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Left-definite Hamiltonian systems and corresponding nested circles

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Abstract: This work aims to construct the Titchmarsh-Weyl $M(\lambda)-$theory for an even-dimensional left-definite Hamiltonian system. For this purpose, we introduce a suitable Lagrange formula and selfadjoint boundary conditions including the spectral parameter λ. Then we obtain circle equations having nesting properties. Using the intersection point belonging to all the circles we share a lower bound for the number of Dirichlet-integrable solutions of the system.

Key words: Left-definite equations, Hamiltonian systems, Weyl's theory

1. Introduction

In this paper, we aim to introduce a lower bound for the number of linearly independent *integrable-square* solutions of the following $2m-$dimensional left-definite Hamiltonian system

$$JY' = [\lambda A + B]\, Y,\ x \in [a, b), \tag{1.1}$$

with the aid of the $M(\lambda)$ matrices and nested-surfaces related with selfadjoint boundary-value problems on some compact subintervals of $[a, b)$, where b is the only singular point of (1.1), λ is a complex parameter with $Im\lambda \neq 0$, $J, A = A(x), B = B(x)$ are $2m \times 2m$ matrices such that

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix},\ A(x) = \begin{bmatrix} P(x) & 0 \\ 0 & 0 \end{bmatrix},\ B(x) = \begin{bmatrix} -B_1(x) & \widetilde{B}^*(x) \\ \widetilde{B}(x) & B_2(x) \end{bmatrix}.$$

Here I is the identity matrix of dimension m, $P^*(x) = P(x)$ is an $m \times m$ matrix, $B_1^*(x) = B_1(x)$, $B_2^*(x) = B_2(x)$ and $\widetilde{B}(x)$ are $m \times m$ matrices such that

$$B_1(x) \geq 0,\ B_2(x) \geq 0.$$

Before passing to the details we shall share some background information on scalar and matrix-differential equations.

The investigation of singular second-order scalar-differential equations has been initiated by Weyl [26] with the aid of his famous limit-point/circle theory. This theory has been rehandled by Titchmarsh [22] and according to Titchmarsh-Weyl theory the following second-order differential equation

$$-(py')' + qy = \lambda wy,\ x \in [a, b), \tag{1.2}$$

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has at least one solution for $Im\lambda \neq 0$ satisfying

$$\int_a^b w\,|y|^2\,dx < \infty,$$

where b is the only singular point of (1.2), p, q, w are real-valued functions such that p^{-1}, q, w are locally integrable functions on $[a, b)$ and $w > 0$. This result is obtained with the aid of the solution χ of the form $\chi = \varphi + m\psi$ and some selfadjoint boundary-value problems constructed on compact subintervals of $[a, b)$, where φ and ψ are the solutions of (1.2) satisfying

$$\begin{aligned} \psi(a,\lambda) &= \sin\alpha, \quad p(a)\psi'(a,\lambda) = -\cos\alpha, \\ \varphi(a,\lambda) &= \cos\alpha, \quad p(a)\varphi'(a,\lambda) = \sin\alpha, \end{aligned}$$

and $0 \leq \alpha < \pi$. Indeed, the selfadjoint boundary conditions

$$\begin{aligned} \cos\alpha y(a) + \sin\alpha p(a)y'(a) = 0, \\ \cos\beta y(c) + \sin\beta p(c)y'(c) = 0, \end{aligned} \tag{1.3}$$

where $0 \leq \alpha, \beta < \pi$ and $a < c < b$, requires the form of m as the following

$$m = m(c,\beta,\lambda) = -\frac{\cot\beta\varphi(c,\lambda) + p(b)\varphi'(c,\lambda)}{\cot\beta\psi(c,\lambda) + p(b)\psi'(c,\lambda)}. \tag{1.4}$$

Now (1.4) implies that there exists a circle equation in the $m-$plane corresponding to the point c and it can be seen that this circle is totally contained in another circle corresponding to the point c_1 for $c_1 < c \leq b$. Consequently these circles have nesting properties.

