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On abstract Cauchy problems in the frame of a generalized Caputo type derivative

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Abstract

In this paper, we consider a class of abstract Cauchy problems in the framework of a generalized Caputo type fractional. We discuss the existence and uniqueness of mild solutions to such a class of fractional differential equations by using properties found in the related fractional calculus, the theory of uniformly continuous semigroups of operators and the fixed point theorem. Moreover, we discuss the continuous dependence on parameters and Ulam stability of the mild solutions. At the end of this paper, we bring forth some examples to endorse the obtained results.

Keywords: Caputo type generalized fractional operators, abstract Cauchy problem, existence, uniqueness, continuous dependence on parameters, stability, uniformly continuous semigroups, fixed point theorems, Mittag-Leffler type function.
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1. Introduction

The fractional calculus is a generalization of ordinary differentiation and integration to an arbitrary order that can be noninteger. Very recently it has been recognized that the fractional calculus arise naturally in various fields of science. The use of fractional calculus in the mathematical modeling of engineering and physical problems has become increasingly popular in recent years. The applications of the fractional calculus in physics were initially undertaken by Abel and Heaviside [1, 2, 3]. The physical interest in fractional calculus is due it's nonlocal behavior, linearity and nature which introduces a history dependence into the system. Examples include material sciences, mechanics, wave propagation, signal processing, system identification, and so on. Heymans and Podlubny [4] showed that in some examples from the eld of viscoelasticity, it is possible to attribute physical meaning to initial conditions expressed in terms of Caputo fractional derivative

$$\left({}^{c}\mathcal{D}^{\alpha,1}_{a^{+},t}\right) g(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} (t-\tau)^{n-\alpha-1}\, g^{(n)}(\tau)\mathrm{d}\tau. \tag{1.1}$$

In consequence, there are several contributions focusing on the different definitions of fractional derivatives such as Riemann-Liouville, Hadamard, Grünwald-Letnikov, Riesz, Caputo, Marchaud, Weyl, Hilfer, Caputo and Fabrizio, Atangana and Baleanu and others; see [5, 6, 7, 8, 9, 10, 11, 12] and the references therein.

In [2], Kiryakova proposed a theory of a generalized fractional calculus and their applications. One of the proposed generalizations of the fractional calculus operators which has wide applications is the ρ–fractional operator. This notion is referred to as the fractional integral which combines the Riemann-Liouville and the Hadamard integral into a single form

In recent contribution, the authors in [13, 2012], introduced a new definition of the fractional derivative. This new derivative has gained widely attention and attracted a large number of scientists in different scientific fields for the exploration of diverse topics. For example, in [14], D. Anderson et.al. studied the properties of the Katugampola fractional derivative with potential application in quantum mechanics.

Nowadays, fractional derivatives have been begun to be applied to real world modeling problems (vertical motion of a falling body problem in a resistant medium and the Malthusian growth equation,.., etc.) and various other problems involving special functions of mathematical physics as well as their extensions and generalizations in one or more variables, one can see, [15, 16, 17, 18, 19, 20]. Some of the most prominent examples are given in a book by Oldham and Spanier [21] (diffusion processes) and the classic papers of Bagley and Torvik [22], and Caputo and Mainardi [23] (these two papers dealing with the modeling of viscoelastic materials). More recent results are described, for example, in the works of Gaul et al. [24] (description of mechanical systems subject to damping), Podlubny[25] (control theory), Poinot and Trigeassou [26] in identification of physical systems and by Delgado et al. [27] in electrical circuits.

Recently, fractional differential equations have received increasing attention because the behavior of many physical systems, such as fluid flows, electrical networks, viscoelasticity, chemical physics, diffusion phenomena, electron-analytical chemistry, biology, and control theory, can be properly described by using the fractional order system theory and so forth (see [6, 28]). For instance, used by Westerlund and Ekstam in modeling of electrical capacitors, by Rossikhin and Shitikova in simulating viscoelastic materials and by Tvazoei et al. in design and practical implementation of controllers.

In this paper, we consider equations of the form

$$\left({}^{c}\mathcal{D}^{\alpha,\rho}_{0^{+},t}\right) v(t,x) = \mathrm{A}v(t,x) + f(t,x,v(t,x)),\ \rho>0,\ t\in[0,T],T<\infty,\ x\in\mathbb{R}. \tag{1.2}$$

A strong motivation for investigating of (1.2) comes from physics. For example, fractional diffusion equations are abstract partial differential equations that involve fractional derivatives in space and time. They are useful to model anomalous diffusion on fractals (physical objects of fractional dimension, like some amorphous semiconductors or strongly porous materials,... etc), where a plume of particles spreads in a different manner than the classical diffusion equation predicts. The time fractional diffusion equation is obtained from the standard diffusion equation by replacing the first-order time derivative with a fractional derivative of order $\alpha \in (0,1)$.

It is natural from the physical point of view to consider a usual Cauchy problem, with the initial condition.

$$v(0,x) = v_0\left(x\right), x \in \mathbb{R}. \tag{1.3}$$

In the physical literature the expression of Caputo fractional derivative (1.1) on the right is used as the basic object for formulating fractional diffusion equations. For example, particles in turbulent flows have long jump step and rapid diffusion speed. In the diffusion equation, one uses $\alpha \in (1,2]$ indicate superdiffusion. So studying such fractional partial differential models will enable us to better understand how the diffusion flux goes from regions of higher concentration to regions of lower concentration.

In many concrete situations, the evolution equation (or the associated linear operator) is given as a (formal) sum of several terms having different physical meaning and different mathematical properties. However, some physical phenomena in nature can be modelled and described by equations and systems with fractional derivatives of $\alpha \in (n, n+1]\,,\ n \in \mathbb{N}$.

In the past few years, the researchers have showed their interest in introducing fractional interpretations of the classical integral transforms, namely, the Laplace and Fourier transforms [29, 30]. In [5, 31, 32, 33, 34], it can be seen that integral transforms like Laplace, Fourier, generalized Laplace and ρ-Laplace were considered as effective tools for obtaining analytic solutions to some classes of fractional differential equations.

Qualitative theory and its applications in physics, engineering, economics, biology and ecology were extensively discussed and demonstrated in [2, 6] and the references therein. Investigating the existence, uniqueness, stability, continuous dependence of data of classical, strong, weak and mild solutions of fractional differential equations have been intensively studied by many researchers in the scientific community, especially in fractional calculus, see [35] and references therein. A weak solution (also called a generalized solution) to an differential equation is a function for which the derivatives may not all exist but which is nonetheless deemed to satisfy the equation in some precisely defined sense. A different way of defining a weak solution of (1.2-1.3) was given by J. Ball [36].

It is well known that one important way to introduce the concept of mild solutions for fractional evolution equations is based on some probability densities and Laplace transform. This method was initialed by El-Borai [37]. The problem of the existence of mild solutions for abstract differential equations with a fractional derivative has been considered in several recent papers, (see [38] and references therein).

The notion of an abstract Cauchy problem has recently been introduced by E. Hille in [39]. As we all know, the main difficulty to study the fractional abstract differential equations is how to obtain a suitable fractional resolvent family generated by the infinitesimal generator A in Banach space. We note that the uniformly continuous semigroups are a subset of strongly continuous semi-groups. In order to overcome this barrier, some authors introduced an α-resolvent family under the fractional derivative and some constraints. The notion of α-resolvent families (or solution operators) is introduced in Li and Zheng [40], see [35, 41, 42, 43, 44, 45, 46, 47, 48]. Another approach to treat abstract equations with fractional derivatives based on the well developed theory of resolvent operators for integral equations, (see [49, 50, 51, 52]).

