_{n,1}^1\|x_n^{(2)}-u_n\|^2+\|\sum_{i=2}^{\ell_2}\gamma_{n,i}^1\prod_{t=1}^{i-1}(1-\gamma_{n,t}^1)(u_n-\Gamma^{i-1}u_n) \\
&\quad -\prod_{t=1}^{\ell_1}(1-\gamma_{n,t}^1)(u_n-\Gamma^{\ell_2}u_n)-\Big[\sum_{i=2}^{\ell_2}\gamma_{n,i}^1\prod_{t=1}^{i-1}(1-\gamma_{n,t}^1(\Gamma^{\ell_2}u_n-\Gamma x_n^{(1)})
\end{aligned}$$

$$
\begin{aligned}
& -\prod_{t=1}^{\ell_2}(1-\gamma^1_{n,t})(\Gamma^{\ell_2}x^{(1)}_n-\Gamma^{\ell_2}u_n)\Big]\|^2 \\
\leq\ & \gamma^1_{n,1}\|x^{(2)}_n-u_n\|^2+\|\sum_{i=2}^{\ell_2}\gamma^1_{n,i}\prod_{t=1}^{i-1}(1-\gamma^1_{n,t})(u_n-\Gamma^{i-1}u_n) \\
& -\prod_{t=1}^{\ell_1}(1-\gamma^1_{n,t})(u_n-\Gamma^{\ell_2}u_n)\|^2+\|\sum_{i=2}^{\ell_2}\gamma^1_{n,i}\prod_{t=1}^{i-1}(1-\gamma^1_{n,t}(\Gamma^{\ell_2}u_n-\Gamma x^{(1)}_n) \\
& -\prod_{t=1}^{\ell_2}(1-\gamma^1_{n,t})(\Gamma^{\ell_2}x^{(1)}_n-\Gamma^{\ell_2}u_n)\|^2, \\
\leq\ & \gamma^1_{n,1}\|x^{(2)}_n-u_n\|^2+\sum_{i=2}^{\ell_2}\gamma^1_{n,i}\prod_{t=1}^{i-1}(1-\gamma^1_{n,t})\|u_n-\Gamma^{i-1}u_n\|^2 \\
& +\prod_{t=1}^{\ell_1}(1-\gamma^1_{n,t})\|u_n-\Gamma^{\ell_2}u_n)\|^2+\sum_{i=2}^{\ell_2}\gamma^1_{n,i}\prod_{t=1}^{i-1}(1-\gamma^1_{n,t})\|-(\Gamma^{i-1}x^{(1)}_n-\Gamma^{i-1}u_n)\|^2 \\
& +\prod_{t=1}^{\ell_2}(1-\gamma^1_{n,t})\|\Gamma^{\ell_2}x^{(1)}_n-\Gamma^{\ell_2}u_n\|^2, \\
\leq\ & \gamma^1_{n,1}\|x^{(2)}_n-u_n\|^2+\sum_{i=2}^{\ell_2}\gamma^1_{n,i}\prod_{t=1}^{i-1}(1-\gamma^1_{n,t})(\rho^i)^2\|x^{(1)}_n-u_n\|^2+\prod_{t=1}^{\ell_2}(1-\gamma^1_{n,t})(\rho^i)^2\|x^{(1)}_n-u_n\|^2 \\
& +\sum_{i=2}^{\ell_2}\gamma^1_{n,i}\prod_{t=1}^{i-1}(1-\gamma^1_{n,t})\|u_n-\Gamma^{i-1}u_n\|^2+\prod_{t=1}^{\ell_2}(1-\gamma^1_{n,t})\|u_n-\Gamma^{\ell_2}u_n)\|^2 \\
\leq\ & \frac{\gamma^1_{n,1}}{1-\Big[\sum_{i=2}^{\ell_2}\gamma^1_{n,i}\prod_{t=1}^{i-1}(1-\gamma^1_{n,t})(\rho^i)^2+\prod_{t=1}^{\ell_2}(1-\gamma^1_{n,t})(\rho^i)^2\Big]}\|x^{(2)}_n-u_n\|^2 \\
& +\frac{\sum_{i=2}^{\ell_2}\gamma^1_{n,i}\prod_{t=1}^{i-1}(1-\gamma^1_{n,t})}{1-\Big[\sum_{i=2}^{\ell_2}\gamma^1_{n,i}\prod_{t=1}^{i-1}(1-\gamma^1_{n,t})(\rho^i)^2+\prod_{t=1}^{\ell_2}(1-\gamma^1_{n,t})(\rho^i)^2\Big]}\|u_n-\Gamma^{\ell_2}u_n\|^2 \\
& +\frac{\prod_{t=1}^{\ell_2}(1-\gamma^1_{n,t})}{1-\Big[\sum_{i=2}^{\ell_2}\gamma^1_{n,i}\prod_{t=1}^{\ell_2}(1-\gamma^1_{n,t})(\rho^i)^2+\prod_{t=1}^{\ell_2}(1-\gamma^1_{n,t})(\rho^i)^2\Big]}\|u_n-\Gamma^{\ell_2}u_n\|^2.
\end{aligned}
$$

