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Existence and stability analysis for Caputo generalized hybrid Langevin differential systems involving three-point boundary conditions

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Abstract

This research inscription gets to grips with two novel varieties of boundary value problems. One of them is a hybrid Langevin fractional differential equation, whilst the other is a coupled system of hybrid Langevin differential equation encapsuling a collective fractional derivative known as the ψ-Caputo fractional operator. Such operators are generated by iterating a local integral of a function with respect to another increasing positive function Ψ. The existence of the solutions of the aforehand equations is tackled by using the Dhage fixed point theorem, whereas their uniqueness is handled using the Banach fixed point theorem. On the top of this, the stability within the scope of Ulam–Hyers of solutions to these systems are also considered. Two pertinent examples are presented to corroborate the reported results.

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1 Introduction

The qualitative analysis of differential systems with noninteger or fractional orders (FDS) has underfonged momentous interest virtue of the broader advancement and umpteen practices of the fractional calculus related to natural phenomena in the real world. The applications of such models can be found in many recent works; the reader can refer to [4, 29],and references therein.

Particularly, the existence, uniqueness, and stability analysis of a solution for FDS have been studied rapidly involving fractional derivatives due to Riemann and Liuoville, Caputo, Hadamard, Katugampola, etc; see, e.g., [1, 2, 9, 13, 16, 28], and the references therein.

Unlike standard fractional derivatives, the so-called generalized (or ψ-) fractional derivatives are introduced by many authors [7, 8, 21, 22]. These researches give rise to many investigations in the field of qualitative analysis in the FDS [3, 14, 26].

More specifically, coupled fractional differential systems are quite important in a variety of problems of applied nature, especially in biosciences. The analytical approaches of such systems also considered and investigated, by means of fixed point theorems, the existence, uniqueness, and stability [17, 20, 27].

Hybrid fractional differential equations are one of the recent investigations in the field of mathematical analysis of the FDS. In hybrid systems the authors used a common fixed point theorem for the product or sum of two operators in Banach spaces [11, 15, 19, 24, 25].

The study of Ulam stability for the FDS has been investigated by many authors. They have discussed various Ulam–Hyers stability problems for different kinds of FDS including Langevin systems by using many techniques; see [5, 10, 12], and the references therein.

Motivated by new developments in ψ-fractional calculus, in the present research, we enquire the existence and uniqueness along with the stability in the sense of Ulam–Hyers for solutions to nonlinear hybrid fractional Langevin equations (HLFDS) described by

$$\begin{cases} {}^{c}\mathbb{D}_{a^+}^{\mu;\psi}[{}^{c}\mathbb{D}_{a^+}^{\nu;\psi}[\frac{\omega(\tau)}{\mathbb{G}(\tau,\omega(\tau))}] - \lambda\omega(\tau)] = \mathbb{F}(\tau,\omega(\tau)), \quad \tau \in J := [a,b], \\ \omega(a) = 0, \qquad {}^{c}\mathbb{D}_{a^+}^{\nu;\psi}[\frac{\omega(\tau)}{\mathbb{G}(\tau,\omega(\tau))}]_{\tau=a} = 0, \qquad \omega(b) = \zeta\omega(\eta), \quad \eta \in (a,b), \end{cases} \tag{1.1}$$

and the coupled system of HLFDS formulated by

$${}^{c}\mathbb{D}_{a^+}^{\mu_i 1;\psi}\left[{}^{c}\mathbb{D}_{a^+}^{\nu_i;\psi}\left[\frac{\omega_i(\tau)}{\mathbb{G}_i(\tau,\omega_1(\tau),\omega_2(\tau))}\right] - \lambda_i\omega_i(\tau)\right] = \mathbb{F}_i\big(\tau,\omega_1(\tau),\omega_2(\tau)\big), \tag{1.2}$$

$\tau \in J$, $i = 1,2$, with boundary conditions

$$\omega_i(a) = 0, \qquad {}^{c}\mathbb{D}_{a^+}^{\nu_i;\psi}\left[\frac{\omega_i(\tau)}{\mathbb{G}_i(\tau,\omega_1(\tau),\omega_2(\tau))}\right]_{\tau=a} = 0, \qquad \omega_i(b) = \zeta_i\omega_i(\eta_i), \tag{1.3}$$

$i = 1,2$, $\eta_i \in (a,b)$, where ${}^{c}\mathbb{D}_{a^+}^{\beta;\psi}$ is the ψ-Caputo fractional derivative of order $\beta \in \{\mu,\mu_i\} \subseteq (0,1]$, $\{\nu,\nu_i\} \subseteq (1,2]$, $i = 1,2$, $\mathbb{F} : J \times \mathbb{R} \longrightarrow \mathbb{R}$ and $\mathbb{G} : J \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$ are given continuous functions such that $\mathbb{F}(\tau,0) = 0$, λ, λ_i, ζ, and ζ_i are all real constants. The nonlinear functions $\mathbb{G}_1,\mathbb{G}_2 : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$ and the functions $\mathbb{F}_1,\mathbb{F}_2 : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are also continuous such that $\mathbb{F}_1(\tau,0,\omega_2) = \mathbb{F}_2(\tau,\omega_1,0) = 0$.

Here is a brief outline of the work. Section 2 provides the definitions and initial results presupposed to prove our key findings. Moreover, we present an auxiliary lemma on the representation of solutions of problem (1.1) and system (1.2)–(1.3). In Sect. 3, we establish the existence of solutions taking advantage of the Dhage fixed point theorem. On the other hand, we discuss the uniqueness of these solutions using the Banach fixed point theorem in Sect. 4. We investigate the stability in the sense of Ulam for the proposed problems in Sect. 5. Finally, we provide two examples to support the acquired results in Sect. 6.

2 Groundwork

To procure our fundamental purposes, at the outset, we scrutinize some auxiliary notions needed in the depth of this work.

Let $\mathfrak{C} = C(J,\mathbb{R})$ be the set of continuous real-valued functions $\omega : J \to \mathbb{R}$, which is clearly a Banach space endowed with the supremum norm

$$\|\omega\| = \sup\big\{\big|\omega(\tau)\big| : \tau \in J\big\}$$

and is a Banach algebra under the multiplication defined by

$$(\omega\varpi)(\tau) = \omega(\tau)\varpi(\tau)$$

for $\omega, \varpi \in \mathfrak{C}$ and $\tau \in J$. Now the product space $\mathbb{E} = \mathfrak{C} \times \mathfrak{C}$ is a vector space under the coordinatewise addition and scalar multiplication. On the product linear space $\mathbb{E}$, define the norm $\|\cdot\|$ by

$$\big\|(\omega_1, \omega_2)\big\| = \|\omega_1\| + \|\omega_2\|.$$

Obviously, the norm pace $(\mathbb{E}, \|(\cdot,\cdot)\|)$ is a Banach space, which can be considered a Banach algebra as well. The multiplication action of two elements of $\mathbb{E}$ is defined as

$$\big((\omega_1, \omega_2)(\varpi_1, \varpi_2)\big)(\tau) = (\omega_1, \omega_2)(\tau)(\varpi_1, \varpi_2)(\tau) = \big(\omega_1(\tau)\varpi_1(\tau), \omega_2(\tau)\varpi_2(\tau)\big) \tag{2.1}$$

for all $\tau \in J$, where $(\omega_1, \omega_2), (\varpi_1, \varpi_2) \in \mathbb{E}$.

Afterward, we start by giving the ψ-fractional integrals and derivatives involved. For more related details, we refer the readers to inspect papers [7, 8] and, more generally, the monograph [23].

Definition 2.1 ([7]) Let $\alpha > 0$, and let an increasing function $\psi : J \longrightarrow \mathbb{R}$ satisfy $\psi'(\tau) \neq 0$ for all $\tau \in J$. We define the left-sided ψ-Riemann–Liouville integral of an integrable function ω on J with respect to ψ as

$$\mathbb{I}_{a^+}^{\alpha;\psi} \omega(\tau) = \frac{1}{\Gamma(\alpha)} \int_a^\tau \psi'(s)\big(\psi(\tau) - \psi(s)\big)^{\alpha-1} \omega(s)\, ds, \tag{2.2}$$

where Γ is the usual Euler gamma function.

Equation (2.2) reduces to the Riemann–Liouville and Hadamard fractional integrals by taking $\psi(\tau) = \tau$ and $\psi(\tau) = \ln\tau$, respectively.

Definition 2.2 ([7]) Let $m \in \mathbb{N}$ with $m = [\alpha] + 1$. The left-sided ψ-Caputo fractional derivative $\omega \in C^m(J, \mathbb{R})$ with respect to a strictly increasing function ψ for all $\tau \in J$ is defined as

$${}^c\mathbb{D}_{a^+}^{\alpha;\psi} \omega(\tau) = \mathbb{I}_{a^+}^{m-\alpha;\psi} \left(\frac{1}{\psi'(\tau)} \frac{d}{dz}\right)^m \omega(\tau).$$

Lemma 2.3 ([7]) *Assuming that* $\alpha, \beta > 0$ *and* $\omega \in L^1(J, \mathbb{R})$, *we get*

$$\mathbb{I}_{a^+}^{\alpha;\psi} \mathbb{I}_{a^+}^{\beta;\psi} \omega(\tau) = \mathbb{I}_{a^+}^{\alpha+\beta;\psi} \omega(\tau), \quad \tau \in J.$$

Lemma 2.4 ([7]) *Let* $\alpha > 0$.

(*i*) *If* $\omega \in C(J, \mathbb{R})$, *then*

$${}^c\mathbb{D}_{a^+}^{\alpha;\psi} \mathbb{I}_{a^+}^{\alpha;\psi} \omega(\tau) = \omega(\tau), \quad \tau \in J.$$

(*ii*) *If* $\omega \in C^m(J,\mathbb{R})$ *and* $m-1<\alpha<m$, *then*

$$\mathbb{I}_{a^+}^{\alpha;\psi}\,{}^c\mathbb{D}_{a^+}^{\alpha;\psi}\omega(\tau)=\omega(\tau)-\sum_{k=0}^{m-1}\frac{(\frac{1}{\psi'(\tau)}\frac{d}{d\tau})^k\omega(a)}{k!}\big[\psi(\tau)-\psi(a)\big]^k,\quad \tau\in J,$$

for some constants $c_k, k=0,1,2,\dots,m-1$.

Lemma 2.5 ([8]) *Let* $\tau>a$, $\alpha\geq 0$, *and* $\beta>0$. *Then*

- $\mathbb{I}_{a^+}^{\alpha;\psi}(\psi(t)-\psi(a))^{\beta-1}(\tau)=\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(\psi(\tau)-\psi(a))^{\alpha+\beta-1}$;
- ${}^c\mathbb{D}_{a^+}^{\alpha;\psi}(\psi(t)-\psi(a))^{\beta-1}(\tau)=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\psi(\tau)-\psi(a))^{\beta-\alpha-1}$;
- ${}^c\mathbb{D}_{a^+}^{\alpha;\psi}(\psi(t)-\psi(a))^{k}(\tau)=0$, *for any* $k=0,\dots,m-1; m\in\mathbb{N}$.

Remark 2.6 It is obvious by Lemma 2.4 and 2.5 that under general boundary conditions, we have

$$\mathbb{I}_{a^+}^{\alpha;\psi}{}^c\mathbb{D}_{a^+}^{\alpha;\psi}\omega(\tau)=\omega(\tau)+\sum_{k=0}^{m-1}c_k\big[\psi(\tau)-\psi(a)\big]^k,\quad \tau\in J,$$

Below we provide some background from the fixed point theory

Definition 2.7 ([6]) A self-operator Ψ on a Banach space $\mathfrak{C}$ is called Lipschitz if there exists a constant $L_\Psi>0$ such that

$$\big\|\Psi(\omega)-\Psi(\varpi)\big\|\leq L_\Psi\|\omega-\varpi\|$$

for all elements $\omega,\varpi\in\mathfrak{C}$. If $L_\Psi<1$, then Ψ is called a contraction.

The following theorem plays a crucial role in the analysis carried out in this work.

Theorem 2.8 ([18]) *Let* $\mathbb{X}$ *be a convex bounded closed set contained in the Banach algebra* $\mathfrak{C}$, *and let operators* $\mathcal{P}:\mathfrak{C}\to\mathfrak{C}$ *and* $\mathcal{R}:\mathbb{X}\to\mathfrak{C}$ *be such that*:

(*S*1) $\mathcal{P}$ *is a Lipschitz map with Lipschitz constant* $L_\mathcal{P}$;
(*S*2) $\mathcal{R}$ *is completely continuous*;
(*S*3) $\omega=\mathcal{P}\omega\mathcal{R}\varpi, \forall\varpi\in\mathbb{X}\Rightarrow\omega\in\mathbb{X}$; *and*
(*S*4) $L_\mathcal{P}M_\mathcal{R}<1$, *where* $M_\mathcal{R}=\|\mathcal{R}(\mathbb{X})\|=\sup\{\|\mathcal{R}\omega\|:\omega\in\mathbb{X}\}$.
Then the operator equation $\omega=\mathcal{P}\omega\mathcal{R}\omega$ *possesses a solution in* $\mathbb{X}$.

Theorem 2.9 ([6]) *A contraction mapping* $\Psi:\mathfrak{C}\to\mathfrak{C}$ *possesses a unique fixed point*.

3 Existence results

In this section, we consider a general type of HLFDS (1.1) and the couple HLFDS (1.2)–(1.3) involving an arbitrary function ψ.

To investigate the existence of solutions for (1.1), we need the following lemma.