Bazhlekova [53] used solution operator to investigate the following fractional abstract Cauchy problem

$$\begin{cases} {}^cD^\alpha_{0^+}u(t) = \mathrm{A}u(t),\ t>0, n=[\alpha]+1, \\ u(0)=x,\ u^{(k)}(0)=0, x\in\mathbb{X},\ k=1,...,n-1, \end{cases} \tag{1.4}$$

where ${}^cD^\alpha_{0^+}$ is the Caputo fractional derivative operator.

The important point is that classical solutions exist if (and, by the definition of D(A), **only if) the initial value** x belongs to D(A). However, modifying slightly the concept of "solution" and requiring differentiability only for $t>0$, we obtain such solutions for each $x\in\mathbb{X}$ as soon as the semigroup $S_\alpha(t)\,(t\geqslant 0)$ is immediately differentiable. This already suggests that different concepts of "solutions" might be useful. The most important one renounces differentiability and substitutes the differential equation by an integral equation. The nuance between these two notions of solutions lies on the one hand in the verification of the differential equation (1.4) and on the other hand in the fact that the classical solution is regular. Indeed, the classical solution satisfies the differential equation while for the soft solution, it is the integral of this one which satisfies the differential equation. This criterion has great importance in the study of existence of mild solutions for (1.4), because arguments to solve (1.4) using fixed points theorems can be applied. Then the problem of finding mild solutions for problem (1.4) is reduced to finding the fixed point.

We assumed that the solution $u \in C^1([0,T];X)$. However, this assumption is too restrictive. Therefore, in order to relax this assumption, we introduce the notion of mild solution.

If A is the infinitesimal generator of a C_0 semigroup which is not differentiable then, in general, if $x \notin$D(A), the fractional abstract Cauchy problem (1.4) does not have a solution. The function $t \longrightarrow S_\alpha(t)x$ is then a "generalized solution" of the fractional abstract problem (1.4) which we will call a mild solution. In general, a mild solution may not be differentiate and hence need not be an classical solution to (1.4). But this notion is known as the most natural one of the generalized notions of solutions to (1.4). For regularity results of mild solutions, (see Martin [54]).

Furthermore, mild solutions have a considerable advantage over weak solutions in the sense that it is directly clear how they should be interpreted, whereas defining weak solutions involves some seemingly arbitrary choices. Which choices to make is not directly evident from modelling considerations. in addition mild solutions have the advantage that their existence can be established via standard methods such as Picard iteration and non-extendibility arguments under much less restrictive conditions than classical or weak solutions.

Motivated by some recent developments in ρ-fractional calculus, in this paper, we can reformulate the fractional partial differential problem (1.2-1.3) as an abstract fractional Cauchy problem

$$\left({}^c\mathcal{D}^{\alpha,\rho}_{0^+}\right) u(t) = \mathrm{A}u(t) + f(t,u(t)),\ \rho > 0,\ t \in [0,T]\,, T < \infty, \tag{1.5}$$

with

$$u(0) = u_0, \tag{1.6}$$

where $\left({}^c\mathcal{D}^{\alpha,\rho}_{0^+}\right)$ denotes a generalized Caputo−type fractional derivative of order $0 < \alpha \leq 1$, $\mathrm{A} : \mathrm{D(A)} \subset \mathbb{X} \longrightarrow \mathbb{X}$ is an infinitesimal generator of bounded linear operator, $S_\alpha(t)\,(t \geqslant 0)$ generated by A defined on a Banach space $(\mathbb{X}, \|.\|)$, $u_0 \in \mathbb{X}$ and $f : [0,T] \times \mathbb{X} \longrightarrow \mathbb{X}$ and obtain the unique mild solution by using the theory of generalized uniformly continuous semigroups of operators and the fixed point theorem.

For instance, in differentiation to show that if A is a bounded operator and f is continuous with $\alpha = 1$, the function u defined by (1.5) is continuously differentiable and satisfies (1.6). However, this function exists under more general hypotheses; so it is useful to introduce the concept of a mild solution. Then a mild solution is a classical solution if and only if it is continuously differentiable.

The definition of the mild solution of the fractional abstract Cauchy problem (1.5-1.6) coincides when $f \equiv 0$ with the definition of $S_\alpha(t)x$ as the mild solution of the corresponding homogeneous equation. It is therefore clear that not every mild solution of (1.5-1.6) is indeed a (classical) solution even in the case $f \equiv 0$. For any $f \in L^1((0,T);\mathbb{X})$, the fractional abstract Cauchy problem (1.5-1.6) has a unique mild solution. Now a natural question to the problem (1.5-1.6) is that under what conditions on f, a mild solution is also a classical solution.

The main contributions of the article are as follows: we introduce some notations, properties, lemmas, definitions of ρ-fractional calculus, we present a slight generalization for Ulam–Hyers theorem which was used in studying the stability and preliminary facts needed in our proofs later are given in the second section. Then, in the third section, We adopt the theory of uniformly continuous operator semigroups, we construct an operator $S_\alpha(t)\,(t \geqslant 0)$ and discuss its properties for use in section 4. In subsection 4.1, we study the existence and uniqueness of mild solutions for (1.5-1.6) by virtue of solution operator method and contraction mapping theorem and local fractional Laplace transform.

In subsection 4.2, we represented the mild solution of problem (1.5-1.6) which is defined in terms of the Mittag-Leffler function by applying the fractional analog of the generalized Duhamel principle.

Next, in subsection 4.3 is, we look at the question as to how the solution u varies when we change the order of the fractional differential operator or the initial values and the dependence on parameters of nonlinear term f is also established.

The last, fourth subsection is,devoted to the stable solution of fractional (1.5-1.6) is provided by using classical technique of nonlinear functional analysis investigated by Ulam, an illustrative examples are presented in the fifth section. Finally, the paper is concluded in section 6.

2. Preliminaries and assumptions

In this section we introduce notations, lemmas, definitions, assumation and preliminary facts which are used throughout this paper.

The generalized fractional integrals are defined by, $n-1<\alpha\leq n$ [9]

$$\left(\mathcal{J}_{a^+}^{\alpha,\rho}\right)g(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\alpha-1}g(\tau)\frac{\mathrm{d}\tau}{\tau^{1-\rho}},$$

where $\Gamma(.)$ is the Euler gamma function.

The corresponding left and right generalized Riemann−type fractional derivatives of g of order α are defined by [10]

$$\left(\mathcal{D}_{a^+}^{\alpha,\rho}\right)g(t)=\frac{\gamma^n}{\Gamma(n-\alpha)}\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{n-\alpha-1}g(\tau)\frac{\mathrm{d}\tau}{\tau^{1-\rho}},\ t\in[a,b]$$

and

$$\left(\mathcal{D}_{b^-}^{\alpha,\rho}\right)g(t)=\frac{(-\gamma)^n}{\Gamma(n-\alpha)}\int_t^b\left(\frac{\tau^\rho-t^\rho}{\rho}\right)^{n-\alpha-1}g(\tau)\frac{\mathrm{d}\tau}{\tau^{1-\rho}},\ t\in[a,b],$$

respectively.