The last inequality and (3.32) imply

$$
\begin{aligned}
\|x^{(1)}_n-u_n\|^2 \leq\ & \frac{\gamma^1_{n,1}}{1-\Big[\sum_{i=2}^{\ell_2}\gamma^1_{n,i}\prod_{t=1}^{i-1}(1-\gamma^1_{n,t})(\rho^i)^2+\prod_{t=1}^{\ell_2}(1-\gamma^1_{n,t})(\rho^i)^2\Big]}\|x^{(2)}_n-u_n\|^2 \\
& +\frac{\sum_{i=2}^{\ell_2}\gamma^1_{n,i}\prod_{t=1}^{i-1}(1-\gamma^1_{n,t})+\prod_{t=1}^{\ell_2}(1-\gamma^1_{n,t})}{1-\Big[\sum_{i=2}^{\ell_2}\gamma^1_{n,i}\prod_{t=1}^{i-1}(1-\gamma^1_{n,t})(\rho^i)^2+\prod_{t=1}^{\ell_2}(1-\gamma^1_{n,t})(\rho^i)^2\Big]} \\
& \times(1+(\rho^i)^2)\|u_n-q\|^2. \qquad (3.45)
\end{aligned}
$$

Using a similar argument as in (3.45), we get the following:

$$\|x_n^{(2)} - u_n\|^2 \leq \frac{\gamma_{n,1}^2}{1 - \Big[\sum_{i=2}^{\ell_3} \gamma_{n,i}^2 \prod_{t=1}^{i-1}(1-\gamma_{n,t}^3)(\rho^i)^2 + \prod_{t=1}^{\ell_3}(1-\gamma_{n,t}^2)(\rho^i)^2\Big]}\|x_n^{(3)} - u_n\|^2 + \frac{\sum_{i=2}^{\ell_3} \gamma_{n,i}^2 \prod_{t=1}^{\ell_3}(1-\gamma_{n,t}^2) + \prod_{t=1}^{\ell_3}(1-\gamma_{n,t}^2)}{1 - \Big[\sum_{i=2}^{\ell_3} \gamma_{n,i}^1 \prod_{t=1}^{i-1}(1-\gamma_{n,t}^2)(\rho^i)^2 + \prod_{t=1}^{\ell_3}(1-\gamma_{n,t}^2)(\rho^i)^2\Big]} \times(1+(\rho^i)^2)\|u_n - q\|^2, \tag{3.46}$$

$$\|x_n^{(3)} - u_n\|^2 \leq \frac{\gamma_{n,1}^3}{1 - \Big[\sum_{i=2}^{\ell_4} \gamma_{n,i}^1 \prod_{t=1}^{i-1}(1-\gamma_{n,t}^3)(\rho^i)^2 + \prod_{t=1}^{\ell_4}(1-\gamma_{n,t}^3)(\rho^i)^2\Big]}\|x_n^{(4)} - u_n\|^2 + \frac{\sum_{i=2}^{\ell_4} \gamma_{n,i}^3 \prod_{t=1}^{i-1}(1-\gamma_{n,t}^3) + \prod_{t=1}^{\ell_4}(1-\gamma_{n,t}^3)}{1 - \Big[\sum_{i=2}^{\ell_4} \gamma_{n,i}^3 \prod_{t=1}^{i-1}(1-\gamma_{n,t}^3)(\rho^i)^2 + \prod_{t=1}^{\ell_4}(1-\gamma_{n,t}^3)(\rho^i)^2\Big]} \times(1+(\rho^i)^2)\|u_n - q\|^2 \tag{3.47}$$

and by continuing the computation up to $(r-2)$ and $(r-1)$ times, we get

$$\|x_n^{(r-2)} - u_n\|^2 \leq \frac{\gamma_{n,1}^{r-2}}{1 - \Big[\sum_{i=2}^{\ell_{r-1}} \gamma_{n,i}^{r-2} \prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})(\rho^i)^2 + \prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2})(\rho^i)^2\Big]}\|x_n^{(r-1)} - u_n\|^2 + \frac{\sum_{i=2}^{\ell_{r-1}} \gamma_{n,i}^{r-2} \prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2}) + \prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2})}{1 - \Big[\sum_{i=2}^{\ell_{r-1}} \gamma_{n,i}^{r-2} \prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})(\rho^i)^2 + \prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2})(\rho^i)^2\Big]} \times(1+(\rho^i)^2)\|u_n - q\|^2 \tag{3.48}$$

