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Research article

Existence of solutions and a numerical scheme for a generalized hybrid class of n-coupled modified ABC-fractional differential equations with an application

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Abstract: In this article, we investigate some necessary and sufficient conditions required for the existence of solutions for mABC-fractional differential equations (mABC-FDEs) with initial conditions; additionally, a numerical scheme based on the the Lagrange's interpolation polynomial is established and applied to a dynamical system for the applications. We also study the uniqueness and Hyers-Ulam stability for the solutions of the presumed mABC-FDEs system. Such a system has not been studied for the mentioned mABC-operator and this work generalizes most of the results studied for the ABC operator. This study will provide a base to a large number of dynamical problems for the existence, uniqueness and numerical simulations. The results are compared with the classical results graphically to check the accuracy and applicability of the scheme.

Keywords: modified ABC-operator; hybrid fractional differential equations; existence of solutions, unique solution; Hyers-Ulam-stability; numerical simulations
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1. Introduction

Numerous studies in science and engineering have focused on the numerical simulations of dynamical systems and their mathematical modelling. The usage of fractional order operators is a practical and extensively explored technique for making generalizations of the classical models. The history of the fractional order operators spans both singular and non-singular kernels as well as local and non-local kernels. Recently, these elements were discussed via some interesting results. The readers can study the remarkable monographs [1].

Experts have investigated the general classes of fractional differential equations (FDEs), including; sequential FDEs, hybrid FDEs, mixed FDEs, and many others that are still unexplored in this field. In the field of nonlinear analysis, perturbation approaches are highly helpful for understanding system dynamics that are modeled by different mathematical techniques. Sometimes a differential equation that represents a specific dynamical system is difficult to solve or evaluate, but, by perturbing the system in some way, it can be studied by using techniques for various features of the results. Dhage [2] has classified the hybrid differential equations in linear and quadratic perturbations by the first and second kinds and has given a detail of its importance in the dynamical studies. He provided a scientific evolution of many forms of perturbation techniques in the theory of differential and integral equations. It was thoroughly investigated for various aspects of the solutions to a special quadratic perturbation of the periodic boundary conditions of second order ordinary differential equations. The existence of extremal positive solutions were established for both Caratheodory and discontinuity conditions, and an existence theorem was demonstrated under mixed generalized Lipschitz and Carath'eodory conditions. Some known results for periodic boundary value issues of second order ordinary nonlinear differential equations were included in his findings as special examples.

The illustration of fractional order hybrid differential equations were then studied by several authors for different fractional order derivatives. For instance, Zhao et al. [3] considered the following second kind of quadratic perturbation problem for the existence and uniqueness of solutions (EUS) in the Riemann-Liouville (R-L) sense of the derivative. Sitho et al. [4] studied fractional integro-differential equations for the EUS and with their applications where the derivative was in the R-L sense. Awadalla and Abuasbeh [5] studied a second-class perturbed sequential FDE for the EUS for Caputo-Hadamard operators. Gul et al. [6] studied a system of hybrid FDEs with the application of their results to the dynamical problems where the operator they used was the Caputo's derivative. Khan et al. [7] investigated a sequential system of hybrid FDEs for the EUS and Hyers-Ulam (HU) stability with the help of the Leray-Schauder and Banach alternative theorems. They used two different types of fractional orders, they are; Caputo's fractional derivative and AB-fractional operator. Losada and Nieto [8], examined sequential FDEs with nonsingular kernel for the EUS. Caputo, M. Fabrizio [9] given the definition of fractional derivative with nonsingular kernel. Atangana and Baleanu [10] given the notion of the AB-fractional derivative with their applications. Al-Refai and Baleanu [11] extended the notion given in [10] and solved the initialization issue in the AB-operator. Dhage et al. [12] and Dhage [13] studied hybrid classes of fractional differential equations for the existence and uniqueness of solutions. Al-Refai [14] given the notion of the inverse operator of fractional order modified AB-operator for the derivative and established some applications. Khan et al. [15] presented some simulations for a disease model based on the imperfect testing issue. Shi and Cui [16] developed Hepatitis C mothel and given the necessary conditions required for their stability. Subramanian [17,18] discussed the existence of solution for a coupled system with sequential fractional operators and integro-differential equations. More related results and techniques can be studied in [19–26].

In [27], Jose it et al. studied the stability comparative analysis on different eco-epidemiological models based on the Stage structure for prey and predator with some impulsive. Etemad et al. [28] developed numerical algorithm for the approximate solutions of a system of coupled fractional thermostat control model by the applications of generalized differential transform technique. Selvam et al. [29] studied Hyers-Ulam Mittag-Leffler stability of discrete fractional order Duffing equation with its application to the inverted pendulum. Zada et al. [30] investigated Ulam-Hyers stability for a class of fractional order impulsive integro-differential equations with boundary conditions.

Inspired from these works, in this paper, we discuss the necessary and sufficient criteria for the existence of solutions mABC-FDEs of hybrid system for the suggested problem:

$$\begin{aligned} &{}^{mABC}\mathbb{D}^{\varrho_i}\Big[w_i(t) - \sum_{i=1}^{n} \mathbb{G}_i(t, w_i(t))\Big] = -\lambda_i^*(t, w_i(t)),\ t \in \mathrm{I} = [0, 1], \\ &w_i(0) = \zeta_i,\ \ \mathbb{G}_i(t, w_i(t))|_{t=0} = 0, \end{aligned} \tag{1.1}$$

where, we have $0 < \varrho_i \leq 1$, $\zeta_i \in \mathbb{R}$, the functions $w_i : \mathrm{I} \to \mathbb{R}$ are continuous where $i = 1, 2, \ldots, n$, λ_i^*, $\mathbb{G}_i : \mathrm{I}\times\mathbb{R} \to \mathbb{R}$, $(i = 1, 2, \ldots, m)$ are continuous and satisfy the Caratheodory assumptions. ${}^{mABC}\mathbb{D}^{\alpha_i}$, the mABC-fractional differential operators for $i = 1, 2, \ldots, n$. To the best of the authors knowledge, there are no studies in the literature that address general hybrid problems of this nature.