Lemma 3.1 *Assume that* $\frac{\mathbb{G}(b)}{\mathbb{G}(\eta)}\frac{\psi(b)-\psi(a)}{\psi(\eta)-\psi(a)}\neq\zeta$. *Let* $\omega\in\mathfrak{C}$ *be a solution for the hybrid Langevin equation*

$$\begin{cases}{}^c\mathbb{D}_{a^+}^{\mu;\psi}\big[{}^c\mathbb{D}_{a^+}^{\nu;\psi}\big[\frac{\omega(\tau)}{\mathbb{G}(\tau)}\big]-\lambda\omega(\tau)\big]=\mathbb{F}(\tau),\quad \tau\in J:=[a,b],\\ \omega(a)=0,\qquad {}^c\mathbb{D}_{a^+}^{\nu;\psi}\big[\frac{\omega(\tau)}{\mathbb{G}(\tau)}\big]_{\tau=a}=0,\qquad \omega(b)=\zeta\omega(\eta),\quad \eta\in(a,b).\end{cases}\tag{3.1}$$

Then it satisfies the following integral equation:

$$\omega(\tau) = \mathbb{G}(\tau)\left[\mathcal{H}(\tau) - \frac{[\mathbb{G}(b)\mathcal{H}(b) - \zeta\mathbb{G}(\eta)\mathcal{H}(\eta)](\psi(\tau)-\psi(a))}{\mathbb{G}(b)(\psi(b)-\psi(a)) - \zeta\mathbb{G}(\eta)(\psi(\eta)-\psi(a))}\right], \tag{3.2}$$

where

$$\mathcal{H}(\tau) = \mathbb{I}_{a^+}^{\mu+\nu;\psi}\mathbb{F}(\tau) + \lambda\mathbb{I}_{a^+}^{\nu;\psi}\omega(\tau).$$

In particular, if $\zeta = \frac{\mathbb{G}(b)}{\mathbb{G}(\eta)}$, *then*

$$\omega(\tau) = \mathbb{G}(\tau)\left[\mathcal{H}(\tau) - \frac{\psi(\tau)-\psi(a)}{\psi(b)-\psi(\eta)}\big[\mathcal{H}(b) - \mathcal{H}(\eta)\big]\right].$$

Proof Applying the μth ψ-Riemann–Liouville fractional integral to both sides of (3.1), by Lemma 2.6 we get

$${}^{c}\mathbb{D}_{a^+}^{\nu;\psi}\left[\frac{\omega(\tau)}{\mathbb{G}(\tau)}\right] = \mathbb{I}_{a^+}^{\mu;\psi}\mathbb{F}(\tau) + \lambda\omega(\tau) + c_0.$$

Benefiting from the first and second boundary conditions, we manifestly obtain that $c_0 = 0$.. Applying the νth ψ-Riemann–Liouville fractional integral once more and using Lemma 2.6 lead to the following integral form:

$$\frac{\omega(\tau)}{\mathbb{G}(\tau)} = \mathcal{H}(\tau) + c_1\big(\psi(\tau)-\psi(a)\big) + c_2. \tag{3.3}$$

From the first boundary condition we have $c_2 = 0$. By the last boundary condition we obtain

$$c_1 = \frac{\zeta\mathbb{G}(\eta)\mathcal{H}(\eta) - \mathbb{G}(b)\mathcal{H}(b)}{\mathbb{G}(b)(\psi(b)-\psi(a)) - \zeta\mathbb{G}(\eta)(\psi(\eta)-\psi(a))}.$$

Substituting these constants into (3.3), we obtain (3.2).

Conversely, it is straightforward to observe that the function in (3.2) satisfies Equation (3.1) and the associated boundary conditions. □

According to Lemma 3.1, we precisely define the notion of a mild solution of (1.1).

Definition 3.2 A function $\omega \in \mathfrak{C}$ is said to be a mild solution of (1.1) if ω fulfills the equation

$$\begin{aligned}\omega(\tau) = {}&\mathbb{G}\big(\tau,\omega(\tau)\big)\\ &\times\left[\mathcal{H}\omega(\tau) - \frac{(\psi(\tau)-\psi(a))[\mathbb{G}(b,\omega(b))\mathcal{H}\omega(b) - \zeta\mathbb{G}(\eta,\omega(\eta))\mathcal{H}\omega(\eta)]}{\mathbb{G}(b,\omega(b))(\psi(b)-\psi(a)) - \zeta\mathbb{G}(\eta,\omega(\eta))(\psi(\eta)-\psi(a))}\right],\end{aligned} \tag{3.4}$$

where

$$\mathcal{H}\omega(\tau) = \mathbb{I}_{a^+}^{\mu+\nu;\psi}\mathbb{F}\big(\tau,\omega(\tau)\big) + \lambda\mathbb{I}_{a^+}^{\nu;\psi}\omega(\tau), \quad \tau \in J. \tag{3.5}$$

We make the following assumptions:

($\mathbb{A}1$) The function $\mathbb{G}: J \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$ is continuous, and there exists a positive function ϕ with supremum $\|\phi\|$ such that

$$\left|\mathbb{G}(\tau,\omega) - \mathbb{G}(\tau,\varpi)\right| \le \phi(\tau)|\omega - \varpi|$$

for all $(\tau,\omega), (\tau,\varpi) \in J \times \mathbb{R}$. Moreover, there exists a constant $\vartheta > 0$ such that

$$\left|\mathbb{G}(b,\omega)\big(\psi(b) - \psi(a)\big) - \zeta\,\mathbb{G}(\eta,\omega)\big(\psi(\eta) - \psi(a)\big)\right| \ge \vartheta > 0 \tag{3.6}$$

for all $\omega \in \mathbb{R}$.

($\mathbb{A}2$) The function $\mathbb{F}: J \times \mathbb{R} \to \mathbb{R}$ is continuous, and there exist a function $p \in C(J,\mathbb{R}^+)$ and a nondecreasing function $\chi \in C([0,\infty),(0,\infty))$ such that

$$\left|\mathbb{F}(\tau,\omega)\right| \le p(\tau)\chi\big(|\omega|\big) \tag{3.7}$$

for all $\tau \in J$ and $\omega \in \mathbb{R}$.

($\mathbb{A}3$) There exists $r > 0$ such that

$$r \ge \frac{\mathbb{G}_0\mathcal{A}_r}{1 - \|\phi\|\mathcal{A}_r}$$

and

$$\|\phi\|\mathcal{A}_r < 1, \tag{3.8}$$

where $\mathbb{G}_0 = \sup_{\tau\in J}|\mathbb{G}(\tau,0)|$, and

$$\begin{aligned}
\mathcal{A}_r = {} & \frac{\|p\|(\psi(b) - \psi(a))^{\mu+\nu}}{\vartheta\Gamma(\mu+\nu+1)}\big[\vartheta + 2\mathbb{G}_0\big(\psi(b) - \psi(a)\big)\big]\chi(r) \\
& + \frac{2(\psi(b) - \psi(a))^{\mu+\nu+1}\|p\|\|\phi\|}{\vartheta\Gamma(\mu+\nu+1)}r\chi(r) \\
& + \frac{|\lambda|(\psi(b) - \psi(a))^{\nu+1}\|\phi\|}{\vartheta\Gamma(\nu+1)}r^2 \\
& + \frac{|\lambda|(\psi(b) - \psi(a))^{\nu}}{\vartheta\Gamma(\nu+1)}\big[\vartheta + 2\mathbb{G}_0\big(\psi(b) - \psi(a)\big)\big]r.
\end{aligned} \tag{3.9}$$

Next, we provide the existence of solutions giving credence to the Dhage fixed point theorem.

Theorem 3.3 *If conditions* ($\mathbb{A}1$)–($\mathbb{A}3$) *are satisfied, then* (1.1) *has at least one mild solution.*

Proof Define the set

$$\mathbb{S} = \big\{\omega \in \mathfrak{C} : \|\omega\| \le r\big\}.$$

Clearly, $\mathbb{S}$ is a closed convex bounded subset of the Banach space $\mathfrak{C}$. In virtue of Definition 3.2, we define two operators $\mathcal{P}:\mathfrak{C}\to\mathfrak{C}$ and $\mathcal{R}:\mathbb{S}\to\mathfrak{C}$ by

$$\mathcal{P}\omega(\tau)=\mathbb{G}\big(\tau,\omega(\tau)\big),\quad \tau\in J,$$

and

$$\mathcal{R}\omega(\tau)=\mathcal{H}\omega(\tau)-\frac{(\psi(\tau)-\psi(a))[\mathbb{G}(b,\omega(b))\mathcal{H}\omega(b)-\zeta\mathbb{G}(\eta,\omega(\eta))\mathcal{H}\omega(\eta)]}{\mathbb{G}(b,\omega(b))(\psi(b)-\psi(a))-\zeta\mathbb{G}(\eta,\omega(\eta))(\psi(\eta)-\psi(a))},\quad \tau\in J.$$

Then the integral equation (3.4) can be written in the operator form as

$$\omega(\tau)=\mathcal{P}\omega(\tau)\mathcal{R}\omega(\tau),\quad \tau\in J.$$

We show that the operators $\mathcal{P}$ and $\mathcal{R}$ satisfy all the conditions of Theorem 2.8.

Step 1: Firstly, we show that $\mathcal{P}$ is Lipschitzian on $\mathfrak{C}$. Let $\omega,\varpi\in\mathfrak{C}$. Then by ($\mathbb{A}2$) we have

$$\begin{aligned}\big|\mathcal{P}\omega(\tau)-\mathcal{P}\varpi(\tau)\big|&=\big|\mathbb{G}\big(\tau,\omega(\tau)\big)-\mathbb{G}\big(\tau,\varpi(\tau)\big)\big|\\&\le\phi(\tau)\big|\omega(\tau)-\varpi(\tau)\big|\end{aligned}$$

for all $\tau\in J$. Taking the supremum over τ, we obtain

$$\|\mathcal{P}\omega-\mathcal{P}\varpi\|\le\|\phi\|\|\omega-\varpi\|$$

for all $\omega,\varpi\in\mathfrak{C}$. Therefore $\mathcal{P}$ is Lipschitzian on $\mathfrak{C}$ with Lipschitz constant $\|\phi\|$.

Step 2: We prove that the operator $\mathcal{R}$ is completely continuous on $\mathbb{S}$. For this purpose, we firstly show that the operator $\mathcal{R}$ is continuous on $\mathfrak{C}$. Let ω_n be a sequence in $\mathbb{S}$ converging to a point $\omega\in\mathbb{S}$. Now by the Lebesgue dominated convergence theorem we obtain

$$\begin{aligned}&\lim_{n\to\infty}\mathcal{R}(\omega_n)(\tau)\\&\quad=\lim_{n\to\infty}\bigg\{\mathcal{H}\omega_n(\tau)\\&\qquad-\frac{(\psi(\tau)-\psi(a))[\mathbb{G}(b,\omega_n(b))\mathcal{H}\omega_n(b)-\zeta\mathbb{G}(\eta,\omega_n(\eta))\mathcal{H}\omega_n(\eta)]}{\mathbb{G}(b,\omega_n(b))(\psi(b)-\psi(a))-\zeta\mathbb{G}(\eta,\omega_n(\eta))(\psi(\eta)-\psi(a))}\bigg\}\\&\quad=\mathcal{H}\omega(\tau)\\&\qquad-\frac{(\psi(\tau)-\psi(a))[\mathbb{G}(b,\omega(b))\mathcal{H}\omega(b)-\zeta\mathbb{G}(\eta,\omega(\eta))\mathcal{H}\omega(\eta)]}{\mathbb{G}(b,\omega(b))(\psi(b)-\psi(a))-\zeta\mathbb{G}(\eta,\omega(\eta))(\psi(\eta)-\psi(a))}\\&\quad=\mathcal{R}(\omega)(\tau)\end{aligned}$$

for all $\tau\in J$. This shows that $\mathcal{R}$ is a continuous operator on $\mathbb{S}$.