and

$$\|x_n^{(r-1)} - u_n\|^2 \leq \frac{\gamma_{n,1}^{r-1}}{1 - \Big[\sum_{i=2}^{\ell_r} \gamma_{n,i}^{r-1} \prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-1})(\rho^i)^2 + \prod_{t=1}^{\ell_r}(1-\gamma_{n,t}^{r-1})(\rho^i)^2\Big]}\|x_n^{(r)} - u_n\|^2 + \frac{\sum_{i=2}^{\ell_r} \gamma_{n,i}^{r-1} \prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-1}) + \prod_{t=1}^{\ell_r}(1-\gamma_{n,t}^{r-1})}{1 - \Big[\sum_{i=2}^{\ell_r} \gamma_{n,i}^{r-1} \prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-1})(\rho^i)^2 + \prod_{t=1}^{\ell_r}(1-\gamma_{n,t}^{r-1})(\rho^i)^2\Big]} \times(1+(\rho^i)^2)\|u_n - q\|^2. \tag{3.49}$$

Putting (3.45)–(3.49) into (3.44) and simplifying using Proposition 2.3 (bearing in mind that $\rho^i \in [0,1)$ so that $0 < (\rho^i)^2 < \rho < 1$), we get

$$\begin{aligned}\|x_{n+1} - u_{n+1}\|^2 \leq\ & ((1-\delta_{n,1}) + \delta_{n,1})((1-\gamma_{n,1}^1) + \gamma_{n,1}^1)((1-\gamma_{n,1}^2) + \gamma_{n,1}^2)((1-\gamma_{n,1}^3) + \gamma_{n,1}^3) \\ & \times \cdots \\ & \times((1-\gamma_{n,1}^{r-2}) + \gamma_{n,1}^{r-2})((1-\gamma_{n,1}^{r-1}) + \gamma_{n,1}^{r-1})\|x_n - u_n\|^2 \\ & +\Bigg\{((1-\gamma_{n,1}^1) + \gamma_{n,1}^1)((1-\gamma_{n,1}^2) + \gamma_{n,1}^2)((1-\gamma_{n,1}^3) + \gamma_{n,1}^3)((1-\gamma_{n,1}^{r-2}) + \gamma_{n,1}^{r-2}) \\ & \times\Bigg[\frac{\sum_{i=2}^{\ell_{r-1}} \gamma_{n,i}^{r-2} \prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})}{1 - \Big[\sum_{i=2}^{\ell_{r-1}} \gamma_{n,i}^{r-2} \prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})(\rho^i)^2 + \prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2})(\rho^i)^2\Big]}\end{aligned}$$