Further, we will construct and apply a numerical formulation by Lagrange's-interpolation-polynomial and an application to a dynamical system will be illustrated. We study the proposed system's uniqueness and Hyers-Ulam stability in terms of solution existence. To assess the validity and application of the scheme, the findings are contrasted with the conventional results. The study of dynamical models benefits greatly from the use of numerical techniques. Recent years have seen the development and application of certain numerical techniques for fractional order operators. For instance, the readers can view the work that was studied in [31–35].

We analyse the EUS, HU-stability, and provides an application to the dynamical problem. Recently, researcher engineers have focused on fractional order operators for system dynamics modelling. Singular and non-singular kernels are currently well studied in literature. It is difficult to decide which operator is the most appropriate, but scientists are continuously looking at different operators for new developments. Such a system (1.1) has not been studied for the mentioned mABC-operator, and this work generalizes most of the results studied for the ABC operator. Also, this work will provide a base for a large number of dynamical problems for the existence, uniqueness and numerical simulations. The proposed problem is very much a complex and n-coupled system which is definitely based on a large number of assumptions for all of the results.

Here, we present some basic notions from the modified ABC calculus which will be used further in the results of the article.

Definition 1.1. *[11, 14] For $\varrho \in (0, 1)$, and $f \in L^1(0, T)$, the modified ABC-derivative is given as follows*

$${}^{mABC}D_0^{\varrho} f(t) = \frac{B(\varrho)}{1-\varrho}[f(t) - E_{\varrho}(-\mu_{\varrho} t^{\varrho}) f(0) \tag{1.2}$$

$$- \quad \mu_\varrho \int_0^t (t-s)^{\varrho-1} E_{\varrho,\varrho}(-\mu_\varrho (t-s)^\varrho) f(s)ds].$$

From this definition, one can easily verify that ${}^{mABC}D_0^\varrho C = 0$ [11]. The corresponding integral is given by:

Definition 1.2. *[11, 14] For $\varrho \in (0,1)$, and $f \in L^1(0,T)$, the modified AB-integral is given as follows*

$$ {}^{mAB}D_0^\varrho f(t) = \frac{B(1-\varrho)}{B(\varrho)}[f(t) - f(0)] + \mu_\varrho [{}^{RL}I_0^\varrho (f(t) - f(0))]. \tag{1.3}$$

Lemma 1.1. *[11] For $f' \in L^1(0,\infty)$, and $\varrho \in (0,1)$, we have*

$$ {}^{mAB}I_0^{\varrho mABC} D_0^\varrho f(t) = f(t) - f(0). \tag{1.4}$$

2. Existence criteria

In literature, the existence of solution and numerical approximations for coupled systems, hybrid FDEs, and more general classes are studied in [17, 18, 27–30, 36–38]. Based on the literature, we can proceed to the following lemma.

Lemma 2.1. *The n-coupled system of the hybrid mABC-FDEs given by* (1.1) *has solutions of the kind*

$$\begin{aligned} w_i &= \zeta_i + \sum_{i=1}^n \mathbb{G}_i(t, w_i(t)) + \frac{1-\varrho_i}{B(\varrho_i)} \lambda_i^*(t, w_i(t)) \\ &+ \frac{\varrho_i}{B(\varrho_i)\Gamma(\varrho_i)} \int_0^t (t-s)^{\varrho_i - 1} \lambda_i^*(s, w_i(s)) ds \\ &- \frac{1-\varrho_i}{B(\varrho_i)} \lambda_i^*(0, w_i(0)) \Big(1 + \frac{\gamma_{\varrho_i}}{\Gamma(\varrho_i+1)} t^{\varrho_i}\Big), \end{aligned} \tag{2.1}$$

for $i = 1, 2, \ldots, n$.

Proof. With the application of $({}^{mAB}I_t^{\alpha_i})$ to the system of mABC-differential equations of orders ϱ_i given in (1.1), for all $i = 1, 2, \ldots, n$, we have

$$ w_i(t) - \sum_{i=1}^m \mathbb{G}_i(t, w_i(t)) - w_i(0) =^{mAB} I^{\varrho_i} \lambda_i^*(t, w_i(t)), \tag{2.2}$$

where $i = 1, 2, \ldots, n$. By the conditions $w_i(0) = \zeta_i$, we get the following solutions

$$ w_i(t) = \zeta_i + \sum_{i=1}^m \mathbb{G}_i(t, w_i(t)) +^{mAB} I^{\varrho_i} \lambda_i^*(t, w_i(t)), \tag{2.3}$$

which is the integral form of our mABC-FDEs

$$\begin{aligned} w_i &= \zeta_i + \sum_{i=1}^n \mathbb{G}_i(t, w_i(t)) + \frac{1-\varrho_i}{B(\varrho_i)} \lambda_i^*(t, w_i(t)) \\ &+ \frac{\varrho_i}{B(\varrho_i)\Gamma(\varrho_i)} \int_0^t (t-s)^{\varrho_i - 1} \lambda_i^*(s, w_i(s)) ds - \frac{1-\varrho_i}{B(\varrho_i)} \lambda_i^*(0, w_i(0)) \Big(1 + \frac{\gamma_{\varrho_i}}{\Gamma(\varrho_i+1)} t^{\varrho_i}\Big). \end{aligned} \tag{2.4}$$

This completes the proof. □

In order to proceed to the main results of the paper, we presume a Banach's space $\mathbb{B} = \{w_i(t) : w_i(t) \in \mathbb{C}([0,1],\mathbb{R})$ for $t \in [0,1]\}$, with the norm $\|w_i\| = \max_{t\in[0,1]} |w_i(t)|$, $i = 1,2,\ldots,n$.