The $m = m(c,\beta,\lambda)-$function given by (1.4) and the corresponding results are obtained for the right-definite equation (1.2) as $w > 0$. However, it is possible, in some sense, to allow w having an arbitrary sign on the given interval. For instance, if one imposes some certain signs on p and q (they are chosen as positive functions) then it is possible to get some results on spectral properties of the equation (1.2). This case is known as left-definite case. However, we shall note that there does not exist a global definition for left-definite equations (see [14], [19], [27]). Among these definitions Krall's approach depends on choosing the coefficients p, q, w all positive functions and the corresponding inner product is given by

$$\langle y, z\rangle = \int_a^c (py'\overline{z}' + qy\overline{z})\,dx - p(c)y'(c)\overline{z}(c) + p(a)y'(a)\overline{z}(a) \tag{1.5}$$

which depends on the boundary conditions at regular (or singular) point $c \leq b$. For the singular problem Krall and Race [15] obtained for $p, q, w > 0$ such that $\nu_1 w \leq q \leq \nu_2 w$, where ν_1, ν_2 are positive constants, that at least one solution of (1.2) should have a finite norm generated by the inner product (1.5). It is better to note that the problem that Krall and Race considered contains both the right and left-definite cases. However, according to Pleijel's idea [19], [20], one may construct a norm by (1.5) without the additional terms and the sign of w can be allowed to be an arbitrary sign on the given interval. Indeed, using (1.2) one obtains for the solution $y(x,\lambda)$ and $z(x,\mu)$ of (1.2) corresponding to the parameters λ and μ, respectively, that

$$(\overline{\mu}-\lambda)\int_a^c (py'\overline{z}'+qy\overline{z})\,dx = \begin{array}{l}\lambda\left(p(c)y(c)\overline{z}'(c)-p(a)y(a)\overline{z}'(a)\right)\\ -\overline{\mu}\left(p(c)y'(c)\overline{z}(c)-p(a)y'(a)\overline{z}(a)\right),\end{array} \tag{1.6}$$

where $a<c<b$, $p,q>0$ and there is no sign restriction on w. Now a selfadjoint problem requires the conditions

$$\begin{array}{c}\lambda\cos\alpha y(a)+\sin\alpha p(a)y'(a)=0,\\ \lambda\cos\beta y(c)+\sin\beta p(c)y'(c)=0,\end{array} \tag{1.7}$$

where $0\le\alpha,\beta<\pi$. $\chi=\varphi+m\psi$ satisfies the second boundary condition in (1.7), where

$$\begin{array}{ll}\psi(a,\lambda)=\frac{1}{\lambda}\sin\alpha, & p(a)\psi'(a,\lambda)=-\cos\alpha,\\ \varphi(a,\lambda)=\frac{1}{\lambda}\cos\alpha, & p(a)\varphi'(a,\lambda)=\sin\alpha,\end{array}$$

if m is of the form

$$m=m(c,\beta,\lambda)=-\frac{\lambda\cot\beta\varphi(c,\lambda)+p(c)\varphi'(c,\lambda)}{\lambda\cot\beta\psi(c,\lambda)+p(c)\psi'(c,\lambda)}. \tag{1.8}$$

Now obviously the form of m given in (1.8) differs from the form given in (1.4).

In this paper instead of considering the scalar equation (1.2) in the left-definite form we will consider the even-dimensional left-definite Hamiltonian system (1.1). We shall note that equation (1.2) and indeed any $rth-$order scalar formally symmetric differential equation can be embedded into an equivalent-dimensional Hamiltonian system [25]. Arbitrary-dimensional right-definite Hamiltonian system has been investigated by Atkinson [2] and valuable contributions on this theory have been shared by Kogan and Rofe-Beketov [12], Hinton and Shaw [7], [8], [9], [10], [11], Krall [13] and the others. Moreover, some results on left-definite matrix-eigenvalue problems and Hamiltonian systems have been studied by Schäfke and Schneider [21], Bennewitz [4], [5], Krall [16] and Vonhoff [24]. Here Krall [16] considered again right/left-definite Hamiltonian system on a regular interval and Uğurlu et al. [23] using Krall's approach investigated a singular right/left-definite Hamiltonian system with the aid of the results obtained for right-definite Hamiltonian system. However, in this work, we will consider only a left-definite singular Hamiltonian system and using Hinton-Shaw and Kralls' approaches we will construct $M(\lambda)-$theory for the left-definite even-dimensional Hamiltonian system that helps us to introduce a lower bound for the number of the linearly independent Dirichlet-integrable solutions of (1.1) and it seems that this is the first work on this theory. However, for the scalar case (1.2) the readers may see the book [6] and the papers [1], [3], [17], [18].

2. Basic results

In this section, we will introduce some basic results on the solutions of (1.1) and corresponding boundary value problems.