$$
\begin{aligned}
& +\frac{\prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2})}{1-\Big[\sum_{i=2}^{\ell_{r-1}}\gamma_{n,i}^{r-2}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})(\rho^i)^2+\prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2})(\rho^i)^2\Big]}\Bigg] \\
& +((1-\gamma_{n,1}^1+\gamma_{n,1}^1))((1-\gamma_{n,1}^2)+\gamma_{n,1}^2)((1-\gamma_{n,1}^3)+\gamma_{n,1}^3) \\
& \times\cdots \\
& \times((1-\gamma_{n,1}^{r-2})+\gamma_{n,1}^{r-2}) \\
& \times\Bigg[\frac{\sum_{i=2}^{\ell_{r-2}}\gamma_{n,i}^{r-3}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-3})}{1-\Big[\sum_{i=2}^{\ell_{r-2}}\gamma_{n,i}^{r-3}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-3})(\rho^i)^2+\prod_{t=1}^{\ell_{r-2}}(1-\gamma_{n,t}^{r-3})(\rho^i)^2\Big]} \\
& +\frac{\prod_{t=1}^{\ell_{r-2}}(1-\gamma_{n,t}^{r-3})}{1-\Big[\sum_{i=2}^{\ell_{r-2}}\gamma_{n,i}^{r-3}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-3})(\rho^i)^2+\prod_{t=1}^{\ell_{r-2}}(1-\gamma_{n,t}^{r-3})(\rho^i)^2\Big]}\Bigg] \\
& +\frac{\sum_{i=2}^{\ell_2}\gamma_{n,i}^1\prod_{t=1}^{i-1}(1-\gamma_{n,t}^1)}{1-\Big[\sum_{i=2}^{\ell_2}\gamma_{n,i}^1\prod_{t=1}^{i-1}(1-\gamma_{n,t}^1)(\rho^i)^2+\prod_{t=1}^{\ell_2}(1-\gamma_{n,t}^1)(\rho^i)^2\Big]} \\
& +\frac{\prod_{t=1}^{\ell_2}(1-\gamma_{n,t}^1)}{1-\Big[\sum_{i=2}^{\ell_2}\gamma_{n,i}^1\prod_{t=1}^{i-1}(1-\gamma_{n,t}^1)(\rho^i)^2+\prod_{t=1}^{\ell_2}(1-\gamma_{n,t}^1)(\rho^i)^2\Big]}\Bigg\} \\
& \times((1-\delta_{n,1})+\delta_{n,1})(1+(\rho^i)^2)\|u_n-q\|^2 \\
\leq\; & [1-(1-\delta_{n,1})(1-\rho)]\|x_n-u_n\|^2 \\
& +\Bigg\{((1-\gamma_{n,1}^1)+\gamma_{n,1}^1)((1-\gamma_{n,1}^2)+\gamma_{n,1}^2)((1-\gamma_{n,1}^3)+\gamma_{n,1}^3)((1-\gamma_{n,1}^{r-2})+\gamma_{n,1}^{r-2}) \\
& \times\Bigg[\frac{\sum_{i=2}^{\ell_{r-1}}\gamma_{n,i}^{r-2}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})}{1-\Big[\sum_{i=2}^{\ell_{r-1}}\gamma_{n,i}^{r-2}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})(\rho^i)^2+\prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2})(\rho^i)^2\Big]} \\
& +\frac{\prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2})}{1-\Big[\sum_{i=2}^{\ell_{r-1}}\gamma_{n,i}^{r-2}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-2})(\rho^i)^2+\prod_{t=1}^{\ell_{r-1}}(1-\gamma_{n,t}^{r-2})(\rho^i)^2\Big]}\Bigg] \\
& +((1-\gamma_{n,1}^1+\gamma_{n,1}^1))((1-\gamma_{n,1}^2)+\gamma_{n,1}^2)((1-\gamma_{n,1}^3)+\gamma_{n,1}^3) \\
& \times\cdots \\
& \times((1-\gamma_{n,1}^{r-2})+\gamma_{n,1}^{r-2}) \\
& \times\Bigg[\frac{\sum_{i=2}^{\ell_{r-2}}\gamma_{n,i}^{r-3}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-3})}{1-\Big[\sum_{i=2}^{\ell_{r-2}}\gamma_{n,i}^{r-3}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-3})(\rho^i)^2+\prod_{t=1}^{\ell_{r-2}}(1-\gamma_{n,t}^{r-3})(\rho^i)^2\Big]} \\
& +\frac{\prod_{t=1}^{\ell_{r-2}}(1-\gamma_{n,t}^{r-3})}{1-\Big[\sum_{i=2}^{\ell_{r-2}}\gamma_{n,i}^{r-3}\prod_{t=1}^{i-1}(1-\gamma_{n,t}^{r-3})(\rho^i)^2+\prod_{t=1}^{\ell_{r-2}}(1-\gamma_{n,t}^{r-3})(\rho^i)^2\Big]}\Bigg] \\
& +\frac{\sum_{i=2}^{\ell_2}\gamma_{n,i}^1\prod_{t=1}^{i-1}(1-\gamma_{n,t}^1)}{1-\Big[\sum_{i=2}^{\ell_2}\gamma_{n,i}^1\prod_{t=1}^{i-1}(1-\gamma_{n,t}^1)(\rho^i)^2+\prod_{t=1}^{\ell_2}(1-\gamma_{n,t}^1)(\rho^i)^2\Big]} \\
& +\frac{\prod_{t=1}^{\ell_2}(1-\gamma_{n,t}^1)}{1-\Big[\sum_{i=2}^{\ell_2}\gamma_{n,i}^1\prod_{t=1}^{i-1}(1-\gamma_{n,t}^1)(\rho^i)^2+\prod_{t=1}^{\ell_2}(1-\gamma_{n,t}^1)(\rho^i)^2\Big]}\Bigg\}
\end{aligned}
$$