Assuming $\mathbb{T}_i : \mathbb{C}([0,1],\mathbb{R}) \to \mathbb{C}([0,1],\mathbb{R})$, with operators for $i = 1,2,\ldots,n$, where

$$\begin{aligned}\mathbb{T}_i w_i(t) &= \zeta_i + \sum_{i=1}^{n} \mathbb{G}_i(t,w_i(t)) + \frac{1-\varrho_i}{B(\varrho_i)}\lambda_i^*(t,w_i(t)) \\ &+ \frac{\varrho_i}{B(\varrho_i)\Gamma(\varrho_i)}\int_0^t (t-s)^{\varrho_i-1}\lambda_i^*(s,w_i(s))ds - \frac{1-\varrho_i}{B(\varrho_i)}\lambda_i^*(0,w_i(0))\Big(1+\frac{\gamma_{\varrho_i}}{\Gamma(\varrho_i+1)}t^{\varrho_i}\Big). \quad (2.5)\end{aligned}$$

By (2.5), all of the fixed points of $\mathbb{T}_i$ are the required solution of the system (1.1).

Lemma 2.2. *Assume that for some $\zeta_i^1, \zeta_i^2 \in \mathbb{R}_e$, and w_i, $\bar{w}_i \in \mathbb{C}$, $t \in [0,k]$, that*

$$|\lambda_i^*(t,w_i) - \lambda_i^*(t,\bar{w}_i)| \le \zeta_i^1 |w_i - \bar{w}_i|, \quad (2.6)$$

$$|\mathbb{G}_i(t,w_i) - \mathbb{G}_i(t,\bar{w}_i)| \le \zeta_i^2 |w_i - \bar{w}_i|, \quad (2.7)$$

and

$$\eta_i = \sum_{i=1}^{n} \zeta_i^2 + \frac{\zeta_i^1}{B(\varrho_i)\Gamma(\varrho_i)}, \quad (2.8)$$

where $\eta_i < 1$, for all $i = 1,2,\ldots,n$; then, there exists a unique solution ofthe n-coupled system mABC-FDEs (1.1).

Proof. We assume that the i values are $i = 1,2,\ldots,n$. Assume that $\sup_{t\in[0,k]} |\mathbb{G}_i(t,0)| = \wp_2 < \infty$, and $\sup_{t\in[0,k]} |\lambda_i^*(t,0)| = \wp_1 < \infty$, $\mathbb{S}_{\eta_i} = \{w_i \in \mathbb{C}([0,k],\mathbb{R}_e) : \|w_i\| < \eta_i\}$, for $k \ge 1$. For $w_i \in \mathbb{S}_{\eta_i}$, and $t \in [0,k]$, we have

$$\begin{aligned}|\lambda_i^*(t,w_i(t))| &= |\lambda_i^*(t,w_i(t)) - \lambda_i^*(t,0) + \lambda_i^*(t,0)| \\ &\le |\lambda_i^*(t,w_i(t)) - \lambda_i^*(t,0)| + |\lambda_i^*(t,0)| \\ &\le \zeta_i^1 |w_i(t)| + |\lambda_i^*(t,0)| \quad (2.9) \\ &\le \zeta_i^1 \eta_i + \wp_1.\end{aligned}$$

Also, for $w_i \in \mathbb{S}_{\eta_i}$, $t \in [0,k]$, we have

$$\begin{aligned}|\mathbb{G}_i(t,w_i(t))| &= |\mathbb{G}_i(t,w_i(t)) - \mathbb{G}_i(t,0) + \mathbb{G}_i(t,0)| \\ &\le |\mathbb{G}_i(t,w_i(t)) - \mathbb{G}_i(t,0)| + |\mathbb{G}_i(t,0)| \\ &\le \zeta_i^2 |w_i(t)| + |\mathbb{G}_i(t,0)| \quad (2.10) \\ &\le \zeta_i^2 \eta_i + \wp_2.\end{aligned}$$

And from (2.5), for $t \geq s$, we have

$$
\begin{aligned}
|\mathbb{T}_i w_i(t)| &= \Big|\zeta_i + \sum_{i=1}^{n} \mathbb{G}_i(t, w_i(t)) + \frac{1-\varrho_i}{B(\varrho_i)}\lambda_i^*(t, w_i(t)) + \frac{\varrho_i}{B(\varrho_i)\Gamma(\varrho_i)}\int_0^t (t-s)^{\varrho_i - 1}\lambda_i^*(s, w_i(s))ds \\
&- \frac{1-\varrho_i}{B(\varrho_i)}\lambda_i^*(0, w_i(0))\Big(1 + \frac{\gamma_{\varrho_i}}{\Gamma(\varrho_i + 1)}t^{\varrho_i}\Big)\Big| \\
&\leq \zeta_i + n(\zeta_i^2\eta_i + \wp_2) + \frac{1-\varrho_i}{B(\varrho_i)}(\zeta_i^1\eta_i + \wp_1) \\
&+ \frac{\varrho_i}{B(\varrho_i)\Gamma(\varrho_i)}\int_0^t (t-s)^{\varrho_i}(\zeta_i^1\eta_i + \wp_1)ds + \frac{1-\varrho_i}{B(\varrho_i)}\lambda_i^*(0, w_i(0))\Big(1 + \frac{\gamma_{\varrho_i}}{\Gamma(\varrho_i + 1)}t^{\varrho_i}\Big) \\
&\leq \zeta_i + n(\zeta_i^2\eta_i + \wp_2) + (\frac{1-\varrho_i}{B(\varrho_i)} + \frac{1}{B(\varrho_i)\Gamma(\varrho_i)})(\zeta_i^1\eta_i + \wp_1)ds + \frac{1-\varrho_i}{B(\varrho_i)}\lambda_i^*\Big(1 + \frac{\gamma_{\varrho_i}}{\Gamma(\varrho_i + 1)}\Big)
\end{aligned}
\tag{2.11}
$$