Eq. (1.1) has the following equivalent form

$$\begin{array}{l}-y_2'+B_1y_1-\widetilde{B}^*y_2=\lambda Py_1,\\ y_1'-\widetilde{B}y_1-B_2y_2=0,\end{array} \tag{2.1}$$

where y_1, y_2 are $m \times 1$ component vector-functions of Y as $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Using (1.1) and (2.1) we obtain for the solutions $Y(x,\lambda)$ and $Z(x,\mu)$ of (1.1) corresponding to the parameters λ and μ, respectively, that

$$\lambda \int_{c_1}^{c_2} Z^* AY dx = -z_1^* y_2 \mid_{c_1}^{c_2} + \int_{c_1}^{c_2} (z_1^* B_1 y_1 + z_2^* B_2 y_2)\, dx \tag{2.2}$$

and

$$\overline{\mu} \int_{c_1}^{c_2} Z^* AY dx = -z_2^* y_1 \mid_{c_1}^{c_2} + \int_{c_1}^{c_2} (z_1^* B_1 y_1 + z_2^* B_2 y_2)\, dx, \tag{2.3}$$

where $[c_1, c_2) \subseteq [a, b)$.

We shall adopt the notation

$$\langle Y, Z \rangle \mid_{c_1}^{c_2} = \int_{c_1}^{c_2} Z^* \begin{bmatrix} B_1 & \\ & B_2 \end{bmatrix} Y dx.$$

From now on we will assume the following definiteness condition

$$\langle Y, Y \rangle \mid_a^b > 0$$

for any nontrivial solution $Y(x,\lambda)$ of (1.1).

Using (2.2) and (2.3) we obtain the Lagrange's formula

$$(\lambda - \overline{\mu}) \langle Y, Z \rangle \mid_{c_1}^{c_2} = [Y_\lambda, Z_\mu](c_2) - [Y_\lambda, Z_\mu](c_1), \tag{2.4}$$

where $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ and

$$[Y_\lambda, Z_\mu] := \begin{bmatrix} \overline{\mu} z_1^* & z_2^* \end{bmatrix} J \begin{bmatrix} \lambda y_1 \\ y_2 \end{bmatrix}.$$

Now we shall impose some selfadjoint boundary conditions to the solutions of (1.1) on regular subintervals of $[a, b)$.

Let α_1, α_2 be some $m \times m$ matrices such that $rank(\alpha_1, \alpha_2) = m$ satisfying

$$\alpha_1 \alpha_1^* + \alpha_2 \alpha_2^* = I, \;\; \alpha_1 \alpha_2^* - \alpha_2 \alpha_1^* = 0,$$

where I is the $m \times m$ identity matrix. We shall consider the following boundary condition at $x = a$

$$\begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} y_1(a) \\ y_2(a) \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \begin{bmatrix} \lambda y_1(a) \\ y_2(a) \end{bmatrix} = 0, \tag{2.5}$$

where $\Lambda := \lambda I$.

Now let β_1, β_2 be some $m \times m$ matrices such that $rank(\beta_1, \beta_2) = m$ satisfying

$$\beta_1\beta_1^* + \beta_2\beta_2^* = I, \; \beta_1\beta_2^* - \beta_2\beta_1^* = 0,$$

and we shall consider the other boundary condition at a regular end point $x = c, \; a < c < b,$ as

$$\begin{bmatrix} \beta_1 & \beta_2 \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} y_1(c) \\ y_2(c) \end{bmatrix} = \begin{bmatrix} \beta_1 & \beta_2 \end{bmatrix} \begin{bmatrix} \lambda y_1(c) \\ y_2(c) \end{bmatrix} = 0. \tag{2.6}$$

First result on the corresponding boundary-value problem can be given as follows.

