mathematics

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Int N-Soft Substructures of Semigroups

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Abstract: The N-soft sets are newly defined structures with many applications in the real world. We aim for combining the semigroup theory and N-soft sets to provide a comprehensive account of the hybrid framework of N-soft Semigroups. In this paper, we define the γ-inclusive set, int N-soft subsemigroups, int N-soft left [right] ideals of S, int N-soft product and int N-soft characteristic function, θ-Generalized int N-soft subsemigroups and θ-Generalized int N-soft left [right] ideals of S. We also discuss some examples and theorems based on the restricted (extended) union, restricted (extended) intersection, and γ-inclusive set.

Keywords: soft set; N-soft set; γ-inclusive set; int soft subsemigroups; int N-soft subsemigroups; int N-soft left ideals; int N-soft product

MSC: 06F05; 03B52; 08A72; 03G10

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Citation: Shabir, M.; Mushtaq, R.; Jawad, M.; Naz, M.; Jarad, F.; Abdeljawad, T. Int N-Soft Substructures of Semigroups. *Mathematics* **2023**, *11*, 267. https://doi.org/10.3390/math11020267

Academic Editor: Gia Sirbiladze

Received: 22 November 2022
Revised: 29 December 2022
Accepted: 30 December 2022
Published: 4 January 2023

1. Introduction

In 1999, Molodtsov [1] coined the concept of soft sets to handle uncertain types of data using parametrization. The parameters can be anything, e.g., indices, the combination of letters, whole sentences, some functions, matrices, and so on. Almost all of the operations on the classical sets were tried to be adapted into the framework of soft sets. Maji et al. [2] presented the operations on soft sets for the first time after [1] and Ali et al. [3] not only refined these operations but also introduced several different operations and extended their properties on soft sets. Another work on the operations of soft sets was presented by Sezgin et al. in [4]. Hence, the theoretical structure of soft sets appeared richer than the crisp set theory.

The concepts of soft sets hybridized with the other theories of uncertainty gave rise to many new structures as in the studies on fuzzy soft sets with applications in decision-making by Maji et al. [5]. Not only the theories of uncertainty but also the algebraic theories integrated with the soft sets had been defined and investigated extensively and several new structures had been worked out so far. A few works included where Ali et al. [6] explored the algebraic structures of soft sets associated with the newly defined operations, and Naz and Shabir studied the algebra of fuzzy soft sets in [7]. Semigroups played a fundamental role in the formation of algebraic structures and a crucial work on soft ideals over semigroups combining both, the theories of semigroups and soft sets with a rigorous approach presented by Shabir and Ali and Ali et al. appeared in [8,9], respectively. Another work on soft binary relations defined and applied them to the semigroups by Feng et al. [10]. In 2010, Çağman and Enginoğlu [11] interpreted soft set theory with the idea of uni–int decision-making, and following those concepts, Jun et al. [12] defined intersectional soft BCK/BCI-ideals. In 2012, Çağman et al. [13] outlined the soft int-group and its applications to group theory. Their work led to the establishment of the ideal theory in semigroups based on intersectional soft sets [14] and introduced the notions of soft intersection semigroups, ideals, and bi-ideals [15]. The theory of soft int-rings and its algebraic applications was given by Cıtak and Çağman, introduced in [16].

Mostly the researchers in the soft set model used binary evaluation either 0 or 1, whereas non-binary evaluation was often used in daily life. These applications focused mainly on the characterization of the objects in a comparable form on which one could make consistent and correct decisions. For the inclusion of ranking purposes, the extended model of the soft set used, called the N-soft set, was introduced by Fatimah et al. in [17]. They argued that for N = 2, the N-soft set was restricted back to the soft set so that the N-soft set was simply an extension of Molodtsov's soft set. They also proposed some new definitions of the operations for these N-soft sets. Additionally, N-soft set conjecture was strengthened and implemented to solve a variety of multi-attribute decision-making challenges as described in [18], including fuzzy N-soft sets [19], N-soft topology [20], intuitionistic fuzzy N-soft rough sets [21], bipolar N-Soft sets [22] and many more.

In 2019, Riaz et al. [20] described further algebraic operations on N-soft sets. With the introduction of these new operations in soft sets, it was vital to study the underlying algebraic structures. Later, Kamachi developed more operations on N-soft sets and highlighted the concepts of N-soft groups, N-soft rings, N-soft fields and N-soft lattices with the derivation of their properties in [23]. A detailed account of the operations of N-soft sets and their lattice-theoretic structure was presented in [24]. We extend these studies to combine the concepts under the approximation operator of intersection and N-soft semigroups. This will give a forehand and a better understanding of their structure for various applications. Motivated by the previous works, a research gap in the hybrid framework of N-soft sets and semigroup theory is identified, leading to the present work. The wider objective of this research may be interpreted as: (1) identification of the new algebraic structures within the framework of N-soft sets; (2) properties of algebraic operations on N-soft sets; (3) modelling of the existing semigroups in view of N-soft sets by providing constructive examples. The authors believe that the generalization of semigroups obtained in this way is significantly broadening the conceptual work, and therefore the is significantly better.

In this paper, we study generalized int N-soft substructures of semigroups. Some basic definitions and notions such as semigroup, N-soft sets, and int-soft subsemigroup are given in Section 2. These definitions will help us to discuss our work. In Section 3 we discuss int N-soft semigroups, int N-soft left [right] ideals of S, int N-soft product and int N-soft characteristic function, θ-Generalized int N-soft subsemigroups and θ-Generalized int N-soft left [right] ideals of S. The restricted and extended operations give rise to new results. At the end, we briefly describe the examples related to the discussion topics.

2. Preliminaries

In this section, we will discuss some basic notions and results of semigroups, N-soft sets, and int-soft subsemigroup, which will be of value for later sections.

2.1. Semigroups

In this subsection, we will give the definitions of substructures of semigroup and some related terms. Our reader is suggested to [9,25] for a detailed account of related topics.

A non-empty set S together with an associative binary operation is called a *semigroup*. A semigroup is called *commutative* if the binary operation defined on it is commutative. An element $e \in S$ is called an *identity element* of S if $es = se = s$ for all s in S.

A non-empty subset T of a semigroup $(S, \cdot)$ is called a *subsemigroup* of S if T is closed under " $\cdot$ ". T is called a *left (right) ideal* of S if $ST \subseteq T$ ($TS \subseteq T$) that is $st \in T$ ($ts \in T$) for all $t \in T$ and $s \in S$. If T is not only a left ideal but also a right ideal, then it is called an *ideal or two-sided ideal* of S.

2.2. Soft Sets

Let U be an initial universe, E be the set of parameters, and $P(U)$ denote the power set of U and A, B be non-empty subsets of E. We give a brief account of soft sets here and refer to [1,3] for details.

A pair $S = (F, A)$ is called a *soft set* over U, where F is a mapping given by F: $A \to P(U)$. For $e \in A$, $F(e)$ may be considered as the set of e-approximate elements of the soft set (F, A).

For two soft sets (F, A) and (G, B) over the same universe U, we say that (F, A) is a *soft subset* of (G, B) if:

(1) $A \subseteq B$ and
(2) $F(e) \subseteq G(e)$ for all $e \in A$.

We denote it as $(F, A) \tilde{\subset} (G, B)$. Two soft sets (F, A) and (G, B) over the same universe U are said to be *equal* if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A). We denote it as $(F, A) \doteq (G, B)$.

A soft set (ϕ, A) over U is called a *relative null soft set* with respect to the parameter set A, denoted by ϕ_A, if $\phi(e) = \phi$ for all $e \in A$. In the same way, the relative null soft set with respect to E is called the *null soft set over* U and is denoted by (ϕ, E). A soft set $(\sqcup, A)$ over U is called a *relative whole soft set with respect to the parameter set* A, denoted by $\sqcup_A$, if $\sqcup(e) = U$ for all $e \in A$. In the same way, the relative whole soft set with respect to E is called the absolute soft set over U and is denoted by $(\sqcup, E)$. The soft set (ϕ, ϕ) is the unique soft set over U with an empty parameter set, denoted by ϕ_ϕ.

2.3. N-Soft Set

We give here some basic definitions and results for N-soft sets.