Next, we prove that the set $\mathcal{R}(\mathbb{S})$ is a uniformly bounded in $\mathbb{S}$. For any $\omega\in\mathbb{S}$ and $\tau\in J$, we have

$$\big|\mathcal{H}\omega(\tau)\big|\le\mathbb{I}_{a^+}^{\mu+\nu;\psi}\big|\mathbb{F}\big(\tau,\omega(\tau)\big)\big|+|\lambda|\mathbb{I}_{a^+}^{\nu;\psi}\big|\omega(\tau)\big|$$

$$\leq \frac{(\psi(\tau)-\psi(a))^{\mu+\nu}\|p\|\chi(r)}{\Gamma(\mu+\nu+1)}+\frac{|\lambda|(\psi(\tau)-\psi(a))^{\nu}r}{\Gamma(\nu+1)}.$$

Therefore

$$\begin{aligned}
&\big|\mathcal{R}(\omega)(\tau)\big| \\
&\quad\leq \big|\mathcal{H}\omega(\tau)\big| \\
&\qquad+\frac{(\psi(\tau)-\psi(a))[|\mathbb{G}(b,\omega(b))||\mathcal{H}\omega(b)|+|\zeta||\mathbb{G}(\eta,\omega(\eta))||\mathcal{H}\omega(\eta)|]}{|\mathbb{G}(b,\omega(b))(\psi(b)-\psi(a))-\zeta\mathbb{G}(\eta,\omega(\eta))(\psi(\eta)-\psi(a))|} \\
&\quad\leq \frac{(\psi(\tau)-\psi(a))^{\mu+\nu}\|p\|\chi(r)}{\Gamma(\mu+\nu+1)}+\frac{|\lambda|(\psi(\tau)-\psi(a))^{\nu}r}{\Gamma(\nu+1)} \\
&\qquad+\frac{(\psi(\tau)-\psi(a))|\mathbb{G}(b,\omega(b))|}{\vartheta} \\
&\qquad\times\left[\frac{(\psi(b)-\psi(a))^{\mu+\nu}\|p\|\chi(r)}{\Gamma(\mu+\nu+1)}+\frac{|\lambda|(\psi(b)-\psi(a))^{\nu}r}{\Gamma(\nu+1)}\right] \\
&\qquad+\frac{|\zeta|(\psi(\tau)-\psi(a))|\mathbb{G}(\eta,\omega(\eta))|}{\vartheta} \\
&\qquad\times\left[\frac{(\psi(\eta)-\psi(a))^{\mu+\nu}\|p\|\chi(r)}{\Gamma(\mu+\nu+1)}+\frac{|\lambda|(\psi(\eta)-\psi(a))^{\nu}r}{\Gamma(\nu+1)}\right] \\
&\quad\leq \frac{\|p\|(\psi(b)-\psi(a))^{\mu+\nu}}{\vartheta\Gamma(\mu+\nu+1)}\big[\vartheta+2\mathbb{G}_0\big(\psi(b)-\psi(a)\big)\big]\chi(r) \\
&\qquad+\frac{2(\psi(b)-\psi(a))^{\mu+\nu+1}\|p\|\|\phi\|}{\vartheta\Gamma(\mu+\nu+1)}r\chi(r) \\
&\qquad+\frac{|\lambda|(\psi(b)-\psi(a))^{\nu+1}\|\phi\|}{\vartheta\Gamma(\nu+1)}r^2 \\
&\qquad+\frac{|\lambda|(\psi(b)-\psi(a))^{\nu}}{\vartheta\Gamma(\nu+1)}\big[\vartheta+2\mathbb{G}_0\big(\psi(b)-\psi(a)\big)\big]r.
\end{aligned} \tag{3.10}$$

Thus $\|\mathcal{R}\omega\|\leq\mathcal{A}_r$ for all $\omega\in\mathbb{S}$ with $\mathcal{A}_r$ given in (3.9). This shows that $\mathcal{R}$ is uniformly bounded on $\mathbb{S}$.

Let $\tau_1,\tau_2\in J$ be such that $\tau_1<\tau_2$. Then for any $\omega\in\mathbb{S}$, by (3.7) we get

$$\begin{aligned}
&\big|\mathcal{H}\omega(\tau_2)-\mathcal{H}\omega(\tau_1)\big| \\
&\quad\leq\int_a^{\tau_1}\frac{\psi'(s)}{\Gamma(\mu+\nu)}\big[\big(\psi(\tau_2)-\psi(s)\big)^{\mu+\nu-1}-\big(\psi(\tau_1)-\psi(s)\big)^{\mu+\nu-1}\big]\big|\mathbb{F}\big(s,\omega(s)\big)\big|\,ds \\
&\qquad+\int_{\tau_1}^{\tau_2}\frac{\psi'(s)}{\Gamma(\mu+\nu)}\big(\psi(\tau_2)-\psi(s)\big)^{\mu+\nu-1}\big|\mathbb{F}\big(s,\omega(s)\big)\big|\,ds \\
&\qquad+|\lambda|\int_a^{\tau_1}\frac{\psi'(s)}{\Gamma(\nu)}\big[\big(\psi(\tau_2)-\psi(s)\big)^{\nu-1}-\big(\psi(\tau_1)-\psi(s)\big)^{\mu+\nu-1}\big]\big|\omega(s)\big|\,ds \\
&\qquad+\int_{\tau_1}^{\tau_2}\frac{\psi'(s)}{\Gamma(\nu)}\big(\psi(\tau_2)-\psi(s)\big)^{\nu-1}\big|\omega(s)\big|\,ds \\
&\quad\leq\frac{[2(\psi(\tau_2)-\psi(\tau_1))^{\mu+\nu}+(\psi(\tau_2)-\psi(a))^{\mu+\nu}-(\psi(\tau_1)-\psi(a))^{\mu+\nu}]\|p\|\chi(r)}{\Gamma(\mu+\nu+1)} \\
&\qquad+\frac{|\lambda|[2(\psi(\tau_2)-\psi(\tau_1))^{\nu}+(\psi(\tau_2)-\psi(a))^{\nu}-(\psi(\tau_1)-\psi(a))^{\nu}]r}{\Gamma(\nu+1)}.
\end{aligned}$$

By similar arguments as in (3.10) we obtain

$$
\begin{aligned}
&\left|\frac{[\mathbb{G}(b,\omega(b))\mathcal{H}\omega(b)-\zeta\mathbb{G}(\eta,\omega(\eta))\mathcal{H}\omega(\eta)]}{\mathbb{G}(b,\omega(b))(\psi(b)-\psi(a))-\zeta\mathbb{G}(\eta,\omega(\eta))(\psi(\eta)-\psi(a))}\right|\\
&\quad\le \frac{|\mathbb{G}(b,\omega(b))|}{\vartheta}\left[\frac{(\psi(b)-\psi(a))^{\mu+\nu}\|p\|\chi(r)}{\Gamma(\mu+\nu+1)}+\frac{|\lambda|(\psi(b)-\psi(a))^{\nu}r}{\Gamma(\nu+1)}\right]\\
&\qquad+\frac{|\zeta||\mathbb{G}(\eta,\omega(\eta))|}{\vartheta}\left[\frac{(\psi(\eta)-\psi(a))^{\mu+\nu}\|p\|\chi(r)}{\Gamma(\mu+\nu+1)}+\frac{|\lambda|(\psi(\eta)-\psi(a))^{\nu}r}{\Gamma(\nu+1)}\right]\\
&\quad\le \frac{\mathbb{G}_0\|p\|(\psi(b)-\psi(a))^{\mu+\nu}\chi(r)}{\vartheta\Gamma(\mu+\nu+1)}+\frac{r\chi(r)(\psi(b)-\psi(a))^{\mu+\nu+1}\|p\|\|\phi\|}{\vartheta\Gamma(\mu+\nu+1)}\\
&\qquad+\frac{|\lambda|r^2(\psi(b)-\psi(a))^{\nu}\|\phi\|}{\vartheta\Gamma(\nu+1)}+\frac{\mathbb{G}_0|\lambda|r(\psi(b)-\psi(a))^{\nu}}{\vartheta\Gamma(\nu+1)}\\
&\quad=\mathfrak{B}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\left|\mathcal{R}(\omega)(\tau_2)-\mathcal{R}(\omega)(\tau_1)\right|\\
&\quad\le \left|\mathcal{H}\omega(\tau_2)-\mathcal{H}\omega(\tau_1)\right|+\left|\psi(\tau_2)-\psi(\tau_1)\right|\\
&\qquad\times\left|\frac{[\mathbb{G}(b,\omega(b))\mathcal{H}\omega(b)-\zeta\mathbb{G}(\eta,\omega(\eta))\mathcal{H}\omega(\eta)]}{\mathbb{G}(b,\omega(b))(\psi(b)-\psi(a))-\zeta\mathbb{G}(\eta,\omega(\eta))(\psi(\eta)-\psi(a))}\right|\\
&\quad\le \frac{[2(\psi(\tau_2)-\psi(\tau_1))^{\mu+\nu}+(\psi(\tau_2)-\psi(a))^{\mu+\nu}-(\psi(\tau_1)-\psi(a))^{\mu+\nu}]\|p\|\chi(r)}{\Gamma(\mu+\nu+1)}\\
&\qquad+\frac{|\lambda|[2(\psi(\tau_2)-\psi(\tau_1))^{\nu}+(\psi(\tau_2)-\psi(a))^{\nu}-(\psi(\tau_1)-\psi(a))^{\nu}]r}{\Gamma(\nu+1)}\\
&\qquad+\mathfrak{B}\left|\psi(\tau_2)-\psi(\tau_1)\right|.
\end{aligned}
$$

This implies

$$
\left|\mathcal{R}(\omega)(\tau_2)-\mathcal{R}(\omega)(\tau_1)\right|\to 0\quad\text{as } \tau_1\to\tau_2
$$

uniformly for all $\omega\in\mathbb{S}$. Thus $\mathcal{R}$ has the equicontinuity specification on the Banach space $\mathfrak{C}$. As a consequence, $\mathcal{R}$ is relatively compact, and thus the Arzelà–Ascoli theorem yields that $\mathcal{R}$ is completely continuous, and, finally, $\mathcal{R}$ is compact on $\mathbb{S}$.

Step 3: Hypothesis (S3) of Theorem 2.8 is satisfied.

Let $\omega\in\mathfrak{C}$ and $\varpi\in\mathbb{S}$ be arbitrary elements such that $\omega=\mathcal{P}\omega\mathcal{R}\varpi$. Then we have

$$
\begin{aligned}
\left|\omega(\tau)\right| &= \left|\mathcal{P}(\omega)(\tau)\mathcal{R}(\omega)(\tau)\right|\\
&\le \left[\left|\mathbb{G}(\tau,\omega)-\mathbb{G}(\tau,0)\right|+\left|\mathbb{G}(\tau,0)\right|\right]\mathcal{A}_r\\
&\le \left[\|\phi\|\|\omega\|+\mathbb{G}_0\right]\mathcal{A}_r.
\end{aligned}
$$

Taking the supremum in the above inequality, we obtain

$$
\|\omega\|\le\frac{\mathbb{G}_0\mathcal{A}_r}{1-\|\phi\|\mathcal{A}_r}\le r.
$$

Thus $\omega\in\mathbb{S}$, and so statement (S3) of Theorem 2.8 follows.

Step 4: At last, we have

$$M_{\mathcal{R}} = \|\mathcal{R}(\mathbb{S})\| = \sup\{\|\mathcal{R}(\omega)\| : \omega \in \mathbb{S}\} \le \mathcal{A}_r.$$

From above estimate we obtain

$$L_{\mathcal{P}} M_{\mathcal{R}} \le \|\phi\| \mathcal{A}_r < 1,$$

and so hypothesis (S4) of Theorem 2.8 is satisfied. Accordingly, the operators $\mathcal{N}$ and $\mathcal{R}$ approve all four statements of Theorem 2.8, and thus the equation $\mathcal{P}(\omega)\mathcal{R}(\omega) = \omega$ possesses a mild solution in $\mathbb{S}$. Consequently, the HLFDS (1.1) involves a mild solution on J. This establishes the required result. □

Remark 3.4 Let $\zeta = \frac{\mathbb{G}(b,\omega(b))}{\mathbb{G}(\eta,\omega(\eta))}$. Then the integral solution (3.4) reduces to the following form:

$$\omega(\tau) = \mathbb{G}\big(\tau, \omega(\tau)\big)\left[\mathcal{H}\omega(\tau) - \frac{\psi(\tau) - \psi(a)}{\psi(b) - \psi(\eta)}\big[\mathcal{H}\omega(b) - \mathcal{H}\omega(\eta)\big]\right],$$

where $\mathcal{H}$ is defined by (3.5). It is easy to rewrite the value of $\mathcal{A}_r$ defined by (3.9) as

$$\begin{aligned} \mathcal{A}_r = {}& \frac{|\lambda|(\psi(b) - \psi(a))^{\nu}}{\Gamma(\nu + 1)}\left(1 + \frac{2(\psi(b) - \psi(a))}{\psi(b) - \psi(\eta)}\right) r \\ & \times \frac{(\psi(b) - \psi(a))^{\mu+\nu}\|p\|}{\Gamma(\mu + \nu + 1)}\left(1 + \frac{2(\psi(b) - \psi(a))}{\psi(b) - \psi(\eta)}\right)\chi(r). \end{aligned}$$

In this case, there is no need to assume condition (3.6).

The proof of the next result follows by the proof of Theorem 3.3 taking into account the modified ideas in assumptions ($\mathbb{A}1$) and ($\mathbb{A}3$) as explained in Remark 3.4.

Corollary 3.5 *Assume that conditions* ($\mathbb{A}1$)–($\mathbb{A}3$) *hold and that* $\zeta = \frac{\mathbb{G}(b,\omega(b))}{\mathbb{G}(\eta,\omega(\eta))}$. *Then the HLFDS* (1.1) *has at least one mild solution defined on* J.

Let us now define the notion of a mild solution of the coupled HLFDS (1.2)–(1.3).

Definition 3.6 An element $(\omega_1, \omega_2) \in \mathbb{E}$ is said to be a mild solution of the coupled HLFDS (1.2)–(1.3) if it satisfies

$$\begin{aligned} \omega_i(\tau) = {}& \mathbb{G}_i\big(\tau, \omega_1(\tau), \omega_2(\tau)\big)\Big[\mathcal{H}_i(\omega_1, \omega_2)(\tau) - \big(\psi(\tau) - \psi(a)\big) \\ & \times \frac{[\mathbb{G}_i(b, \omega_1(b), \omega_2(b))\mathcal{H}_i(\omega_1, \omega_2)(b) - \zeta_i \mathbb{G}_i(\eta_i, \omega_1(\eta_i), \omega_2(\eta_i))\mathcal{H}_i(\omega_1, \omega_2)(\eta_i)]}{\mathbb{G}_i(b, \omega_1(b), \omega_2(b))(\psi(b) - \psi(a)) - \zeta_i \mathbb{G}_i(\eta_i, \omega_1(\eta_i), \omega_2(\eta_i))(\psi(\eta_i) - \psi(a))}\Big], \end{aligned} \tag{3.11}$$

$i = 1, 2$, where

$$\mathcal{H}_i(\omega_1, \omega_2)(\tau) = \mathbb{I}_{a^+}^{\mu_i+\nu_i;\psi} \mathbb{F}_i\big(\tau, \omega_1(\tau), \omega_2(\tau)\big) + \lambda_i \mathbb{I}_{a^+}^{\nu_i;\psi} \omega_i(\tau). \tag{3.12}$$

To start verifying the next result, the following assumptions are further required.