$$\times((1-\delta_{n,1})+\delta_{n,1})(1+(\rho^i)^2)\|u_n-q\|^2. \tag{3.50}$$

We set

$$\sigma_n = (1-\delta_{n,1})(1-\rho) \tag{3.51}$$

and

$$\begin{aligned}
\tau_n \;=\; & \Big\{((1-\gamma^1_{n,1})+\gamma^1_{n,1})((1-\gamma^2_{n,1})+\gamma^2_{n,1})((1-\gamma^3_{n,1})+\gamma^3_{n,1})((1-\gamma^{r-2}_{n,1})+\gamma^{r-2}_{n,1}) \\
& \times\Bigg[\frac{\sum_{i=2}^{\ell_{r-1}}\gamma^{r-2}_{n,i}\prod_{t=1}^{i-1}(1-\gamma^{r-2}_{n,t})}{1-\Big[\sum_{i=2}^{\ell_{r-1}}\gamma^{r-2}_{n,i}\prod_{t=1}^{i-1}(1-\gamma^{r-2}_{n,t})(\rho^i)^2+\prod_{t=1}^{\ell_{r-1}}(1-\gamma^{r-2}_{n,t})(\rho^i)^2\Big]} \\
& +\frac{\prod_{t=1}^{\ell_{r-1}}(1-\gamma^{r-2}_{n,t})}{1-\Big[\sum_{i=2}^{\ell_{r-1}}\gamma^{r-2}_{n,i}\prod_{t=1}^{i-1}(1-\gamma^{r-2}_{n,t})(\rho^i)^2+\prod_{t=1}^{\ell_{r-1}}(1-\gamma^{r-2}_{n,t})(\rho^i)^2\Big]}\Bigg] \\
& +((1-\gamma^1_{n,1}+\gamma^1_{n,1}))((1-\gamma^2_{n,1})+\gamma^2_{n,1})((1-\gamma^3_{n,1})+\gamma^3_{n,1}) \\
& \times\cdots \\
& \times((1-\gamma^{r-2}_{n,1})+\gamma^{r-2}_{n,1}) \\
& \times\Bigg[\frac{\sum_{i=2}^{\ell_{r-2}}\gamma^{r-3}_{n,i}\prod_{t=1}^{i-1}(1-\gamma^{r-3}_{n,t})}{1-\Big[\sum_{i=2}^{\ell_{r-2}}\gamma^{r-3}_{n,i}\prod_{t=1}^{i-1}(1-\gamma^{r-3}_{n,t})(\rho^i)^2+\prod_{t=1}^{\ell_{r-2}}(1-\gamma^{r-3}_{n,t})(\rho^i)^2\Big]} \\
& +\frac{\prod_{t=1}^{\ell_{r-2}}(1-\gamma^{r-3}_{n,t})}{1-\Big[\sum_{i=2}^{\ell_{r-2}}\gamma^{r-3}_{n,i}\prod_{t=1}^{i-1}(1-\gamma^{r-3}_{n,t})(\rho^i)^2+\prod_{t=1}^{\ell_{r-2}}(1-\gamma^{r-3}_{n,t})(\rho^i)^2\Big]}\Bigg] \\
& +\frac{\sum_{i=2}^{\ell_2}\gamma^1_{n,i}\prod_{t=1}^{i-1}(1-\gamma^1_{n,t})}{1-\Big[\sum_{i=2}^{\ell_2}\gamma^1_{n,i}\prod_{t=1}^{i-1}(1-\gamma^1_{n,t})(\rho^i)^2+\prod_{t=1}^{\ell_2}(1-\gamma^1_{n,t})(\rho^i)^2\Big]} \\
& +\frac{\prod_{t=1}^{\ell_2}(1-\gamma^1_{n,t})}{1-\Big[\sum_{i=2}^{\ell_2}\gamma^1_{n,i}\prod_{t=1}^{i-1}(1-\gamma^1_{n,t})(\rho^i)^2+\prod_{t=1}^{\ell_2}(1-\gamma^1_{n,t})(\rho^i)^2\Big]}\Big\} \\
& \times((1-\delta_{n,1})+\delta_{n,1})(1+(\rho^i)^2)\|u_n-q\|^2.
\end{aligned} \tag{3.52}$$

Then, we have (from (3.50)) that

$$\|x_{n+1}-u_{n+1}\|^2 \le (1-\sigma_n)\|x_n-u_n\|^2+\tau_n. \tag{3.53}$$

From Lemma 2.3 and (3.53), we obtain

$$\lim_{n\to\infty}\|x_n-u_n\| = 0. \tag{3.54}$$

Since $\|u_n-q\| \le \|x_n-u_n\|+\|x_n-q\|$, it follows from the assumption $\lim_{n\to\infty}\|x_n-q\| = 0$ and (3.54) that $\lim_{n\to\infty}\|u_n-q\| = 0$.

Since $(a) \Rightarrow (b)$ and $(b) \Rightarrow (a)$, it follows that the convergence of the implicit IH-multistep iterative scheme (3.1) is equivalent to the convergence of the implicit IH-Mann iterative scheme (3.7) when applied to the general class of the map (3.56). This completes the proof. □

The corollaries below are immediate consequences of Theorem 3.3.