Definition 1 ([17]). *Let $R_N = \{0, 1, 2, ..., N-1\}$ be a set of ordered grades where $N \in \{2, 3, ...\}$. Then (F, A, N) is an N-soft set if $F : A \to 2^{U \times R_N}$, with the property that for each $e \in A$, there exists a unique $(u, r_e) \in U \times R_N$ such that $(u, r_e) \in F(e)$ where $u \in U$, $r_e \in R_N$. We will write $F(e)(u) = r_e$ as a short for $(u, r_e) \in F(e)$.*

Definition 2 ([17]). *Two N-soft sets (F, A, N) and (G, B, N') over the same universe U are said to be equal if and only if $F = G$, $A = B$ and $N = N'$. We denote it as $(F, A, N) \doteq (G, B, N')$.*

Definition 3 ([17]). *The extended union of two N-soft sets (F, A, N_1) and (G, B, N_2) over the fixed universe U is denoted by $(L, A \cup B, max(N_1, N_2)) = (F, A, N_1) \cup_\xi (G, B, N_2)$, and is defined by*

$$L(e)(u) = \begin{cases} F(e)(u) & \text{if } e \in A - B \\ G(e)(u) & \text{if } e \in B - A \\ r_e & \text{if } e \in A \cap B \end{cases} \tag{1}$$

where $r_e = max(r_e^1, r_e^2)$, $r_e^1 = F(e)(u)$ and $r_e^2 = G(e)(u)$ for all $u \in U$.

Definition 4 ([17]). *The extended intersection of two N-soft sets (F, A, N_1) and (G, B, N_2) over the fixed universe U is denoted by $(J, A \cup B, max(N_1, N_2)) = (F, A, N_1) \cap_\xi (G, B, N_2)$, and is defined by*

$$J(e)(u) = \begin{cases} F(e)(u) & \text{if } e \in A - B \\ G(e)(u) & \text{if } e \in B - A \\ r_e & \text{if } e \in A \cap B \end{cases} \tag{2}$$

where $r_e = min(r_e^1, r_e^2)$, $r_e^1 = F(e)(u)$ and $r_e^2 = G(e)(u)$ for all $u \in U$.

Definition 5 ([17]). *The restricted union of two N-soft sets (F, A, N_1) and (G, B, N_2) over the fixed universe U is denoted by $(K, A \cap B, max(N_1, N_2)) = (F, A, N_1) \cup_\Re (G, B, N_2)$ and is de-*

fined as, for all $e \in A \cap B$ and $u \in U$, $r_e = K(e)(u)$, where $r_e = max(r_e^1, r_e^2)$, $r_e^1 = F(e)(u)$ and $r_e^2 = G(e)(u)$. If $A \cap B = \phi$, then $(F, A, N_1) \cup_{\Re} (G, B, N_2) = (\phi, \phi, N)$.

Definition 6 ([17]). *The restricted intersection of two N-soft sets (F, A, N_1) and (G, B, N_2) over the fixed universe U is denoted by $(H, A \cap B, min(N_1, N_2)) = (F, A, N_1) \cap_{\Re} (G, B, N_2)$ and is defined as, for all $e \in A \cap B$ and $u \in U$, $r_e = H(e)(u)$, where $r_e = min(r_e^1, r_e^2)$, $r_e^1 = F(e)(u)$ and $r_e^2 = G(e)(u)$. If $A \cap B = \phi$, then $(F, A, N_1) \cap_{\Re} (G, B, N_2) = (\phi, \phi, N)$.*

2.4. Int Soft Subsemigroup

Let S be a semigroup and U be a non-empty set.

Definition 7 ([26]). *A soft set (F, S) over U is called an int-soft subsemigroup over U if it satisfies, for all x, $y \in S$*

$$F(xy) \supseteq F(x) \cap F(y).$$

Definition 8 ([26]). *If a soft set (F, S) over U satisfies the following assertion that is for all x, $y \in S$ there exists $\theta \in P(U)$*

$$F(xy) \supseteq \theta \cap F(x) \cap F(y).$$

then we say that (F, S) is a θ-generalized int-soft subsemigroup over U.

Definition 9 ([26]). *A soft set (F, S) over U is called an int-soft left (right) ideal over U if it satisfies, for all x, $y \in S$*

$$F(xy) \supseteq F(y) \quad (F(xy) \supseteq F(x)).$$

If a soft set (F, S) over U is both an int-soft left ideal and an int-soft right ideal over U, we say that (F, S) is an int-soft two-sided ideal over U.

3. Int N-Soft Substructures of Semigroups

In this section, we will introduce the basic notions and results of int N-soft subsemigroup and int N-soft left [right] ideals of S. The parameter set of an N-soft set, which we will use in this section is a semigroup, whereas the universe set is any set. We discuss the notions and properties of int N-soft product and int N-soft characteristic function. We also discuss the θ-generalized int N-soft subsemigroup and investigate several properties.

3.1. Int N-Soft Subsemigroup

Let U be an initial universe and S be a semigroup considered as a set of parameters. Let (F, S, N) be an N-soft set.

Definition 10. *Let A and B be any subsets of the semigroup S. Then, the multiplication of A and B is defined by*

$$AB = \{ab \in S \mid a \in A \text{ and } b \in B\}.$$

Definition 11. *For any subset γ of $U \times R_N$ such that for each $u \in U$ there exists a unique number $n \leq$ N-1, for which (u,n) $\in \gamma$ we write $\gamma(u) = n$, the γ-inclusive set of an N-soft set (F, S, N) over U is denoted and defined by*

$$F_\gamma = \{x \in S : \gamma(u) \leq F(x)(u) \text{ for all } u \in U\}.$$

By $\gamma \subseteq U \times R_N$ we always mean it satisfies the condition given above.

Definition 12. *An N-soft set (F, S, N) over U is called an int N-soft subsemigroup of S over U if for all $x, y \in S$ and $u \in U$, we have*

$$F(xy)(u) \geq min\{F(x)(u), F(y)(u)\}.$$

Example 1. *Let S = {a,b,c,d} be a semigroup as defined in the Cayley Table:*

S	a	b	c	d
a	a	a	a	a
b	a	b	a	a
c	a	a	c	a
d	a	b	a	d

and let U = {u_1,u_2,u_3,u_4,u_5} be the universe. Consider the 5-soft set $(F, S, 5)$ over U, as given in Table 1.

Table 1. The 5-Soft Set of $(F, S, 5)$.

$(F, S, 5)$	a	b	c	d
u_1	4	4	2	3
u_2	4	3	2	2
u_3	3	3	3	2
u_4	3	2	1	1
u_5	4	2	2	0

Consider $\gamma = \{(u_1, 2), (u_2, 2), (u_3, 2), (u_4, 2), (u_5, 2)\}$. Then, $F_\gamma = \{a, b\}$. Simple calculations show that $(F, S, 5)$ is an int 5-soft subsemigroup over U.

Lemma 1. *An N-soft set (F, S, N) over U is an int N-soft subsemigroup over U if and only if the non-empty γ-inclusive set of (F, S, N) is a subsemigroup of S for all $\gamma \subseteq U \times R_N$.*

Proof. Assume an N-soft set (F, S, N) over U is an int N-soft subsemigroup over U. Then, we show that the non-empty γ-inclusive set of (F, S, N) is a subsemigroup of S for all $\gamma \subseteq U \times R_N$. For this, let $x, y \in F_\gamma$. Then, by definition of γ-inclusive, we have

$$\gamma(u) \leq F(x)(u) \; for \; all \; u \in U$$

and

$$\gamma(u) \leq F(y)(u) \; for \; all \; u \in U.$$

Since, $\gamma(u) \leq F(x)(u)$ and $\gamma(u) \leq F(y)(u)$, we have

$$\gamma(u) \leq min\{F(x)(u), F(y)(u)\}.$$

Since (F, S, N) is an int N-soft subsemigroup over U, we have for all $x, y \in S$ and $u \in U$,

$$\gamma(u) \leq min\{F(x)(u), F(y)(u)\} \leq F(xy)(u).$$

As $\gamma(u) \leq F(xy)(u)$, we have $xy \in F_\gamma$.

Hence, the non empty γ-inclusive set of (F, S, N) is a subsemigroup of S.

Conversely, assume that F_γ is a subsemigroup of S for all γ. Let $x, y \in S$ and $u_o \in U$ be such that

$$F(xy)(u_o) < min \{F(x)(u_o), F(y)(u_o)\}.$$

Construct a subset γ of $U \times R_N$ such that

$$\gamma(u) = 0 \; if \; u \neq u_o \; and \; \gamma(u_o) = min \{F(x)(u_o), F(y)(u_o)\}.$$

Then $x, y \in F_\gamma$ but $xy \notin F_\gamma$ is a contradiction.
Hence, $F(xy)(u) \geq min\{F(x)(u), F(y)(u)\}$. □

Theorem 1. *If (F, S, N_1) and (G, S, N_2) are two int N-soft subsemigroups of a semigroup S over U. Then, their restricted (extended) intersection is an int N-soft subsemigroup of S.*

Proof. Let $(F, S, N_1) \cap_{\Re} (G, S, N_2) = (H, S, min(N_1, N_2))$, where for $x \in S$ and $u \in U$, we have

$$H(x)(u) = min\{F(x)(u), G(x)(u)\}.$$

As (F, S, N_1) and (G, S, N_2) are int N-soft subsemigroups of S over U, we have for all $x, y \in S$ and $u \in U$

$$F(xy)(u) \geq min\{F(x)(u), F(y)(u)\}.$$

$$G(xy)(u) \geq min\{G(x)(u), G(y)(u)\}.$$

Now

$$\begin{aligned} H(xy)(u) &= min\{F(xy)(u), G(xy)(u)\}. \\ &\geq min\ \{min\{F(x)(u), F(y)(u)\}, min\{G(x)(u), G(y)(u)\}\}. \\ &= min\ \{min\{F(x)(u), G(x)(u)\}, min\{F(y)(u), G(y)(u)\}\}. \\ &= min\{H(x)(u), H(y)(u)\}. \end{aligned}$$

Hence, $(H, S, min(N_1, N_2))$ is an int N-soft subsemigroup of S over U. □

The converse of the above theorem may not be true which is explained in the following example.