The left-sided generalized Caputo−type fractional derivative of g of order α is defined by[11]

$$\left({}^c\mathcal{D}_{a^+}^{\alpha,\rho}\right)g(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{n-\alpha-1}\frac{(\gamma^n g)(\tau)\mathrm{d}\tau}{\tau^{1-\rho}}=\mathcal{J}_{a^+}^{n-\alpha,\rho}(\gamma^n g)(t).$$

Analogous formula can be offered for the right fractional derivative as follows

$$\left({}^c\mathcal{D}_{b-}^{\alpha,\rho}\right)g(t)=\frac{1}{\Gamma(n-\alpha)}\int_t^b\left(\frac{\tau^\rho-t^\rho}{\rho}\right)^{n-\alpha-1}\frac{((-\gamma)^n g)(\tau)\mathrm{d}\tau}{\tau^{1-\rho}}\mathcal{J}_{b-}^{n-\alpha,\rho}((-\gamma)^n g)(t).$$

For $\alpha>0$ and $\rho>0$, we have [11]

$$\left({}^c\mathcal{D}_{a^+}^{\alpha,\rho}\right)g(t)=\mathcal{D}_{a^+}^{\alpha,\rho}g(t)-\sum_{k=0}^{n-1}\frac{\gamma^k g(a)}{\Gamma(k+1-\alpha)}\left(\frac{t^\rho}{\rho}-\frac{a^\rho}{\rho}\right)^{k-\alpha},$$

where $\gamma=t^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}t}$.

Lemma 2.1. *[11]*
i. *Let $g\in AC_\gamma^n[a,b]$ or $\mathcal{C}_\gamma^n[a,b]$ and $\alpha\in\mathbb{C}$. Then,*

$$\mathcal{J}_{a^+}^{\alpha,\rho}\left({}^c\mathcal{D}_{a^+}^{\alpha,\rho}\right)g(t)=g(t)-\sum_{k=0}^{n-1}\frac{(\gamma^k g)(a)}{k!}\left(\frac{t^\rho}{\rho}-\frac{a^\rho}{\rho}\right)^k.$$

ii. For $\alpha>0;\ \beta>0;\ 1\leqslant p<\infty;\ a\in(0,\infty);\ \rho,\ c\in\mathbb{R};\ \rho\geqslant c.$

$$\mathcal{J}_{a^+}^{\alpha,\rho}\mathcal{J}_{a^+}^{\beta,\rho}g=\mathcal{J}_{a^+}^{\alpha+\beta,\rho}g;\ g\in\mathbb{X}_c^p(a,b).$$

iii. For all $\alpha\in(n-1,n]$ and $\beta\geqslant0$ the relation holds

$$\mathcal{J}_{a^+}^{\alpha+\beta,\rho}g(t)=\mathcal{J}_{a^+}^{\beta+n,\rho}\mathcal{D}_{a^+}^{n-\alpha,\rho}g(t).$$

Lemma 2.2. *[11] For $\beta > -1$, one has*

$$\left(\mathcal{J}^{\alpha,\rho}_{a^+,t}\right)\left[\left(\frac{\tau^\rho}{\rho}-\frac{a^\rho}{\rho}\right)^\beta\right]=\frac{\Gamma(1+\beta)}{\Gamma(1+\alpha+\beta)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\beta+\alpha}$$

and

$$\left({}^c\mathcal{D}^{\alpha,\rho}_{a^+,t}\right)\left[\left(\frac{\tau^\rho}{\rho}-\frac{a^\rho}{\rho}\right)^\beta\right]=\frac{\Gamma(1+\beta)}{\Gamma(1-\alpha+\beta)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\beta-\alpha}.$$

Lemma 2.3. *[55]Let $\alpha \geqslant 0$, $n=[\alpha]+1$ and $g\in AC^n_\gamma[a,b]$, where $0<a<b<\infty$. Then*

$$g(t)=g(a)+\frac{\left({}^c\mathcal{D}^{\alpha,\rho}_{a^+}\right)g(\xi)}{\Gamma(\alpha+1)}\left(\frac{t^\rho}{\rho}-\frac{a^\rho}{\rho}\right)^\alpha\ ;\ a\le\xi\le t\le b.$$

Lemma 2.4. *[56]For $0<\alpha<1, 1/p+1/q\le 1+\alpha$,if $g\in\left(\mathcal{J}^{\alpha,\rho}_{a^+}\right)(X_q)$ and $h\in\left(\mathcal{J}^{\alpha,\rho}_{b^-}\right)(X_p)$,*

$$\int_a^b h(\tau)\left(\mathcal{D}^{1-\alpha,\rho}_{a^+,\tau}\right)g(\tau)\frac{\mathrm{d}\tau}{\tau^{1-\rho}}=\int_a^b g(\tau)\left(\mathcal{D}^{1-\alpha,\rho}_{\tau,b^-}\right)h(\tau)\frac{\mathrm{d}\tau}{\tau^{1-\rho}}.$$

Definition 2.5. [56]Let $h,\ g\in L^1(\mathbb{R}^+)$, we define the product of convolution g and h by

$$(h*_\rho g)(t)=\int_0^t h\left(\left(\rho\left(\frac{t^\rho}{\rho}-\frac{\tau^\rho}{\rho}\right)\right)^{\frac{1}{\rho}}\right)g(\tau)\frac{\mathrm{d}\tau}{\tau^{1-\rho}}.$$

Definition 2.6. [33]Let $g:[0,\infty)\longrightarrow\mathbb{R}^+$ be a real valued function. The ρ-Laplace transform of g is defined by

$$\mathcal{L}_\rho\{g(t)\}(\tau)=\int_0^{+\infty}\exp\left(-\tau\frac{t^\rho}{\rho}\right)g(t)\frac{\mathrm{d}t}{t^{1-\rho}},$$

for all values of τ.

Theorem 2.7. *[33]Let $\alpha>0$ and g be a piecewise continuous function on each interval $[0,t]$ and of ρ exponential order $e^{c\frac{t^\rho}{\rho}}$. Then the Laplace transform formula for the Caputo type generalized fractional integral is defined by*

$$\mathcal{L}_\rho\left\{\mathcal{J}^{\alpha,\rho}_{a^+}g(t)\right\}(\tau)=\tau^{-\alpha}\mathcal{L}_\rho\{g(t)\},\ \tau>c.$$

Lemma 2.8. *[32]Let $\Re(\beta)>\Re(\alpha)>0$ and $(\frac{\lambda}{\tau^\alpha})<1$. then*

$$\mathcal{L}_\rho\left\{\left(\frac{t^\rho}{\rho}\right)^{\beta-1}E_{\alpha,\beta}\left(\lambda\left(\frac{t^\rho}{\rho}\right)^\alpha\right)\right\}(\tau)=\frac{\tau^{\alpha-\beta}}{\tau^\alpha-\lambda},$$

where the two parameter Mittag–Leffler function $E_{\alpha,\beta}$ is given by

$$E_{\alpha,\beta}(z)=\sum_{k=0}^\infty\frac{z^k}{\Gamma(\alpha k+\beta)}=\frac{1}{2\pi i}\int_{\mathcal{H}}\frac{t^{\alpha-\beta}e^t}{t^\alpha-z}\mathrm{d}t,\ \alpha,\beta>0,\ z\in\ \mathbb{C},$$

where the path of integration $\mathcal{H}$ is a loop which starts and ends at $-\infty$, and encircles the circles disc $|t|\le|z|^{1/\alpha}$ in the positive sense: $|\arg(t)|\le\pi$ on $\mathcal{H}$.

