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Original article

Monotonicity and positivity analyses for two discrete fractional-order operator types with exponential and Mittag–Leffler kernels

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A B S T R A C T

The discrete analysed fractional operator technique was employed to demonstrate positive findings concerning the Atangana-Baleanu and discrete Caputo-Fabrizo fractional operators. Our tests utilized discrete fractional operators with orders between $1 < \varphi < 2$, as well as between $1 < \varphi < \frac{3}{2}$. We employed the initial values of Mittag–Leffler functions and applied the principle of mathematical induction to ensure the positivity of the discrete fractional operators at each time step. As a result, we observed a significant impact of the positivity of these operators on $(\nabla\mathbb{Q})(\tau)$ within $\mathbb{N}_{\mathfrak{p}_0+1}$ according to the Riemann–Liouville interpretation. Furthermore, we established a correlation between the discrete fractional operators based on the Liouville-Caputo and Riemann–Liouville definitions. In addition, we emphasized the positivity of $(\nabla\mathbb{Q})(\tau)$ in the Liouville-Caputo sense by utilizing this relationship. Two examples are presented to validate the results.

1. Introduction

In the past decade, discrete fractional calculus has found applications in various fields including applied mathematics, physics, medicine, and chemistry (refer to, for instance, Goodrich and Peterson (2015); Atici et al. (2020); Atici et al. (2017); Iqbal et al. (2023a); Shah et al. (2022)). However, there has been a significant resurgence of interest in discrete fractional calculus operators towards the end of the 20th century and the beginning of the 21st century. This renewed interest stems from the development of novel fractional calculus operators (see Kilbas et al., 2006; Srivastava, 2021a; Srivastava, 2021b; Iqbal et al., 2023b; Shah

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et al., 2023), such as the Riemann–Liouville and Liouville-Caputo fractional operators (see Abdeljawad, 2011), Caputo-Fabrizo fractional operators with an exponential kernel (see Abdeljawad and Baleanu, 2017a; Abdeljawad et al., 2017), Atangana-Baleanu fractional operators with the Mittag–Leffler kernel (see Abdeljawad and Madjidi, 2017; Abdeljawad, 2018), and other generalized fractional operators (see Srivastava, 2021a; Srivastava, 2021b; Mohammed and Abdeljawad, 2020).

Discrete fractional operators serve as mathematical tools that generalize and expand upon the principles of differentiation and integration by accommodating non-integer orders. In contrast to conventional calculus, which solely deals with integer orders, discrete fractional operators operate on discrete data points, facilitating the examination of non-smooth or irregularly sampled signals. These operators provide a mechanism to describe and comprehend intricate systems that exhibit fractal or self-similar characteristics. By capturing signal behavior across various scales, discrete fractional operators enable the analysis of phenomena that cannot be sufficiently elucidated through traditional integer-order calculus. Their utility extends to diverse domains, including signal processing, image analysis, time series analysis, and fractional differential equations, among others. Ongoing research in the field of discrete fractional operators focuses on refining their properties, devising efficient computational algorithms, and exploring novel applications in emerging fields.

In the last few years, the search for analyses of the discrete fractional operators that are close to the monotonicity of the function, that is to say, satisfying the delta or nabla positivity of the function, has received a lot of attention, see for example (Goodrich, 2014; Du et al., 2016; Goodrich, 2016; Dahal and Goodrich, 2021). The theory of positivity and monotonicity analyses of the discrete fractional operators on Riemann–Liouville differences was started by by Dahal and Goodrich in Dahal and Goodrich (2014). After that, many authors followed their idea to investigate further results for Riemann–Liouville difference operators, for example in Atici and Uyanik (2015), Dahal and Goodrich (2014), Goodrich and Lizama (2020), Goodrich and Lyons (2020). These results have been developed and applied on other types of discrete fractional operators (see, for example, Abdeljawad and Abdalla, 2017; Mohammed et al., 2021 on the Liouville-Caputo, Mohammed and Baleanu, 2022; Abdeljawad, 2017 on the Caputo-Fabrizo, and Mohammed et al., 2022a; Abdeljawad and Baleanu, 2017b; Suwan et al., 2018 on the Atangana-Baleanu fractional difference operators). Furthermore, many researchers have considered the relationship between sequential fractional difference operators and the positivity and monotonicity of their corresponding functions in both the Riemann–Liouville sense and the Liouville-Caputo sense (see, for example, Dahal and Goodrich, 2019; Dahal et al., 2021; Goodrich et al., 2021a; Mohammed et al., 2022b; Goodrich et al., 2021b; Goodrich, 2017).

In a very recent work, Jia et al. Jia et al. (2015) demonstrated a connection between the positivity of the Riemann–Liouville fractional difference and the monotonicity of the function involved. Based on these results, we will find a relationship between the positivity of the fractional differences and their corresponding functions in the Riemann–Liouville sense. In addition, we will establish these results in the Liouville-Caputo sense by using the relationship between the Riemann–Liouville sense and the Liouville-Caputo sense of the Caputo-Fabrizo and Atangana-Baleanu fractional differences, which we will prove in this article as well.

The organization of the study is as follows: Section 2 provides an introduction to the preliminaries and notations that will be utilized. The theory of discrete fractional calculus has been briefly discussed. However, even though some generalizations of relationships exist for lower order $0 < \varphi < 1$, a study in the discrete Caputo-Fabrizo and Atangana-Baleanu fractional settings are missing for the higher orders. For that reason, in this section, we make these relationships between the discrete fractional operators with respect to their corresponding Riemann–Liouville and Liouville-Caputo operators. In Section 3, we present our main results on the discrete Caputo-Fabrizo and Atangana-Baleanu fractional operators for functions defined on $\mathbb{N}_{\mathrm{p}_0}$. Lastly, the conclusion of the study presented in this article is provided in Section 4.