($\mathbb{B}1$) The function $\mathbb{G}_i : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$ is continuous, and there exists a positive function ϕ_i with supremum $\|\phi_i\|$such that

$$\left|\mathbb{G}_i(\tau,\omega_1,\omega_2) - \mathbb{G}_i(\tau,\varpi_1,\varpi_2)\right| \leq \|\phi_i\|\big(|\omega_1 - \varpi_1| + |\omega_2 - \varpi_2|\big)$$

for all $\tau \in J, i = 1,2$, and $\omega_1,\omega_2,\varpi_1,\varpi_2 \in \mathbb{R}$. Moreover, there exists a positive constant ϑ_i such that

$$\begin{aligned}&\left|\mathbb{G}_i\big(b,\omega_1(b),\omega_2(b)\big)\big(\psi(b)-\psi(a)\big) - \zeta_i\mathbb{G}_i\big(\eta_i,\omega_1(\eta_i),\omega_2(\eta_i)\big)\big(\psi(\eta_i)-\psi(a)\big)\right| \\ &\quad\geq \vartheta_i \end{aligned}\tag{3.13}$$

for $i = 1,2$ and $\omega_1,\omega_2 \in \mathbb{R}$.

($\mathbb{B}2$) The function $\mathbb{F}_i : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, and there exist a function $p_i \in C(J,\mathbb{R}^+)$ and a nondecreasing function $\chi_i C([0,\infty),(0,\infty))$ such that

$$\left|\mathbb{F}_i(\tau,\omega_1,\omega_2)\right| \leq p_i(\tau)\chi_i\big(|\omega_1| + |\omega_2|\big)$$

for all $\tau \in J$ and $\omega_1,\omega_2 \in \mathbb{R}$.

($\mathbb{B}3$) There exist $\rho,\rho_i > 0$ such that

$$\rho \geq \frac{\mathbb{G}_{0,1}\mathcal{A}_{\rho_1} + \mathbb{G}_{0,i}\mathcal{A}_{\rho_2}}{1 - \|\phi_1\|\mathcal{A}_{\rho_1} - \|\phi_2\|\mathcal{A}_{\rho_2}},$$

where $\mathbb{G}_{0,i} = \sup_{\tau\in J}|\mathbb{G}_i(\tau,0,0)|(i = 1,2)$, and

$$\begin{aligned}\mathcal{A}_{\rho_i} &= \frac{\|p_i\|(\psi(b)-\psi(a))^{\mu_i+\nu_i}}{\vartheta_i\Gamma(\mu_i+\nu_i+1)}\big[\vartheta_i + 2\mathbb{G}_{0,i}\big(\psi(b)-\psi(a)\big)\big]\chi_i(\rho_1+\rho_2) \\ &\quad+ \frac{2(\psi(b)-\psi(a))^{\mu_i+\nu_i+1}\|p_i\|\|\phi_i\|}{\vartheta_i\Gamma(\mu_i+\nu_i+1)}\rho_i\chi_i(\rho_1+\rho_2) \\ &\quad+ \frac{|\lambda_i|(\psi(b)-\psi(a))^{\nu_i+1}\|\phi_i\|}{\vartheta_i\Gamma(\nu_i+1)}\rho_i^2 \\ &\quad+ \frac{|\lambda_i|(\psi(b)-\psi(a))^{\nu_i}}{\vartheta_i\Gamma(\nu_i+1)}\big[\vartheta_i + 2\mathbb{G}_{0,i}\big(\psi(b)-\psi(a)\big)\big]\rho_i, \\ \mathcal{A}_\rho &= \mathcal{A}_{\rho_1} + \mathcal{A}_{\rho_2}, \qquad \|\phi\| = \|\phi_1\| + \|\phi_2\|.\end{aligned}\tag{3.14}$$

The next result is an analogue of Theorem 3.3 in the coupled form, and hence we will not go into details in the proof.

Theorem 3.7 *Suppose that hypotheses* ($\mathbb{B}1$)–($\mathbb{B}4$) *hold. If*

$$\|\phi\|\mathcal{A}_\rho < 1,\tag{3.15}$$

then the coupled HLFDS (1.2)–(1.3) *possesses a mild solution on* J.

Proof Consider a subset $\mathbb{X}$ of the Banach space $\mathbb{E}$ given by

$$\mathbb{X} = \big\{\omega = (\omega_1,\omega_2) \in \mathbb{E} : \|\omega_i\| \leq \rho_i;\ \rho \geq \rho_1 + \rho_2\big\}.$$

Evidently, $\mathbb{X}$ is a convex, bounded, and closed set contained in the Banach space $\mathfrak{C}\times\mathfrak{C}=\mathbb{E}$. Define the operators $\mathcal{P}=(\mathcal{P}_1,\mathcal{P}_2):\mathbb{E}\to\mathbb{E}$ and $\mathcal{R}=(\mathcal{R}_1,\mathcal{R}_2):\mathbb{X}\to\mathbb{E}$ by

$$\mathcal{P}_i(\omega_1,\omega_2)(\tau)=\mathbb{G}_i\big(\tau,\omega_1(\tau),\omega_2(\tau)\big),$$

where

$$\begin{aligned}&\mathcal{R}_i(\omega_1,\omega_2)(\tau)\\&\quad=\mathcal{H}_i(\omega_1,\omega_2)(\tau)-\big(\psi(\tau)-\psi(a)\big)\\&\qquad\times\frac{[\mathbb{G}_i(b,\omega_1(b),\omega_2(b))\mathcal{H}_i(\omega_1,\omega_2)(b)-\zeta_i\mathbb{G}_i(\eta_i,\omega_1(\eta_i),\omega_2(\eta_i))\mathcal{H}_i(\omega_1,\omega_2)(\eta_i)]}{\mathbb{G}_i(b,\omega_1(b),\omega_2(b))(\psi(b)-\psi(a))-\zeta_i\mathbb{G}_i(\eta_i,\omega_1(\eta_i),\omega_2(\eta_i))(\psi(\eta_i)-\psi(a))}\end{aligned}$$

for $\tau\in J$, $i=1,2$. In this case the coupled system of the given hybrid integral equation (3.11) can be represented in the framework of a system of operator equations as

$$\mathcal{P}(\omega_1,\omega_2)(\tau)\mathcal{R})\omega_1,\omega_2)(\tau)=(\omega_1,\omega_2)(\tau),\quad \tau\in J,$$

which further, taking into account the multiplication given in (2.1), reduces to

$$\big(\mathcal{P}_1(\omega_1,\omega_2)(\tau)\mathcal{R}_1(\omega_1,\omega_2)(\tau),\mathcal{P}_2(\omega_1,\omega_2)(\tau)\mathcal{R}_2(\omega_1,\omega_2)(\tau)\big)=(\omega_1,\omega_2)(\tau)$$

for $\tau\in J$. This further implies that

$$\mathcal{P}_i(\omega_1,\omega_2)(\tau)\mathcal{R}_i(\omega_1,\omega_2)(\tau)=\omega_i(\tau),\quad \tau\in J, i=1,2.$$

In the following steps, we demonstrate that the operators $\mathcal{P}$ and $\mathcal{R}$ follow the statements of Theorem 2.8.

Step 1: We first show that $\mathcal{P}=(\mathcal{P}_1,\mathcal{P}_2)$ is Lipschitzian on $\mathbb{E}$ with Lipschitz constant $\|\phi\|=\|\phi_1\|+\|\phi_2\|$. Let $\omega=(\omega_1,\omega_2),\varpi=(\varpi_1,\varpi_2)\in\mathbb{E}$ be arbitrary. Then using ($\mathbb{B}2$), we have

$$\begin{aligned}&\big|\mathcal{P}_i(\omega_1,\omega_2)(\tau)-\mathcal{P}_i(\varpi_1,\varpi_2)(\tau)\big|\\&\quad=\big|\mathbb{G}_i\big(\tau,\omega_1(\tau),\omega_2(\tau)\big)-\mathbb{G}_i\big(\tau,\varpi_1(\tau),\varpi_2(\tau)\big)\big|\\&\quad\le\|\phi_i\|\big(\big|\omega_1(\tau)-\varpi_1(\tau)\big|+\big|\omega_2(\tau)-\varpi_2(\tau)\big|\big)\end{aligned}$$

for all $\tau\in J$, $i=1,2$. Taking the supremum norm over τ, we get that

$$\big\|\mathcal{P}_i(\omega_1,\omega_2)-\mathcal{P}_i(\varpi_1,\varpi_2)\big\|\le\|\phi_i\|\big(\|\omega_1-\varpi_1\|+\|\omega_2-\varpi_2\|\big)$$

for all $i=1,2$, $\omega,\varpi\in\mathbb{E}$. Accordingly, by the definition of operator $\mathcal{P}$ we get

$$\begin{aligned}&\|\mathcal{P}\omega-\mathcal{P}\varpi\|\\&\quad=\big\|\big(\mathcal{P}_1(\omega_1,\omega_2),\mathcal{P}_2(\omega_1,\omega_2)\big)-\big(\mathcal{P}_1(\varpi_1,\varpi_2),\mathcal{P}_2(\varpi_1,\varpi_2)\big)\big\|\\&\quad=\big\|\big(\mathcal{P}_1(\omega_1,\omega_2)-\mathcal{P}_1(\varpi_1,\varpi_2),\mathcal{P}_2(\omega_1,\omega_2)-\mathcal{P}_2(\varpi_1,\varpi_2)\big)\big\|\\&\quad=\big\|\mathcal{P}_1(\omega_1,\omega_2)-\mathcal{P}_1(\varpi_1,\varpi_2)\big\|+\big\|\mathcal{P}_2(\omega_1,\omega_2)-\mathcal{P}_2(\varpi_1,\varpi_2)\big\|\end{aligned}$$

$$
\begin{aligned}
&\leq \|\phi_1\|\big(\|\omega-\varpi_1\|+\|\omega_2-\varpi_2\|\big)+\|\phi_2\|\big(\|\omega-\varpi_1\|+\|\omega_2-\varpi_2\|\big)\\
&=\big(\|\phi_1\|+\|\phi_2\|\big)\big(\|\omega-\varpi_1\|+\|\omega_2-\varpi_2\|\big)\\
&=\|\phi\|\|\omega-\varpi\|
\end{aligned}
$$

for all $\omega,\varpi\in\mathbb{E}$. As a consequence, $\mathcal{P}=(\mathcal{P}_1,\mathcal{P}_2)$ is a Lipschitz map subject to constant

$$
\|\phi\|=\|\phi_1\|+\|\phi_2\|>0.
$$

Step 2: Now we show that $\mathcal{R}=(\mathcal{R}_1,\mathcal{R}_2)$ is continuous and compact operator from $\mathbb{X}$ into $\mathbb{E}$. To deduce the continuity of $\mathcal{R}$, let $\{\omega_{1,n},\omega_{2,n}\}_{n\in\mathbb{N}}$ be a sequence of points of $\mathbb{X}$ tending to $(\omega_1,\omega_2)\in\mathbb{X}$. Then the Lebesgue dominated convergence theorem yields

$$
\begin{aligned}
&\lim_{n\to\infty}\mathcal{R}_i(\omega_{1,n},\omega_{2,n})(\tau)\\
&\quad=\lim_{n\to\infty}\{\mathcal{H}_i(\omega_{1,n},\omega_{2,n})(\tau)-\big(\psi(\tau)-\psi(a)\big)\\
&\qquad\times\frac{[\mathbb{G}_i(b,\omega_{1,n}(b),\omega_{2,n}(b))\mathcal{H}_i(\omega_1,\omega_2)(b)-\zeta\mathbb{G}_i(\eta,\omega_{1,n}(\eta),\omega_{2,n}(\eta))\mathcal{H}_i(\omega_1,\omega_2)(\eta)]}{\mathbb{G}_i(b,\omega_{1,n}(b),\omega_{2,n}(b))(\psi(b)-\psi(a))-\zeta\mathbb{G}_i(\eta,\omega_{1,n}(\eta),\omega_{2,n}(\eta))(\psi(\eta)-\psi(a))}\\
&\quad=\mathcal{R}_i(\omega_1,\omega_2)(\tau),\quad i=1,2,\tau\in J.
\end{aligned}
$$