Lemma 2.1. *Let $Y(x,\lambda)$ be an eigenfunction of the problem (1.1), (2.5), (2.6) corresponding to the eigenvalue λ. Then λ should be real.*

Proof First we shall note that (2.5) and (2.6) can be handled as

$$\begin{bmatrix} \lambda y_1(a) \\ y_2(a) \end{bmatrix} = Kv, \; \begin{bmatrix} \lambda y_1(c) \\ y_2(c) \end{bmatrix} = Lv, \tag{2.7}$$

where v is a $2m \times 1$ vector and

$$K = \begin{bmatrix} 0 & \alpha_2^* \\ 0 & -\alpha_1^* \end{bmatrix}, \; L = \begin{bmatrix} \beta_2^* & 0 \\ -\beta_1^* & 0 \end{bmatrix}.$$

A direct calculation shows that

$$K^*JK = L^*JL = 0. \tag{2.8}$$

On the other side (2.4) implies that

$$2iIm\lambda \left\langle Y,Y \right\rangle |_a^c = v^* \left(L^*JL - K^*JK\right) v. \tag{2.9}$$

(2.8) and (2.9) complete the proof. □

Let $\mathcal{U}(x,\lambda), \; Im\lambda \neq 0,$ be an $2m \times 2m$ fundamental solution of (1.1) satisfying

$$\mathcal{U}(a,\lambda) = \begin{bmatrix} \lambda^{-1}\alpha_1^* & -\lambda^{-1}\alpha_2^* \\ \alpha_2^* & \alpha_1^* \end{bmatrix}.$$

We shall consider the partition of $\mathcal{U}(x,\lambda)$ as follows

$$\mathcal{U} = \begin{bmatrix} \Theta & \Phi \end{bmatrix} = \begin{bmatrix} \Theta_1 & \Phi_1 \\ \Theta_2 & \Phi_2 \end{bmatrix},$$

where $\Theta = \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix}, \; \Phi = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}$ are $2m \times m$ matrix-function such that $\Theta_1, \Theta_2, \Phi_1, \Phi_2$ are $m \times m$ matrix-functions. Note that Φ satisfies the condition (2.5).

(2.4) and a direct calculation gives the following.

Lemma 2.2. *Following equation holds*

$$\mathcal{U}^*(c,\overline{\lambda}) \begin{bmatrix} \Lambda^* & 0 \\ 0 & I \end{bmatrix} J \begin{bmatrix} \Lambda & 0 \\ 0 & I \end{bmatrix} \mathcal{U}(c,\lambda) = J, \; Im\lambda \neq 0.$$

Lemma 2.3. *Let* $\Delta = \{\lambda : \lambda \text{ is an eigenvalue of } (1.1),(2.5),(2.6)\}$. *Then* Δ *is denumerable. Let* λ_k *denote the members of* Δ, *where* k *belongs to a subset of the set of nonnegative integers. Then the series*

$$\sum_{\lambda_k \neq 0} |\lambda_k|^{-1-\epsilon}$$

converges for any $\epsilon > 0$.

Proof Any solution $V(x,\lambda)$, $Im\lambda \neq 0$, of (1.1) can be represented as

$$V(x,\lambda) = \begin{bmatrix} \lambda\alpha_1 & \alpha_2 \\ -\lambda\alpha_2 & \alpha_1 \end{bmatrix} \mathcal{U}(x,\lambda)V(a,\lambda). \tag{2.10}$$

Using the boundary conditions given in (2.7) and (2.10) we get that

$$\left\{ L - \begin{bmatrix} \Lambda & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \lambda\alpha_1 & \alpha_2 \\ -\lambda\alpha_2 & \alpha_1 \end{bmatrix} \mathcal{U}(c,\lambda) \begin{bmatrix} \Lambda^{-1} & 0 \\ 0 & I \end{bmatrix} K \right\} v = 0. \tag{2.11}$$

Hence for $v \neq 0$ we get from (2.11) that

$$\det \left\{ L - \begin{bmatrix} \Lambda & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \lambda\alpha_1 & \alpha_2 \\ -\lambda\alpha_2 & \alpha_1 \end{bmatrix} \mathcal{U}(c,\lambda) \begin{bmatrix} \Lambda^{-1} & 0 \\ 0 & I \end{bmatrix} K \right\} = 0$$

which coincides with the eigenvalues of the problem (1.1), (2.5), (2.6) and hence the eigenvalues should be a discrete subset of the real-line.

(1.1) and Gronwall's inequality imply that

$$\mathcal{U}(c,\lambda) = O(\exp(const.\,|\lambda|))$$

and hence the proof is completed. □

3. Nested circles

In this section, we will construct circle equations and show that these circles have nesting properties.

Let us consider the following $2m \times m$ matrix-function for $Im\lambda \neq 0$

$$\Psi(x,\lambda) = \mathcal{U}(x,\lambda) \begin{bmatrix} I \\ M \end{bmatrix}, \; x \in [a,b), \tag{3.1}$$

where M is an $m \times m$ matix. Note that Ψ is a solution of (1.1).