2. Definitions, preliminaries and other necessary tools

In this section, we delve into multiple definitions of the Atangana-Baleanu and discrete Caputo-Fabrizo fractional operators. Additionally, we provide a set of remarks that are crucial for establishing the main results (for more comprehensive information, refer to Abdeljawad (2018); Abdeljawad and Baleanu (2017a); Abdeljawad et al. (2017); Abdeljawad and Madjidi (2017)).

$$\mathscr{F}_{\mathrm{p}_0,\kappa} := \left\{\mathrm{Q};\ \mathrm{Q} : \mathbb{N}_{\mathrm{p}_0-\kappa} \to \mathbb{R} \quad \text{with } \kappa \in \mathbb{N}_0,\ \mathrm{p}_0 \in \mathbb{R}\right\},$$

where $\mathbb{N}_{\mathrm{p}_0} := \{\mathrm{p}_0, \mathrm{p}_0 + 1, \cdots\}$.

Definition 2.1 (*see Abdeljawad, 2018; Abdeljawad and Baleanu, 2017a*). Let $\mathrm{Q} \in \mathscr{F}_{\mathrm{p}_0,0}$ and $\mathscr{C}(\varphi) > 0$ is a multiplier. Then the following operators:

$$\left({}^{ABC}_{\ \mathrm{p}_0}\nabla^{\varphi}\mathrm{Q}\right)(\tau) := \frac{\mathscr{C}(\varphi)}{-\varphi+1}\sum_{r=\mathrm{p}_0+1}^{\tau}(\nabla_r\mathrm{Q})(r)\mathrm{E}_{\overline{\varphi}}(\xi, 1+\tau-r) \quad \{\tau \in \mathbb{N}_{\mathrm{p}_0+1}\}, \tag{2.1}$$

and

$$\left({}^{CFC}_{\ \mathrm{p}_0}\nabla^{\varphi}\mathrm{Q}\right)(\tau) := \mathscr{C}(\varphi)\sum_{r=\mathrm{p}_0+1}^{\tau}(\nabla_r\mathrm{Q})(r)(-\varphi+1)^{\tau-r} \quad \{\tau \in \mathbb{N}_{\mathrm{p}_0+1}\} \tag{2.2}$$

are called the discrete Atangana-Baleanu and Caputo-Fabrizo fractional operators in the Liouville-Caputo sense, respectively. Also, the following operators:

$$\left({}^{ABR}_{\ \mathrm{p}_0}\nabla^{\varphi}\mathrm{Q}\right)(\tau) := \frac{\mathscr{C}(\varphi)}{-\varphi+1}\nabla_{\tau}\sum_{r=\mathrm{p}_0+1}^{\tau}\mathrm{Q}(r)\mathrm{E}_{\overline{\varphi}}(\xi, \tau-r+1) \quad \{\tau \in \mathbb{N}_{\mathrm{p}_0+1}\}, \tag{2.3}$$

and

$$\left({}^{CFR}_{\ \mathrm{p}_0}\nabla^{\varphi}\mathrm{Q}\right)(\tau) := \mathscr{C}(\varphi)\nabla_{\tau}\sum_{r=\mathrm{p}_0+1}^{\tau}\mathrm{Q}(r)(-\varphi+1)^{\tau-r} \quad \{\tau \in \mathbb{N}_{\mathrm{p}_0+1}\} \tag{2.4}$$

are called the discrete Atangana-Baleanu and Caputo-Fabrizo fractional operators in the Riemann–Liouville sense, respectively.

The above definitions have been generalized by Abdeljawad et al. (2017), and by Abdeljawad and Madjidi (2017), as follows.

Definition 2.2 (*see Abdeljawad et al., 2017; Abdeljawad and Madjidi, 2017*). For $\mathrm{Q} \in \mathscr{F}_{\mathrm{p}_0,\kappa}$ with $\kappa < \varphi \leqq \kappa + 1$, the discrete Atangana-Baleanu and Caputo-Fabrizo fractional difference operators can be expressed as follows:

$$\begin{aligned}&\left({}^{CFC}_{\ \mathrm{p}_0}\nabla^{\varphi}\mathrm{Q}\right)(\tau) = \left({}^{CFC}_{\ \mathrm{p}_0}\nabla^{\varphi-\kappa}\nabla^{\kappa}\mathrm{Q}\right)(\tau) \quad \text{and}\\ &\left({}^{ABC}_{\ \mathrm{p}_0}\nabla^{\varphi}\mathrm{Q}\right)(\tau) = \left({}^{ABC}_{\ \mathrm{p}_0}\nabla^{\varphi-\kappa}\nabla^{\kappa}\mathrm{Q}\right)(\tau),\end{aligned} \tag{5}$$

in the Liouville-Caputo sense, and

$$\begin{aligned}&\left({}^{CFR}_{\ \mathrm{p}_0}\nabla^{\varphi}\mathrm{Q}\right)(\tau) = \left({}^{CFR}_{\ \mathrm{p}_0}\nabla^{\varphi-\kappa}\nabla^{\kappa}\mathrm{Q}\right)(\tau), \quad \text{and}\\ &\left({}^{ABR}_{\ \mathrm{p}_0}\nabla^{\varphi}\mathrm{Q}\right)(\tau) = \left({}^{ABR}_{\ \mathrm{p}_0}\nabla^{\varphi-\kappa}\nabla^{\kappa}\mathrm{Q}\right)(\tau),\end{aligned} \tag{6}$$

in the Riemann–Liouville sense, for each $\tau \in \mathbb{N}_{p_0+1}$.