In particular

$$E_1(z)=\exp(z),\ E_{1,2}(z)=\frac{\exp(z)-1}{z}\text{ and }E_{1/2}\left(z^{1/2}\right)=\exp(z)\left[1+erf\left(z^{1/2}\right)\right],$$

where *erf* (z) error function.

The following lemmas give asymptotic formula and estimate of the behaviour of the Mittag–Leffler functions.

Lemma 2.9. *[57]If $0 < \alpha < 2$ and $\beta > 0$; then, for $|z| \longrightarrow +\infty$,*
i. *If $|\arg(z)| \leq \frac{1}{2}\alpha\pi$,*

$$E_{\alpha,\beta}(z) \simeq \frac{1}{\alpha} z^{\frac{(1-\beta)}{\alpha}} \exp(z^{\frac{1}{\alpha}}) + \epsilon_{\alpha,\beta}(z).$$

ii. *If $|\arg(-z)| < (1 - \frac{1}{2}\alpha)\pi$,*

$$E_{\alpha,\beta}(z) = \mathcal{E}_{\alpha,\beta}(z),$$

where

$$\mathcal{E}_{\alpha,\beta}(z) = -\sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\alpha - \beta n)} + \mathcal{O}(|z|^{-N}); |z| \longrightarrow \infty,$$

for some $N \in \mathbb{N} - \{1\}$.

Lemma 2.10. *Let $0 < \alpha < 1$ and $\beta > 0$. Then*

$$E_{\alpha,\beta}\left(\omega\left(\frac{t^\rho}{\rho}\right)^\alpha\right) \leq \mathcal{C}_{\alpha,\beta}\left(1 + \omega^{\frac{1-\beta}{\alpha}}\right)\left(1 + \left(\frac{t^\rho}{\rho}\right)^{1-\beta}\right) \exp\left((\omega^{\frac{1}{\alpha}} \frac{t^\rho}{\rho}\right); \omega \geqslant 0; t \geqslant 0. \tag{2.1}$$

Proof. For $\omega = 0$ and all $t \geqslant 0$, the inequality is trivially satisfed. Fix $0 < \alpha < 2, \beta > 0$ and $T > 0$. Choose an arbitrarily large $T > 0$.

Case 1: For all $t > (\frac{T}{\omega})^{\frac{1}{\alpha}}$, follows that there exists a constant $\mathcal{C}_1 > 0$ such that

$$\begin{aligned} E_{\alpha,\beta}\left(\omega\left(\frac{t^\rho}{\rho}\right)^\alpha\right) &\leq \mathcal{C}_1\left(\omega\left(\frac{t^\rho}{\rho}\right)^\alpha\right)^{\frac{1-\beta}{\alpha}} \exp\left(\left(\omega\left(\frac{t^\rho}{\rho}\right)^\alpha\right)^{\frac{1}{\alpha}}\right) \\ &= \mathcal{C}_1\left(\omega^{\frac{1-\beta}{\alpha}}\left(\frac{t^\rho}{\rho}\right)^{1-\beta}\right) \exp\left(\omega^{\frac{1}{\alpha}}\left(\frac{t^\rho}{\rho}\right)\right) \\ &\leq \mathcal{C}_1\left(\left(1 + \omega^{\frac{1-\beta}{\alpha}}\right)\left(1 + \frac{t^\rho}{\rho}\right)^{1-\beta}\right) \exp\left(\omega^{\frac{1}{\alpha}}\left(\frac{t^\rho}{\rho}\right)\right). \end{aligned}$$

Case 2: For $t \in [0, (\frac{T}{\omega})^{\frac{1}{\alpha}}]$, there exists a constant $\mathcal{C}_2 > 0$ such that

$$E_{\alpha,\beta}\left(\omega\left(\frac{t^\rho}{\rho}\right)^\alpha\right) \leq \mathcal{C}_2\left(\left(1 + \omega^{\frac{1-\beta}{\alpha}}\right)\left(1 + \frac{t^\rho}{\rho}\right)^{1-\beta}\right) \exp\left(\omega^{\frac{1}{\alpha}}\left(\frac{t^\rho}{\rho}\right)\right).$$

Taking $\mathcal{C}_{\alpha,\beta} = \max\{\mathcal{C}_1; \mathcal{C}_2\}$ we obtain the inequality (2.1). □

Lemma 2.11. *[58] If u, v are non negative and integrable functions on (a, T), as the function v is not decreasing over (a, T), and $w \in L^1(a, T)$; $a \geq 0$, it senses from*

$$u(t) \leq v(t) + \int_a^t b(\tau)u(\tau)\mathrm{d}\tau,$$

then

$$u(t) \leq v(t) \exp(\int_a^t b(\tau)\mathrm{d}\tau).$$

Theorem 2.12. *[59] Let B_r be closed, convex and nonempty subset of a Banach space $\mathbb{X}$. Let $\Psi : B_r \to B_r$ be a continuous mapping such that $\Psi(B_r)$ is a relatively compact subset of $\mathbb{X}$. Then Ψ has at least one fixed point in B_r.*

For the study of Hyers-Ulam-Rassias and generalized Ulam-Hyers-Rassias stabilities of the equation (1.5) on a compact interval $[a,T]$, we will adapt such definitions [60, 61, 62].

Let $\epsilon > 0$ and $\Phi : [a,T] \to [0,\infty)$ be a continuous function and consider the following inequalities

$$\left\| \left({}^{c}\mathcal{D}^{\alpha,\rho}_{0^+}\right) \tilde{u}(t) - \mathrm{A}\tilde{u}(t) - f(t,\tilde{u}(t)) \right\| \leq \epsilon, \tag{2.2}$$

$$\left\| \left({}^{c}\mathcal{D}^{\alpha,\rho}_{0^+}\right) \tilde{u}(t) - \mathrm{A}\tilde{u}(t) - f(t,\tilde{u}(t)) \right\| \leq \epsilon\Phi\left(t\right), \tag{2.3}$$

and

$$\left\| \left({}^{c}\mathcal{D}^{\alpha,\rho}_{0^+}\right) \tilde{u}(t) - \mathrm{A}\tilde{u}(t) - f(t,\tilde{u}(t)) \right\| \leq \Phi\left(t\right). \tag{2.4}$$