Corollary 3.2. *Let H be a real Hilbert space, E a nonempty closed and convex subset of H and $\Gamma : E \longrightarrow E$, with $q \in F(\Gamma)$ satisfying the following condition:*

$$||\Gamma^i x - q|| \leq \rho^i ||x - y||, \tag{3.55}$$

where $\rho^i \in [0, 1)$. Let $u_0 = z_0 = w_0 \in E$, then the following are equivalent.

(a) (i) Implicit IH-Mann iterative scheme $\{u_n\}_{n=1}^{\infty}$ defined by (3.7) converges to q;
(ii) Implicit IH-Ishikawa iterative scheme $\{z_n\}_{n=1}^{\infty}$ defined by (3.6) converges to q;
(b) (i) Implicit IH-Mann iterative scheme $\{u_n\}_{n=1}^{\infty}$ defined by (3.7) converges to q;
(ii) Implicit IH-Noor iterative scheme $\{z_n\}_{n=1}^{\infty}$ defined by (3.5) converges to q.

Proof. The proof of Corollary 3.4 is similar to that of Theorem 3.3. This completes the proof. □

Corollary 3.3. *Let H be a real Hilbert space, E a nonempty closed and convex subset of H and $\Gamma : E \longrightarrow E$, with $q \in F(\Gamma)$ satisfying the following condition:*

$$||\Gamma^i x - q|| \leq \rho^i ||x - y||, \tag{3.56}$$

where $\rho^i \in [0, 1)$. Let $u_0 = z_0 = w_0 x_0 \in E$, then the following are equivalent.

(i) Implicit IH-Mann iterative scheme $\{u_n\}_{n=1}^{\infty}$ defined by (3.7) converges to q;
(ii) Implicit IH-Ishikawa iterative scheme $\{z_n\}_{n=1}^{\infty}$ defined by (3.6) converges to q;
(iii) Implicit IH-Noor iterative scheme $\{u_n\}_{n=1}^{\infty}$ defined by (3.5) converges to q;
(iv) Implicit IH-multistep iterative scheme $\{z_n\}_{n=1}^{\infty}$ defined by (3.1) converges to q.

4. Applications, numerical examples and open problem

Implicit iterations could be seen in application for problems involving recurrent neural network (RNN) analysis. Indeed, neural networks are a class of nonlinear approximation functions and stable states which is established in recurrent auto-associative neural network using iterations. Here, we analyze the convergence speed of implicit iterations in an RNN and several important results will be studied for decreasing and increasing functions. The results obtained possess multifaceted real life applications and in particular can be helpful to design the inner product kernel of support vector machines with a faster convergence rate (for further study about RNNs, we refer any interested reader to [44]).

Now, we demonstrate the equivalence of convergence between the implicit IH-multistep iterative scheme (3.1) and other implicit IH-type [implicit IH-Noor (3.5), implicit IH-Ishikawa (3.6), implicit IH-Mann (3.7)] iterative schemes with the help of computer programs in Matlab. We shall consider increasing and decreasing functions for the demonstration of our results as shown in the tables below.

4.1. Example of increasing function

Let $\Gamma : [6, 8] \longrightarrow [6, 8]$ be defined by $\Gamma x = \dfrac{x}{2} + 3$. Then, Γ is an increasing function with the fixed point $q = 6.000000$. Using the initial values $x_0 = w_0 = z_0 = u_0 = 7.000000$ and $\delta_{n,i} = \gamma^j_{n,i} = 1 - \dfrac{1}{n}$ for $j = 1, 2, 3, \cdots, r - 2, n \geq 2$ for all iterative schemes. The equivalent of the iterative schemes considered for the fixed point $q = 6.000000$ are as shown below in the Table 1.

Table 1. Numerical example for decreasing function $\Gamma x = \frac{x}{2} + 3$.