Remark 2.1. The above definitions contain the one-parameter Mittag–Leffler function $E_\alpha(z)$, which is defined here as follows (see Mohammed and Abdeljawad, 2020; see also Srivastava, 2021a; Srivastava, 2021b for much more general families of Mittag–Leffler type functions):

$$\mathrm{E}_{\overline{\varphi}}(\xi,\tau) := \sum_{k=0}^{\infty} \xi^k \frac{\tau^{\overline{k\varphi}}}{\Gamma(k\varphi+1)} =: E_{\overline{\varphi}}\left(\xi\tau^{\overline{\varphi}}\right) \tag{7}$$

for any $\xi \in \mathbb{R}$ such that $|\xi| < 1$, $\varphi, \tau \in \mathbb{C}$ with $\mathrm{Re}(\varphi) > 0$. and $\tau^{\overline{\varphi}}$ is the rising function defined by

$$\tau^{\overline{\varphi}} = \frac{\Gamma(\tau+\varphi)}{\Gamma(\tau)}, \tag{8}$$

for φ in $\mathbb{R}$ and τ in $\mathbb{R} \setminus \{\cdots,-2,-1,0\}$.

On the other hand, in view of Mohammed et al. (2022b, Remark 2.2), we have some initial values for $\xi = \frac{-\varphi+1}{-\varphi+2}$ and $1 < \varphi < \frac{3}{2}$:

- $\mathrm{E}_{\overline{\varphi-1}}(\xi,0) = 1$,
- $\mathrm{E}_{\overline{\varphi-1}}(\xi,1) = 2-\varphi$,
- $\mathrm{E}_{\overline{\varphi-1}}(\xi,2) = \varphi(2-\varphi)^2$,
- $\mathrm{E}_{\overline{\varphi-1}}(\xi,3) = \frac{2-\varphi}{2}\left[(\varphi-1)^3(2\varphi-3) - 3(\varphi-1)^2 + 2\right]$.

According to this, together with Fig. 1, one can observe that

$$0 < \mathrm{E}_{\overline{\varphi-1}}(\xi,\tau) < 1$$

for each $1 < \varphi < \frac{3}{2}$ and $\tau = 1,2,3,\cdots$, and it is monotonically decreasing for each $1 < \varphi < \frac{3}{2}$ and $\tau = 0,1,2,\cdots$.

By establishing a correlation between the interpretations of the Riemann–Liouville and Liouville-Caputo concepts, a connection can be derived between the discrete Caputo-Fabrizio and Atangana-Baleanu fractional operators.

Proposition 2.1. *Assume that* $\mathrm{Q} \in \mathscr{F}_{p_0,0}$. *Then, for* $\kappa < \varphi \leqq \kappa + 1$, *it is asserted that*

$$\left({}^{CFR}_{\ \mathrm{p}_0}\nabla^{\varphi}\mathrm{Q}\right)(\tau) = \left({}^{CFC}_{\ \mathrm{p}_0}\nabla^{\varphi}\mathrm{Q}\right)(\tau) + \frac{\mathscr{C}(\varphi-\kappa)}{\kappa+-\varphi+1}(\kappa+-\varphi+1)^{\tau-\mathrm{p}_0}\left(\nabla^{\kappa}\mathrm{Q}\right)(\mathrm{p}_0).$$

Furthermore, for $\kappa < \varphi \leqq \kappa + \frac{1}{2}$, *it is asserted that*

$$\left({}^{ABR}_{\ \mathrm{p}_0}\nabla^{\varphi}\mathrm{Q}\right)(\tau) = \left({}^{ABC}_{\ \mathrm{p}_0}\nabla^{\varphi}\mathrm{Q}\right)(\tau) + \frac{\mathscr{C}(\varphi-\kappa)}{\kappa+-\varphi+1}\mathrm{E}_{\overline{\varphi-\kappa}}(\xi_\kappa,\tau-\mathrm{p}_0)\left(\nabla^{\kappa}\mathrm{Q}\right)(\mathrm{p}_0),$$

for each $\tau \in \mathbb{N}_{\mathrm{p}_0+1}$, *where*

$$\xi_\kappa = -\frac{\varphi-\kappa}{\kappa+-\varphi+1}.$$

Proof. The first part of the proof can be deduced by referencing Definition 2.2 and Abdeljawad (2018, Proposition 8). Similarly, the second part of the proof can be established by relying on Definition 2.2 and Abdeljawad (2018, Theorem 10). □

Remark 2.2. We note that, for $\xi = -\frac{\varphi}{-\varphi+1}$ with $0 < \varphi < \frac{1}{2}$, it is known that $|\xi| < 1$. Moreover, since

$$0 < \varphi - \kappa \leqq \frac{1}{2},$$

we see that $|\xi_\kappa| < 1$ for

$$\xi_\kappa = -\frac{\varphi-\kappa}{\kappa+-\varphi+1}$$

with

$$\kappa < \varphi \leqq \kappa + \frac{1}{2}.$$

3. Main results

Within this section, we shall provide the proofs for the positivity theorems. which are stated as Theorem 3.1 and Theorem 3.2 below. To do this, we first need to two lemmas, one for the discrete Caputo-Fabrizo operators and the other one for the Atangana-Baleanu operators.