This implies that $\mathbb{T}_i\mathbb{S}_{\eta_i} \subset \mathbb{S}_{\eta_i}$. Furthermore, we assume that $w_l,\, w_j \in \mathbb{C}([0,1], \mathbb{R}_e)$and $k \geq 1$ for $t \geq s \in [0,1]$; one have

$$
\begin{aligned}
|\mathbb{T}_i w_i(t) - \mathbb{T}_i v_i(t)| &= \Big|\zeta_i + \sum_{i=1}^{n} \mathbb{G}_i(t, w_i(t)) + \frac{1-\varrho_i}{B(\varrho_i)}\lambda_i^*(t, w_i(t)) \\
&+ \frac{\varrho_i}{B(\varrho_i)\Gamma(\varrho_i)}\int_0^t (t-s)^{\varrho_i - 1}\lambda_i^*(s, w_i(s))ds - \frac{1-\varrho_i}{B(\varrho_i)}\lambda_i^*(0, w_i(0))\Big(1 + \frac{\gamma_{\varrho_i}}{\Gamma(\varrho_i + 1)}t^{\varrho_i}\Big) \\
&- \Big[\zeta_i + \sum_{i=1}^{n} \mathbb{G}_i(t, v_i(t)) + \frac{1-\varrho_i}{B(\varrho_i)}\lambda_i^*(t, v_i(t)) \\
&+ \frac{\varrho_i}{B(\varrho_i)\Gamma(\varrho_i)}\int_0^t (t-s)^{\varrho_i - 1}\lambda_i^*(s, v_i(s))ds - \frac{1-\varrho_i}{B(\varrho_i)}\lambda_i^*(0, v_i(0))\Big(1 + \frac{\gamma_{\varrho_i}}{\Gamma(\varrho_i + 1)}t^{\varrho_i}\Big)\Big]\Big| \\
&\leq \sum_{i=1}^{n} \zeta_i^2|w_i - v_i| + \frac{1}{B(\varrho_i)\Gamma(\varrho_i)}\zeta_i^1|w_i - v_i| + \frac{1-\varrho_i}{B(\varrho_i)}\frac{1+\gamma_{\varrho_i}}{\Gamma(\varrho_i + 1)}\zeta_i^1|w_i - v_i| \\
&\leq (\sum_{i=1}^{n} \zeta_i^2 + \frac{1}{B(\varrho_i)\Gamma(\varrho_i)}\zeta_i^1 + \frac{1-\varrho_i}{B(\varrho_i)}\frac{1+\gamma_{\varrho_i}}{\Gamma(\varrho_i + 1)}\zeta_i^1)|w_i - v_i|.
\end{aligned}
\tag{2.12}
$$

For $\eta_i < 1$, where η_i's are given by (2.8). This implies that the operators $\mathbb{T}_i$ are contractions. By the help of Banach's fixed point theorem the hybrid system of mABC-FDEs (1.1) has a unique solution which are the fixed points of the operators $\mathbb{T}_i$, where $i = 1, 2, 3, \ldots, n$. □

Theorem 2.1. *Assume that the conditions of the Lemma 2.2 are satisfied, then, there is a solution of the hybrid m-coupled-system mABC-FDEs given by* (1.1).

Proof. By the assumptions of the conditions in Lemma 2.2, we have that $\mathbb{T}_i$ are bounded for

i=1, 2, . . . , n, and, for t_1, $t_2 \in [0, k]$ with $t_2 > t_1$, and $k \leq 1$, consider

$$
\begin{aligned}
|\mathbb{T}_i w(t_2) - \mathbb{T}_i w(t_1)| = \Big| \zeta_i + \sum_{i=1}^{n} \mathbb{G}_i(t_2, w(t_2)) + \frac{1-\varrho_i}{B(\varrho_i)} \lambda_i^*(t_2, w(t_2)) \\
+ \frac{\varrho_i}{B(\varrho_i)\Gamma(\varrho_i)} \int_0^{t_2} (t_2 - s)^{\varrho_i - 1} \lambda_i^*(s, w_i(s)) ds \\
- \frac{1-\varrho_i}{B(\varrho_i)} \lambda_i^*(0, w_i(0)) \Big(1 + \frac{\gamma_{\varrho_i}}{\Gamma(\varrho_i + 1)} t_2^{\varrho_i}\Big) \\
- \Big[\zeta_i + \sum_{i=1}^{n} \mathbb{G}_i(t_1, w(t_1)) + \frac{1-\varrho_i}{B(\varrho_i)} \lambda_i^*(t_1, w(t_1)) \\
+ \frac{\varrho_i}{B(\varrho_i)\Gamma(\varrho_i)} \int_0^{t_1} (t_1 - s)^{\varrho_i - 1} \lambda_i^*(s, v_i(s)) ds - \frac{1-\varrho_i}{B(\varrho_i)} \lambda_i^*(0, w_i(0)) \Big(1 + \frac{\gamma_{\varrho_i}}{\Gamma(\varrho_i + 1)} t_1^{\varrho_i}\Big) \Big] \Big| \\
\leq \sum_{i=1}^{n} \Big| \mathbb{G}_i(t_2, w_i(t_2)) - \mathbb{G}_i(t_1, w_i(t_1)) \Big| + \frac{1-\varrho_i}{B(\varrho_i)} \Big| \lambda_i^*(t_2, w(t_2)) - \lambda_i^*(t_1, w(t_1)) \Big| \\
+ \frac{1}{B(\varrho_i)\Gamma(\varrho_i)} \Big| t_2^{\varrho_i} - t_2^{\varrho_i} \Big| (\xi_i^1 \eta_i + \wp_1) - \frac{1-\varrho_i}{B(\varrho_i)} \lambda_i^*(0, w_i(0)) \frac{\gamma_{\varrho_i}}{\Gamma(\varrho_i + 1)} \Big| t_2^{\varrho_i} - t_2^{\varrho_i} \Big|.
\end{aligned}
\tag{2.13}
$$