Definition 2.13. Problem (1.5-1.6) is Ulam-Hyers stable if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $\tilde{u} \in C([a,T],\mathbb{R})$ of the inequality (2.2) there exists a mild solution $u \in C([a,T],\mathbb{R})$ of problem (1.5-1.6) with

$$|\tilde{u}(t) - u(t)| \leq \epsilon c_f.$$

Definition 2.14. The equation (1.5-1.6) is Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $\tilde{u} \in$C$([a,T],\mathbb{R})$ of the inequality (2.3) there exists a mild solution $u \in C([a,T],\mathbb{R})$ of equation (1.5-1.6) with

$$|\tilde{u}(t) - u(t)| \leq \epsilon c_f \Phi\left(t\right).$$

Definition 2.15. The equation (1.5-1.6) is generalized Ulam-Hyers-Rassias stable with respect to Φ if there exists $c_f > 0$ such that for each solution $\tilde{u} \in C[a,T]$ of the inequation (2.4) there exists a mild solution $u \in C([a,T],\mathbb{R})$ of the equation (1.5-1.6) with

$$|\tilde{u}(t) - u(t)| \leq c_f \Phi\left(t\right).$$

Remark 2.16. for every $\epsilon > 0$, a function $\tilde{u} \in C[a,T]$ is a solution of of the inequality (2.2), where $\Phi(t) \geq 0$ if and only if there exists a function $g \in C([0,T],\mathbb{R})$ (which depend on $\tilde{u}$) such that

(i) $|g(t)| \leq \epsilon, \ \forall t \in [0,T].$

(ii) $\left({}^{c}D^{\alpha,\rho}_{0+,t}\right)[\tilde{u}] = \mathrm{A}\tilde{u}(t) + f(t,\tilde{u}(t)) + g(t).$

The remaining portion of the paper, we make use of the next assumptions:

$\mathbf{A}_1$ There exist a constant $L > 0$ such that

$$|f(t,u) - f(t,v)| \leq L|u - v|, \text{ for each } t \in [0,T] \text{ and all } u,v \in \mathbb{R}.$$

$\mathbf{A}_2$ There exists an increasing function $p(t) \in (\mathcal{C}[0,T],\mathbb{R}^+)$, for any $t \in [0,T]$,

$$|f(t,u)| \leq p(t)\frac{|u|}{1+|u|}, \ u \in \mathbb{R}.$$

$\mathbf{A}_3$ There exists an increasing function $\Phi(t) \in (\mathcal{C}[0,T],\mathbb{R}^+)$ and there exists $l_\Phi > 0$ such that for any $t \in [0,T]$,

$$\left| \left(\mathcal{J}^{1,\rho}_{0^+}\right)[\Phi] \right| \leq l_\Phi \Phi(t), \ \rho > 0.$$

$\mathbf{A}_4$ There exists an increasing function $q(\tau), \ \Phi(t) \in (\mathcal{C}[0,T],\mathbb{R}^+)$ such that for any $t \in [0,T]$,

$$|f(t,u)| \leq q(\tau)\Phi(\tau), \ u \in \mathbb{R}.$$

For the sake of brevity the notation $f := f(u) \equiv f(t,u)$ and $S_{\alpha,\rho}(t,\tau) := S_\alpha\left(\left(\rho\left(\frac{t^\rho}{\rho} - \frac{\tau^\rho}{\rho}\right)\right)^{\frac{1}{\rho}}\right)$is introduced here.

Next, we turn our attention to the semigroup associated with A.

3. Uniformly continuous solution operators of bounded linear operators

Throughout the paper,$(\mathbb{X}; \|.\|)$ is a Banach space, and $BL(\mathbb{X})$ is the space of all bounded linear operators on $\mathbb{X}$. A is a bounded linear operator on $\mathbb{X}$.

We denote the domain, range, resolvent set, spectrum set and resolvent of the operator A, by $\mathrm{D}(\mathrm{A}), \mathcal{R}(\mathrm{A}), \varrho(\mathrm{A}), \sigma(\mathrm{A}$ and $R(\lambda^{\alpha}, \mathrm{A})(\lambda \in \varrho(\mathrm{A}))$, respectively.

For $0 \leq w < \frac{\alpha\pi}{2}$ we denote the sector with angle w by

$$\sum\Big|_{w} = \{\lambda \in \mathbb{C} : \lambda \neq 0,\ |\arg \lambda| < w\}.$$

We assume that A is a sectorial operator of angle $w \in \left[0, \frac{\alpha}{2}\pi\right)$.

In order to study the nonlinear problem (1.5-1.6), we first consider the associated homogeneous linear problem

$$\begin{cases} \left({}^{c}\mathcal{D}_{0^+}^{\alpha,\rho}\right) u(t) = \mathrm{A}u(t),\ t \in [0, T], \\ u(0) = x,\ x \in \mathbb{X}. \end{cases} \tag{3.1}$$

Definition 3.1. A semigroup $S_\alpha(t)\,(t \geq 0)$ on a Banach space $\mathbb{X}$ is called uniformly continuous (or norm continuous) if $t \in \mathbb{R}_+ \longrightarrow S_\alpha(t) \in BL(\mathbb{X})$ of bounded linear operators on $\mathbb{X}$ is continuous with respect to the uniform operator topology on $BL(\mathbb{X})$.

For well-posed problems we define $S_\alpha(t)x := u(t)$, where $u(t)$ is the solution of (3.1), and call the operator function $S_\alpha(t)$ the solution operator of (3.1).

When A is linear and bounded, uniformly continuous solution operator for (3.1) is defined in terms of the corresponding integral equation

$$u(t) = x + \left(\mathcal{J}_{0^+}^{\alpha;\rho}\right) \mathrm{A}u(t). \tag{3.2}$$

The problem (3.1) is well-posed if and only if the integral equation (3.2) is well-possed.

Definition 3.2. Let $\alpha > 0$. A family $\{S_\alpha(t)\}_{t\geq 0} \subset \mathrm{D}(\mathrm{A})$ of linear and bounded operators on Banach space $\mathbb{X}$ is called a uniformly continuous solution operator for (3.1) if the following conditions are satisfied:
(**i**) $S_\alpha(t)$ is a uniformly continuous function for $t \geq 0$.
(**ii**) $S_\alpha(0) = \mathrm{I_d}$ and

$$\lim_{t \longrightarrow 0} \|S_\alpha(t) - \mathrm{I_d}\| = 0,$$

where $\mathrm{I_d}$ is identity operator on $\mathbb{X}$.
(**iii**) $\mathrm{A}S_\alpha(t)x = S_\alpha(t)\mathrm{A}x$; for all $x \in \mathbb{X}, t \geq 0$.
(**iv**) $S_\alpha(t)x$ is a solution of (3.2) for all $x \in \mathbb{X}, t \geq 0$, *i.e.*, the resolvent equation

$$S_\alpha(t)x = x + \left(\mathcal{J}_{0^+}^{\alpha;\rho}\right) S_\alpha(t)\mathrm{A}x,$$

holds for all $t \geq 0$.