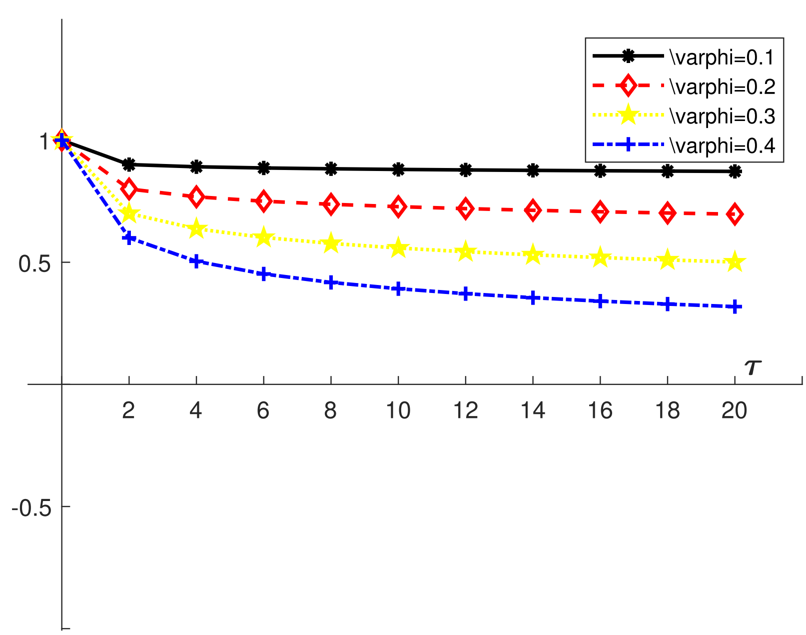

Fig. 1. Graph of $\mathrm{E}_{\overline{\varphi-1}}(\xi,\tau)$ for $\varphi \in (1, 1.5)$ and $\tau = 1,2,\cdots,20$.

Lemma 3.1. *Assume that* $Q \in \mathscr{F}_{p_0,0}, 1 < \varphi < 2$ *and* $\left({}^{CFR}_{p_0}\nabla^{\varphi}Q\right)(\tau) \geqslant q0 \ \forall\ \tau \in \mathbb{N}_{p_0+1}$. *Then*

$$(\nabla Q)(\tau) \geqslant q(\varphi-1)\left\{(-\varphi+2)^{\tau-p_0-2}(\nabla Q)(p_0+1) + \sum_{r=p_0+2}^{\tau-1}(\nabla_r Q)(r)(-\varphi+2)^{\tau-r-1}\right\},$$

for each $\tau \in \mathbb{N}_{p_0+2}$.

Proof. From Definition 2.1 with $1 < \varphi < 2$, one can see, for each $\tau \in \mathbb{N}_{p_0+2}$, that

$$\begin{aligned}\left({}^{CFR}_{p_0}\nabla^{\varphi}Q\right)(\tau) &= \mathscr{C}(\varphi-1)\left\{\sum_{r=p_0+1}^{\tau}(\nabla_r Q)(r)(-\varphi+2)^{\tau-r} - \sum_{r=p_0+1}^{\tau-1}(\nabla_r Q)(r)(-\varphi+2)^{\tau-r-1}\right\}\\ &= \mathscr{C}(\varphi-1)\left\{(\nabla Q)(\tau) + \sum_{r=p_0+1}^{\tau-1}(\nabla_r Q)(r)\left[(-\varphi+2)^{\tau-r} - (-\varphi+2)^{\tau-r-1}\right]\right\}\\ &= \mathscr{C}(\varphi-1)\Big\{(\nabla Q)(\tau) + (-\varphi+2)(-\varphi+1)^{\tau-p_0-2}(\nabla Q)(p_0+1)\\ &\quad +(-\varphi+1)\sum_{r=p_0+2}^{\tau-1}(\nabla_r Q)(r)(-\varphi+2)^{\tau-r-1}\Big\},\end{aligned} \tag{3.1}$$

which, together with the fact that $\mathscr{C}(\varphi-1) > 0$ and the assumption that $\left({}^{CFR}_{p_0}\nabla^{\varphi}Q\right)(\tau) \geqslant q0$, can be rearranged to the derive the desired results. □

Corollary 3.1. *Assume that* $Q \in \mathscr{F}_{p_0,0}, 1 < \varphi < 2$ *and*

$$\left({}^{CFC}_{p_0}\nabla^{\varphi}Q\right)(\tau) \geqslant q - \frac{\mathscr{C}(\varphi-1)}{-\varphi+2}(-\varphi+2)^{\tau-p_0}(\nabla Q)(p_0)$$

for each $\tau \in \mathbb{N}_{p_0+1}$. *Then*

$$(\nabla Q)(\tau) \geqslant q(-1+\varphi)\left\{(2-\varphi)^{\tau-p_0-2}(\nabla Q)(1+p_0) + \sum_{r=p_0+2}^{\tau-1}(\nabla_r Q)(r)(-\varphi+2)^{\tau-r-1}\right\},$$

for each $\tau \in \mathbb{N}_{p_0+2}$.