This implies that as $t_2 \to t_1$, we have $\mathbb{T}_i w(t_2) \to \mathbb{T}_i w(t_1)$. This implies that $|\mathbb{T}_i w_i(t_2) - \mathbb{T}_i w_i(t_1)| \to 0$ as $t_2 \to t_1$. Hence, $\mathbb{T}_i$ are equicontinuous for $i = 1, 2, \ldots, n$ and $s \leq t$. Furthermore, for $u \in \{u \in \mathbb{C}([0, k], \mathbb{R}_e) : u = \hbar \mathbb{T}_i(u),\ for \hbar \in [0, 1]\}$, we have

$$
\begin{aligned}
\|w_i\| &= \max_{t \in I} |\mathbb{T}_i w_i(t)| = \Big| \zeta_i + \sum_{i=1}^{n} \mathbb{G}_i(t, w_i(t)) + \frac{1-\varrho_i}{B(\varrho_i)} \lambda_i^*(t, w_i(t)) \\
&+ \frac{\varrho_i}{B(\varrho_i)\Gamma(\varrho_i)} \int_0^{t} (t - s)^{\varrho_i - 1} \lambda_i^*(s, w_i(s)) ds - \frac{1-\varrho_i}{B(\varrho_i)} \lambda_i^*(0, w_i(0)) \Big(1 + \frac{\gamma_{\varrho_i}}{\Gamma(\varrho_i + 1)} t^{\varrho_i}\Big) \Big| \\
&\leq \zeta_i + \sum_{i=1}^{n} (\xi_i^2 \|w_i\| + \wp_2) + \frac{1-\varrho_i}{B(\varrho_i)} (\xi_i^1 \|w_i\| + \wp_1) \\
&+ \frac{(\xi_i^1 \|w_i\| + \wp)}{B(\varrho_i)\Gamma(\varrho_i)} + \frac{1-\varrho_i}{B(\varrho_i)} \lambda_i^*(0, w_i(0)) \Big(1 + \frac{\gamma_{\varrho_i}}{\Gamma(\varrho_i + 1)}\Big) \\
&= \lambda_i^1 + \lambda_i^2 \|w_i\|.
\end{aligned}
\tag{2.14}
$$

We have that

$$
\lambda_i^1 = \zeta_i + \wp_2 + \frac{1-\varrho_i}{B(\varrho_i)} \wp_1 + \frac{\wp}{B(\varrho_i)\Gamma(\varrho_i)} + \frac{1-\varrho_i}{B(\varrho_i)} \lambda_i^*(0, w_i(0)) \Big(1 + \frac{\gamma_{\varrho_i}}{\Gamma(\varrho_i + 1)}\Big), \tag{2.15}
$$

and

$$
\lambda_i^{*2} = \sum_{i=1}^{n} \xi_i^{\;2} + \frac{1-\varrho_i}{B(\varrho_i)} \xi_i^1 + \frac{\xi_i^1}{B(\varrho_i)\Gamma(\varrho_i)}. \tag{2.16}
$$

For $i = 1, 2, \ldots, n$, with the help of (2.14)–(2.16), we have

$$
\|w_i\| \leq \frac{\lambda_i^{*1}}{1 - \lambda_i^{*2}}, \tag{2.17}
$$

for $i=1,2,\ldots,n$. Thus, Leray-Schauder's alternative theorem is satisfied; hence, (1.1) has a solution. □

3. Hyers-Ulam stability

This section is reserved for the HU-stability of the n-coupled-system (2.5). For this, consider the definition:

Definition 3.1. *The coupled integral system* (2.5) *is HU-stable, if for some* $\zeta_i>0$*, we have* $\Delta_i>0$*, with* w_i *satisfying*

$$\|w_i-\mathbb{T}_iw_i\|_1<\Delta_i, \tag{3.1}$$

with $\bar{w}_i(t)$ *of the coupled-system* (2.5) *with*

$$\bar{w}_i(t)=\mathbb{T}_i\bar{w}_i(t), \tag{3.2}$$

and

$$\|w_i-\bar{w}_i\|<\Delta_i\zeta_i, \tag{3.3}$$

where $i=1,2,\ldots,n$*.*

Theorem 3.1. *Assume the conditions of Lemma 2.2, the* (2.5) *is HU stable, equivalently; the n-coupled hybrid-system of mABC-FDEs given by* (1.1) *is stable.*

Proof. Assume that $w_i\in\mathbb{C}$ for $i=1,2,\ldots.$ with the property (3.1) and let $w_i^*\in\mathbb{C}$ for the coupled-system (1.1) satisfying w (2.5), implies that

$$\begin{aligned}
|\mathbb{T}_iw_i(t)-\mathbb{T}_iw_i^*(t)| &= \Big|\zeta_i+\sum_{i=1}^{n}\mathbb{G}_i(t,w_i(t))+\frac{1-\varrho_i}{B(\varrho_i)}\lambda_i^*(t,w_i(t))\\
&+\frac{\varrho_i}{B(\varrho_i)\Gamma(\varrho_i)}\int_0^t(t-s)^{\varrho_i-1}\lambda_i^*(s,w_i(s))ds-\frac{1-\varrho_i}{B(\varrho_i)}\lambda_i^*(0,w_i(0))\Big(1+\frac{\gamma_{\varrho_i}}{\Gamma(\varrho_i+1)}t^{\varrho_i}\Big)\\
&-\Big[\zeta_i+\sum_{i=1}^{n}\mathbb{G}_i(t,w_i^*(t))+\frac{1-\varrho_i}{B(\varrho_i)}\lambda_i^*(t,w_i^*(t))\\
&+\frac{\varrho_i}{B(\varrho_i)\Gamma(\varrho_i)}\int_0^t(t-s)^{\varrho_i-1}\lambda_i^*(s,w_i^*(s))ds\\
&-\frac{1-\varrho_i}{B(\varrho_i)}\lambda_i^*(0,w_i^*(0))\Big(1+\frac{\gamma_{\varrho_i}}{\Gamma(\varrho_i+1)}t^{\varrho_i}\Big)\Big]\Big|\\
&\leq\sum_{i=1}^{n}\zeta_i^2|w_i-w_i^*|+\frac{1}{B(\varrho_i)\Gamma(\varrho_i)}\zeta_i^1|w_i-w_i^*|+\frac{1-\varrho_i}{B(\varrho_i)}\frac{1+\gamma_{\varrho_i}}{\Gamma(\varrho_i+1)}\zeta_i^1|w_i-w_i^*|\\
&\leq(\sum_{i=1}^{n}\zeta_i^2+\frac{1}{B(\varrho_i)\Gamma(\varrho_i)}\zeta_i^1+\frac{1-\varrho_i}{B(\varrho_i)}\frac{1+\gamma_{\varrho_i}}{\Gamma(\varrho_i+1)}\zeta_i^1)|w_i-w_i^*|.
\end{aligned} \tag{3.4}$$