Hence $\mathcal{R}(\omega_{1,n},\omega_{2,n})=(\mathcal{R}_1(\omega_{1,n},\omega_{2,n}),\mathcal{R}_2(\omega_{1,n},\omega_{2,n}))$ converges to $\mathcal{R}(\omega_1,\omega_2)$ pointwise on J. Next, we prove the compactness of $\mathcal{R}$ on $\mathbb{X}$. Firstly, to ensure the uniform boundedness, applying $(\mathbb{B}2)$, for $(\omega_1,\omega_2)\in\mathbb{X}$, we get

$$
\begin{aligned}
\big|\mathcal{H}_i(\omega_1,\omega_2)(x)\big|&\leq\mathbb{I}_{a^+}^{\mu_i+\nu_i;\psi}\big|\mathbb{F}_i\big(x,\omega_1(x),\omega_2(x)\big)\big|+|\lambda_i|\mathbb{I}_{a^+}^{\nu_i;\psi}\big|\omega_i(x)\big|\\
&\leq\frac{(\psi(x)-\psi(a))^{\mu_i+\nu_i}\|p_i\|\chi_i(\rho_1+\rho_2)}{\Gamma(\mu_i+\nu_i+1)}+\frac{|\lambda_i|(\psi(x)-\psi(a))^{\nu_i}\rho_i}{\Gamma(\nu_i+1)}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\big\|\mathcal{R}_i(\omega_1,\omega_2)\big\|\\
&\quad\leq\frac{\|p_i\|(\psi(b)-\psi(a))^{\mu_i+\nu_i}}{\vartheta_i\Gamma(\mu_i+\nu_i+1)}\big[\vartheta_i+2\mathbb{G}_{0,i}\big(\psi(b)-\psi(a)\big)\big]\chi_i(\rho_1+\rho_2)\\
&\qquad+\frac{2(\psi(b)-\psi(a))^{\mu_i+\nu_i+1}\|p_i\|\|\phi_i\|}{\vartheta_i\Gamma(\mu_i+\nu_i+1)}\rho_i\chi_i(\rho_1+\rho_2)\\
&\qquad+\frac{|\lambda_i|(\psi(b)-\psi(a))^{\nu_i+1}\|\phi_i\|}{\vartheta_i\Gamma(\nu_i+1)}\rho_i^2\\
&\qquad+\frac{|\lambda_i|(\psi(b)-\psi(a))^{\nu_i}}{\vartheta_i\Gamma(\nu_i+1)}\big[\vartheta_i+2\mathbb{G}_{0,i}\big(\psi(b)-\psi(a)\big)\big]\rho_i
\end{aligned}
$$

for all $(\omega_1,\omega_2)\in\mathbb{X}$. Hence $\mathcal{R}_i$ is a uniformly bounded operator by the upper bound $\mathcal{A}_{\rho_i}$ on $\mathbb{X}$. Accordingly, $\mathcal{R}$ is a uniformly bounded operator on $\mathbb{X}$, because

$$
\begin{aligned}
\big\|\mathcal{R}(\omega_1,\omega_2)(\tau)\big\|&=\big\|\mathcal{R}_1(\omega_1,\omega_2)(\tau)\big\|+\big\|\mathcal{R}_2(\omega_1,\omega_2)(\tau)\big\|\\
&\leq\mathcal{A}_{\rho_1}+\mathcal{A}_{\rho_2}\leq\mathcal{A}_\rho<\infty.
\end{aligned}
$$

Next, to confirm the equicontinuity of $\mathcal{R}$, let $(\omega_1,\omega_2) \in \mathbb{X}$ be an arbitrary point, and let $\tau_1,\tau_2 \in J$ be such that $\tau_1 < \tau_2$. Then we have

$$\begin{aligned} &\bigl|\mathcal{R}_i(\omega_1,\omega_2)(\tau_1) - \mathcal{R}_i(\omega_1,\omega_2)(\tau_2)\bigr| \\ &\quad\leq \frac{[2(\psi(\tau_2)-\psi(\tau_1))^{\mu_i+\nu_i} + (\psi(\tau_2)-\psi(a))^{\mu_i+\nu_i} - (\psi(\tau_1)-\psi(a))^{\mu_i+\nu_i}]\|p_i\|\chi_i(\rho_1+\rho_2)}{\Gamma(\mu_i+\nu_i+1)} \\ &\qquad + \frac{|\lambda_i|[2(\psi(\tau_2)-\psi(\tau_1))^{\nu_i} + (\psi(\tau_2)-\psi(a))^{\nu_i} - (\psi(\tau_1)-\psi(a))^{\nu_i}]\rho_i}{\Gamma(\nu_i+1)} \\ &\qquad + \mathfrak{B}_i\bigl|\psi(\tau_2)-\psi(\tau_1)\bigr| \\ &\quad\to 0 \quad \text{as } \tau_1 \to \tau_2, \end{aligned}$$

where

$$\begin{aligned} \mathfrak{B}_i = {}& \frac{\mathbb{G}_{0,i}\|p_i\|(\psi(b)-\psi(a))^{\mu_i+\nu_i}\chi_i(\rho_1+\rho_2)}{\vartheta_i\Gamma(\mu_i+\nu_i+1)} \\ &+ \frac{\rho_i\chi_i(\rho_1+\rho_2)(\psi(b)-\psi(a))^{\mu_i+\nu_i+1}\|p_i\|\|\phi_i\|}{\vartheta_i\Gamma(\mu_i+\nu_i+1)} \\ &+ \frac{|\lambda_i|\rho_i^2(\psi(b)-\psi(a))^{\nu_i}\|\phi_i\|}{\vartheta_i\Gamma(\nu_i+1)} + \frac{\mathbb{G}_{0,i}|\lambda_i|\rho_i(\psi(b)-\psi(a))^{\nu_i}}{\vartheta_i\Gamma(\nu_i+1)}. \end{aligned}$$

Hence it follows that

$$\bigl|\mathcal{R}(\omega_1,\omega_2)(\tau_1) - \mathcal{R}(\omega_1,\omega_2)(\tau_2)\bigr| \to 0 \quad \text{as } \tau_1 \to \tau_2$$

uniformly for all $(\omega_1,\omega_2) \in \mathbb{X}$. Thus $\mathcal{R}$ is equicontinuous on the Banach space $\mathbb{E}$. As a consequence, $\mathcal{R}$ is relatively compact, and thus the Arzelà–Ascoli theorem yields that $\mathcal{R}$ is completely continuous, and, finally, $\mathcal{R}$ is compact on $\mathbb{X}$.

Step 3: We now proceed to demonstrate the third condition (S3) of Theorem 2.8. Let (ϖ_1,ϖ_2) be an element of $\mathbb{X}$ such that

$$(\omega_1,\omega_2) = \bigl(\mathcal{P}_1(\omega_1,\omega_2)\mathcal{R}_1(\varpi_1,\varpi_2), \mathcal{P}_2(\omega_1,\omega_2)\mathcal{R}_2(\varpi_1,\varpi_2)\bigr).$$

Then, for $i = 1,2$, we obtain

$$\begin{aligned} |(\omega_i(\tau)| &= \bigl|\mathcal{P}_i(\omega_1,\omega_2)(\tau)\mathcal{R}_i(\varpi_1,\varpi_2)(\tau)\bigr| \\ &\leq \bigl[\|\phi_i\|\bigl(\|\omega_1\| + \|\omega_2\|\bigr) + \mathbb{G}_{0,i}\bigr]\mathcal{A}_{\rho_i}. \end{aligned}$$

Condition (3.15) implies that $\|\phi_1\|\mathcal{A}_{\rho_1} + \|\phi_2\|\mathcal{A}_{\rho_2} < 1$. Therefore

$$\|\omega_1\| + \|\omega_2\| \leq \frac{\mathbb{G}_{0,1}\mathcal{A}_{\rho_1} + \mathbb{G}_{0,i}\mathcal{A}_{\rho_2}}{1 - \|\phi_1\|\mathcal{A}_{\rho_1} - \|\phi_2\|\mathcal{A}_{\rho_2}}.$$

As $\|(\omega_1,\omega_2)\| = \|\omega_1\| + \|\omega_2\|$, we have that $\|(\omega_1,\omega_2)\| \leq \rho$. Thus $(\omega_1,\omega_2) \in \mathbb{X}$, and so statement (S3) of Theorem 2.8 follows.

Step 4: At last, we have

$$M_{\mathcal{R}} = \bigl\|\mathcal{R}(\mathbb{X})\bigr\| = \sup\bigl\{\bigl\|\mathcal{R}(\omega_1,\omega_2)\bigr\| : (\omega_1,\omega_2) \in \mathbb{X}\bigr\}$$

$$
\begin{aligned}
&= \sup\{\|\mathcal{R}_1(\omega_1,\omega_2)\| + \|\mathcal{R}_2(\omega_1,\omega_2)\| : (\omega_1,\omega_2) \in \mathbb{X}\} \\
&\leq \mathcal{A}_{\rho_1} + \mathcal{A}_{\rho_2} = \mathcal{A}_\rho.
\end{aligned}
$$

From this estimate by (3.15) we obtain

$$
L_{\mathcal{P}} M_{\mathcal{R}} \leq \|\phi\| \mathcal{A}_\rho < 1,
$$

and so hypothesis (S4) of Theorem 2.8 is satisfied. Accordingly, the operators $\mathcal{P}$ and $\mathcal{R}$ approve all four statements of Theorem 2.8, and thus the equation $\mathcal{P}(\omega_1,\omega_2)\mathcal{R}(\omega_1,\omega_2) = (\omega_1,\omega_2)$ possesses a mild solution in $\mathbb{X}$. Consequently, the coupled HLFDS (1.2)–(1.3) involves a mild solution on J. This finishes the proof. □

Remark 3.8 Let $\zeta_i = \frac{\mathbb{G}_i(b,\omega_1(b),\omega_2(b))}{\mathbb{G}_i(\eta,\omega_1(\eta),\omega_2(\eta))}$, $i = 1,2$. Then the integral solution (3.11) reduces to the following form:

$$
\begin{aligned}
\omega_i(\tau) &= \mathbb{G}_i\big(\tau,\omega_1(\tau),\omega_2(\tau)\big) \\
&\quad \times \left[\mathcal{H}_i\big(\omega_1(\tau),\omega_2(\tau)\big) - \frac{\psi(\tau)-\psi(a)}{\psi(b)-\psi(\eta)}\big[\mathcal{H}_i\big(\omega_1(b),\omega_2(b)\big) - \mathcal{H}_i\big(\omega_1(\eta),\omega_2(\eta)\big)\big]\right],
\end{aligned}
\tag{3.16}
$$

where $\mathcal{H}_i$ is defined by (3.12). As in Remark 3.4, we modify the value of $\mathcal{A}_{\rho_i}$ by (3.14) as

$$
\begin{aligned}
\mathcal{A}_{\rho_i} &= \frac{|\lambda_i|(\psi(b)-\psi(a))^{\nu_i}}{\Gamma(\nu_i+1)}\left(1 + \frac{2(\psi(b)-\psi(a))}{\psi(b)-\psi(\eta)}\right)\rho_i \\
&\quad \times \frac{(\psi(b)-\psi(a))^{\mu_i+\nu_i}\|p_i\|}{\Gamma(\mu_i+\nu_i+1)}\left(1 + \frac{2(\psi(b)-\psi(a))}{\psi(b)-\psi(\eta)}\right)\chi_i(\rho_1+\rho_2).
\end{aligned}
$$

In this case, there is no need to assume condition (3.13).

The proof of the next result follows by the proof of Theorem 3.7 taking into account the modified ideas in assumptions ($\mathbb{B}1$) and ($\mathbb{B}3$), as explained in Remark 3.8.

Corollary 3.9 *Suppose that hypotheses* ($\mathbb{B}1$)–($\mathbb{B}3$) *hold. Furthermore, if*

$$
\|\phi\| \mathcal{A}_\rho < 1,
$$

then the coupled HLFDS (1.2)–(1.3) *possesses a mild solution on* J.

4 Uniqueness of the solution

It is known that the uniqueness of the solution of a nonlinear differential equation can be obtained Theorem 2.9 (the Banach fixed point theorem). Unfortunately, it is hard to get a contraction principle for the integral system (3.11) even though assuming the boundedness and Lipschitz conditions for nonlinear functions. For simplicity, we consider hereafter the HLFDS (1.1) and the coupled HLFDS (1.2)–(1.3) in the cases of $\zeta = \frac{\mathbb{G}(b,\omega(b))}{\mathbb{G}(\eta,\omega(\eta))}$ and $\zeta_i = \frac{\mathbb{G}_i(b,\omega_1(b),\omega_2(b))}{\mathbb{G}_i(\eta,\omega_1(\eta),\omega_2(\eta))}$, $i = 1,2$, respectively, as in Corollaries 3.5 and 3.9 and Remarks 3.4 and 3.8.

We further make the following assumption for the next result:

($\mathbb{C}1$) The function $\mathbb{G}: J \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$ is continuous, and there exists a function $\phi \in C(J, \mathbb{R}^+)$ with supremum $\|\phi\|$ such that

$$\left|\mathbb{G}(\tau,\omega) - \mathbb{G}(\tau,\varpi)\right| \leq \phi(\tau)|\omega - \varpi|$$

for all $(\tau,\omega), (\tau,\varpi) \in J \times \mathbb{R}$. Moreover, there is a constant $k_{\mathbb{G}} > 0$ such that

$$\left|\mathbb{G}(\tau,\omega)\right| \leq k_{\mathbb{G}}$$

for all $(\tau,\omega) \in J \times \mathbb{R}$.

($\mathbb{C}2$) The function $\mathbb{F}: J \times \mathbb{R} \to \mathbb{R}$ is continuous, and there exists a function $p \in C(J, \mathbb{R}^+)$ with supremum $\|p\|$ such that

$$\left|\mathbb{F}(\tau,\omega) - \mathbb{F}(\tau,\varpi)\right| \leq p(\tau)|\omega - \varpi|$$

for all $(\tau,\omega), (\tau,\varpi) \in J \times \mathbb{R}$. Moreover, there is a constant $k_{\mathbb{F}} > 0$ such that

$$\left|\mathbb{F}(\tau,\omega)\right| \leq k_{\mathbb{F}}$$

for all $(\tau,\omega) \in J \times \mathbb{R}$.