$\Psi(x,\lambda)$ satisfies the boundary condition (2.6) if M is of the form

$$M = M_c(\beta_1,\beta_2,\lambda) = -\left(\lambda\beta_1\Phi_1(c,\lambda) + \beta_2\Phi_2(c,\lambda)\right)^{-1}\left(\lambda\beta_1\Theta_1(c,\lambda) + \beta_2\Theta_2(c,\lambda)\right). \tag{3.2}$$

We shall note that $\left(\lambda\beta_1\Phi_1(c,\lambda) + \beta_2\Phi_2(c,\lambda)\right)^{-1}$ exists as otherwise λ with $Im\lambda \neq 0$ would be an eigenvalue of a selfadjoint boundary-value problem.

Consider the following expression

$$\mathcal{E}(M_c) := \left[\begin{array}{cc} I & M^* \end{array}\right] \mathcal{U}^*(c,\lambda) \left[\begin{array}{cc} \Lambda^* & 0 \\ 0 & I \end{array}\right] (J/i) \left[\begin{array}{cc} \Lambda & 0 \\ 0 & I \end{array}\right] \mathcal{U}(c,\lambda) \left[\begin{array}{c} I \\ M \end{array}\right]$$

and we shall adopt the following notation

$$\mathcal{U}^*(c,\lambda) \left[\begin{array}{cc} \Lambda^* & 0 \\ 0 & I \end{array}\right] (J/i) \left[\begin{array}{cc} \Lambda & 0 \\ 0 & I \end{array}\right] \mathcal{U}(c,\lambda) = \varepsilon \left[\begin{array}{cc} \mathbb{K} & \mathbb{L}^* \\ \mathbb{L} & \mathbb{N} \end{array}\right], \tag{3.3}$$

where $\varepsilon = 1$ when $Im\lambda > 0$ and $\varepsilon = -1$ when $Im\lambda < 0$. Therefore $\mathcal{E}(M)$ can also be represented as the following

$$\mathcal{E}(M_c) = \varepsilon \left[\begin{array}{cc} I & M^* \end{array}\right] \left[\begin{array}{cc} \mathbb{K} & \mathbb{L}^* \\ \mathbb{L} & \mathbb{N} \end{array}\right] \left[\begin{array}{c} I \\ M \end{array}\right]. \tag{3.4}$$

If M is of the form (3.2) we get the equation

$$\mathcal{E}(M_c) = 0. \tag{3.5}$$

Lemma 3.1. *We have the following*

$$\mathbb{N} = 2\left|Im\lambda\right| \int_a^c \Phi^* \left[\begin{array}{cc} B_1 & 0 \\ 0 & B_2 \end{array}\right] \Phi dx.$$

Proof (3.3) implies the form of $\mathbb{N}$ as

$$\varepsilon\mathbb{N} = \Phi^*(c,\lambda) \left[\begin{array}{cc} \Lambda^* & 0 \\ 0 & I \end{array}\right] (J/i) \left[\begin{array}{cc} \Lambda & 0 \\ 0 & I \end{array}\right] \Phi(c,\lambda). \tag{3.6}$$

On the other side a direct calculation gives that

$$\Phi^*(a,\lambda) \left[\begin{array}{cc} \Lambda^* & 0 \\ 0 & I \end{array}\right] (J/i) \left[\begin{array}{cc} \Lambda & 0 \\ 0 & I \end{array}\right] \Phi(a,\lambda) = 0. \tag{3.7}$$

Then (2.4), (3.6), (3.7) give the result. □

Corollary 3.2. (i) $\mathbb{N} > 0$,

(ii) *as* c *increases* $\mathbb{N}$ *increases.*

Expanding (3.5) we obtain the following form

$$(M_c - C)^* R_1^{-2} (M_c - C) = R_2^2, \tag{3.8}$$

where $C = \mathbb{N}^{-1}\mathbb{L}$, $R_1 = \mathbb{N}^{-1/2}$ and $R_2 = (\mathbb{L}^*\mathbb{N}^{-1}\mathbb{L} - \mathbb{K})^{1/2}$.