For $\eta_i<1$, where η_i's are given by (2.8), for $i=1,2,\ldots,n$. By the (3.1), (3.2) and (3.4), consider the following norm

$$\|w_i-\bar{w}_i^*\| \;=\; \|w_i-\mathbb{T}_iw_i+\mathbb{T}_iw_i-\bar{w}_i^*\|$$

$$\leq \|w_i - \mathbb{T}_i w_i\| + \|\mathbb{T}_i w_i - \mathbb{T}_i \bar{w}_i^*\| \tag{3.5}$$
$$\leq \Delta_i + \eta_i \|w_i - \bar{w}_i^*\|,$$

where $i = 1, 2, \ldots, m$. Furthermore,

$$\|w_i - \bar{w}_i^*\| \leq \frac{\Delta_i}{1-\eta_i}, \tag{3.6}$$

with $\zeta_i = \frac{1}{1-\eta_i}$. Hence, the coupled system (2.5) is stable. This further implies the stability of the coupled Hybrid mABC-FDEs system (1.1). □

4. Numerical scheme

Consider the equation given in (2.5), with w_i the fixed points for $i = 1, 2, \ldots, n$. We have

$$\begin{aligned} w_i(t) &= \zeta_i + \sum_{i=1}^{n} \mathbb{G}_i(t, w_i(t)) + \frac{1-\varrho_i}{B(\varrho_i)} \lambda_i^*(t, w_i(t)) \\ &+ \frac{\varrho_i}{B(\varrho_i)\Gamma(\varrho_i)} \int_0^t (t-s)^{\varrho_i - 1} \lambda_i^*(s, w_i(s)) ds \\ &- \frac{1-\varrho_i}{B(\varrho_i)} \lambda_i^*(0, w_i(0)) \Big(1 + \frac{\gamma_{\varrho_i}}{\Gamma(\varrho_i + 1)} t^{\varrho_i}\Big). \end{aligned} \tag{4.1}$$

We are producing a numerical scheme for this system with the help of Lagrange's interpolation polynomials.

Replacing t by t_{n+1}, we have

$$\begin{aligned} w_i(t_{n+1}) &= \zeta_i + \sum_{i=1}^{n} \mathbb{G}_i(t_n, w_i(t_n)) + \frac{1-\varrho_i}{B(\varrho_i)} \lambda_i^*(t_n, w_i(t_n)) + \frac{\varrho_i}{B(\varrho_i)\Gamma(\varrho_i)} \int_0^{t_{n+1}} (t_{n+1}-s)^{\varrho_i - 1} \lambda_i^*(s, w_i(s)) ds \\ &- \frac{1-\varrho_i}{B(\varrho_i)} \lambda_i^*(0, w_i(0)) \Big(1 + \frac{\gamma_{\varrho_i}}{\Gamma(\varrho_i + 1)} t_n^{\varrho_i}\Big). \end{aligned} \tag{4.2}$$

By the Lagrange's interpolation, we have

$$\begin{aligned} \lambda_i^*(t, w_i(t)) &= \frac{\lambda_i^*(t_k, w_i(t_k)(t-t_{k-1})}{t_k - t_{k-1}} - \frac{\lambda_i^*(t_{k-1}, w_i(t_{k-1}))(t-t_k)}{t_k - t_{k-1}} \\ &= \frac{\lambda_i^*(t_k, w_i(t_k))(t-t_{k-1})}{h} - \frac{\lambda_i^*(t_{k-1}, w_i(t_{k-1})(t-t_k)}{h}. \end{aligned} \tag{4.3}$$

By the help of (4.2) and (4.3), we have

$$\begin{aligned} w_i(t_{k+1}) &= \zeta_i + \sum_{i=1}^{n} \mathbb{G}_i(t_k, w_i(t_k)) + \frac{1-\varrho_i}{B(\varrho_i)} \lambda_i^*(t_k, w_i(t_k)) \\ &+ \frac{\varrho_i}{B(\varrho_i)\Gamma(\varrho_i)} \Sigma_{i=1}^{n} \Big[\frac{\lambda_i^*(t_i, u(t_i))}{h} \int_{t_k}^{t_{k+1}} (\zeta - t_{i-1})(t_{n+1} - \zeta)^{\varrho_1 - 1} d\zeta \\ &- \frac{\lambda_i^*(t_{i-1}, w_i(t_{i-1}))}{h} \int_{t_k}^{t_{n+1}} (\zeta - t_i)(t_{n+1} - \zeta)^{\varrho_1 - 1} d\zeta \Big] \end{aligned} \tag{4.4}$$

$$- \frac{1-\varrho_i}{B(\varrho_i)}\lambda_i^*(0, w_i(0))\Big(1 + \frac{\gamma_{\varrho_i}}{\Gamma(\varrho_i+1)}t_k^{\varrho_i}\Big).$$

Solving the integrals, we get

$$\begin{aligned} w_{k+1} &= \zeta_i + \sum_{j=1}^{k} \mathbb{G}_i(t_k, w_i(t_k)) + \frac{1-\varrho_i}{B(\varrho_i)}\lambda_i^*(t_k, w_i(t_k)) \\ &+ \frac{\varrho_1 h^{\varrho_1}}{\Gamma(\varrho_1+2)}\Sigma_{j=1}^{k}\Big[\lambda_i^*(t_i, w_i(t_i))\Big((k-j+1)^{\varrho_1}(k+2-j+\varrho_1) \\ &- (k-i)^{\varrho_1}(k+2-j+2\varrho_1)\Big) - \lambda_i^*(t_{i-1}, w_{i-1})\Big((k-i+1)^{\varrho_1+1} \\ &- (k-j+1+\varrho_1)(k-i)^{\varrho_1}\Big)\Big] - \frac{1-\varrho_i}{B(\varrho_i)}\lambda_i^*(0, w_i(0))\Big(1 + \frac{\gamma_{\varrho_i}}{\Gamma(\varrho_i+1)}(kh)^{\varrho_i}\Big). \end{aligned} \tag{4.5}$$

One can see some more useful numerical schemes in previous works [36–38].