Example 2. *Let S = {a,b,c,d} be a semigroup as defined in the Cayley Table:*

S	a	b	c	d
a	a	a	a	a
b	a	b	a	a
c	a	a	c	c
d	a	a	d	d

and let $U = \{u_1, u_2, u_3, u_4\}$ be the universe. Consider $(F, S, 3)$ and $(G, S, 4)$ any two N-soft sets over U, as given in Tables 2 and 3.

Table 2. The 3-Soft Set of $(F, S, 3)$.

$(F, S, 3)$	a	b	c	d
u_1	2	2	1	2
u_2	1	2	1	2
u_3	1	1	2	1
u_4	0	1	0	0

Table 3. The 4-Soft Set of $(G, S, 4)$.

$(G, S, 4)$	a	b	c	d
u_1	1	3	0	1
u_2	1	2	2	1
u_3	1	1	3	2
u_4	2	3	3	3

Since,

$$F(bd)(u_2) \ngeq min\{F(b)(u_2), F(d)(u_2)\}.$$

$$G(cb)(u_2) \ngeq min\{G(c)(u_2), G(b)(u_2)\}.$$

Hence, $(F, S, 3)$ and $(G, S, 4)$ are not int N-soft subsemigroups over U. We know that

$$(F, S, 3) \cap_{\Re} (G, S, 4) = (H, S, min(3, 4)) = (H, S, 3).$$

as given in Table 4.

Table 4. The 3-Soft Set of $(H, S, 3)$.

$(H, S, 3)$	a	b	c	d
u_1	1	2	0	1
u_2	1	2	1	1
u_3	1	1	2	1
u_4	0	1	0	0

Simple calculations show that $(H, S, 3)$ is an int 3-soft subsemigroup over U, where $(F, S, 3)$ and $(G, S, 4)$ are not int N-soft subsemigroups over U.

In general, the N-soft restricted union of two int N-soft subsemigroups over U is not an int N-soft subsemigroup over U, which is explained in the following example.

Example 3. *Let S = {a,b,c,} be a semigroup as defined in the Cayley Table:*

S	a	b	c
a	a	a	a
b	a	b	a
c	a	a	c

and let U = {u_1,u_2,u_3,u_4} be the universe. Consider the N-soft sets $(F, S, 3)$ and $(G, S, 4)$ over U, as given in Tables 5 and 6.

Table 5. The 3-Soft Set of $(F, S, 3)$.

$(F, S, 3)$	a	b	c
u_1	2	2	1
u_2	2	0	2
u_3	1	1	2
u_4	2	2	1

Table 6. The 4-Soft Set of $(F, S, 4)$.

$(G, S, 4)$	a	b	c
u_1	3	3	3
u_2	2	3	1
u_3	1	2	1
u_4	2	0	2

Simple calculations show that $(F, S, 3)$ and $(G, S, 4)$ are int N-soft subsemigroups over U. We know that

$$(F, S, 3) \cup_{\Re} (G, S, 4) = (K, S, max(3, 4)) = (K, S, 4).$$

given in Table 7.

Table 7. The 4-Soft Set of $(K, S, 4)$.

$(K, S, 4)$	*a*	*b*	*c*
u_1	3	3	3
u_2	2	3	2
u_3	1	2	2
u_4	2	2	2

Since, $K(bc)(u_3) \ngeq min\{K(b)(u_3), K(c)(u_3)\}$.
Hence, $(K, S, 4)$ *is not an int 4-soft subsemigroup over U.*

3.2. Int N-Soft Left [Right] Ideals of S

In this subsection, we define the int N-soft left [right] ideal of semigroups and study their basic properties.

Definition 13. *An N-soft set* (F, S, N) *over U is called an int N-soft left [right] ideal of S over U if for all* $x, y \in S$ *and* $u \in U$*, we have*

$$F(xy)(u) \geq F(y)(u) \ [F(xy)(u) \geq F(x)(u)].$$

An N-soft set (F, S, N) *over U is called an int N-soft two-sided ideal or simply an int N-soft ideal of S over U if it is both an int N-soft left ideal and an int N-soft right ideal of S over U.*

Example 4. *Let S = {a,b,c,d} be a semigroup, as defined in the Cayley Table:*

S	*a*	*b*	*c*	*d*
a	*a*	*a*	*a*	*a*
b	*a*	*b*	*a*	*a*
c	*a*	*a*	*c*	*a*
d	*a*	*b*	*a*	*d*

and let U = $\{u_1, u_2, u_3, u_4, u_5\}$ *be the universe. Consider the 5-soft set* $(F, S, 5)$ *over U, as given in Table 8.*

Table 8. The 5-Soft Set of $(F, S, 5)$.

$(F, S, 5)$	*a*	*b*	*c*	*d*
u_1	4	4	2	3
u_2	4	3	2	2
u_3	3	3	3	2
u_4	3	2	1	1
u_5	4	2	2	0

Simple calculations show that $(F, S, 5)$ *is an int N-soft subsemigroup over U.*

Moreover, we can check that $(F, S, 5)$ *is an int 5-soft two-sided ideal of S over U. Hence,* (F, S, N) *is an int N-soft two-sided ideal over U.*

Remark 1. *Obviously, every int N-soft left [right] ideal over U is an int N-soft subsemigroup over U.*

However, the converse is not true, which is explained in the following example.

Example 5. *Let S = {a,b,c} be a semigroup as defined in the Cayley Table:*

S	a	b	c
a	a	a	a
b	a	b	b
c	a	b	c

and let U = $\{u_1, u_2, u_3, u_4\}$ be the universe. Consider the 4-soft set $(F, S, 4)$ over U, as given in Table 9.

Table 9. The 4-Soft Set of $(F, S, 4)$.

$(F, S, 4)$	a	b	c
u_1	3	1	3
u_2	2	2	3
u_3	2	2	2
u_4	3	3	3

Simple calculations show that $(F, S, 4)$ is an int 4-soft subsemigroup over U. Moreover, we can calculate that

$$F(bc)(u_1) = F(b)(u_1) \ngeq F(c)(u_1).$$

$$F(cb)(u_1) = F(b)(u_1) \ngeq F(c)(u_1).$$

Thus $(F, S, 4)$ is not an int 4-soft left ideal nor int 4-soft right ideal over U.

Theorem 2. *An N-soft set (F, S, N) over U is an int N-soft left [right] ideal of S over U if and only if the non-empty γ-inclusive set of (F, S, N) is a left [right] ideal of S for all subsets γ of $U \times R_N$.*

Proof. Assume an N-soft set (F, S, N) over U is an int N-soft left ideal over U. Then, we show that the non-empty γ-inclusive set of (F, S, N) is a left ideal of S for all $\gamma \subseteq U \times R_N$. For this, let $y \in F_\gamma$. Then, by definition of γ-inclusive, we have

$$\gamma(u) \leq F(y)(u) \ for \ all \ u \in U.$$

Since (F, S, N) is an int N-soft left ideal over U, we have for all $x \in S$ and $u \in U$,

$$\gamma(u) \leq F(y)(u) \leq F(xy)(u).$$

As $\gamma(u) \leq F(xy)(u)$, we have $xy \in F_\gamma$.

Hence, the non empty γ-inclusive set of (F, S, N) is a left ideal of S.

Conversely, assume that F_γ is a left ideal of S for all γ. Let $x, y \in S$ and $u_o \in U$ be such that

$$F(xy)(u_o) < F(y)(u_o).$$

Construct a subset γ of $U \times R_N$ such that

$$\gamma(u) = 0 \ if \ u \neq u_o \ and \ \gamma(u_o) = F(y)(u_o).$$

Then $y \in F_\gamma$ but $xy \notin F_\gamma$, a contradiction.

Hence, $F(xy)(u) geq F(y)(u)$.