Proof. The deduction of the outcome can be made directly by referring to Proposition 2.1 and Lemma 3.1. □

Lemma 3.2. *Suppose that* $Q \in \mathscr{F}_{p_0,0}, 1 < \varphi < \frac{3}{2}, \xi_1 = -\frac{\varphi-1}{-\varphi+2}$ *and* $\left({}^{ABR}_{p_0}\nabla^{\varphi}Q\right)(\tau) \geqslant q0$ *for each* $\tau \in \mathbb{N}_{p_0+1}$. *Then*

$$\begin{aligned}(\nabla Q)(\tau) \geqslant q\frac{1}{-\varphi+2}\Big\{&\left[\mathsf{E}_{\overline{\varphi-1}}(\xi_1,\tau-p_0-1) - \mathsf{E}_{\overline{\varphi-1}}(\xi_1,\tau-p_0)\right](\nabla Q)(1+p_0)\\ &+ \sum_{r=p_0+2}^{\tau-1}\left[\mathsf{E}_{\overline{\varphi-1}}(\xi_1,\tau-r) - \mathsf{E}_{\overline{\varphi-1}}(\xi_1,\tau-r+1)\right](\nabla_r Q)(r)\Big\},\end{aligned}$$

for each $\tau \in \mathbb{N}_{p_0+2}$.

Proof. According to Definition 2.1 when $1 < \varphi < \frac{3}{2}$, we find for each $\tau \in \mathbb{N}_{p_0+2}$ that

$$\begin{aligned}\left({}^{ABR}_{p_0}\nabla^{\varphi}Q\right)(\tau) &= \tfrac{\mathscr{C}(\varphi-1)}{-\varphi+2}\left\{\sum_{r=p_0+1}^{\tau}\mathsf{E}_{\overline{\varphi-1}}(\xi_1,\tau-r+1)(\nabla_r Q)(r) - \sum_{r=p_0+1}^{\tau-1}\mathsf{E}_{\overline{\varphi-1}}(\xi_1,\tau-r)(\nabla_r Q)(r)\right\}\\ &= \tfrac{\mathscr{C}(\varphi-1)}{2-\varphi}\left\{(-\varphi+2)(\nabla Q)(\tau) + \sum_{r=p_0+1}^{\tau-1}\left[\mathsf{E}_{\overline{\varphi-1}}(\xi_1,\tau-r+1) - \mathsf{E}_{\overline{\varphi-1}}(\xi_1,\tau-r)\right](\nabla_r Q)(r)\right\}\\ &= \tfrac{\mathscr{C}(\varphi-1)}{2-\varphi}\Big\{(-\varphi+2)(\nabla Q)(\tau) + \left[\mathsf{E}_{\overline{\varphi-1}}(\xi_1,\tau-p_0) - \mathsf{E}_{\overline{\varphi-1}}(\xi_1,\tau-p_0-1)\right](\nabla Q)(p_0+1)\\ &\quad + \sum_{r=p_0+2}^{\tau-1}\left[\mathsf{E}_{\overline{\varphi-1}}(\xi_1,\tau-r+1) - \mathsf{E}_{\overline{\varphi-1}}(\xi_1,\tau-r)\right](\nabla_r Q)(r)\Big\}.\end{aligned} \tag{2}$$

Considering this together with the fact that $\mathscr{C}(\varphi-1) > 0$ and the assumption $\left({}^{ABR}_{p_0}\nabla^{\varphi}Q\right)(\tau) \geqslant q0$, we arrive at the intended result. □

Corollary 3.2. *Assume that* $Q \in \mathscr{F}_{p_0,0}, 1 < \varphi < \frac{3}{2}, \xi_1 = -\frac{\varphi-1}{-\varphi+2}$ *and* $\left({}^{ABC}_{p_0}\nabla^{\varphi}Q\right)(\tau) \geqslant q - \frac{\mathscr{C}(\varphi-1)}{-\varphi+2}\mathsf{E}_{\overline{\varphi-1}}(\xi_1,\tau-p_0)(\nabla Q)(p_0)$ *for each* $\tau \in \mathbb{N}_{p_0+1}$. *Then, we have*

$$\begin{aligned}(\nabla Q)(\tau) \geqslant q\frac{1}{2-\varphi}\Big\{&\left[\mathsf{E}_{\overline{\varphi-1}}(\xi_1,\tau-p_0) - \mathsf{E}_{\overline{\varphi-1}}(\xi_1,\tau-p_0-1)\right](\nabla Q)(p_0+1)\\ &+ \sum_{r=p_0+2}^{\tau-1}\left[\mathsf{E}_{\overline{\varphi-1}}(\xi_1,\tau-r+1) - \mathsf{E}_{\overline{\varphi-1}}(\xi_1,\tau-r)\right](\nabla_r Q)(r)\Big\},\end{aligned}$$

for each $\tau \in \mathbb{N}_{p_0+2}$.

Proof. The proof can be straightforwardly derived from the application of Proposition 2.1 and Lemma 3.2. □

Theorem 3.1. *Suppose that* $Q \in \mathscr{F}_{p_0,0}$, $1 < \varphi < 2$ *and* $\left({}^{CFR}_{p_0}\nabla^{\varphi}Q\right)(\tau) \geqslant q0 \ \forall\ \tau \in \mathbb{N}_{p_0+1}$. *Then* $(\nabla Q)(\tau) \geqslant q0 \ \forall\ \tau \in \mathbb{N}_{p_0+1}$.