4.1. Numerical results

Assume that $\lambda_1^* = \mu - \lambda^*\mathbb{S}(t) - \mu\mathbb{S}(t)$, $\lambda_2^* = (1-\psi)\lambda^*\mathbb{S}(t) + \psi\lambda^*(1 - I - \mathbb{P} - \mathbb{T}) - (\mu + \sigma + \epsilon + d)\mathbb{I}$, $\lambda_3^* = \epsilon\mathbb{I} + \rho\mathbb{T} - (\mu + \delta + \wp)\mathbb{P}$, and $\lambda_4^* = \wp\mathbb{P} - (\mu + \rho + \theta)\mathbb{T}$, with $\mathbb{G}_i = 0$, for $i = 1, \ldots, m$, the sense of derivative as the modified fractional differential operator, we get the following hepatitis C model. For a detail, one can see the work in [16].

$$\begin{cases} {}_0^{mABC}\mathbb{D}_t^{\nu}\mathbb{S}(t) = \mu - \mu\mathbb{S}(t) - \lambda^*\mathbb{S}(t), \\ {}_0^{mABC}\mathbb{D}_t^{\nu}\mathbb{I}(t) = (1-\psi)\lambda^*\mathbb{S}(t) - (\mu + \sigma + \epsilon + d)\mathbb{I} + \psi\lambda^*(1 - I - \mathbb{P} - \mathbb{T}), \\ {}_0^{mABC}\mathbb{D}_t^{\nu}\mathbb{P}(t) = \epsilon\mathbb{I} - (\mu + \delta + \wp)\mathbb{P} + \rho\mathbb{T}, \\ {}_0^{mABC}\mathbb{D}_t^{\nu}\mathbb{T}(t) = \wp\mathbb{P} - (\mu + \theta + \rho)\mathbb{T}. \end{cases} \tag{4.6}$$

In this mABC system, reinfection is taken into account. Eight categories are used to categorise the population as a whole: sensitive to infection $\mathbb{S}(t)$, acutely infected $\mathbb{I}$, consistently (severely) infected $\mathbb{P}$, eliminated $\mathbb{R}$, acute reinfection $\mathbb{V}$, chronic reinfection $\mathbb{W}$, medication for severe infection $\mathbb{T}$, and medication for severe reinfection $\mathbb{Q}$. The parameters mean that: μ is for the natural deaths, σ is the rate of recovery from acute infection, δ is the rate of recovery from the severe infection, d is the death rate by the acute infection and ϵ is the progression rate severe illness.

This model is numerically examined for different fractional orders to see the importance of the m-ABC-differential operator and also to provide an illustration of the numerical scheme (4.5).

In Figure 1, we have given a joint numerical solution of the model (4.6) for the classical order. In order to illustrate the role of fractional orders, we further give three more graphs of joint solutions. In Figure 2, presents the numerical solution of the model (4.6) for the fractional order 0.985 and Figure 3, Figure 4 are numerical solutions of the model (4.6) for the orders 0.965 and 0.945. We can observe that the fractional orders play vital role in the solution of the problems and for each fractional order there is a unique solution.

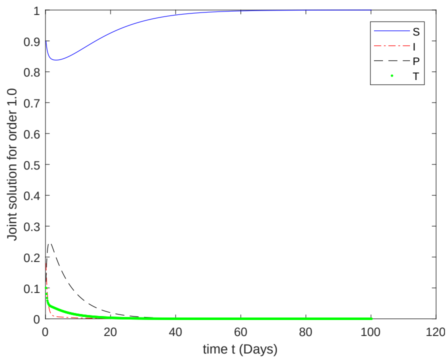

Figure 1. Numerical solution of (4.6) for the classical order 1.0.

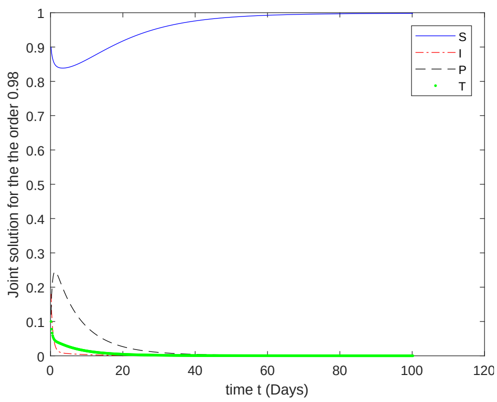

Figure 2. Numerical solution of (4.6) for the fractional order 0.985.

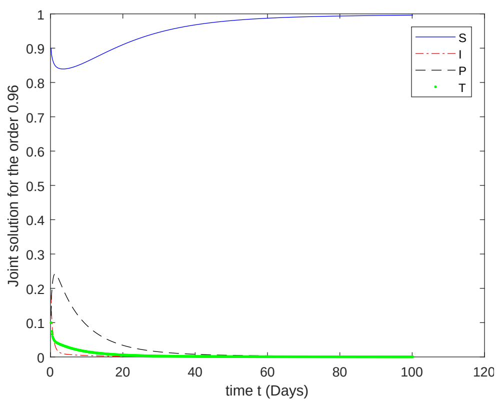

Figure 3. Numerical solution of (4.6) for the fractional order 0.965.

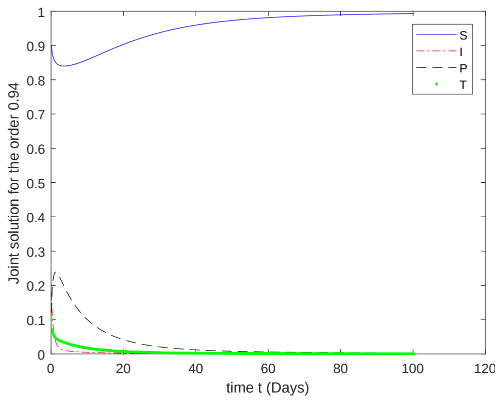

Figure 4. Joint solution of the model (4.6) at fractional order 0.945.