Definition 3.3. An operator A is said to belong $\mathcal{SG}_\alpha(M, w)$ if A generates a solution operator $S_\alpha(t)$ satisfying

$$\| S_\alpha(t) \| \leq M e^{\omega \frac{t^\rho}{\rho}},\ M \geq 1, \omega \geq 0,\ t > 0. \tag{3.3}$$

Denote

$$\mathcal{SG}_\alpha(w) = \cup \{\mathcal{SG}_\alpha(M, w) : M \geq 1\}.$$

The relation between the solution operator $S_\alpha(t)$ and its infinitesimal generator A is characterized by

Lemma 3.4. *For $\alpha > 0$, the infinitesimal generator* A *of a uniformly continuous solution operator $S_\alpha(t)$ $(t \geq 0)$ for (3.1) is defined by*

$$\mathrm{A}x = \Gamma(1+\alpha) \lim_{t \longrightarrow 0^+} \frac{S_\alpha(t)x - x}{\left(\frac{t^\rho}{\rho}\right)^\alpha}, \text{ for all } x \in \mathrm{D}(\mathrm{A}), \tag{3.4}$$

where

$$\mathrm{D}(\mathrm{A}) = \left\{ x \in \mathbb{X} : \lim_{t \longrightarrow 0^+} \frac{S_\alpha(t)x - x}{\left(\frac{t^\rho}{\rho}\right)^\alpha} \text{ exists} \right\}.$$

Proof. For all $x \in \mathbb{X}$, $u(t) = S_\alpha(t)x$ i.e by (3.1), we have

$$\left({}^c\mathcal{D}_{0^+}^{\alpha,\rho}\right) S_\alpha(t)x = \mathrm{A}S_\alpha(t)x, \tag{3.5}$$

for which this limits exists, and the generator A could also be defined as

$$\left({}^c\mathcal{D}_{0^+}^{\alpha,\rho}\right) S_\alpha(t)x \mid_{t=0} = \mathrm{A}S_\alpha(0)x = \mathrm{A}x. \tag{3.6}$$

On the other hand, by Lemma 2.3, we have

$$S_\alpha(t)x = S_\alpha(0)x + \frac{\left({}^c\mathcal{D}_{0^+}^{\alpha,\rho}\right) S_\alpha(\xi)x}{\Gamma(\alpha+1)} \left(\frac{t^\rho}{\rho}\right)^\alpha, \ 0 \leq \xi \leq t; \ t \in (0,T).$$

So,

$$\Gamma(\alpha+1) \frac{S_\alpha(t)x - x}{\left(\frac{t^\rho}{\rho}\right)^\alpha} = \left({}^c\mathcal{D}_{0^+}^{\alpha,\rho}\right) S_\alpha(\xi)x, \ 0 \leq \xi \leq t; \ t \in (0,T), \tag{3.7}$$

We claim that

$$\Gamma(\alpha+1) \lim_{t \longrightarrow 0} \frac{S_\alpha(t)x - x}{\left(\frac{t^\rho}{\rho}\right)^\alpha} = \lim_{\xi \longrightarrow 0} \left({}^c\mathcal{D}_{0^+}^{\alpha,\rho}\right) S_\alpha(\xi)x \text{ as } \xi \longrightarrow 0,$$

□

The relation between the solution operator $S_\alpha(t)$ and the resolvent operators of A is characterized by:

Theorem 3.5. *Assume* $\mathrm{A} \in \mathcal{SG}_\alpha(M,w)$, *and let $S_\alpha(t)$ be the corresponding solution operator. Then for $\Re(\lambda) > w$, the operator $(\lambda^\alpha \mathrm{I_d} - \mathrm{A})$ inversible and*

$$R(\lambda^\alpha, \mathrm{A})x = \lambda^{1-\alpha} \mathcal{L}_\rho \left(S_\alpha(t)x\right)(\lambda) \text{ for } x \in \mathbb{X}, \tag{3.8}$$

where $R(\lambda^\alpha, \mathrm{A}) = (\lambda^\alpha \mathrm{I_d} - \mathrm{A})^{-1}$ stands for the resolvent operator of A, *that is*

$$\{\lambda^\alpha : \Re(\lambda) > w\} \subseteq \varrho(\mathrm{A}).$$

Proof. Assume $\mathrm{A} \in \mathcal{SG}_\alpha(M,w)$, so, for everything $\Re(\lambda) > w$, and $x \in \mathrm{D}(\mathrm{A})$; we define

$$\mathcal{R}(\lambda)x = \mathcal{L}_\rho \left(S_\alpha(t)x\right)(\lambda) = \int_0^{+\infty} \exp\left(-\lambda \frac{t^\rho}{\rho}\right) S_\alpha(t)x \frac{\mathrm{d}t}{t^{1-\rho}}; \rho > 0. \tag{3.9}$$

By Definition 3.2-**iv,** we give

$$\begin{aligned} \mathcal{R}(\lambda)x &= \int_0^{+\infty} \exp\left(-\lambda \frac{t^\rho}{\rho}\right) S_\alpha(t)x \frac{\mathrm{d}t}{t^{1-\rho}} \\ &= \int_0^{+\infty} e^{\left(-\lambda \frac{t^\rho}{\rho}\right)} x \frac{\mathrm{d}t}{t^{1-\rho}} + \int_0^{+\infty} e^{\left(-\lambda \frac{t^\rho}{\rho}\right)} \left[\frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} \mathrm{A}S_\alpha(\tau)x \frac{\mathrm{d}\tau}{\tau^{1-\rho}}\right] \frac{\mathrm{d}t}{t^{1-\rho}} \\ &= \frac{x}{\lambda} + \lambda^{-\alpha} \mathrm{A}\mathcal{L}_\rho \left(S_\alpha(t)x\right)(\lambda). \end{aligned}$$

So this implies that

$$\mathcal{R}(\lambda)x = \frac{x}{\lambda} + \lambda^{-\alpha}\mathrm{A}\mathcal{R}(\lambda)x.$$

Therefore

$$\mathcal{R}(\lambda) = \lambda^{\alpha-1}(\lambda^{\alpha}\mathrm{I_d} - \mathrm{A})^{-1} \tag{3.10}$$

and like $S_\alpha(t)$ is switched with A.

Denote

$$R(\lambda^{\alpha}, \mathrm{A}) = (\lambda^{\alpha}\mathrm{I_d} - \mathrm{A})^{-1}.$$

The following integral representation of the resolvent of the infinitesimal generator A of $S_\alpha(t)$, valid in the right half-plane

$$R(\lambda^{\alpha}, \mathrm{A}) = (\lambda^{\alpha}\mathrm{I_d} - \mathrm{A})^{-1} = \lambda^{1-\alpha}\int_0^{+\infty} \exp\left(-\lambda\frac{t^\rho}{\rho}\right) S_\alpha(t)\frac{\mathrm{d}t}{t^{1-\rho}}, \tag{3.11}$$

with $\{\lambda^\alpha : \Re\lambda > w\} \subseteq \varrho(\mathrm{A})$. □

Theorem 3.6. *Let $\alpha > 0$. Then $\mathrm{A} \in \mathcal{SG}_\alpha(M,\omega)$ and the corresponding solution operator is continuous in the uniform topology if and only if* A *is a bounded linear operator.*

Proof. The proof is divided into two parts, necessity and suffeciency.
Necessary: Let $\mathrm{A} \in \mathcal{SG}_\alpha(M,\omega)$ and take $\lambda > \omega > 0$. then (3.8) implies

$$\lambda^{\alpha-1}R(\lambda^{\alpha}, \mathrm{A}) - \lambda^{-1}\mathrm{I_d} = \int_0^{+\infty} \exp\left(-\lambda\frac{t^\rho}{\rho}\right)[S_\alpha(t) - \mathrm{I_d}]\frac{\mathrm{d}t}{t^{1-\rho}}, \tag{3.12}$$

when $\mu(t) :=\| S_\alpha(t) - \mathrm{I_d} \|$ is continuous on $t \geq 0$ and $\mu(0) = 0$. Using (3.1), we obatin

$$\mu(t) :=\| S_\alpha(t) - \mathrm{I_d} \| \leq \| S_\alpha(t) + \mathrm{I_d} \| \leq Me^{\omega\frac{t^\rho}{\rho}} + 1.$$