Proof. We proceed with the proof by mathematical induction. By using Definition (6) for $1 < \varphi < 2$ at $\tau = p_0+1$, one can have

$$\begin{aligned}\left({}^{CFR}_{p_0}\nabla^{\varphi}Q\right)(p_0+1) &= \left({}^{CFR}_{p_0}\nabla^{\varphi-1}\nabla Q\right)(p_0+1)\\ &= \mathscr{C}(\varphi-1)\left\{\sum_{r=1+p_0}^{1+p_0}(\nabla Q)(r)(-\varphi+2)^{p_0+1-r} - \sum_{r=1+p_0}^{p_0}(\nabla Q)(r)(2-\varphi)^{p_0-r}\right\}\\ &= \mathscr{C}(\varphi-1)(\nabla Q)(p_0+1),\end{aligned}$$

which implies that $(\nabla Q)(p_0+1) \geqslant q0$ by the hypothesis and the fact that $\mathscr{C}(\varphi-1) > 0$. At $\tau = p_0+2$, we have

$$\begin{aligned}\left({}^{CFR}_{p_0}\nabla^{\varphi}Q\right)(p_0+2) &= \mathscr{C}(\varphi-1)\left\{\sum_{r=1+p_0}^{2+p_0}(\nabla Q)(r)(-\varphi+2)^{p_0+2-r} - \sum_{r=1+p_0}^{1+p_0}(\nabla Q)(r)(-\varphi+2)^{p_0+1-r}\right\}\\ &= \mathscr{C}(\varphi-1)\{(-\varphi+1)(\nabla Q)(1+p_0) + (\nabla Q)(2+p_0)\} \geqslant q0,\end{aligned}$$

which leads to

$$(\nabla Q)(p_0+2) \geqslant q\underbrace{(\varphi-1)}_{>0}\,\underbrace{(\nabla Q)(p_0+1)}_{\geqslant q0} \geqslant q0.$$

Suppose now that $\kappa \geqslant q2$ and that $(\nabla Q)(p_0+\imath) \geqslant q0$ for $\imath = 2, 3, \cdots, \kappa$. Then, by using Lemma 3.1 at $\tau = p_0+\kappa+1$, we have

$$\begin{aligned}(\nabla Q)(p_0+\kappa+1) &\geqslant q\underbrace{(\varphi-1)}_{>0}\left\{\underbrace{(2-\varphi)^{\kappa-1}}_{>0}\underbrace{(\nabla Q)(1+p_0)}_{\geqslant q0} + \sum_{r=p_0+2}^{p_0+\kappa}\underbrace{(\nabla_r Q)(r)}_{\geqslant q0 \text{ by inductive assumption}}\underbrace{(-\varphi+2)^{\kappa+p_0-r}}_{>0}\right\}\\ &\geqslant q0,\end{aligned}$$

which completes the proof of the induction process. Hence, the proof is done. □

Corollary 3.3. *Suppose* $Q \in \mathscr{F}_{p_0,0}, 1 < \varphi < 2$ *and* $\left({}^{CFC}_{p_0}\nabla^{\varphi}Q\right)(\tau) \geqslant q - \frac{\mathscr{C}(\varphi-1)}{-\varphi+2}(-\varphi+2)^{\tau-p_0}(\nabla Q)(p_0) \ \forall\ \tau \in \mathbb{N}_{p_0+1}$. *Then,* $(\nabla Q)(\tau) \geqslant q0 \ \forall\ \tau \in \mathbb{N}_{p_0+1}$.

Theorem 3.2. *Suppose that* $Q \in \mathscr{F}_{p_0,0}$, $1 < \varphi < \frac{3}{2}, \xi_1 = -\frac{\varphi-1}{-\varphi+2}$ *and* $\left({}^{ABR}_{p_0}\nabla^{\varphi}Q\right)(\tau) \geqslant q0 \ \forall\ \tau \in \mathbb{N}_{p_0+1}$. *Then* $(\nabla Q)(\tau) \geqslant q0 \ \forall\ \tau \in \mathbb{N}_{p_0+1}$.

Proof. Again, we use mathematical induction to proceed with the proof of Theorem 3.2. Indeed, by using Definition (6) for $1 < \varphi < \frac{3}{2}$ at $\tau = p_0+1$, and Remark 2.1, we see that

$$\begin{aligned}\left({}^{ABR}_{p_0}\nabla^{\varphi}Q\right)(p_0+1) &= \left({}^{CFR}_{p_0}\nabla^{\varphi-1}\nabla Q\right)(p_0+1)\\ &= \tfrac{\mathscr{C}(\varphi-1)}{-\varphi+2}\left\{\sum_{r=1+p_0}^{1+p_0}(\nabla Q)(r)\mathsf{E}_{\overline{\varphi-1}}(\xi_1,p_0+2-r) - \sum_{r=1+p_0}^{p_0}(\nabla Q)(r)\mathsf{E}_{\overline{\varphi-1}}(\xi_1,p_0-r+1)\right\}\\ &= \tfrac{\mathscr{C}(\varphi-1)}{-\varphi+2}(\nabla Q)(p_0+1)\mathsf{E}_{\overline{\varphi-1}}(\xi_1,1) = \mathscr{C}(\varphi-1)(\nabla Q)(p_0+1) \geqslant q0,\end{aligned}$$

which implies that $(\nabla \mathrm{Q})(\mathrm{p}_0+1) \geqslant q0$, where we have used the fact that $\mathscr{C}(\varphi-1) > 0$. Moreover, at $\tau = \mathrm{p}_0+2$, we have