n	IH-MANN	IH- ISHIKAWA	IH-NOOR	IH-MULTI-STEP
1	7.000000	7.000000	7.000000	7.000000
2	6.800000	6.400000	6.200000	6.100000
3	6.640000	6.160000	6.040000	6.010000
4	6.512000	6.064000	6.008000	6.001000
5	6.409600	6.025600	6.001600	6.000100
6	6.327680	6.010240	6.000320	6.000001
7	6.262144	6.004096	6.000064	6.000000
8	6.209715	6.001638	6.000013	6.000000
9	6.167772	6.000655	6.000003	6.000000
10	6.134218	6.000262	6.000001	6.000000
12	6.107374	6.000105	6.000000	6.000000
13	6.085899	6.000042	6.000000	6.000000
14	6.068719	6.000017	6.000000	6.000000
15	6.054976	6.000007	6.000000	6.000000
16	6.043980	6.000003	6.000000	6.000000
17	6.035184	6.000001	6.000000	6.000000
18	6.028147	6.000000	6.000000	6.000000
19	6.022518	6.000000	6.000000	6.000000
20	6.018014	6.000000	6.000000	6.000000
21	6.014412	6.000000	6.000000	6.000000
22	6.011529	6.000000	6.000000	6.000000
23	6.009223	6.000000	6.000000	6.000000
24	6.007379	6.000000	6.000000	6.000000
25	6.005903	6.000000	6.000000	6.000000
26	6.004722	6.000000	6.000000	6.000000
27	6.003778	6.000000	6.000000	6.000000
28	6.003022	6.000000	6.000000	6.000000
29	6.002418	6.000000	6.000000	6.000000
30	6.001934	6.000000	6.000000	6.000000
31	6.001547	6.000000	6.000000	6.000000
32	6.001238	6.000000	6.000000	6.000000
33	6.000990	6.000000	6.000000	6.000000
34	6.000792	6.000000	6.000000	6.000000
35	6.000634	6.000000	6.000000	6.000000

Continued on next page

n	IH-MANN	IH-ISHIKAWA	IH-NOOR	IH-MULTI-STEP
36	6.000507	6.000000	6.000000	6.000000
37	6.000406	6.000000	6.000000	6.000000
38	6.000325	6.000000	6.000000	6.000000
39	6.000260	6.000000	6.000000	6.000000
40	6.000208	6.000000	6.000000	6.000000
41	6.000166	6.000000	6.000000	6.000000
42	6.000133	6.000000	6.000000	6.000000
43	6.000106	6.000000	6.000000	6.000000
44	6.000085	6.000000	6.000000	6.000000
45	6.000068	6.000000	6.000000	6.000000
46	6.000054	6.000000	6.000000	6.000000
47	6.000044	6.000000	6.000000	6.000000
48	6.000035	6.000000	6.000000	6.000000
49	6.000028	6.000000	6.000000	6.000000
50	6.000022	6.000000	6.000000	6.000000
51	6.000018	6.000000	6.000000	6.000000
52	6.000014	6.000000	6.000000	6.000000
53	6.000011	6.000000	6.000000	6.000000
54	6.000009	6.000000	6.000000	6.000000
55	6.000007	6.000000	6.000000	6.000000
56	6.000006	6.000000	6.000000	6.000000
57	6.000005	6.000000	6.000000	6.000000
58	6.000004	6.000000	6.000000	6.000000
59	6.000003	6.000000	6.000000	6.000000
60	6.000002	6.000000	6.000000	6.000000
61	6.000002	6.000000	6.000000	6.000000
62	6.000002	6.000000	6.000000	6.000000
63	6.000001	6.000000	6.000000	6.000000
64	6.000001	6.000000	6.000000	6.000000
65	6.000001	6.000000	6.000000	6.000000
65	6.000001	6.000000	6.000000	6.000000
65	6.000000	6.000000	6.000000	6.000000

4.2. Example of decreasing function

Let $\Gamma : [0,1] \longrightarrow [0,1]$ be defined by $\Gamma x = (1-x)^2$. Then, Γ is an increasing function with the fixed point $q = 0.381996$. Using the initial values $x_0 = w_0 z_0 u_0 = 7.000000$ and $\delta_{n,i} = \gamma^j_{n,i} = 1 - \frac{1}{n}$ for $j = 1, 2, 3, \cdots, r-2, n \geq 2$ for all iterative schemes. The equivalent of the iterative schemes considered for the fixed point $q = 0.381996$ are as shown below in Table 2.

Table 2. Numerical example for decreasing function $\Gamma x = (1 - x)^2$.

n	IH-MANN	IH-ISHIKAWA	IH-NOOR	IH-MULTI-STEP
1	0.700000	0.700000	0.700000	0.700000
2	0.483425	0.413628	0.391780	0.385002
3	0.413628	0.385002	0.382256	0.381994
4	0.391780	0.382256	0.381975	0.381966
5	0.385002	0.381994	0.381966	0.381966
6	0.382904	0.381969	0.381966	0.381966
7	0.382256	0.381966	0.381966	0.381966
8	0.382056	0.381966	0.381966	0.381966
9	0.381994	0.381966	0.381966	0.381966
10	0.381975	0.381966	0.381966	0.381966
12	0.381969	0.381966	0.381966	0.381966
13	0.381967	0.381966	0.381966	0.381966
14	0.381966	0.381966	0.381966	0.381966
15	0.381966	0.381966	0.381966	0.381966