Similarly an N-soft set (F, S, N) over U is an int N-soft right ideal of S over U if and only if the non-empty γ-inclusive set of (F, S, N) is a right ideal of S for all subsets γ of $U \times R_N$. □

Corollary 1. *An N-soft set (F, S, N) over U is an int N-soft two-sided ideal of S over U if and only if the non-empty γ-inclusive set of (F, S, N) is a two-sided ideal of S for all subsets γ of $U \times R_N$.*

Theorem 3. *If (F, S, N_1) and (G, S, N_2) are any two int N-soft left [right] ideals of S over U. Then, their restricted (extended) intersection is an int N-soft left [right] ideal of S.*

Proof. Let $(F,S,N_1) \cap_{\Re} (G,S,N_2) = (H,S,min(N_1,N_2))$, where for $x \in S$ and $u \in U$, we have

$$H(x)(u) \ = \ min\{F(x)(u), G(x)(u)\}.$$

As (F,S,N_1) and (G,S,N_2) are int N-soft left ideals of S over U, we have for all $x,y \in S$ and $u \in U$,

$$F(xy)(u) \geq \ F(y)(u).$$
$$G(xy)(u) \geq \ G(y)(u).$$

Now

$$\begin{aligned} H(xy)(u) \ = \ & min\{F(xy)(u), G(xy)(u)\}. \\ & \geq min \ \{F(y)(u), G(y)(u)\}. \\ & = \ H(y)(u)\}. \end{aligned}$$

Hence, $(H,S,min(N_1,N_2))$ is an int N-soft left ideal of S over U.

Similarly, if (F,S,N_1) and (G,S,N_2) are two int N-soft right ideals of S over U, then their restricted (extended) intersection is an int N-soft right ideal of S. □

Corollary 2. *If (F,S,N_1) and (G,S,N_2) are any two int N-soft two sided ideals of S over U. Then, their restricted (extended) intersection is an int N-soft two-sided ideal of S.*

Theorem 4. *If (F,S,N_1) and (G,S,N_2) are any two int N-soft left [right] ideals of S over U. Then, their restricted (extended) union is an int N-soft left [right] ideal of S.*

Proof. Let $(F,S,N_1) \cup_{\Re} (G,S,N_2) = (K,S,max(N_1,N_2))$, where for $x \in S$ and $u \in U$, we have

$$K(x)(u) \ = \ max\{F(x)(u), G(x)(u)\}.$$

As (F,S,N_1) and (G,S,N_2) are int N-soft left ideals of S over U, we have for all $x,y \in S$ and $u \in U$,

$$F(xy)(u) \geq \ F(y)(u).$$
$$G(xy)(u) \geq \ G(y)(u).$$

Now

$$\begin{aligned} K(xy)(u) \ = \ & max\{F(xy)(u), G(xy)(u)\}. \\ & \geq max \ \{F(y)(u), G(y)(u)\}. \\ & = \ K(y)(u)\}. \end{aligned}$$

Hence, $(K,S,max(N_1,N_2))$ is an int N-soft left ideal of S over U.

Similarly, if (F,S,N_1) and (G,S,N_2) are two int N-soft right ideals of S over U, then their restricted (extended) union is an int N-soft right ideal of S. □

Corollary 3. *If (F,S,N_1) and (G,S,N_2) are any two int N-soft two-sided ideals of S over U, then their restricted (extended) union is an int N-soft two-sided ideal of S.*

3.3. Int N-Soft Product and Int N-Soft Characteristic Function

In this subsection, we define int N-soft product and int N-soft characteristic function and study their properties.

Definition 14. *The int N-soft product of any two N-soft sets* (F, S, N_1) *and* (G, S, N_2) *over the common universe U is denoted by* $(F \circ G, S, min(N_1, N_2)) = (F, S, N_1) \circ (G, S, N_2)$ *and is defined by*

$$(F \circ G)(x)(u) = \begin{cases} \max_{x=yz} \{min(F(y)(u), G(z)(u))\} & \text{if } \exists\, y, z \in S \text{ such that } x = yz, \\ 0 & otherwise, \end{cases} \tag{3}$$

for all $x \in S$ *and* $u \in U$.

Example 6. *Let S = {a,b,c,d} be a semigroup as defined in the Cayley Table:*

S	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

and let U = $\{u_1, u_2, u_3, u_4, u_5\}$ *be the universe. Consider the N-soft sets* $(F, S, 6)$ *and* $(G, S, 4)$ *over U, as given in Tables 10 and 11.*

$$(F \circ G)(b)(u) = \{max\{min\{F(c)(u), G(c)(u)\}, min\{F(d)(u), G(c)(u)\}, min\{F(d)(u), G(d)(u)\}\}\}.$$

$$(F \circ G)(b)(u) = \{(u_1, 2), (u_2, 2), (u_3, 3), (u_4, 2), (u_5, 1)\}.$$

$$(F \circ G)(d)(u) = \{(u_1, 0), (u_2, 0), (u_3, 0), (u_4, 0), (u_5, 0)\}.$$

Table 10. The 6-Soft Set of $(F, S, 6)$.

$(F, S, 6)$	a	b	c	d
u_1	3	3	4	0
u_2	2	3	4	1
u_3	4	2	5	1
u_4	5	4	2	3
u_5	3	1	0	4

Table 11. The 4-Soft Set of $(G, S, 4)$.

$(G, S, 4)$	a	b	c	d
u_1	3	1	2	0
u_2	2	0	2	1
u_3	2	1	3	2
u_4	1	0	1	2
u_5	1	3	1	1

Hence, int N-soft product of $(F, S, 6)$ *and* $(G, S, 4)$*, is given in Table 12.*

Table 12. The 4-Soft Set of $(G, S, 4)$.

$(F \circ G, S, 4)$	a	b	c	d
u_1	3	2	0	0
u_2	2	2	0	0
u_3	3	3	0	0
u_4	2	2	0	0
u_5	3	1	0	0

Proposition 1. *Let* (F_1, S, N_1), (F_2, S, N_2), (G_1, S, N_3) *and* (G_2, S, N_4) *be any N-soft sets over* U. *If*

$$(F_1, S, N_1) \subseteq (G_1, S, N_3) \text{ and } (F_2, S, N_2) \subseteq (G_2, S, N_4),$$

then

$$(F_1 \circ F_2, S, min(N_1, N_2)) \subseteq (G_1 \circ G_2, S, min(N_3, N_4)).$$

Proof. Let $x \in S$. If x is not expressed as $x = yz$ for $y, z \in S$, then clearly

$$(F_1 \circ F_2)(x)(u) = 0 = (G_1 \circ G_2)(x)(u).$$

Hence, $(F_1 \circ F_2, S, min(N_1, N_2)) \subseteq (G_1 \circ G_2, S, min(N_3, N_4))$.

Suppose that there exists $y, z \in S$ such that $x = yz$. Then,

$$\begin{aligned}(F_1 \circ F_2)(x)(u) &= \max_{x=yz}\{min(\ddot{F}_1(y)(u), \ddot{F}_2(z)(u))\}. \\ &\leq \max_{x=yz}\{min(\ddot{G}_1(y)(u), \ddot{G}_2(z)(u))\}. \\ &= (G_1 \circ G_2)(x)(u).\end{aligned}$$

Therefore $(F_1 \circ F_2, S, min(N_1, N_2)) \subseteq (G_1 \circ G_2, S, min(N_3, N_4))$. □

Theorem 5. *An N-soft set* (F, S, N) *over* U *is an int N-soft subsemigroup over* U *if and only if*

$$(F \circ F, S, N) \subseteq (F, S, N).$$

Proof. Assume that $(F \circ F, S, N) \subseteq (F, S, N)$. Let $x, y \in S$. Then we have for all $u \in U$

$$F(xy)(u) \geq (F \circ F)(xy)(u) > min\,(F(x)(u), F(y)(u)).$$

Thus, (F, S, N) over U is an int N-soft subsemigroup over U.

Conversely, suppose that (F, S, N) is an int N-soft subsemigroup over U. Then, for all $u \in U$, we have

$$F(x)(u) \geq min\,(F(y)(u), F(z)(u)).$$

for all $x \in S$ with $x = yz$. Thus

$$\begin{aligned}F(x)(u) &\geq \max_{x=yz}\{min(F(y)(u), F(z)(u))\}. \\ &= (F \circ F)(x)(u).\end{aligned}$$

for all $x \in S$. Hence $(F \circ F, S, N) \subseteq (F, S, N)$. □

Theorem 6. *Let* (F, S, N_1) *and* (G, S, N_2) *be any two N-soft sets over* U. *If* (F, S, N_1) *is an int N-soft left ideal over* U, *then so is the int N-soft product* $(F \circ G, S, min(N_1, N_2))$.