$$\begin{aligned}\left({}^{ABR}_{\mathrm{p}_0}\nabla^{\varphi}\mathrm{Q}\right)(\mathrm{p}_0+2) &= \tfrac{\mathscr{C}(\varphi-1)}{-\varphi+2}\left\{\sum_{r=1+\mathrm{p}_0}^{2+\mathrm{p}_0}(\nabla\mathrm{Q})(r)\mathrm{E}_{\overline{\varphi-1}}(\xi_1,\mathrm{p}_0-r+3)-\sum_{r=1+\mathrm{p}_0}^{\mathrm{p}_0+1}(\nabla\mathrm{Q})(r)\mathrm{E}_{\overline{\varphi-1}}(\xi_1,\mathrm{p}_0-r+2)\right\}\\ &= \tfrac{\mathscr{C}(\varphi-1)}{-\varphi+2}\left\{(\nabla\mathrm{Q})(1+\mathrm{p}_0)\mathrm{E}_{\overline{\varphi-1}}(\xi_1,2)+(\nabla\mathrm{Q})(2+\mathrm{p}_0)\mathrm{E}_{\overline{\varphi-1}}(\xi_1,1)-(\nabla\mathrm{Q})(1+\mathrm{p}_0)\mathrm{E}_{\overline{\varphi-1}}(\xi_1,1)\right\}\\ &= \mathscr{C}(\varphi-1)\left\{-(\varphi-1)^2(\nabla\mathrm{Q})(1+\mathrm{p}_0)+(\nabla\mathrm{Q})(2+\mathrm{p}_0)\right\} \geqslant q0,\end{aligned}$$

which implies that

$$(\nabla\mathrm{Q})(2+\mathrm{p}_0) \geqslant q\,\underbrace{(\varphi-1)^2}_{>0}\,\underbrace{(\nabla\mathrm{Q})(1+\mathrm{p}_0)}_{\geqslant q0} \geqslant q0.$$

Assume that $\kappa \geqslant q2$ and that $(\nabla\mathrm{Q})(\mathrm{p}_0+\imath) \geqslant q0$ for $\imath = 2,3,\cdots,\kappa$. Then, by making use of Lemma 3.2 at $\tau = \mathrm{p}_0+\kappa+1$, we get

$$\begin{aligned}(\nabla\mathrm{Q})(\mathrm{p}_0+\kappa+1) &\geqslant q\underbrace{\frac{1}{-\varphi+2}}_{>0}\left\{\underbrace{\left[\mathrm{E}_{\overline{\varphi-1}}(\xi_1,\kappa)-\mathrm{E}_{\overline{\varphi-1}}(\xi_1,\kappa+1)\right]}_{>0}\underbrace{(\nabla\mathrm{Q})(\mathrm{p}_0+1)}_{\geqslant q0}\right.\\ &\quad\left.+\sum_{r=\mathrm{p}_0+2}^{\mathrm{p}_0+\kappa}\left[\underbrace{\mathrm{E}_{\overline{\varphi-1}}(\xi_1,\mathrm{p}_0+\kappa-r+1)-\mathrm{E}_{\overline{\varphi-1}}(\xi_1,\mathrm{p}_0+\kappa-r+2)}_{>0}\right]\underbrace{(\nabla_r\mathrm{Q})(r)}_{\geqslant q0 \text{ by inductive assumption}}\right\}\\ &\geqslant q0,\end{aligned}$$

which completes the induction process, where we used that $\mathrm{E}_{\overline{\varphi-1}}(\xi_1,\kappa)$ is monotonically decreasing according to Remark 2.1. Hence, the proof is complete. □

Corollary 3.4. *Suppose* $Q \in \mathscr{F}_{p_0,0}, 1 < \varphi < \frac{3}{2}, \xi_1 = -\frac{\varphi-1}{2-\varphi}$ *and* $\left({}^{ABC}_{p_0}\nabla^{\varphi}Q\right)(\tau) \geqslant q - \frac{\mathscr{C}(\varphi-1)}{2-\varphi}\mathrm{E}_{\overline{\varphi-1}}(\xi_1,\tau-p_0)(\nabla Q)(p_0) \quad \forall \quad \tau \in \mathbb{N}_{p_0+1}$. *Then,* $(\nabla Q)(\tau) \geqslant q0\ \forall\ \tau \in \mathbb{N}_{p_0+1}$.

Once theoretical results are successfully derived, it is important to check their accuracy. For this reason, we briefly consider two examples as a verification for the main theorems. It is noteworthy that these examples are computed using the MATLAB software.

Example 3.1. By considering Eq. (3.1) for $\tau := \mathrm{p}_0+3$, we get

$$\begin{aligned}\left({}^{CFR}_{\mathrm{p}_0}\nabla^{\varphi}\mathrm{Q}\right)(\mathrm{p}_0+3) &= \mathscr{C}(\varphi-1)\{(\nabla\mathrm{Q})(\mathrm{p}_0+3)+(2-\varphi)(1-\varphi)(\nabla\mathrm{Q})(\mathrm{p}_0+1)\\ &\quad\left.+(1-\varphi)\sum_{r=\mathrm{p}_0+2}^{\mathrm{p}_0+2}(\nabla_r\mathrm{Q})(r)(2-\varphi)^{\mathrm{p}_0+2-r}\right\}.\end{aligned}$$

For $a = 0$, it follows that

$$\begin{aligned}\left({}^{CFR}_{0}\nabla^{\varphi}\mathrm{Q}\right)(3) &= \mathscr{C}(\varphi-1)\left\{(\nabla\mathrm{Q})(3)+(2-\varphi)(-\varphi+1)(\nabla\mathrm{Q})(1)+(1-\varphi)\sum_{r=2}^{2}(\nabla_r\mathrm{Q})(r)(2-\varphi)^{2-r}\right\}\\ &= \mathscr{C}(\varphi-1)\{(\nabla\mathrm{Q})(3)+(2-\varphi)(1-\varphi)(\nabla\mathrm{Q})(1)+(1-\varphi)(\nabla\mathrm{Q})(2)\}\\ &= \mathscr{C}(\varphi-1)\{\mathrm{Q}(3)-\mathrm{Q}(2)+(2-\varphi)(1-\varphi)[\mathrm{Q}(1)-\mathrm{Q}(0)]+(1-\varphi)[\mathrm{Q}(2)-\mathrm{Q}(1)]\}.\end{aligned}$$