Remark 4.1. *(a) Using Table 1, it is observed that for the increasing function* $\Gamma x = \frac{x}{2} + 3$*, the convergence of the implicit multistep-IH iterative scheme* (3.1) *to the fixed point 6.000000 is equivalent to the convergence of other implicit IH-type [implicit IH-Noor (IIHN)* (3.5)*, implicit IH-Ishikawa (IHII)* (3.6) *and implicit IH-Mann (IHM)* (3.7)*] iterative schemes to the same fixed point of 6.000000.*

(b) Using Table 2, it is observed that, for the decreasing function $\Gamma x = (1 - x)^2$ *, the convergence of the implicit IH-multistep iterative scheme* (3.1) *to the fixed point 0.381966 is equivalent to the convergence of other implicit IH-type [implicit IH-Noor (IIHN)* (3.5)*, implicit IH-Ishikawa (IIHI)* (3.6) *and implicit IH-Mann (IIHM)* (3.7)*] iterative schemes to the same fixed point of 0.381966.*

Remark 4.2. *Despite the remarkable results obtained in the papers studied (and their various inclusions), the implications of the "sum conditions" (that is, the condition that* $\sum_{r=0}^{\ell_1} \alpha_{n,r} = 1, \sum_{s=0}^{\ell_{t+1}} \alpha_{n,s}^t = 1$ *and* $\sum_{s=0}^{\ell_u} \alpha_{n,t}^{u-1} = 1$*, where* $\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq \ell_u$ *for each u,* $\alpha_{n,s}^t, \alpha_{n,0}, \neq 0$ *for each t,* $\alpha_{n,s}^t \in [0, 1]$ *and* ℓ_1 *and* ℓ_u *are fixed integers for each u) are quite enormous . For instance, the sum condition implies that*

(1) for large $\ell_u, u \geq 1$*, one has to choose different points of the sequences* $\{\alpha_{n,i}\}_{n=0}^{\infty}$ *that would guarantee instant generation of such a finite family of control sequences such that* $\sum_{r=0}^{\ell_1} \alpha_{n,r} = 1, \sum_{s=0}^{\ell_{t+1}} \alpha_{n,s}^t = 1$ *and* $\sum_{s=0}^{\ell_u} \alpha_{n,t}^{u-1} = 1$*, which might be almost impossible and*

(2) one has to make adequate provisions for the computational time and memory space for the computation and storage of the bulky and complex task of generating $\sum_{r=0}^{\ell_1} \alpha_{n,r} = 1, \sum_{s=0}^{\ell_{t+1}} \alpha_{n,s}^t = 1$ *and* $\sum_{s=0}^{\ell_u} \alpha_{n,t}^{u-1} = 1$*, which invariably leads to enormous computational cost.*

Unlike the papers studied, the iterative schemes used to obtain our results do not require sum conditions. Consequently, our iterative schemes are more efficient in application as compared to *several other iterative techniques studied in this area.*

Remark 4.3. *The following areas are still open:*

(i) The results obtained in this paper are in the setting of real Hilbert spaces. However, there are other spaces more general than Hilbert spaces. Hence, it becomes necessary to ask if Propositions (2.3) and (2.4) could be proved in those other spaces so as to generalize the results in this paper.
(ii) The results in this paper are for a finite family of a general class of contractive-type maps. Again, is it possible to prove Propositions (2.3) and (2.4) for the case of an infinite family of maps so as to extend the results in this paper?
(ii) In this paper, the speed of convergence of the iterative schemes was only considered for different IH-type implicit iteration methods. Relating the speed of convergence of the iterative methods studied in the paper to other implicit iterative methods studied in literature is still open.

5. Conclusions

In this paper, we studied the set of fixed points and considered iterative schemes of the IH-type in order to obtain approximate fixed points of contractive-type mappings for which we have proven strong convergence theorems without any imposition of sum conditions on the control parameters.

Further, we showed that IH-Mann, IH-Ishikawa, IH-Noor and IH-multistep iteration techniques defined with the help of contractive-type mappings are equivalent. Also, we demonstrated the rate of convergence for the various iteration schemes considered and discovered that the IH-multistep iterative scheme converges faster than the rest of the iterative schemes for increasing and decreasing functions.

Finally, an affirmative answer has been provided for Question 1.2 and the numerical examples considered in this paper justified our claim on the equivalence results obtained. These results show that our implicit IH-type hybrid iterative schemes (for which no imposition of any sum condition is required) have better potentials for further applications than some other iterative schemes considered so far in this area.

Conflict of interest

The authors declare no conflicts of interest.

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