Lemma 3.3. *We have the following*

$$\varepsilon\mathbb{L} = 2Im\lambda \int_a^c \Phi^* \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \Theta dx - iI.$$

Proof From (3.3) we get that

$$\mathbb{L} = \Phi^*(c,\lambda) \begin{bmatrix} \Lambda^* & 0 \\ 0 & I \end{bmatrix} (J/i) \begin{bmatrix} \Lambda & 0 \\ 0 & I \end{bmatrix} \Theta(c,\lambda). \tag{3.9}$$

Moreover, a direct calculation shows that

$$\Phi^*(a,\lambda) \begin{bmatrix} \Lambda^* & 0 \\ 0 & I \end{bmatrix} (J/i) \begin{bmatrix} \Lambda & 0 \\ 0 & I \end{bmatrix} \Theta(a,\lambda) = -iI. \tag{3.10}$$

Now (2.4), (3.9) and (3.10) complete the proof. □

Lemma 3.4. $\mathbb{L}^*\mathbb{N}^{-1}\mathbb{L} - \mathbb{K} = \overline{\mathbb{N}}^{-1} > 0,$ *where* $\overline{\mathbb{N}}^{-1} = \mathbb{N}^{-1}(\overline{\lambda}).$

Proof Using Lemma 2.2 we obtain that

$$\left\{ J \begin{bmatrix} \Lambda & 0 \\ 0 & I \end{bmatrix} \mathcal{U}(x,\overline{\lambda}) \right\} \left\{ -J\mathcal{U}^*(x,\lambda) \begin{bmatrix} \Lambda^* & 0 \\ 0 & I \end{bmatrix} \right\} = \mathcal{I}, \tag{3.11}$$

where $\mathcal{I}$ denotes the identity matrix of dimension $2m$. Multiplying by J from the left of (3.11) we get that

$$\begin{bmatrix} \Lambda & 0 \\ 0 & I \end{bmatrix} \mathcal{U}(x,\overline{\lambda}) J \mathcal{U}^*(x,\lambda) \begin{bmatrix} \Lambda^* & 0 \\ 0 & I \end{bmatrix} = J.$$

Hence

$$\begin{aligned} J \quad &= \mathcal{U}^*(x,\lambda) \begin{bmatrix} \Lambda^* & 0 \\ 0 & I \end{bmatrix} J \begin{bmatrix} \Lambda & 0 \\ 0 & I \end{bmatrix} \mathcal{U}(x,\overline{\lambda}) = \mathcal{U}^*(x,\lambda) \begin{bmatrix} \Lambda^* & 0 \\ 0 & I \end{bmatrix} \times \\ &\left\{ -J \begin{bmatrix} \Lambda & 0 \\ 0 & I \end{bmatrix} \mathcal{U}(x,\lambda) J \mathcal{U}^*(x,\overline{\lambda}) \begin{bmatrix} \Lambda^* & 0 \\ 0 & I \end{bmatrix} \right\} J \begin{bmatrix} \Lambda & 0 \\ 0 & I \end{bmatrix} \mathcal{U}(x,\overline{\lambda}) \end{aligned}$$

or

$$\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} = - \begin{bmatrix} \mathbb{K} & \mathbb{L}^* \\ \mathbb{L} & \mathbb{N} \end{bmatrix} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} \overline{\mathbb{K}} & \overline{\mathbb{L}}^* \\ \overline{\mathbb{L}} & \overline{\mathbb{N}} \end{bmatrix}. \tag{3.12}$$

Using (3.12) we obtain that

$$\overline{\mathbb{N}}^{-1} = \mathbb{L}^*\mathbb{N}^{-1}\mathbb{L} - \mathbb{K}$$

and this completes the proof. □

Corollary 3.5. (i) $R_2 = \overline{R}_1,\ Im\lambda \neq 0.$

(ii) $\lim_{c\to b} R_1(c,\lambda) = R_b(\lambda) \geq 0,\ \lim_{c\to b} R_2(c,\lambda) = R_b(\overline{\lambda}) \geq 0.$

Theorem 3.6. *As* $c \to b$ $\mathcal{E}(M_c) = 0$ *are nested.*

Proof The interior of the circle $\mathcal{E}(M_c) = 0$ is described by

$$\varepsilon\Psi^*(c,\lambda) \begin{bmatrix} \Lambda^* & 0 \\ 0 & I \end{bmatrix} (J/i) \begin{bmatrix} \Lambda & 0 \\ 0 & I \end{bmatrix} \Psi(c,\lambda) \leq 0. \tag{3.13}$$

On the other side, (3.13) is equivalent to the following

$$2\left|Im\lambda\right|\int\limits_a^c \Psi^* \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \Psi dx \pm \left(M^* - M\right)/i \leq 0. \tag{3.14}$$

Let us choose a point c_1 which is smaller than c. Then M_c is contained in the circle corresponding to the point c_1. This completes the proof. □

Theorem 3.7. $\lim_{c\to b} C(c,\lambda) = C_b(\lambda)$.