Let's consider $\mathrm{Q}(0)=\frac{1}{2}, \mathrm{Q}(1)=1, \mathrm{Q}(2)=\frac{3}{2}, \mathrm{Q}(3)=2$ and $\varphi=\frac{3}{2}$, we see that

$$\begin{aligned}\left({}^{CFR}_{0}\nabla^{1.5}\mathrm{Q}\right)(3) &= \mathscr{C}(0.5)\{0.5+(-0.5)(0.5)(0.5)+(-0.5)(0.5)\}\\ &= 0.1250\,\mathscr{C}(0.5) > 0.\end{aligned}$$

Thus, clearly, Theorem 3.1 confirms that $(\nabla\mathrm{Q})(3) > 0$.

Example 3.2. By using Eq. (2) for $t := \mathrm{p}_0+3$, we see that

$$\begin{aligned}\left({}^{ABR}_{\mathrm{p}_0}\nabla^{\varphi}\mathrm{Q}\right)(\mathrm{p}_0+3) &= \mathscr{C}(\varphi-1)\left\{(\nabla\mathrm{Q})(\mathrm{p}_0+3)+\tfrac{1}{2-\varphi}\left[\mathrm{E}_{\overline{\varphi-1}}(\xi,3)-\mathrm{E}_{\overline{\varphi-1}}(\xi,2)\right](\nabla\mathrm{Q})(1+\mathrm{p}_0)\right.\\ &\quad\left.+\tfrac{1}{2-\varphi}\sum_{r=2+\mathrm{p}_0}^{2+\mathrm{p}_0}\left[\mathrm{E}_{\overline{\varphi-1}}(\xi,\mathrm{p}_0+4-r)-\mathrm{E}_{\overline{\varphi-1}}(\xi,\mathrm{p}_0+3-r)\right](\nabla_r\mathrm{Q})(r)\right\}.\end{aligned}$$

For $\mathrm{p}_0 = 0$, it follows that

$$\begin{aligned}\left({}^{ABR}_{0}\nabla^{\varphi}\mathrm{Q}\right)(3) &= \mathscr{C}(\varphi-1)\left\{(\nabla\mathrm{Q})(3)+\tfrac{1}{2}(\varphi-1)^2(2\varphi^2-5\varphi+2)(\nabla\mathrm{Q})(1)\right.\\ &\quad\left.+\tfrac{1}{-\varphi+2}\sum_{r=2}^{2}\left[\mathrm{E}_{\overline{\varphi-1}}(\xi,4-r)-\mathrm{E}_{\overline{\varphi-1}}(\xi,3-r)\right](\nabla_r\mathrm{Q})(r)\right\}\\ &= \mathscr{C}(\varphi-1)\left\{(\nabla\mathrm{Q})(3)+\tfrac{1}{2}(\varphi-1)^2(2\varphi^2-5\varphi+2)(\nabla\mathrm{Q})(1)-(\varphi-1)^2(\nabla\mathrm{Q})(2)\right\}\\ &= \mathscr{C}(\varphi-1)\left\{\mathrm{Q}(3)-\mathrm{Q}(2)+\tfrac{1}{2}(\varphi-1)^2(2\varphi^2-5\varphi+2)[\mathrm{Q}(1)-\mathrm{Q}(0)]-(\varphi-1)^2[\mathrm{Q}(2)-\mathrm{Q}(1)]\right\}.\end{aligned}$$

By considering $\mathrm{Q}(0)=\frac{1}{2}, \mathrm{Q}(1)=1, \mathrm{Q}(2)=\frac{3}{2}, \mathrm{Q}(3)=2$ and $\varphi=1.4$, we find that

$$\left({}^{ABR}_{0}\nabla^{1.4}\mathrm{Q}\right)(3) = 0.3768\,\mathscr{C}(0.4) > 0.$$

Thus, obviously, Theorem 3.2 confirms that $(\nabla\mathrm{Q})(3) > 0$.

4. Conclusion

In this paper, the discrete analysed fractional operator technique has been successfully applied to find positivity results to the discrete Caputo-Fabrizo and Atangana-Baleanu fractional operators with exponential and Mittag–Leffler kernels, respectively. As a result, general forms of $(\nabla\mathrm{Q})(\tau)$ on $\tau \in \mathbb{N}_{\mathrm{p}_0+1}$ in terms of the discrete Caputo-Fabrizo fractional operators with the exponential kernel and the Atangana-Baleanu fractional operators with the Mittag–Leffler kernel have been obtained in summation representations in Lemma 3.1 and Lemma 3.2, respectively. These new general forms might be potentially useful in the study of $(\nabla\mathrm{Q})(\tau)$ to be positive for each time step τ in $\mathbb{N}_{\mathrm{p}_0+1}$ by the induction procedure as we have done in Theorem 3.1 and Theorem 3.2 in the Riemann–Liouville sense. Particularly, due to a strong relationship between the discrete fractional operators in the sense of Liouville-Caputo and Riemann–Liouville (as in Proposition 2.1), the positivity of $(\nabla\mathrm{Q})(\tau)$ has been highlighted for those operators in the Liouville-Caputo sense as in Corollaries 3.1,3.2,3.3,3.4. Finally, two examples (Examples 3.1 and Example 3.2) have been solved to demonstrate the validity of our main results.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Supplementary material

Supplementary data associated with this article can be found, in the online version, at https://doi.org/10.1016/j.jksus.2023.102794.

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