($\mathbb{C}3$) The constant $\mathcal{B} < 1$, where

$$\begin{aligned}\mathcal{B} = {}& \frac{|\lambda|\|\phi\|(\psi(b) - \psi(a))}{\Gamma(\nu+1)} \\ & \times \left[\left(\psi(b) - \psi(a)\right)^{\nu-1} + \frac{(\psi(b) - \psi(a))^{\nu} + (\psi(\eta) - \psi(a))^{\nu}}{\psi(b) - \psi(\eta)}\right] \\ & + \frac{(k_{\mathbb{G}}(\|p\| + |\lambda|) + k_{\mathbb{F}}\|\phi\|)(\psi(b) - \psi(a))}{\Gamma(\mu+\nu+1)} \\ & \times \left[\left(\psi(b) - \psi(a)\right)^{\mu+\nu-1} + \frac{(\psi(b) - \psi(a))^{\mu+\nu} + (\psi(\eta) - \psi(a))^{\mu+\nu}}{\psi(b) - \psi(\eta)}\right].\end{aligned}$$

We start with the HLFDS (1.1) and establish the first uniqueness result.

Theorem 4.1 *Assume that* ($\mathbb{C}1$)–($\mathbb{C}3$) *hold. Then there exists a unique mild solution of the HLFDS* (1.1) *on* J.

Proof Let $\mathcal{Q}: \mathfrak{C} \to \mathfrak{C}$ be the operator defined as

$$\mathcal{Q}\omega(\tau) = \mathbb{G}\big(\tau,\omega(\tau)\big)\left[\mathcal{H}\omega(\tau) - \frac{\psi(\tau) - \psi(a)}{\psi(b) - \psi(\eta)}\big[\mathcal{H}\omega(b) - \mathcal{H}\omega(\eta)\big]\right], \quad \tau \in J.$$

Then $\mathcal{Q}$ is well defined and continuous due to the continuity of $\mathbb{G}$ and $\mathcal{H}$. For ω, $\varpi \in \mathfrak{C}$, by ($\mathbb{C}2$) we obtain

$$\begin{aligned}&\left|\mathcal{H}\omega(\tau) - \mathcal{H}\varpi(\tau)\right| \\ &\leq \mathbb{I}_{a^+}^{\mu+\nu;\psi}\left|\mathbb{F}\big(\tau,\omega(\tau)\big) - \mathbb{F}\big(\tau,\varpi(\tau)\big)\right| + \lambda\mathbb{I}_{a^+}^{\nu;\psi}\left|\omega(\tau) - \varpi(\tau)\right|\end{aligned} \tag{4.1}$$

$$\leq \left(\frac{(\psi(\tau) - \psi(a))^{\mu+\nu} \|p\|}{\Gamma(\mu+\nu+1)} + \frac{|\lambda|(\psi(\tau) - \psi(a))^{\nu}}{\Gamma(\nu+1)} \right) \|\omega - \varpi\|$$

and

$$\left|\mathcal{H}\omega(\tau)\right| \leq \frac{k_{\mathbb{F}}(\psi(\tau) - \psi(a))^{\mu+\nu}}{\Gamma(\mu+\nu+1)} + \frac{|\lambda|(\psi(\tau) - \psi(a))^{\nu}}{\Gamma(\nu+1)}. \tag{4.2}$$

Applying the triangle inequality, we get

$$\begin{aligned}
&\left|\mathcal{Q}\omega(\tau) - \mathcal{Q}\varpi(\tau)\right| \\
&\quad\leq \left|\mathbb{G}\big(\tau,\omega(\tau)\big)\mathcal{H}\omega(\tau) - \mathbb{G}\big(\tau,\varpi(\tau)\big)\mathcal{H}\varpi(\tau)\right| \\
&\qquad+ \frac{\psi(\tau) - \psi(a)}{\psi(b) - \psi(\eta)}\left|\mathbb{G}\big(\tau,\varpi(\tau)\big)\mathcal{H}\varpi(b) - \mathbb{G}\big(\tau,\omega(\tau)\big)\mathcal{H}\omega(b)\right| \\
&\qquad+ \frac{\psi(\tau) - \psi(a)}{\psi(b) - \psi(\eta)}\left|\mathbb{G}\big(\tau,\omega(\tau)\big)\mathcal{H}\omega(\eta) - \mathbb{G}\big(\tau,\varpi(\tau)\big)\mathcal{H}\varpi(\eta)\right| \\
&\quad\leq \left|\mathbb{G}\big(\tau,\omega(\tau)\big)\right|\left|\mathcal{H}\omega(\tau) - \mathcal{H}\varpi(\tau)\right| \\
&\qquad+ \left|\mathcal{H}\varpi(\tau)\right|\left|\mathbb{G}\big(\tau,\omega(\tau)\big) - \mathbb{G}\big(\tau,\varpi(\tau)\big)\right| \\
&\qquad+ \frac{\psi(\tau) - \psi(a)}{\psi(b) - \psi(\eta)}\left|\mathbb{G}\big(\tau,\varpi(\tau)\big)\right|\left|\mathcal{H}\varpi(b) - \mathcal{H}\omega(b)\right| \\
&\qquad+ \frac{\psi(\tau) - \psi(a)}{\psi(b) - \psi(\eta)}\left|\mathcal{H}\omega(b)\right|\left|\mathbb{G}\big(\tau,\varpi(\tau)\big) - \mathbb{G}\big(\tau,\omega(\tau)\big)\right| \\
&\qquad+ \frac{\psi(\tau) - \psi(a)}{\psi(b) - \psi(\eta)}\left|\mathbb{G}\big(\tau,\omega(\tau)\big)\right|\left|\mathcal{H}\omega(\eta) - \mathcal{H}\varpi(\eta)\right| \\
&\qquad+ \frac{\psi(\tau) - \psi(a)}{\psi(b) - \psi(\eta)}\left|\mathcal{H}\varpi(\eta)\right|\left|\mathbb{G}\big(\tau,\omega(\tau)\big) - \mathbb{G}\big(\tau,\varpi(\tau)\big)\right|.
\end{aligned}$$

By (4.1) and (4.2), using assumptions ($\mathbb{C}1$)–($\mathbb{C}2$), we deduce that

$$\begin{aligned}
&\left|\mathcal{Q}\omega(\tau) - \mathcal{Q}\varpi(\tau)\right| \\
&\quad\leq k_{\mathbb{G}}\left(\frac{(\psi(\tau) - \psi(a))^{\mu+\nu} \|p\|}{\Gamma(\mu+\nu+1)} + \frac{|\lambda|(\psi(\tau) - \psi(a))^{\nu}}{\Gamma(\nu+1)} \right) \|\omega - \varpi\| \\
&\qquad+ \|\phi\|\left(\frac{k_{\mathbb{F}}(\psi(\tau) - \psi(a))^{\mu+\nu}}{\Gamma(\mu+\nu+1)} + \frac{|\lambda|(\psi(\tau) - \psi(a))^{\nu}}{\Gamma(\nu+1)} \right) \|\omega - \varpi\| \\
&\qquad+ \frac{k_{\mathbb{G}}[\psi(\tau) - \psi(a)]}{\psi(b) - \psi(\eta)}\left(\frac{(\psi(b) - \psi(a))^{\mu+\nu} \|p\|}{\Gamma(\mu+\nu+1)} + \frac{|\lambda|(\psi(b) - \psi(a))^{\nu}}{\Gamma(\nu+1)} \right) \|\omega - \varpi\| \\
&\qquad+ \frac{\|\phi\|[\psi(\tau) - \psi(a)]}{\psi(b) - \psi(\eta)}\left(\frac{k_{\mathbb{F}}(\psi(b) - \psi(a))^{\mu+\nu}}{\Gamma(\mu+\nu+1)} + \frac{|\lambda|(\psi(b) - \psi(a))^{\nu}}{\Gamma(\nu+1)} \right) \|\omega - \varpi\| \\
&\qquad+ \frac{k_{\mathbb{G}}[\psi(\tau) - \psi(a)]}{\psi(b) - \psi(\eta)}\left(\frac{(\psi(\eta) - \psi(a))^{\mu+\nu} \|p\|}{\Gamma(\mu+\nu+1)} + \frac{|\lambda|(\psi(\eta) - \psi(a))^{\nu}}{\Gamma(\nu+1)} \right) \|\omega - \varpi\| \\
&\qquad+ \frac{\|\phi\|[\psi(\tau) - \psi(a)]}{\psi(b) - \psi(\eta)}\left(\frac{k_{\mathbb{F}}(\psi(\eta) - \psi(a))^{\mu+\nu}}{\Gamma(\mu+\nu+1)} + \frac{|\lambda|(\psi(\eta) - \psi(a))^{\nu}}{\Gamma(\nu+1)} \right) \|\omega - \varpi\|.
\end{aligned}$$

Taking the supremum over J and simplifying lead to

$$\|\mathcal{Q}\omega - \mathcal{Q}\varpi\| \leq \mathcal{B}\|\omega - \varpi\|.$$

Hypothesis ($\mathbb{C}3$) allows us to apply the Banach fixed point theorem (Theorem 2.9), which finishes the proof. □

The uniqueness of the mild solution for the coupled HLFDS (1.2)–(1.3) can be achieved by the same arguments as in Theorem 4.1. Hence we omit the proof. Let us first introduce the assumptions that will be used for the next result.

($\mathbb{D}1$) The function $\mathbb{G}_i : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$ is continuous, and there exists a function $\phi_i \in C(J, \mathbb{R}^+)$ with supremum $\|\phi_i\|$ such that

$$\left|\mathbb{G}_i(\tau,\omega_1,\omega_2) - \mathbb{G}_i(\tau,\varpi_1,\varpi_2)\right| \le \phi_i(\tau)\left[|\omega_1 - \varpi_1| + |\omega_2 - \varpi_2|\right]$$

for all $(\tau,\omega_1,\omega_2),(\tau,\varpi_1,\varpi_2) \in J \times \mathbb{R} \times \mathbb{R}$, $i = 1,2$. Moreover, there is a constant $k_{\mathbb{G}_i} > 0$ such that

$$\left|\mathbb{G}_i(\tau,\omega_1,\omega_2)\right| \le k_{\mathbb{G}_i}$$

for all $(\tau,\omega_1,\omega_2) \in J \times \mathbb{R} \times \mathbb{R}$, $i = 1,2$.

($\mathbb{D}2$) The function $\mathbb{F}_i : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, and there exists a function $p_i \in C(J, \mathbb{R}^+)$ with supremum $\|p_i\|$ such that

$$\left|\mathbb{F}_i(\tau,\omega_1,\omega_2) - \mathbb{F}_i(\tau,\varpi_1,\varpi_2)\right| \le p_i(\tau)\left[|\omega_1 - \varpi_1| + |\omega_2 - \varpi_2|\right]$$

for all $(\tau,\omega_1,\omega_2),(\tau,\varpi_1,\varpi_2) \in J \times \mathbb{R} \times \mathbb{R}$, $i = 1,2$. Moreover, there is a constant $k_{\mathbb{F}_i} > 0$ such that

$$\left|\mathbb{F}(\tau,\omega_1,\omega_2)\right| \le k_{\mathbb{F}_i}$$

for all $(\tau,\omega_1,\omega_2) \in J \times \mathbb{R} \times \mathbb{R}$, $i = 1,2$.

($\mathbb{D}3$) The constant $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2 < 1$, where

$$\begin{aligned} \mathcal{B}_i = {} & \frac{|\lambda_i|\|\phi_i\|(\psi(b) - \psi(a))}{\Gamma(\nu_i + 1)} \\ & \times \left[\left(\psi(b) - \psi(a)\right)^{\nu_i - 1} + \frac{(\psi(b) - \psi(a))^{\nu_i} + (\psi(\eta_i) - \psi(a))^{\nu_i}}{\psi(b) - \psi(\eta_i)}\right] \\ & + \frac{(k_{\mathbb{G}_i}(\|p_i\| + |\lambda_i|) + k_{\mathbb{F}_i}\|\phi_i\|)(\psi(b) - \psi(a))}{\Gamma(\mu_i + \nu_i + 1)} \\ & \times \left[\left(\psi(b) - \psi(a)\right)^{\mu_i + \nu_i - 1} + \frac{(\psi(b) - \psi(a))^{\mu_i + \nu_i} + (\psi(\eta_i) - \psi(a))^{\mu_i + \nu_i}}{\psi(b) - \psi(\eta_i)}\right]. \end{aligned} \tag{4.3}$$

Theorem 4.2 *Assume that* ($\mathbb{D}1$)–($\mathbb{D}3$) *hold. Then there exists a unique mild solution of the coupled HLFDS* (1.2)–(1.3) *on* J.

5 Ulam–Hyers stability

In this section, we study the Ulam–Hyers and generalized Ulam–Hyers stability of the coupled HLFDS (1.2)–(1.3). Once we obtain the stability for the coupled HLFDS (1.2)–(1.3), then it will be satisfied for the HLFDS (1.1).

For this, let $\varepsilon > 0$, and let $\Phi : J \to \mathbb{R}^+$ be a continuous function. Consider the following inequality:

$$\left| {}^c\mathbb{D}_{a^+}^{\mu_{i1};\psi}\left[{}^c\mathbb{D}_{a^+}^{\nu_i;\psi}\left[\frac{\omega_i(\tau)}{\mathbb{G}_i(\tau,\omega(\tau))} \right] - \lambda_i \omega_i(\tau) \right] - \mathbb{F}_i\big(\tau,\omega(\tau)\big) \right| \leq \varepsilon \tag{5.1}$$

for $\tau \in J$, $\omega = (\omega_1, \omega_2) \in \mathbb{E}$, $i = 1, 2$.