Proof Using Corollary 3.2, Lemma 3.4, Corollary 3.5, and (3.8) we may introduce the following equation

$$\left(R_1^{-1}(M_c - C)\overline{R}_1^{-1}\right)^* \left(R_1^{-1}(M_c - C)\overline{R}_1^{-1}\right) = I,$$

and hence

$$M_c = C + R_1 U \overline{R}_1, \tag{3.15}$$

where U is a unitary matrix.

Let C_{c_1} and C_{c_2} be the centers of the circles $\mathcal{E}(M_{c_1}) = 0$ and $\mathcal{E}(M_{c_2}) = 0$, respectively. Using (3.15) we may write the equations

$$M_{c_1} = C_{c_1} + R_1(c_1)U_1\overline{R}_1(c_1)$$

and

$$M_{c_2} = C_{c_2} + R_1(c_2)U_2\overline{R}_1(c_2). \tag{3.16}$$

We have seen for $c_1 < c_2 \leq b$ that the circle $\mathcal{E}(M_{c_2}) = 0$ associated with the point c_2 is totally contained in the circle $\mathcal{E}(M_{c_1}) = 0$ associated with the point c_1. Therefore (3.16) can be written as

$$M_{c_2} = C_{c_1} + R_1(c_2)V_1\overline{R}_1(c_2), \tag{3.17}$$

where V_1 is a contractive matrix. Using (3.16) and (3.17) we get that

$$V_1 = R_1^{-1}(c_1)\left(C_{c_2} - C_{c_1} + R_1(c_2)U_2\overline{R}_1(c_2)\right)\overline{R}_1^{-1}(c_1). \tag{3.18}$$

(3.18) shows that there exists a mapping F from the unit ball into itself defined by $F(U_2) = V_1$ so that (3.18) can also be represented as

$$F(U_2) = R_1^{-1}(c_1)\left(C_{c_2} - C_{c_1} + R_1(c_2)U_2\overline{R}_1(c_2)\right)\overline{R}_1^{-1}(c_1). \tag{3.19}$$

F is a continuous mapping. Indeed, from (3.19) one obtains the equation

$$F(U_2) - F(V_1) = R_1^{-1}(c_1)R_1(c_2)\left(U_2 - V_1\right)\overline{R}_1(c_2)\overline{R}_1^{-1}(c_1).$$

Hence F has a fixed point by Brauwer's fixed point theorem. Replacing U_2 and V_1 by U we get that

$$\left\|C_{c_1} - C_{c_2}\right\| \leq \left\|R_1(c_1)\right\| \left\|\overline{R}_1(c_2) - \overline{R}_1(c_1)\right\| + \left\|\overline{R}_1(c_2)\right\| \left\|R_1(c_1) - R_1(c_2)\right\|.$$

Consequently, the centers constitute a Cauchy sequence and converge.

Using Lemma 3.1 and Lemma 3.3 we obtain the form of the center C_c as

$$C_c = -\left(2Im\lambda \int_a^c \Phi^* \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \Phi dx\right)^{-1} \left(2Im\lambda \int_a^c \Phi^* \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \Theta dx - iI\right)$$

and as we have seen that the limit $\lim_{c\to b} C_c = C_b$ exists. This completes the proof. □

4. Dirichlet-integrable solutions

We say that a solution $Y(x,\lambda)$ of (1.1) is Dirichlet-integrable on $[a,b)$ if the inequality

$$\int_a^b Y^*(x,\lambda) \begin{bmatrix} B_1(x) & 0 \\ 0 & B_2(x) \end{bmatrix} Y(x,\lambda)dx < \infty$$

holds.

From Corollary 3.5 and Theorem 3.7, we may infer that the limiting *point*

$$M_b = C_b + R_b U \overline{R}_b \tag{4.1}$$

is well-defined and exists.

Now we may introduce the following.

Theorem 4.1. *Let M_b be the matrix defined by (4.1) and $\Psi(x,\lambda)$, $Im\lambda \neq 0$, be of the form*

$$\Psi(x,\lambda) = \mathcal{U}(x,\lambda) \begin{bmatrix} I \\ M_b \end{bmatrix}.$$

Then $\Psi(x,\lambda)$ is Dirichlet-integrable on $[a,b)$.