Proof. Let $x, y \in S$. If $y = ab$ for some $a, b \in S$, then $xy = x(ab) = (xa)b$ and for all $u \in U$, we have

$$\begin{aligned}(F \circ G)(y)(u) &= \max_{y=ab}\{min\{F(a)(u), G(b)(u)\}\}. \\ &\leq \max_{xy=(xa)b}\{min\{F(xa)(u), G(b)(u)\}\}. \\ &\leq \max_{xy=cb}\{min(F(c)(u), G(b)(u))\}. \\ &= (F \circ G)(xy)(u).\end{aligned}$$

If y is not expressible as $y = ab$ for all $a, b \in S$, then for all $u \in U$, we have

$$(F \circ G)(y)(u) \ = \ 0 \leq \ (F \circ G)(xy)(u).$$

Thus $(F \circ G)(y)(u) \leq \ (F \circ G)(xy)(u)$. for all $x, y \in S$, and so $(F \circ G, S, min(N_1, N_2))$ is an int N-soft left ideal over U. □

Corollary 4. *Let (F, S, N_1) and (G, S, N_2) be any two N-soft sets over U. If (G, S, N_2) is an int N-soft right ideal over U, then so is the int N-soft product $(F \circ G, S, min(N_1, N_2))$.*

Theorem 7. *If (F, S, N_1) is an int N-soft right ideal over U and (G, S, N_2) is an int N-soft left ideal over U, then*

$$(F, S, N_1) \circ \ (G, S, N_2) \subseteq \ (F, S, N_1) \cap_{\Re} \ (G, S, N_2).$$

Proof. We know that

$$(F, S, N_1) \circ \ (G, S, N_2) \ = \ (F \circ G, S, min(N_1, N_2)).$$

and

$$(F, S, N_1) \cap_{\Re} \ (G, S, N_2) \ = \ (H, S, min(N_1, N_2)).$$

Let $x \in S$, if x is not expressible as $x = ab$ for $a, b \in S$. Then

$$(F \circ G)(x)(u) \ = \ 0 \leq \ min\{F(x)(u), G(x)(u)\}.$$

Assume that there exists $a, b \in S$ such that $x = ab$. Then

$$\begin{aligned} (F \circ G)(x)(u) \quad &= \max_{x=ab} \{min\{F(a)(u), G(b)(u)\}\}. \\ &\leq \ \max_{x=ab} \{min\{F(ab)(u), G(ab)(u)\}\}. \\ &= \ min\{F(x)(u), G(x)(u)\} \ = \ H(x)(u). \end{aligned}$$

in any case, we have

$$(F, S, N_1) \circ \ (G, S, N_2) \subseteq \ (F, S, N_1) \cap_{\Re} \ (G, S, N_2).$$

□

Definition 15. *For a non-empty subset A of S, defines N-soft characteristic function χ_A as follows,*

$$\chi_A \ : \ S \ \rightarrow \ P(U \ \times \ R_N).$$

For each $u \in U$ and $x \in A$, we have

$$\chi_A(x)(u) = \begin{cases} N-1 & \text{if} \ \ x \ \in \ A, \\ 0 & \quad otherwise. \end{cases} \tag{4}$$

Then (χ_A, S, N) is an N-soft set over U, which is called N-soft characteristic set.

Example 7. *Let S = {a, b, c, d} be a semigroup as defined in the Cayley Table:*

S	a	b	c	d
a	a	a	a	a
b	a	b	a	a
c	a	a	c	a
d	a	b	a	d

Let A = {a,b} be a non-empty subset of S and let $U = \{u_1,u_2,u_3,u_4,u_5\}$ be the universe. Then, the 5-soft characteristic set over U, is given in Table 13.

Table 13. The 5-Soft Set of $(\chi_A, S, 5)$.

$(\chi_A, S, 5)$	***a***	***b***	***c***	***d***
u_1	4	4	0	0
u_2	4	4	0	0
u_3	4	4	0	0
u_4	4	4	0	0
u_5	4	4	0	0

Theorem 8. *For any non-empty subset A of S, the following are equivalent,*

(1) A is a left [right] ideal of S.
(2) An N-soft characteristic set (χ_A, S, N) is an int N-soft left [right] ideal over U.

Proof. Assume that A is a left ideal of S. For any $x, y \in S$. If y $\notin A$ then for all $u \in U$, we have

$$\chi_A(xy)(u) \geq 0 = \chi_A(y)(u).$$

If $y \in A$, then x$y \in A$ since A is a left ideal of S. Thus, for all $u \in U$, we have

$$\chi_A(xy)(u) = N - 1 = \chi_A(y)(u).$$

Therefore, (χ_A, S, N) is an int N-soft left ideal over U.

Conversely, suppose that (χ_A, S, N) is an int N-soft left ideal over U. Let $x \in S$ and $y \in A$. Then, for all $u \in U$, we have

$$\chi_A(y)(u) = N - 1.$$

Hence, $\chi_A(xy)(u) \geq \chi_A(y)(u) = N - 1$.
That is $\chi_A(xy)(u) = N - 1$.
Thus $xy \in A$ and therefore A is a left ideal of S. □

Corollary 5. *For any non-empty subset A of S, the following are equivalent,*

(1) A is a two-sided ideal of S.
(2) An N-soft characteristic set (χ_A, S, N) is an int N-soft two sided ideal over U.

Theorem 9. *For a non-empty subset T of S, the following are equivalent,*

(1) T is a subsemigroup of S.
(2) An N-soft characteristic set (χ_T, S, N) is an int N-soft subsemigroup over U.

Proof. The proof is similar to the proof of Theorem 8. □

Theorem 10. *Let (χ_A, S, N) and (χ_B, S, N) be any N-soft characteristic sets over U, where A and B are non-empty subsets of S. Then, the following properties hold*

(1) $(\chi_A, S, N) \cap_{\Re} (\chi_B, S, N) = (\chi_{A\cap B}, S, N)$.
(2) $(\chi_A, S, N) \circ (\chi_B, S, N) = (\chi_{AB}, S, N)$.

Proof.

(1) We know that

$$(\chi_A, S, N) \cap_{\Re} (\chi_B, S, N) = (\chi_A \cap_{\Re} \chi_B, S, N).$$

Let $x \in S$, if $x \in A \cap B$. Then $x \in A$ and $x \in B$. Thus, we have

$$(\chi_A \cap_{\Re} \chi_B)(x)(u) = min\{\chi_A(x)(u), \chi_B(x)(u)\} = N-1 = \chi_{A\cap B}(x)(u).$$

If $x \notin A \cap B$, then $x \notin A$ or $x \notin B$. Hence, we have

$$(\chi_A \cap_{\Re} \chi_B)(x)(u) = min\{\chi_A(x)(u), \chi_B(x)(u)\} = 0 = \chi_{A\cap B}(x)(u).$$

Therefore, $(\chi_A, S, N) \cap_{\Re} (\chi_B, S, N) = (\chi_{A\cap B}, S, N)$.

(2) For any $x \in S$, suppose $x \in AB$. Then, there exist $a \in A$ and $b \in B$ such that $x = ab$. Thus, we have

$$\begin{aligned}(\chi_A \circ \chi_B)(x)(u) &= \max_{x=yz}\{min\{\chi_A(y)(u), \chi_B(z)(u)\}\}. \\ &\geq min\{\chi_A(a)(u), \chi_B(b)(u)\} = N-1.\end{aligned}$$

Since $x \in AB$, we obtain $\chi_{AB}(x)(u) = N-1$.

Now, suppose $x \notin AB$, then $x \neq ab$ for all $a \in A$ and $b \in B$. If $x = yz$ for some $y, z \in S$ then $y \notin A$ or $z \notin B$. Thus,

$$(\chi_A \circ \chi_B)(x)(u) = \max_{x=yz}\{min\{\chi_A(y)(u), \chi_B(z)(u)\}\} = 0 = \chi_{AB}(x)(u).$$

If $x \neq yz$ for all $x, y \in S$, then

$$(\chi_A \circ \chi_B)(x)(u) = 0 = \chi_{AB}(x)(u).$$

In any case, we have

$$(\chi_A, S, N) \circ (\chi_B, S, N) = (\chi_{AB}, S, N).$$

□

3.4. θ-Generalized Int N-Soft Subsemigroup

In this subsection, we define θ-generalized int N-soft subsemigroup and study their several properties.

Definition 16. *For a subset θ of $U \times R_N$ such that for each $u \in U$ there exists a unique number $m \in \{1, 2, \ldots, N-1\}$, for which $(u, m) \in \theta$, we write $\theta(u) = m$, an N-soft set (F, S, N) is called a θ-generalized int N-soft subsemigroup over U, if for all $x, y \in S$ and $u \in U$, we have*

$$F(xy)(u) \geq min\{\theta(u), min\{F(x)(u), F(y)(u)\}\}.$$

By $\theta \subseteq U \times R_N$ we always mean that it satisfies the condition given above.

Obviously, every int N-soft subsemigroup is a θ-generalized int N-soft subsemigroup but the converse is not true, which is shown in the following example.