Definition 5.1 ([10]) The coupled HLFDS (1.2)–(1.3) is Ulam–Hyers stable if there exists $c > 0$ such that for each $\varepsilon > 0$ and for each solution $\omega \in \mathbb{E}$ of (5.1)–(1.3), there exists a solution $\varpi \in \mathbb{E}$ of (1.2)–(1.3) with

$$\|\omega - \varpi\| \leq c\varepsilon.$$

Definition 5.2 ([10]) The coupled HLFDS (1.2)–(1.3) is generalized Ulam–Hyers stable if there exists $\sigma \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $\sigma(0) = 0$ such that for each $\varepsilon > 0$ and for each solution $\omega \in \mathbb{E}$ of (5.1)–(1.3), there exists a solution $\varpi \in \mathbb{E}$ of (1.2)–(1.3) with

$$\|\omega - \varpi\| \leq \sigma(\varepsilon).$$

Remark 5.3 It is clear that Definition 5.1 for $\sigma(\varepsilon) = c\varepsilon$ leads to Definition 5.2, but the converse is not true in general.

Remark 5.4 ([10]) A function $\omega \in \mathbb{E}$ is a solution of inequality (5.1)–(1.3) if and only if there exists a function $g_i \in C(J, \mathbb{R})$ (which depends on ω) such that

- $|g_i(\tau)| \leq \varepsilon, \tau \in J$;
- ${}^cD_{a^+}^{\mu_{i1};\psi}[{}^c\mathbb{D}_{a^+}^{\nu_i;\psi}[\frac{\omega_i(\tau)}{\mathbb{G}_i(\tau,\omega(\tau))}] - \lambda_i\omega_i(\tau)] = \mathbb{F}_i(\tau, \omega(\tau)) + g_i(\tau), \tau \in J$;
- $\omega_i(a) = 0, {}^c\mathbb{D}_{a^+}^{\nu_i;\psi}[\frac{\omega_i(\tau)}{\mathbb{G}_i(\tau,\omega(\tau))}]_{\tau=a} = 0, \omega_i(b) = \zeta_i\omega_i(\eta_i), i = 1, 2.$

For simplification of equations in the next result, we denote

$$\begin{aligned} \mathcal{C}_i &= \frac{\|\phi_i\|}{\Gamma(\mu_i + \nu_i + 1)} \frac{(\psi(b) - \psi(a))^{\mu_i+\nu_i+1}}{\psi(b) - \psi(\eta_i)}, \\ \mathcal{D}_i &= \frac{k_{\mathbb{G}_i}(\psi(b) - \psi(a))}{\Gamma(\mu_i + \nu_i + 1)} \\ &\quad \times \left(\big(\psi(b) - \psi(a)\big)^{\mu_i+\nu_i-1} + \frac{(\psi(b) - \psi(a))^{\mu_i+\nu_i} + (\psi(\eta_i) - \psi(a))^{\mu_i+\nu_i}}{\psi(b) - \psi(\eta_i)} \right), \end{aligned}$$

and recall the constants $\mathcal{B}_i$, $i = 1, 2$, defined by (4.3). Now we introduce the first result.

Theorem 5.5 *Suppose $\mathcal{C}_1 + \mathcal{C}_2 < \frac{1}{2}$ and $\mathcal{B}_1 + \mathcal{B}_2 < \frac{1}{2}$, and let hypotheses* ($\mathbb{D}1$)–($\mathbb{D}3$) *be satisfied. Then the coupled HLFDS* (1.2)–(1.3) *is generalized Ulam–Hyers stable.*

Proof Let $\varepsilon > 0$, and let $\omega = (\omega_1, \omega_2) \in \mathbb{E}$ be a solution of (5.1)–(1.3). Then by Lemma 3.1 and Remark 5.4 there exists a function $g_i \in C(J, \mathbb{R})$ satisfying $|g_i(\tau)| \leq \varepsilon$ such that

$$\omega_i(\tau) = \mathbb{G}_i\big(\tau, \omega_1(\tau), \omega_2(\tau)\big) \tag{5.2}$$

$$\times\left[\mathcal{F}_i(\omega_1,\omega_2)(\tau)-\frac{\psi(\tau)-\psi(a)}{\psi(b)-\psi(\eta_i)}\big(\mathcal{F}_i(\omega_1,\omega_2)(b)-\mathcal{F}_i(\omega_1,\omega_2)(\eta_i)\big)\right],$$

where

$$\mathcal{F}_i(\omega_1,\omega_2)(\tau)=\mathbb{I}_{a^+}^{\mu_i+\nu_i;\psi}\big[\mathbb{F}_i\big(\tau,\omega_1(\tau),\omega_2(\tau)\big)+g_i(\tau)\big]+\lambda_i\mathbb{I}_{a^+}^{\nu_i;\psi}\omega_i(\tau).$$

Let $\varpi\in\mathbb{E}$ be a solution of the coupled HLFDS (1.2)–(1.3). Then it satisfies the integral equation (3.16). Using ($\mathbb{D}2$), we have

$$\begin{aligned}&\big|\mathcal{F}_i(\omega_1,\omega_2)(\tau)-\mathcal{H}_i(\varpi_1,\varpi_2)(\tau)\big|\\&\quad\le\mathbb{I}_{a^+}^{\mu_i+\nu_i;\psi}\big|\mathbb{F}_i\big(\tau,\omega_1(\tau),\omega_2(\tau)\big)-\mathbb{F}_i\big(\tau,\varpi_1(\tau),\varpi_2(\tau)\big)\big|+\mathbb{I}_{a^+}^{\mu_i+\nu_i;\psi}\big|g_i(\tau)\big|\\&\qquad+|\lambda_i|\mathbb{I}_{a^+}^{\nu_i;\psi}\big|\omega_i(\tau)-\varpi_i(\tau)\big|\\&\quad\le\frac{\|p_i\|(\psi(\tau)-\psi(a))^{\mu_i+\nu_i}}{\Gamma(\mu_i+\nu_i+1)}\|\omega-\varpi\|+\frac{\varepsilon(\psi(\tau)-\psi(a))^{\mu_i+\nu_i}}{\Gamma(\mu_i+\nu_i+1)}\\&\qquad+\frac{|\lambda_i|(\psi(\tau)-\psi(a))^{\mu_i+\nu_i}}{\Gamma(\mu_i+\nu_i+1)}\|\omega-\varpi\|\end{aligned}\tag{5.3}$$

and

$$\begin{aligned}\big|\mathcal{H}_i(\omega_1,\omega_2)(\tau)\big|&\le\frac{k_{\mathbb{F}}(\psi(\tau)-\psi(a))^{\mu_i+\nu_i}}{\Gamma(\mu_i+\nu_i+1)}+\frac{|\lambda_i|(\psi(\tau)-\psi(a))^{\nu_i}}{\Gamma(\nu_i+1)},\\\big|\mathcal{F}_i(\omega_1,\omega_2)(\tau)\big|&\le\frac{(k_{\mathbb{F}}+\varepsilon)(\psi(\tau)-\psi(a))^{\mu_i+\nu_i}}{\Gamma(\mu_i+\nu_i+1)}+\frac{|\lambda_i|(\psi(\tau)-\psi(a))^{\nu_i}}{\Gamma(\nu_i+1)}.\end{aligned}\tag{5.4}$$

Applying the triangle inequality, we obtain

$$\begin{aligned}&\big|\omega_i(\tau)-\varpi_i(\tau)\big|\\&\quad\le\big|\mathbb{G}_i\big(\tau,\omega_1(\tau),\omega_2(\tau)\big)\big|\big|\mathcal{F}_i(\omega_1,\omega_2)(\tau)-\mathcal{H}_i(\varpi_1,\varpi_2)(\tau)\big|\\&\qquad+\big|\mathcal{H}_i(\varpi_1,\varpi_2)(\tau)\big||\mathbb{G}_i\big(\tau,\omega_1(\tau),\omega_2(\tau)\big)-\mathbb{G}_i\big(\tau,\varpi_1(\tau),\varpi_2(\tau)\big)\\&\qquad+\left|\frac{\psi(\tau)-\psi(a)}{\psi(b)-\psi(\eta_i)}\right|)\mathcal{H}_i(\varpi_1,\varpi_2)(b)-\mathcal{F}_i(\omega_1,\omega_2)(b)||\mathbb{G}_i\big(\tau,\varpi_1(\tau),\varpi_2(\tau)\big)\\&\qquad+\left|\frac{\psi(\tau)-\psi(a)}{\psi(b)-\psi(\eta_i)}\right|\mathbb{G}_i\big(\tau,\varpi_1(\tau),\varpi_2(\tau)\big)-\mathbb{G}_i\big(\tau,\omega_1(\tau),\omega_2(\tau)\big)||\mathcal{F}_i(\omega_1,\omega_2)(b)\\&\qquad+\left|\frac{\psi(\tau)-\psi(a)}{\psi(b)-\psi(\eta_i)}\right|\mathcal{F}_i(\omega_1,\omega_2)(\eta_i)-\mathcal{H}_i(\varpi_1,\varpi_2)(\eta_i)||\mathbb{G}_i\big(\tau,\omega_1(\tau),\omega_2(\tau)\big)\\&\qquad+\left|\frac{\psi(\tau)-\psi(a)}{\psi(b)-\psi(\eta_i)}\right|\mathbb{G}_i\big(\tau,\omega_1(\tau),\omega_2(\tau)\big)-\mathbb{G}_i\big(\tau,\varpi_1(\tau),\varpi_2(\tau)\big)|\big|\mathcal{H}_i(\varpi_1,\varpi_2)(\eta_i)\big|.\end{aligned}$$

By ($\mathbb{D}1$), ($\mathbb{D}2$), (5.3), and (5.4) we have

$$\begin{aligned}&\big|\omega_i(\tau)-\varpi_i(\tau)\big|\\&\quad\le k_{\mathbb{G}_i}\bigg(\frac{\varepsilon(\psi(\tau)-\psi(a))^{\mu_i+\nu_i}}{\Gamma(\mu_i+\nu_i+1)}+\frac{\|p_i\|(\psi(\tau)-\psi(a))^{\mu_i+\nu_i}}{\Gamma(\mu_i+\nu_i+1)}\|\omega-\varpi\|\\&\qquad+\frac{|\lambda_i|(\psi(\tau)-\psi(a))^{\mu_i+\nu_i}}{\Gamma(\mu_i+\nu_i+1)}\|\omega-\varpi\|\bigg)\end{aligned}$$

$$
\begin{aligned}
&+\|\phi_i\|\left(\frac{k_{\mathbb{F}}(\psi(\tau)-\psi(a))^{\mu_i+\nu_i}}{\Gamma(\mu_i+\nu_i+1)}+\frac{|\lambda_i|(\psi(\tau)-\psi(a))^{\nu_i}}{\Gamma(\nu_i+1)}\right)\|\omega-\varpi\| \\
&+k_{\mathbb{G}_i}\frac{\psi(\tau)-\psi(a)}{\psi(b)-\psi(\eta_i)}\left(\frac{\varepsilon(\psi(b)-\psi(a))^{\mu_i+\nu_i}}{\Gamma(\mu_i+\nu_i+1)}\right. \\
&\left.+\left(\frac{\|p_i\|(\psi(b)-\psi(a))^{\mu_i+\nu_i}}{\Gamma(\mu_i+\nu_i+1)}+\frac{|\lambda_i|(\psi(b)-\psi(a))^{\mu_i+\nu_i}}{\Gamma(\mu_i+\nu_i+1)}\right)\|\omega-\varpi\|\right) \\
&+\|\phi_i\|\frac{\psi(\tau)-\psi(a)}{\psi(b)-\psi(\eta_i)} \\
&\times\left(\frac{(k_{\mathbb{F}}+\varepsilon)(\psi(b)-\psi(a))^{\mu_i+\nu_i}}{\Gamma(\mu_i+\nu_i+1)}+\frac{|\lambda_i|(\psi(b)-\psi(a))^{\nu_i}}{\Gamma(\nu_i+1)}\right)\|\omega-\varpi\| \\
&+k_{\mathbb{G}_i}\frac{\psi(\tau)-\psi(a)}{\psi(b)-\psi(\eta_i)}\left(\frac{\varepsilon(\psi(\eta_i)-\psi(a))^{\mu_i+\nu_i}}{\Gamma(\mu_i+\nu_i+1)}\right. \\
&\left.+\left(\frac{\|p_i\|(\psi(\eta_i)-\psi(a))^{\mu_i+\nu_i}}{\Gamma(\mu_i+\nu_i+1)}+\frac{|\lambda_i|(\psi(\eta_i)-\psi(a))^{\mu_i+\nu_i}}{\Gamma(\mu_i+\nu_i+1)}\right)\|\omega-\varpi\|\right) \\
&+\|\phi_i\|\frac{\psi(\tau)-\psi(a)}{\psi(b)-\psi(\eta_i)} \\
&\times\left(\frac{k_{\mathbb{F}}(\psi(\tau)-\psi(a))^{\mu_i+\nu_i}}{\Gamma(\mu_i+\nu_i+1)}+\frac{|\lambda_i|(\psi(\tau)-\psi(a))^{\nu_i}}{\Gamma(\nu_i+1)}\right)\|\omega-\varpi\|.
\end{aligned}
$$

Simplifications lead to

$$\|\omega-\varpi\|\leq 2(\mathcal{D}_1+\mathcal{D}_2)\left(\frac{\varepsilon}{1-\varepsilon}\right),\quad \varepsilon<1.$$

The generalized Ulam–Hyers stability condition is satisfied if we assume that $\sigma(\varepsilon)=2(\mathcal{D}_1+\mathcal{D}_2)(\frac{\varepsilon}{1-\varepsilon})$ and $\sigma(0)=0$. This completes the proof. □

Remark 5.6 The appearance of ε in the denominator is due to the term $|\mathcal{F}_i(\omega_1,\omega_2)(\tau)|$ estimated in (5.4). This implies that there is no guarantee to ensure the coupled HLFDS (1.2)–(1.3) is Ulam–Hyers stable using the conditions of Theorem 5.5.