Fix $\epsilon > 0$ and take $\delta > 0$ such that $\mu(t) \leq \epsilon$ if $t \in [0,\delta]$ then

$$\begin{aligned}
\| \quad & \lambda^{\alpha-1}R(\lambda^{\alpha}, \mathrm{A}) - \lambda^{-1}\mathrm{I_d} \| \leq \int_0^{+\infty} \exp\left(-\lambda\frac{t^\rho}{\rho}\right)\mu(t)\frac{\mathrm{d}t}{t^{1-\rho}} \\
\leq \quad & \int_0^{\delta} \exp\left(-\lambda\frac{t^\rho}{\rho}\right)\mu(t)\frac{\mathrm{d}t}{t^{1-\rho}} + \int_\delta^{+\infty} \exp\left(-\lambda\frac{t^\rho}{\rho}\right)\mu(t)\frac{\mathrm{d}t}{t^{1-\rho}} \\
\leq \quad & \frac{\epsilon}{\lambda} + \mathcal{O}\left(\frac{1}{\lambda}\right) < 1, \lambda \longrightarrow +\infty,
\end{aligned}$$

when $\mu(t)$ sur $[0;\delta]$ and $\forall\epsilon > 0, \exists\delta > 0 : \mu(t) \leq \epsilon$; and

$$\begin{aligned}
\int_\delta^{+\infty} \exp\left(-\lambda\frac{t^\rho}{\rho}\right)\mu(t)\frac{\mathrm{d}t}{t^{1-\rho}} \quad & \leq \quad \int_\delta^{+\infty} \exp\left(-\lambda\frac{t^\rho}{\rho}\right)\left[Me^{\omega\frac{t^\rho}{\rho}} + 1\right]\frac{\mathrm{d}t}{t^{1-\rho}} \\
& \leq \quad M\int_\delta^{+\infty} \exp\left(-\lambda\frac{t^\rho}{\rho}\right)e^{\omega\frac{t^\rho}{\rho}}\frac{\mathrm{d}t}{t^{1-\rho}} + \int_\delta^{+\infty} \exp\left(-\lambda\frac{t^\rho}{\rho}\right)\frac{\mathrm{d}t}{t^{1-\rho}} \\
& \leq \quad M\int_\delta^{+\infty} e^{-(\lambda-\omega)\frac{t^\rho}{\rho}}\frac{\mathrm{d}t}{t^{1-\rho}} + \int_\delta^{+\infty} \exp\left(-\lambda\frac{t^\rho}{\rho}\right)\frac{\mathrm{d}t}{t^{1-\rho}} \\
& \leq \quad \frac{1}{\lambda-\omega}e^{-(\lambda-\omega)\frac{\delta^\rho}{\rho}} + \frac{1}{\lambda}e^{-\lambda\frac{\delta^\rho}{\rho}}.
\end{aligned}$$

Thus

$$\| \lambda^{\alpha}R(\lambda^{\alpha}, \mathrm{A}) - \mathrm{I_d} \| \leq \epsilon + \mathcal{O}\left(\frac{1}{\lambda}\right) < 1.$$

Hence $\lambda^{\alpha} R(\lambda^{\alpha}, \mathrm{A})$ has a bounded inverse. that is $R(\lambda^{\alpha}, \mathrm{A})^{-1} = (\lambda^{\alpha}\mathrm{I_d} - \mathrm{A})$ is bounded, thus A is bounded.
Sufficiency: let A is bounded and putting

$$S_{\alpha}(t) = E_{\alpha}\left(\left(\frac{t^{\rho}}{\rho}\right)^{\alpha} \mathrm{A}\right) \equiv \sum_{n=0}^{\infty} \frac{\left(\frac{t^{\rho}}{\rho}\right)^{\alpha n} \mathrm{A}^{n}}{\Gamma(1+\alpha n)}, \tag{3.13}$$

the right hand side of (3.13) converges in norm for every $t \geq 0$;

$$\begin{aligned}
\frac{\left(\frac{t^{\rho}}{\rho}\right)^{\alpha(n+1)}}{\Gamma(1+\alpha(1+n))} \frac{\Gamma(1+\alpha n)\mathrm{A}^{n+1}}{\left(\frac{t^{\rho}}{\rho}\right)^{\alpha n} \mathrm{A}^{n}} &= \frac{\Gamma(\alpha n+1)}{\Gamma(\alpha(n+1)+1)} \left(\frac{t^{\rho}}{\rho}\right)^{\alpha} \mathrm{A} \\
&= \frac{\Gamma(\alpha n+1)}{(\alpha n+\alpha)...(\alpha n+1)\Gamma(\alpha n+1)} \left(\frac{t^{\rho}}{\rho}\right)^{\alpha} \mathrm{A}.
\end{aligned}$$

Thus

$$\lim_{n \longrightarrow +\infty} \frac{\mathrm{A}}{(\alpha n+\alpha)...(\alpha n+1)} = 0$$

and definies a bounded linear operator $S_{\alpha}(t)$

$$\| S_{\alpha}(t) \| \leq \sum_{n=0}^{\infty} \frac{\| \mathrm{A} \|^{n} \, (\frac{t^{\rho}}{\rho})^{\alpha n}}{\Gamma(1+\alpha n)} = E_{\alpha}\left(\| \mathrm{A} \| \left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right). \tag{3.14}$$

If $0 < \alpha \leq 1, \; t \geqslant 0$ then

$$|\arg\left(\| \mathrm{A} \| \left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right)| \leq \frac{1}{2}\alpha\pi$$

$$E_{\alpha}\left(\| \mathrm{A} \| \left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right) = \frac{1}{\alpha}\exp\left(\| \mathrm{A} \| \left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right) + \epsilon_{\alpha,1}\left(\| \mathrm{A} \| \left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right)$$

and

$$\epsilon_{\alpha,1}\left(\| \mathrm{A} \| \left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right) = \sum_{n=1}^{N-1} \frac{\left(\| \mathrm{A} \| \left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right)^{-n}}{\Gamma(1-\alpha n)} + \mathcal{O}\left(|\arg\left(\| \mathrm{A} \| \left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right)^{-N}|\right).$$

The continuity of the Mittg−Leffler function in $t \geq 0$ imply that, if $w \geqslant 0$, there is a constant $\mathcal{C}$ such that

$$E_{\alpha}\left(w\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right) \leq \mathcal{C}e^{w^{\frac{1}{\alpha}} \frac{t^{\rho}}{\rho}}; \; t \geqslant 0; \; \alpha \in (0,1), \tag{3.15}$$

therefore (3.14) and (3.15) imply

$$\| S_{\alpha}(t) \| \leq \mathcal{C}\exp\left(\left(\frac{t^{\rho}}{\rho}\right) \| \mathrm{A} \|^{\frac{1}{\alpha}}\right). \tag{3.16}$$

Now we can apply (3.16) and obtain again the estimate (3.3), so, implies that $S_{\alpha}(t)$ is exponentially bounded.