Example 8. *Let S = {a, b, c, d} be a semigroup as defined in the Cayley Table:*

S	a	b	c	d
a	a	a	c	c
b	b	b	d	d
c	a	a	c	c
d	b	b	d	d

and let $U = \{u_1, u_2, u_3, u_4\}$ *be the universe. Consider* $(F, S, 4)$ *N-soft sets over* U*, as given in Table 14.*

$$F(ad)(u_1) = F(c)(u_1) \ngeq min\{F(a)(u_1), F(d)(u_1)\}.$$

Table 14. The 4-Soft Set of $(F, S, 4)$.

$(F, S, 4)$	a	b	c	d
u_1	3	2	1	3
u_2	2	1	2	2
u_3	3	3	2	3
u_4	2	3	2	2

Hence $(F, S, 4)$ *is not an int N-soft subsemigroups over* U.

Consider $\theta = \{(u_1,1),(u_2,1),(u_3,2),(u_4,3)\}$. *The simple calculations show that* $(F, S, 4)$ *is a* θ*-generalized int N-soft subsemigroup over* U.

Theorem 11. *An N-soft set* (F, S, N) *over* U *is a* θ*-generalized int N-soft subsemigroups over* U *if and only if the non-empty* γ*-inclusive set of* (F, S, N) *is a subsemigroup of* S *for all* $\gamma \subseteq U \times R_N$ *with* $\gamma(u) \leq \theta(u)$.

Proof. Assume that (F, S, N) is a θ-generalized int N-soft subsemigroups over U. Let $x, y \in F_\gamma$ where $\gamma(u) \leq \theta(u)$. Then, by definition of the γ-inclusive set, we have

$$\gamma(u) \ \leq \ F(x)(u) \ \ and \ \ \gamma(u) \ \leq \ F(y)(u) \ \ for\ all\ u \ \in U.$$

It follows that

$$\gamma(u) \ \leq \ min\{F(x)(u), F(y)(u)\}.$$

Since the int N-soft set over U is a θ-generalized int N-soft subsemigroups over U, we have for all u $\in U$

$$F(xy)(u) \ \geq \ min\{\theta(u), \ min\{F(x)(u), F(y)(u)\}\} \ \geq \ min\{\theta(u), \gamma(u)\} \ = \ \gamma(u).$$

As $\gamma(u) \ \leq \ F(xy)(u)$, we have $xy \ \in \ F_\gamma$.

Hence, γ-inclusive set of (F, S, N) is a subsemigroup of S for all γ with $\gamma(u) \leq \theta(u)$.

Conversely, assume that F_γ is a subsemigroup of S for all γ with $\gamma(u) \leq \theta(u)$. Let $x, y \in S$ and $u_o \in U$ be such that

$$F(xy)(u_o) \ < \ min\{\theta(u), \ min\{F(x)(u_o), F(y)(u_o)\}\}.$$

Construct a subset γ of $U \times R_N$ such that $\gamma(u) \ = \ 0 \, if \ u \neq \ u_o \ and \ \gamma(u_o) = min\{\theta(u_o), min\{F(x)(u_o), F(y)(u_o)\}\}$. Then, $\gamma(u) \leq \ \theta(u)$.

Then $x, y \in F_\gamma$ but $xy \notin F_\gamma$, a contradiction.

Hence, $F(xy)(u) \ \geq \ min\{\theta(u), \ min\{F(x)(u), F(y)(u)\}\}$.

Therefore, (F, S, N) is a θ-Generalized int N-soft subsemigroups over U. □

Theorem 12. *If* (F, S, N_1) *and* (G, S, N_2) *are two* θ*-Generalized int N-soft subsemigroups of a semigroup* S *over* U*. Then, their restricted (extended) intersection is a* θ*-Generalized int N-soft subsemigroup of* S.

Proof. Let $(F, S, N_1) \cap_{\Re} (G, S, N_2) = (H, S, min(N_1, N_2))$,
where for $x \in S$ and $u \in U$, we have

$$H(x)(u) \ = \ min\{F(x)(u), G(x)(u)\}.$$

As (F, S, N_1) and (G, S, N_2) are θ-Generalized int N-soft subsemigroups of S over U, we have for all $x, y \in S$ and $u \in U$

$$F(xy)(u) \geq min\{\theta(u), min\{F(x)(u), F(y)(u)\}\}.$$
$$G(xy)(u) \geq min\{\theta(u), min\{G(x)(u), G(y)(u)\}\}.$$

Now

$$H(xy)(u) = min\{F(xy)(u), G(xy)(u)\}.$$
$$\geq min\ \{min\{\theta(u), min\{F(x)(u), F(y)(u)\}\}, min\{\theta(u), min\{G(x)(u), G(y)(u)\}\}\}.$$
$$= min\ \{\theta(u), min\{\{min\{F(x)(u), G(x)(u)\}, min\{F(y)(u), G(y)(u)\}\}\}.$$
$$= min\{\theta(u), min\{H(x)(u), H(y)(u)\}.$$

Hence, $(H, S, min(N_1, N_2))$ is a θ-Generalized int N-soft subsemigroup of S over U. □

In general, the N-soft restricted union of two θ-Generalized int N-soft subsemigroups over U is not a θ-Generalized int N-soft subsemigroup over U, which is explained in the following example.

Example 9. *Let S = {a,b,c,} be a semigroup as defined in the Cayley Table:*

S	a	b	c
a	a	a	a
b	a	b	a
c	a	a	c

and let U = {u_1,u_2,u_3} be the universe. Consider $(F, S, 4)$ and $(G, S, 3)$ N-soft sets over U, as given in Tables 15 and 16.

Table 15. The 4-Soft Set of $(F, S, 4)$.

$(F, S, 4)$	a	b	c
u_1	3	3	3
u_2	2	3	1
u_3	1	2	1

Table 16. The 3-Soft Set of $(G, S, 3)$.

$(G, S, 3)$	a	b	c
u_1	2	2	1
u_2	2	0	2
u_3	1	1	2

We know that

$$(F, S, 4) \cup_{\Re} (G, S, 3) = (K, S, max(4, 3)) = (K, S, 4).$$

given in Table 17.

Table 17. The 4-Soft Set of $(K, S, 4)$.

$(K, S, 4)$	a	b	c
u_1	3	3	3
u_2	2	3	2
u_3	1	2	2

Consider $\theta = \{(u_1, 2), (u_2, 1), (u_3, 2)\}$.

Simple calculations show that $(F, S, 4)$ *and* $(G, S, 3)$ *are* θ*-Generalized int N-soft subsemigroups over* U*. However,* $K(bc)(u_3) \ngeq min\{\theta(u_3), min\{K(b)(u_3), K(c)(u_3)\}\}$.

Hence, $(K, S, 4)$ *is not a* θ*-Generalized int 4-soft subsemigroup over* U.

Theorem 13. *For every* θ, $\vartheta \in P(U \times R_N)$*, if* $\vartheta(u) \leq \theta(u)$ *then every* θ*-Generalized int N-soft subsemigroup is a* ϑ*-Generalized int N-soft subsemigroup.*

Proof. Let $\theta, \vartheta \in P(U \times R_N)$ be such that $\vartheta(u) \leq \theta(u)$. Let (F, S, N) be a θ-generalized int N-soft subsemigroup over U. For any $x, y \in S$, we have

$$\begin{aligned} F(xy)(u) &\geq min\{\theta(u), min\{F(x)(u), F(y)(u)\}\}. \\ &\geq min\{\vartheta(u), min\{F(x)(u), F(y)(u)\}\}. \end{aligned}$$

Therefore, (F, S, N) is a ϑ-Generalized int N-soft subsemigroup over U. □

Theorem 14. *If* (F, S, N) *over* U *is a* θ*-generalized int N-soft subsemigroup over* U*, then the set*

$$S_a = \{x \in S : F(x)(u) \geq min(\theta(u), F(a)(u)) \; for \; all \; u \in U\}$$

is a subsemigroup of S *for all* $a \in S$.

Proof. Assume that (F, S, N) is a θ-generalized int N-soft subsemigroup over U. Then, for any $x, y \in S$ and $u \in U$, we have

$$F(xy)(u) \geq min\{\theta(u), \; min\{F(x)(u), F(y)(u)\}\}.$$

Let $x, y \in S_a$, we have

$$min\{\theta(u), F(a)(u)\} \leq F(x)(u) \; for \; a \in S$$

and

$$min\{\theta(u), F(a)(u)\} \leq F(y)(u) \; for \; a \in S.$$

it follows that,

$$min\{\theta(u), F(a)(u)\} \leq min\{F(x)(u), F(y)(u)\}.$$

Since,

$$F(xy)(u) \geq min\{\theta(u), min\{F(x)(u), F(y)(u)\}\}.$$

$$F(xy)(u) \geq min\{\theta(u), min\{\theta(u), F(a)(u)\}.$$

$$F(xy)(u) \geq min\{\theta(u), F(a)(u)\}.$$

Thus, by definition xy $\in S_a$.