Proof Using (4.1) we may consider the circle $\mathcal{E}(M_b) = 0$. For $Im\lambda > 0$ we get that M_b is contained in another circle $\mathcal{E}(M_c) = 0$, where $c < b$. Hence a direct calculation shows that

$$2Im\lambda \int_a^c \Psi^*(x,\lambda) \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \Psi(x,\lambda)dx \leq (M_b - M_b^*)/i. \tag{4.2}$$

(4.2) shows that the term $(M_b - M_b^*)/2iIm\lambda$ is an upper bound for the Dirichlet integral and passing to the limit as $c \to b$ we complete the proof for $Im\lambda > 0$.

For the case $Im\lambda < 0$ the proof can be introduced similarly and hence the proof is completed. □

Theorem 4.2. *There exist at least ν, $m \leq \nu \leq 2m$, Dirichlet-integrable solutions of (1.1), where $\nu = \min(rankR_b, rank\overline{R}_b)$.*

Proof Let $\Psi_1(x,\lambda)$ and $\Psi_2(x,\lambda)$ be $2m \times m$ matrix functions with $Im\lambda \neq 0$ defined by $\mathcal{U}(x,\lambda)\begin{bmatrix} I \\ C_b \end{bmatrix}$ and $\mathcal{U}(x,\lambda)\begin{bmatrix} I \\ M_b \end{bmatrix}$, respectively, where $M_b = C_b + R_b U \overline{R}_b$ and U is a unitary matrix. Hence we have

$$\begin{bmatrix} \Psi_1(x,\lambda) & \Psi_2(x,\lambda) \end{bmatrix} = \mathcal{U}(x,\lambda) \begin{bmatrix} I & I \\ C_b & M_b \end{bmatrix}. \tag{4.3}$$

The matrix appearing at the most right hand-side of (4.3) can be handled as the following

$$\begin{bmatrix} I & I \\ C_b & M_b \end{bmatrix} = \begin{bmatrix} I & 0 \\ C_b & R_b U \overline{R}_b \end{bmatrix} \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}. \tag{4.4}$$

Using Krall's results ([13], p. 671) we obtain from (4.4) that

$$rank \begin{bmatrix} I & I \\ C_b & M_b \end{bmatrix} = m + \min(rank R_b, rank \overline{R}_b). \tag{4.5}$$

Note that the right hand-side of (4.4) and $\mathcal{U}(x,\lambda)$ are invertible. Hence (4.3) and (4.5) complete the proof. □

Finally, we shall share a result for the location of the additional Dirichlet-integrable solutions of (1.1).

Theorem 4.3. *Let $\eta_1(c) \leq ... \leq \eta_m(c)$ be the eigenvalues of $\mathbb{N}$ and let exactly ν solutions of (1.1) be Dirichlet-integrable, where $m \leq \nu \leq 2m$. Then the values $\lim_{c\to b} \eta_1(c), ..., \lim_{c\to b} \eta_{m-\nu}(c)$ remain finite and the others go to infinity for $Im\lambda \neq 0$.*

Proof Let ξ_c be a unit eigenvector of $\mathbb{N}$ corresponding to the eigenvalue $\eta(c)$ and set $\Psi = \Phi\xi_c$. Then one gets for $Im\lambda \neq 0$ that

$$2iIm\lambda \int_a^c \Psi^* \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \Psi dx = \xi_c^* \Phi^*(c,\lambda) J \Phi(c,\lambda)\xi_c = i\varepsilon\eta(c),$$

where $\varepsilon = \begin{cases} 1, & Im\lambda > 0 \\ -1, & Im\lambda < 0 \end{cases}$. Hence

$$\int_a^c \Psi^* \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \Psi dx = \frac{\eta(c)}{2\,|Im\lambda|} < \frac{const.}{2\,|Im\lambda|}. \tag{4.6}$$

We shall choose a convergent subsequence of $\{\xi_c\}$ as $c \to b$ and we shall construct a solution $\Psi = \Phi\xi$ which is Dirichlet-integrable by (4.6). However, from Theorem 4.1 $\Psi = \mathcal{U}\begin{bmatrix} I \\ M_b \end{bmatrix}$ constitutes m of such solutions. Hence this completes the proof. □

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