In Figure 5, the dynamics for $S(t)$ are expressed for different fractional orders ν=1.0, 0.985, 0.965, 0.945. There is a gradual decrease in the population of the class with respect to the decrease in the order of the derivative. While, in Figure 6, the dynamics for the $I(t)$ are expressed for different fractional orders $\nu = 1.0$, 0.985, 0.965, 0.945. There is a gradual increase in the population of the class with respect to the decrease in the order of the derivative. Similarly, in Figure 7, the dynamics for $P(t)$ are presented for the fractional orders $\nu = 1.0$, 0.985, 0.965, 0.945, where we can observe a gradual increase in the population of the class with respect to the decrease in the orders of the derivative.

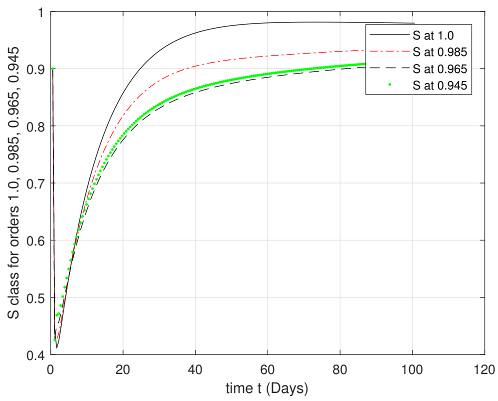

Figure 5. S(t) for different fractional orders in comparison with the fractional orders.

In Figure 5, the simulation shows the dynamics of the $S(t)$ for different orders of the mABC operator. All of the results behaved similarly with a slight difference. This shows the beauty of the mABC operator which possess both the integer order results as well as novel solutions for the fractional orders.

The dynamics of infection are simulated in Figure 6 which shows that that the rate of infection decreased in a few days. This behaviour can also be observed for the fractional order system which is affirms the applicability of the mABC operator and the numerical scheme based on the interpolation polynomial.

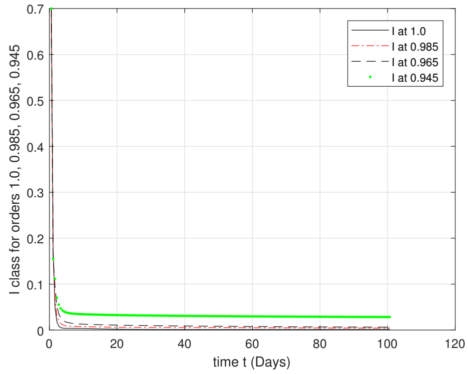

Figure 6. I(t) for different fractional orders in comparison with the fractional orders.

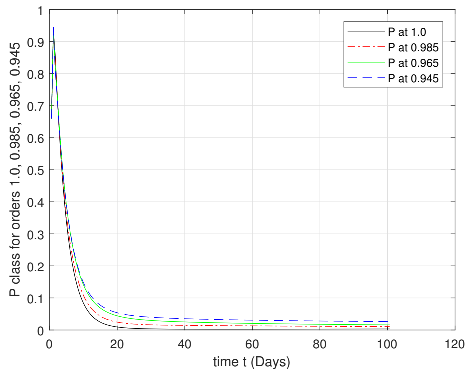

Figure 7. P(t) for different fractional orders in comparison with the fractional orders.

Finally, in the Figure 8, the dynamics for the $T(t)$ is expressed for 120 days for different fractional orders $\nu = 1.0, 0.985, 0.965, 0.945$. There is a gradual increase in the population of the class with respect to the decrease in the order of the derivative. We have compared our results with the classical order graphically. The fractional derivative allows us for more solutions and information regarding the suggested model.

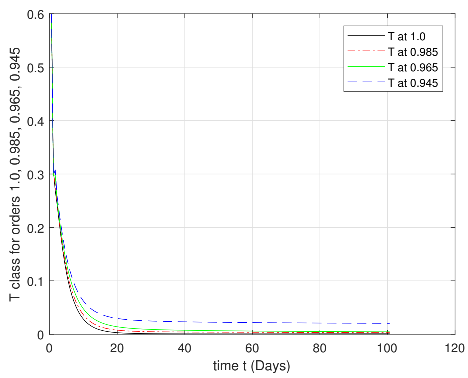

Figure 8. T(t) for different fractional orders in comparison with the fractional orders.

5. Conclusions

In this paper, the authors discussed a general coupled system of Hybrid mABC-FDEs for EUS and HU stability. In our proposed problem, we considered the newly established modified ABC operator. This new operator has more benefits than the preexisting ABC operator including the initialization and the well-posedness of the new operator. We believe that this operator has opened a new gateway to the scholars for the research work. For the EUS, we obtained help from the literature and applied the fixed point theorems. The HU stability was illustrated on the basis of preexisting literature. A new numerical algorithm is obtained with the use of Lagrange's interpolation polynomial and was applied to a hepatitis C mathematical model. We observed that the results are more realistic and that the scheme can be further utilized for the study of dynamical problems.

We have provided a combined numerical solution of the model (4.6) for the classical order in Figure 1. We have also provided three additional graphs of joint solutions to further highlight the function of fractional orders. The numerical solution of the model (4.6) for the fractional order 0.985 is shown in Figure 2 and the numerical solutions for the orders 0.965 and 0.945 are shown in Figures 3 and 4. We can see that fractional orders are crucial to obtaining solutions, and that each fractional order has a different answer.

The presumed problem (1.1) is a very much complex and a system of n-coupled mABC-FDEs. For the existence of solutions, Leray Shauder's technique was adopted and HU-stability was analyzed. The readers may reconsider the presumed problem with help of other fixed point approaches for the existence of unique solutions and multi solutions. They may also develop the new numerical schemes via other techniques.

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Conflict of interest

The authors declare no conflict of interest.

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