Hence, S_a is a subsemigroup of S for all a $\in S$. □

3.5. θ-Generalized Int N-Soft Left [Right] Ideals of S

In this subsection, we define θ-generalized int N-soft left [right] ideals of S and study their several properties.

Definition 17. *For a subset θ of $U \times R_N$ such that for each $u \in U$ there exists a unique number $m \in \{1, 2, \ldots, N-1\}$, for which $(u,m) \in \theta$, we write $\theta(u) = m$, an N-soft set (F, S, N) is called a θ-generalized int N-soft left [right] ideal of S over U, if for all $x, y \in S$ and $u \in U$, we have*

$$F(xy)(u) \geq min\{\theta(u), F(y)(u)\} \; [F(xy)(u) \geq min\{\theta(u), F(x)(u)\}].$$

An N-soft set (F, S, N) over U is called a θ-generalized int N-soft two-sided ideal or simply a θ-generalized int N-soft ideal of S over U if it is both a θ-generalized int N-soft left ideal and a θ-generalized int N-soft right ideal of S over U.

By $\theta \subseteq U \times R_N$ we always mean it satisfies the condition given above.

Obviously, every int N-soft left [right] ideal of S is a θ-generalized int N-soft left [right] ideal of S, but the converse is not true, which is shown in the following example.

Example 10. *Let S = {a,b,c} be a semigroup as defined in the Cayley Table:*

S	a	b	c
a	a	a	a
b	a	b	b
c	a	b	c

and let U = {u_1,u_2,u_3} be the universe. Consider $(F, S, 5)$ an N-soft set over U, as given in Table 18.

$$F(bc)(u_1) = F(b)(u_1) \ngeq F(c)(u_1) \quad and \quad F(cb)(u_1) = F(b)(u_1) \ngeq F(c)(u_1).$$

Table 18. The 5-Soft Set of $(F, S, 5)$.

$(F, S, 5)$	*a*	*b*	*c*
u_1	4	2	4
u_2	3	3	4
u_3	3	3	3

Hence, $(F, S, 5)$ is not an int N-soft left [right] ideal of S over U.

Consider θ = {(u_1,1),(u_2,1),(u_3,2)}. The simple calculations show that $(F, S, 5)$ is a θ-generalized int N-soft two-sided ideal of S over U.

Theorem 15. *An N-soft set (F, S, N) over U is a θ-generalized int N-soft left [right] ideal of S over U if and only if the non-empty γ-inclusive set of (F, S, N) is a left [right] ideal of S for all $\gamma \subseteq U \times R_N$ with $\gamma(u) \leq \theta(u)$.*

Proof. Assume that (F, S, N) is a θ-generalized int N-soft left ideal of S over U. Let $y \in F_\gamma$ where $\gamma(u) \leq \theta(u)$. Then, by definition of the γ-inclusive set, we have

$$\gamma(u) \leq F(y)(u) \; for \; all \; u \in U.$$

Since an int N-soft set of S over U is a θ-generalized int N-soft left ideal of S over U, we have for all u $\in U$

$$F(xy)(u) \geq min\{\theta(u), F(y)(u)\} \geq min\{\theta(u), \gamma(u)\} = \gamma(u).$$

As $\gamma(u) \leq F(xy)(u)$ we have $xy \in F_\gamma$.

Hence, the non-empty γ-inclusive set of (F, S, N) is a left ideal of S for all γ with $\gamma(u) \leq \theta(u)$.

Conversely, assume that F_γ is a left ideal of S for all γ with $\gamma(u) \leq \theta(u)$. Let $x, y \in S$ and $u_o \in U$ be such that

$$F(xy)(u_o) < min\{\theta(u), F(y)(u_o)\}.$$

Construct a subset γ of $U \times R_N$ such that $\gamma(u) = 0\, if\ u \neq u_o\ and\ \gamma(u_o) = min\{\theta(u_o), F(y)(u_o)\}$. Then $\gamma(u) \leq \theta(u)$.

Then $y \in F_\gamma$ but $xy \notin F_\gamma$, a contradiction.

Hence, $F(xy)(u) \geq min\{\theta(u), F(y)(u)\}$.

Therefore, (F, S, N) is a θ-Generalized int N-soft left ideal of S over U.

Similarly, an N-soft set (F, S, N) over U is a θ-generalized int N-soft right ideal of S over U if and only if the non-empty γ-inclusive set of (F, S, N) is a right ideal of S for all $\gamma \subseteq U \times R_N$ with $\gamma(u) \leq \theta(u)$. □

Corollary 6. *An N-soft set (F, S, N) over U is a θ-generalized int N-soft two sided ideal of S over U if and only if the non-empty γ-inclusive set of (F, S, N) is a two sided ideal of S for all $\gamma \subseteq U \times R_N$ with $\gamma(u) \leq \theta(u)$.*

Theorem 16. *If (F, S, N_1) and (G, S, N_2) are any two θ-Generalized int N-soft left [right] ideals of S over U. Then, their restricted (extended) intersection is a θ-Generalized int N-soft left [right] ideal of S.*

Proof. Let $(F, S, N_1) \cap_{\Re} (G, S, N_2) = (H, S, min(N_1, N_2))$, where for $x \in S$ and $u \in U$, we have

$$H(x)(u) = min\{F(x)(u), G(x)(u)\}.$$

As (F, S, N_1) and (G, S, N_2) are θ-Generalized int N-soft left ideals of S over U, we have for all $x, y \in S$ and $u \in U$,

$$F(xy)(u) \geq min\{\theta(u), F(y)(u)\}.$$
$$G(xy)(u) \geq min\{\theta(u), G(y)(u)\}.$$

Now

$$\begin{aligned} H(xy)(u) &= min\{F(xy)(u), G(xy)(u)\}. \\ &\geq min\ \{min\{\theta(u), F(y)(u)\}, min\{\theta(u), G(y)(u)\}\}. \\ &= min\ \{\theta(u), min\{F(y)(u), G(y)(u)\}\}. \\ &= min\{\theta(u), H(y)(u)\}. \end{aligned}$$

Hence, $(H, S, min(N_1, N_2))$ is a θ-Generalized int N-soft left ideal of S over U.

Similarly, if (F, S, N_1) and (G, S, N_2) are two θ-Generalized int N-soft right ideals of a semigroup S over U, then their restricted (extended) intersection is a θ-Generalized int N-soft right ideal of S. □

Corollary 7. *If (F, S, N_1) and (G, S, N_2) are any two θ-Generalized int N-soft two-sided ideals of S over U, then their restricted (extended) intersection is a θ-Generalized int N-soft two-sided ideal of S.*

Theorem 17. *If (F, S, N_1) and (G, S, N_2) are any two θ-Generalized int N-soft left [right] ideals of S over U. Then, their restricted (extended) union is a θ-Generalized int N-soft left [right] ideal of S.*

Proof. Let $(F, S, N_1) \cup_{\Re} (G, S, N_2) = (K, S, max(N_1, N_2))$, where for $x \in S$ and $u \in U$, we have

$$K(x)(u) = max\{F(x)(u), G(x)(u)\}.$$

As (F, S, N_1) and (G, S, N_2) are θ-Generalized int N-soft left ideals of S over U, we have for all $x, y \in S$ and $u \in U$,

$$F(xy)(u) \geq min\{\theta(u), F(y)(u)\}.$$
$$G(xy)(u) \geq min\{\theta(u), G(y)(u)\}.$$

Presently,

$$\begin{aligned} K(xy)(u) &= max\{F(xy)(u), G(xy)(u)\}. \\ &\geq max\ \{min\{\theta(u), F(y)(u)\}, min\{\theta(u), G(y)(u)\}\}. \\ &= min\ \{\theta(u), max\{F(y)(u), G(y)(u)\}\}. \\ &= min\{\theta(u), K(y)(u)\}. \end{aligned}$$

Hence, $(K, S, max(N_1, N_2))$ is a θ-Generalized int N-soft left ideal of S over U.

Similarly, if (F, S, N_1) and (G, S, N_2) are two θ-Generalized int N-soft right ideals of a semigroup S over U, then their restricted (extended) union is a θ-Generalized int N-soft right ideal of S. □

Corollary 8. *If (F, S, N_1) and (G, S, N_2) are any two θ-Generalized int N-soft two sided ideals of S over U, then their restricted (extended) union is a θ-Generalized int N-soft two-sided ideal of S.*

Theorem 18. *If (F, S, N) over U is a θ-generalized int N-soft left [right] ideal of S over U, then the set*

$$S_a = \{x \in S : F(x)(u) \geq min(\theta(u), F(a)(u))\ for\ all\ u \in U\}$$

is a left [right] ideal of S for all $a \in S$.