The next result can be proved similarly to Theorem 5.5.

Theorem 5.7 *Let $\mathcal{C}<\frac{1}{2}$ and $\mathcal{B}<\frac{1}{2}$, and let hypotheses* ($\mathbb{C}$1)–($\mathbb{C}$3) *be satisfied. Then the HLFDS* (1.1) *is generalized Ulam–Hyers stable.*

6 Examples

In this section, to illustrate our results, we consider two examples.

Example 6.1 Consider the HLFDS

$$
\begin{cases}
{}^{c}\mathbb{D}_{0^+}^{0.5;\psi}[{}^{c}\mathbb{D}_{0^+}^{1.5;\psi}[\frac{\omega(\tau)}{2+\tau\sin\omega(\tau)}]-0.001\omega(\tau)]=0.001e^{-\tau}\sqrt{|\omega(\tau)|}+0.001e^{-\tau}, \\
\omega(0)=0,\qquad {}^{c}\mathbb{D}_{0^+}^{1.5;\psi}[\frac{\omega(\tau)}{2+\tau\sin\omega(\tau)}]_{\tau=0}=0,\qquad \omega(1)=2\omega(0.5).
\end{cases}
\tag{6.1}
$$

The function $\mathbb{G}(\tau,\omega(\tau))=2+\tau\sin\omega(\tau)$, $\tau\in[0,1]$, is nonzero Lipschitz continuous such that $\phi(\tau)=\tau$ with supremum 1, and $\mathbb{G}_0=2$. If we choose $\psi(\tau)=\tau^2+\tau$, $\tau\in[0,1]$, then

$$\left|\mathbb{G}(b,\omega)\big(\psi(b)-\psi(a)\big)-\zeta\mathbb{G}(\eta,\omega)\big(\psi(\eta)-\psi(a)\big)\right|\geq\frac{7}{4}=\vartheta.$$

Thus condition ($\mathbb{A}1$) holds.

The function $\mathbb{F}(\tau,\omega(\tau))=0.001e^{-\tau}\sqrt{|\omega(\tau)|}+0.001e^{-\tau}$ satisfies ($\mathbb{A}2$) such that $p(\tau)=0.001e^{-\tau}$ has supremum 0.001 on $[0,1]$, and $\chi(r)=\sqrt{r}+1, r\geq 0$, is nondecreasing. In the last condition, we have

$$\mathcal{A}_1=0.045705,\qquad \frac{2\mathcal{A}_1}{1-\mathcal{A}_1}=0.096<1.$$

All hypotheses ($\mathbb{A}1$)–($\mathbb{A}3$) are satisfied. Then Theorem 3.3 ensures the existence of at least one nonzero mild solution of the HLFDS (6.1).

Example 6.2 Consider the coupled HLFDS

$$\begin{cases} {}^{c}\mathbb{D}^{0.5;\psi}_{1^+}[{}^{c}\mathbb{D}^{1.5;\psi}_{1^+}[\frac{\omega_1(\tau)}{0.07+0.001\tau\sin\omega_1(\tau)+0.001\tau^2\sin\omega_2(\tau)}]-\omega_1(\tau)] \\ \quad =0.01e^{-\tau}\frac{|\omega_1(\tau)|+|\omega_2(\tau)|}{1+|\omega_1(\tau)|+|\omega_2(\tau)|},\quad \tau\in[1,e], \\ {}^{c}\mathbb{D}^{0.3;\psi}_{1^+}[{}^{c}\mathbb{D}^{1.7;\psi}_{1^+}[\frac{\omega_1(\tau)}{0.07+0.001\tau\sin\omega_1(\tau)+0.001\tau^2\sin\omega_2(\tau)}]-\omega_1(\tau)] \\ \quad =0.01e^{-\tau}\frac{|\omega_1(\tau)|+|\omega_2(\tau)|}{1+|\omega_1(\tau)|+|\omega_2(\tau)|},\quad \tau\in[1,e], \\ \omega_1(0)=0, {}^{c}\mathbb{D}^{1.5;\psi}_{1^+}[\frac{\omega_1(\tau)}{0.07+0.001\tau\sin\omega_1(\tau)+0.001\tau^2\sin\omega_2(\tau)}]_{\tau=0}=0, \\ \omega_1(e)=\zeta\omega_1(2), \\ \omega_2(0)=0, {}^{c}\mathbb{D}^{1.7;\psi}_{1^+}[\frac{\omega_2(\tau)}{0.07+0.001\tau\sin\omega_1(\tau)+0.001\tau^2\sin\omega_2(\tau)}]_{\tau=0}=0, \\ \omega_2(e)=\zeta\omega_2(2). \end{cases} \tag{6.2}$$

The functions $\mathbb{G}_1(\tau,\omega_1(\tau),\omega_2(\tau))=\mathbb{G}_2(\tau,\omega_1(\tau),\omega_2(\tau))=0.07+0.001\tau\sin\omega_1(\tau)+0.001\tau^2\sin\omega_2(\tau)$ are nonzero Lipschitz continuous such that $\|\phi_i\|=0.001e^2\cong 0.0074$, $k_{\mathbb{G}_i}=0.07+0.001e+0.001e^2\cong 0.08$, and $\zeta_i=\frac{0.07+0.001e\sin\omega_1(e)+0.001e^2\sin\omega_2(e)}{0.07+0.002\sin\omega_1(2)+0.004\sin\omega_2(2)}$.

The functions $\mathbb{F}_1(\tau,\omega_1(\tau),\omega_2(\tau))=\mathbb{F}_2(\tau,\omega_1(\tau),\omega_2(\tau))=0.01e^{-\tau}\frac{|\omega_1(\tau)|+|\omega_2(\tau)|}{1+|\omega_1(\tau)|+|\omega_2(\tau)|}$ are Lipschitzian with common constants $\|p_i\|=0.01$ and $k_{\mathbb{F}_i}=0.01$. If $\psi(\tau)=\ln\tau$ (then the fractional derivative becomes Hadamard derivative), then we obtain

$$\mathcal{B}_1\cong 0.27,\qquad \mathcal{B}_2\cong 0.2644.$$

This implies that $\mathcal{B}<1$, and therefore all hypotheses ($\mathbb{D}1$)–($\mathbb{D}3$) of Theorem 4.2 are satisfied. Thus there exists a unique nonzero mild solution of the coupled HLFDS (6.2).

Moreover, we can find that

$$\mathcal{C}_i=0.0121,\qquad \mathcal{D}_i=0.233,\quad i=1,2.$$

Thus by Theorem 5.5 we deduce that the coupled HLFDS (6.2) is generalized Ulam–Hyers stable.

7 Conclusion

In this paper, we considered the existence, uniqueness, and Ulam–Hyers stability of solutions for a novel class of hybrid Langevin fractional differential systems subject to three-point boundary conditions in view of the ψ-Caputo derivatives. The obtained results are derived by using the Dhage and Banach fixed point theorems. We considered two systems: one is a hybrid Langevin fractional differential system, and the other is a coupled hybrid Langevin fractional differential system. Finally, we introduce two examples to validate our theoretical results. The obtained results are new and generalize many existing results in the literature. This field is active in research, and hence we recommend to continue in this line of studying to more qualitative analysis of such systems and using generalized fractional derivatives. One direction of future investigations can be performed on other fractional models using different fractional derivatives and multipoint boundary conditions.

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No additional information is available for this paper.

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Declarations

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The authors declare no conflict of interest.

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References

1. Abdeljawad, T., Baleanu, D., Jarad, F.: Existence and uniqueness theorem for a class of delay differential equations with left and right Caputo fractional derivatives. J. Math. Phys. **49**, 083507 (2008)
2. Abdeljawad, T., Jarad, F., Baleanu, D.: On the existence and the uniqueness theorem for fractional differential equations with bounded delay within Caputo derivatives. Sci. China Ser. A, Math. **51**, 1775–1786 (2008)
3. Abdo, M.S., Panchal, S.K., Saeed, A.M.: Fractional boundary value problem with ψ-Caputo fractional derivative. Proc. Indian Acad. Sci. Math. Sci. **129**(5), 65 (2019)
4. Abdulwasaa, M.A., Abdo, M.S., Shah, K., Nofal, T.A., Panchal, S.K., Kawale, S.V., Abdel-Atyf, A.: Fractal-fractional mathematical modeling and forecasting of new cases and deaths of Covid-19 epidemic outbreaks in India. Results Phys. **20**, 103702 (2021)
5. Adjabi, Y., Samei, M.E., Matar, M.M., Alzabut, J.: Langevin differential equation in frame of ordinary and Hadamard fractional derivatives under three point boundary conditions. AIMS Math. **6**(3), 2796–2843(2021)
6. Agarwal, R.P., Meehan, M., O'Regan, D.: Fixed Point Theory and Applications. Cambridge University Press, Cambridge (2001)
7. Almeida, R.: A Caputo fractional derivative of a function with respect to another function. Commun. Nonlinear Sci. Numer. Simul. **44**, 460–481 (2017)

8. Almeida, R.: Functional differential equations involving the ψ-Caputo fractional derivative. Fractal Fract. **4**(2), 29 (2020)
9. Baghani, H., Alzabut, J., Nieto, J.J.: Further results on the existence of solutions for generalized fractional Basset–Boussinesq–Oseen equation. Iran. J. Sci. Technol., Trans. A, Sci. **44**, 1461–1467 (2020)
10. Baitiche, Z., Derbazi, C., Matar, M.M.: Ulam stability for nonlinear-Langevin fractional differential equations involving two fractional orders in the ψ-Caputo sense. Appl. Anal. **101**, 4866–4881(2022)
11. Baleanu, D., Etemad, S., Rezapour, S.: On a fractional hybrid integro-differential equation with mixed hybrid integral boundary value conditions by using three operators. Alex. Eng. J. (2020). https://doi.org/10.1016/j.aej.2020.04.053
12. Berhail, A., Tabouche, N., Matar, M.M., Alzabut, J.: On nonlocal integral and derivative boundary value problem of nonlinear Hadamard Langevin equation with three different fractional orders. Bol. Soc. Mat. Mex. **26**, 303–318 (2020)
13. Boutiara, A.: Multi-term fractional q-difference equations with q-integral boundary conditions via topological degree theory. Commun. Optim. Theory **2021**, Article ID 1 (2021)
14. Boutiara, A., Abdo, M.S., Benbachir, M.: Existence results for ψ-Caputo fractional neutral functional integro-differential equations with finite delay. Turk. J. Math. **44**, 2380–2401 (2020)
15. Boutiara, A., Benbachir, M., Guerbati, K.: Hilfer fractional hybrid differential equations with multi-point boundary hybrid conditions. Int. J. Mod. Math. Sci. **19**(1), 17–33 (2021)
16. Boutiara, A., Guerbati, K., Benbachir, M.: Caputo–Hadamard fractional differential equation with three-point boundary conditions in Banach spaces. AIMS Math. **5**(1), 259–272 (2020)
17. Chalishajar, D., Kumar, A.: Existence, uniqueness and Ulam's stability of solutions for a coupled system of fractional differential equations with integral boundary conditions. Mathematics **6**(6), Article ID 96 (2018)
18. Dhage, B.C.: On a fixed point theorem in Banach algebras with applications. Appl. Math. Lett. **18**, 273–280 (2005)
19. Dhage, B.C.: Basic results in the theory of hybrid differential equations with mixed perturbation of second type. Funct. Differ. Equ. **19**, 87–106 (2012)
20. Dhage, B.C., Dhage, S.B., Buvaneswari, K.: Existence of mild solutions of nonlinear boundary value problems of coupled hybrid fractional integro differential equations. J. Fract. Calc. Appl. **10**(2), 191–206 (2019)
21. Jarad, F., Abdeljawad, T., Baleanu, D.: On the generalized fractional derivatives and their Caputo modification. J. Nonlinear Sci. Appl. **10**, 2607–2619 (2017)
22. Katugampola, U.N.: A new approach to generalized fractional derivatives. Bull. Math. Anal. Appl. **6**(4), 1–15 (2014)
23. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North Holland Mathematics Studies, vol. 203. Elsevier, Amsterdam (2006)
24. Matar, M.M.: Qualitative properties of solution for hybrid nonlinear fractional differential equations. Afr. Math. **30**(7), 1169–1179(2019)
25. Matar, M.M.: Existence of solution for fractional neutral hybrid differential equations with finite delay. Rocky Mt. J. Math. **50**(6), 2141–2148(2020)
26. Matar, M.M., Abbas, M.I., Alzabut, J., Kaabar, M.K.A., Etemad, S., Rezapour, S.: Investigation of the p-Laplacian nonperiodic nonlinear boundary value problem via generalized Caputo fractional derivatives. Adv. Differ. Equ. **2021**, Article ID 68 (2021)
27. Matar, M.M., Alzabut, J., Jonnalagadda, J.M.: A coupled system of nonlinear Caputo–Hadamard Langevin equations associated with nonperiodic boundary conditions. Math. Methods Appl. Sci. **44**(3), 2650–2670(2021)
28. Matar, M.M., Lubbad, A.A., Alzabut, J.: On p-Laplacian boundary value problems involving Caputo–Katugampula fractional derivatives. Math. Methods Appl. Sci. (2020). https://doi.org/10.1002/mma.6534
29. Thabet, S.T.M., Abdo, M.S., Shah, K., Abdeljawad, T.: Study of transmission dynamics of Covid-19 mathematical model under ABC fractional order derivative. Results Phys. **19**, 103507 (2020)