Proof. Assume that (F, S, N) is a θ-generalized int N-soft left ideal of S over U. Then for any $x, y \in S$ and $u \in U$, we have

$$F(xy)(u) \geq min\{\theta(u), F(y)(u)\}.$$

Let $y \in S_a$, we have

$$min\{\theta(u), F(a)(u)\} \leq F(y)(u)\ for\ a \in S.$$

Since,

$$F(xy)(u) \geq min\{\theta(u), F(y)(u)\}.$$
$$F(xy)(u) \geq min\{\theta(u), min\{\theta(u), F(a)(u)\}.$$
$$F(xy)(u) \geq min\{\theta(u), F(a)(u)\}.$$

So by definition xy $\in S_a$.

Hence, S_a is a left ideal of S for all a $\in S$.

Similarly, if (F, S, N) over U is a θ-generalized int N-soft right ideal of S over U, then the set

$$S_a = \{x \in S : F(x)(u) \geq min(\theta(u), F(a)(u))\ for\ all\ u \in U\}$$

is a right ideal of S for all a $\in S$. □

Corollary 9. *If (F, S, N) over U is a θ-generalized int N-soft two-sided ideal of S over U, then the set*

$$S_a = \{x \in S : F(x)(u) \geq min(\theta(u), F(a)(u))\ for\ all\ u \in U\}$$

is a two-sided ideal of S for all $a \in S$.

4. Conclusions

In this study, we describe the concepts of int N-soft subsemigroups, int N-soft left [right] ideals of S, int N-soft product and int N-soft characteristic function, θ-Generalized int N-soft subsemigroups and θ-Generalized int N-soft left [right] ideals of S. We also discuss some examples and theorems based on restricted (extended) union, restricted (extended) intersection, and γ-inclusive set. The notions presented in this article may be proposed for the application of N-soft sets in diverse fields of real-world containing uncertain data. Other set-theoretic operations such as set union, set difference, symmetric difference, and complementation may also be used to study various algebraic structures in the context of work by Zhan et al. in [27]. Motivated by this work, the specific algebraic structures of N -soft sets can be applied to decision-making problems and sound techniques for incorporating various criteria can be developed. Therefore, it will allow new perspectives for future work based on the algebraic structure of N-soft semigroups.

Author Contributions: Conceptualization, R.M.; Methodology, M.S. and R.M.; Software, M.S. and R.M.; Validation, M.S., R.M., M.J., M.N., F.J. and T.A.; Formal analysis, M.S. and R.M.; Investigation, M.S. and R.M.; Resources, M.J.; Data curation, R.M.; Writing—original draft, M.S. and R.M.; Writing—review & editing, M.J., M.N., F.J. and T.A.; Supervision, M.S.; Project administration, M.J., M.N., F.J. and T.A.; Funding acquisition, M.J., M.N., F.J. and T.A. All authors have read and agreed to the published version of the manuscript.

Funding: The author T. Abdeljawad would like to thank Prince Sultan University for paying the APC and for the support through the TAS research lab.

Data Availability Statement: Not applicable.

Acknowledgments: The author T. Abdeljawad would like to thank Prince Sultan University for the paying the APC and for the support through the TAS research lab.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Molodtsov, D. Soft set theory—First results. *Comput. Math. Appl.* **1999**, *37*, 19–31. [CrossRef]
2. Maji, P.K.; Biswas, R.; Roy, A.R. Soft set theory. *Comput. Math. Appl.* **2003**, *45*, 555–562. [CrossRef]
3. Ali, M.I.; Feng, F.; Liu, X.; Min, W.K.; Shabir, M. On some new operations in soft set theory. *Comput. Math. Appl.* **2009**, *57*, 1547–1553. [CrossRef]
4. Sezgin, A.; Atagün, A.O. On operations of soft sets. *Comput. Math. Appl.* **2011**, *61*, 1457–1467. [CrossRef]
5. Roy, A.R.; Maji, P.K. A fuzzy soft set theoretic approach to decision-making problems. *J. Comput. Appl. Math.* **2007**, *203*, 412–418. [CrossRef]
6. Ali, M.I.; Shabir, M.; Naz, M. Algebraic structures of soft sets associated with new operations. *Comput. Math. Appl.* **2011**, *61*, 2647–2654. [CrossRef]
7. Naz, M.; Shabir, M. Fuzzy soft sets and their algebraic structures. *World Appl. Sci. J. (Spec. Issue Appl. Math.)* **2013**, *22*, 45–61.
8. Shabir, M.; Ali, M.I. Soft ideals and generalized fuzzy ideals in semigroups. *New Math. Nat. Comput.* **2009**, *5*, 599–615. [CrossRef]
9. Ali, M.I.; Shabir, M.; Shum, K.P. On soft ideals over semigroups. *Southeast Asian Bull. Math.* **2010**, *34*, 595–610.
10. Feng, F.; Ali, M.I.; Shabir, M. Soft relations applied to semigroups. *Filomat* **2013**, *27*, 1183–1196. [CrossRef]
11. Çağman, N.; Enginoğlu, S. Soft set theory and uni–int decision making. *Eur. J. Oper. Res.* **2010**, *207*, 848–855. [CrossRef]
12. Jun, Y.B.; Lee, K.J.; Roh, E.H. Intersectional soft BCK/BCI-ideals. *Ann. Fuzzy Math. Inform.* **2012**, *4*, 1–7.
13. Çağman, N.; Çıtak, F.; Aktaş, H. Soft int-group and its applications to group theory. *Neural Comput. Appl.* **2012**, *21*, 151–158. [CrossRef]
14. Song, S.Z.; Kim, H.S.; Jun, Y.B. Ideal theory in semigroups based on intersectional soft sets. *Sci. World J.* **2014**, *2014*, 136424. [CrossRef] [PubMed]
15. Sezer, A.S.; Çağman, N.; Atagün, A.O.; Ali, M.I.; Türkmen, E. Soft intersection semigroups, ideals and bi-ideals; a new application on semigroup theory I. *Filomat* **2015**, *29*, 917–946. [CrossRef]
16. Cıtak, F.; Çağman, N. Soft int-rings and its algebraic applications. *J. Intell. Fuzzy Syst.* **2015**, *28*, 1225–1233. [CrossRef]
17. Fatimah, F.; Rosadi, D.; Hakim, R.B.; Alcantud, J.C.R. N-soft sets and their decision making algorithms. *Soft Comput.* **2018**, *22*, 3829–3842. [CrossRef]
18. Khan, M.J.; Kumam, P.; Liu, P.; Kumam, W.; Ashraf, S. A Novel Approach to Generalized Intuitionistic Fuzzy Soft Sets and Its Application in Decision Support System. *Mathematics* **2019**, *7*, 742. [CrossRef]

19. Akram, M.; Adeel, A.; Alcantud, J.C.R. Fuzzy N-soft sets: A novel model with applications. *J. Intell. Fuzzy Syst.* **2018**, *35*, 4757–4771. [CrossRef]
20. Riaz, M.; Çağman, N.; Zareef, I.; Aslam, M. N-soft topology and its applications to multi-criteria group decision making. *J. Intell. Fuzzy Syst.* **2019**, *36*, 6521–6536. [CrossRef]
21. Akram, M.; Ali, G.; Alcantud, J.C.R. New decision-making hybrid model: Intuitionistic fuzzy N-soft rough sets. *Soft Comput.* **2019**, *23*, 9853-9868. [CrossRef]
22. Kamacı, H.; Petchimuthu, S. Bipolar N-soft set theory with applications. *Soft Comput.* **2020**, *24*, 16727–16743. [CrossRef]
23. Kamaci, H. Introduction to *N*-soft algebraic structures. *Turk. J. Math.* **2020**, *44*, 2356–2379. [CrossRef]
24. Shabir, M.; Mushtaq, R.; Naz, M. An algebraic approach to N-soft sets with application in decision-making using TOPSIS. *J. Intell. Fuzzy Syst.* **2021**, *41*, 819–839. [CrossRef]
25. Kehayopulu, N.; Tsingelis, M. Regular ordered semigroups in terms of fuzzy subsets. *Inf. Sci.* **2006**, *176*, 3675–3693. [CrossRef]
26. Lee, J.H.; Kong, I.S.; Kim, H.S.; Jung, J.U. Generalized int-soft subsemigroups. *Ann. Fuzzy Math. Inform.* **2014**, *8*, 869–887.
27. Zhan, J.; Çağman, N.; Sezgin Sezer, A. Applications of soft union sets to hemirings via SU-h-ideals. *J. Intell. Fuzzy Syst. Appl. Eng. Technol.* **2014**, *26*, 1363–1370. [CrossRef]