Moreaver, hence $\mathrm{A} \in \mathcal{SG}_{\alpha}\left(\| \mathrm{A} \|^{\frac{1}{\alpha}}\right)$ and $S_{\alpha}(t)$ is the corresponding solution operator, $j = n-1$. From the definition it is clear that if $S_{\alpha}(t)$ is a uniformly continuous semigroup of bounded linear operators then

$$\lim_{t \downarrow \tau} \| S_{\alpha}(t) - S_{\alpha}(\tau) \| = 0 \tag{3.17}$$

and

$$\| S_\alpha(t) - \mathrm{I_d} \| \leq \sum_{n=1}^{\infty} \frac{\| \mathrm{A} \|^n \left(\frac{t^\rho}{\rho}\right)^{\alpha n}}{\Gamma(1+\alpha n)} = \left(\frac{t^\rho}{\rho}\right)^{\alpha} \| \mathrm{A} \| E_{\alpha,\alpha+1}\left(\| \mathrm{A} \| \left(\frac{t^\rho}{\rho}\right)^{\alpha}\right) \longrightarrow 0.$$

Therefore $\lim_{t\downarrow 0} \| S_\alpha(t) - \mathrm{I_d} \| = 0$, i.e., the solution operator is uniformly continuous. □

Theorem 3.7. *Let $S_\alpha(t)$ and $\tilde{S}_\alpha(t)$ be exponential bounded uniformly continuous solution operators with infinitesimal generators A and B, respectively. If $\mathrm{A} = \mathrm{B}$ then $S_\alpha(t) = \tilde{S}_\alpha(t)$, for every $t \geq 0$.*

Proof. Since $S_\alpha(t)$ is exponential bounded there exist constants $M_1 \geq 1$ and $\omega_1 \geq 0$ such that

$$\| S_\alpha(t) \| \leq M_1 e^{\omega_1 \frac{t^\rho}{\rho}}, \; t \geq 0.$$

Then for $\Re\lambda > \omega_1$ and $x \in \mathbb{X}$, we have

$$\lambda^{\alpha-1} R(\lambda^\alpha, \mathrm{A})x = \int_0^{+\infty} \exp\left(-\lambda \frac{t^\rho}{\rho}\right) S_\alpha(t) x \frac{\mathrm{d}t}{t^{1-\rho}}, \tag{3.18}$$

where $R(\lambda^\alpha, \mathrm{A}) = (\lambda^\alpha \mathrm{I_d} - \mathrm{A})^{-1}$ stands for the resolvent operator of A.

Similarly, for $\tilde{S}_\alpha(t)$ there exists ω_2 such that for for $\Re(\lambda) > \omega_2$ and $x \in \mathbb{X}$ we have

$$\| \tilde{S}_\alpha(t) \| \leq M_2 e^{\omega_2 \frac{t^\rho}{\rho}}, \; M_2 \geq 1, \; t \geq 0$$

and

$$\lambda^{\alpha-1} R(\lambda^\alpha, \mathrm{B})x = \int_0^{+\infty} \exp\left(-\lambda \frac{t^\rho}{\rho}\right) \tilde{S}_\alpha(t) x \frac{\mathrm{d}t}{t^{1-\rho}}. \tag{3.19}$$

From (3.18) and (3.19), $S_\alpha(t) = \tilde{S}_\alpha(t)$ follows from the uniqueness of the ρ-Laplace transform.

Below, we establish a connection between the solution operator and the fractional abstract equation (1.5-1.6). □

Theorem 3.8. *Let $0 < \alpha \leq 1$ and $S_\alpha(t)\,(t \geq 0)$ be a uniformly continuous solution operator satisfying (3.3). Then*
(i) *There exists a unique bounded linear operator A such that*

$$S_\alpha(t) = E_\alpha\left(\left(\frac{t^\rho}{\rho}\right)^{\alpha} \mathrm{A}\right), \; t \geqslant 0.$$

(ii) *The operator A in* ***(i)*** *is the infinitesimal generator of solution operator $S_\alpha(t)$.*
(iii) *For every $t \geq 0$,*

$${}^c\mathcal{D}^{\alpha,\rho}_{0+} S_\alpha(t) = \mathrm{A} S_\alpha(t) = S_\alpha(t)\mathrm{A}.$$

Proof.
(i) For $\Re(\lambda) > 0$ and $x \in \mathbb{X}$ we have:

$$\lambda^{\alpha-1} R(\lambda^\alpha, \mathrm{A})x = \int_0^{+\infty} \exp\left(-\lambda \frac{t^\rho}{\rho}\right) S_\alpha(t) x \frac{\mathrm{d}t}{t^{1-\rho}}, \rho > 0. \tag{3.20}$$

where $\lambda^{\alpha-1} R(\lambda^\alpha, \mathrm{A}) = \lambda^{\alpha-1}(\lambda^\alpha \mathrm{I_d} - \mathrm{A})^{-1}$ is $\rho-$ laplace transform of a familly $(S_\alpha(t))_{\alpha \geq 0}$ i.e:

$$\mathcal{L}_\rho \{S_\alpha(t)\}(\lambda) = \lambda^{\alpha-1}(\lambda^\alpha \mathrm{I_d} - \mathrm{A})^{-1} \tag{3.21}$$

and, we have

$$\mathcal{L}_\rho \left\{ E_\alpha\left(\left(\frac{t^\rho}{\rho}\right)^{\alpha} \mathrm{A}\right) \right\}(\lambda) = \lambda^{\alpha-1}(\lambda^\alpha \mathrm{I_d} - \mathrm{A})^{-1}. \tag{3.22}$$

Comparaison (3.21) and (3.22), we have

$$S_\alpha(t) = E_\alpha\left(\left(\frac{t^\rho}{\rho}\right)^\alpha \mathrm{A}\right).$$

(**ii**) Fix $0 < \alpha \leq 1$. From Theorem 3.6 we know that the infinitesimal generator of $S_\alpha(t)$ is a bounded linear operator A. Also, A is the infinitesimal generator of $E_\alpha\left(\mathrm{A}\left(\frac{t^\rho}{\rho}\right)^\alpha\right)$.

(**iii**) For every $x \geq 0$,

$$\mathrm{A}x = \Gamma(1+\alpha)\lim_{\tau\longrightarrow 0}\frac{S_\alpha(\tau)x - x}{\left(\frac{\tau^\rho}{\rho}\right)^\alpha}.$$

So we have

$$\begin{aligned}
\mathrm{A}S_\alpha(t)x &= \Gamma(1+\alpha)\lim_{\tau\longrightarrow 0}\frac{S_\alpha(\tau)S_\alpha(t)x - S_\alpha(t)x}{\left(\frac{\tau^\rho}{\rho}\right)^\alpha}\\
&= \Gamma(1+\alpha)\lim_{\tau\longrightarrow 0}\frac{S_\alpha(t)\left(S_\alpha(\tau)x - x\right)}{\left(\frac{\tau^\rho}{\rho}\right)^\alpha}\\
&= S_\alpha(t)\Gamma(1+\alpha)\lim_{\tau\longrightarrow 0}\frac{S_\alpha(\tau)x - x}{\left(\frac{\tau^\rho}{\rho}\right)^\alpha} = S_\alpha(t)\mathrm{A}x.
\end{aligned}$$

On the other hand, we are going to watch

$${}^c\mathcal{D}^{\alpha,\rho}_{0^+}S_\alpha(t)x = \mathrm{A}S_\alpha(t)x.$$

By Theorem 3.8-i-ii and Lemma 2.2, we deduce that

$$\begin{aligned}
\left({}^c\mathcal{D}^{\alpha,\rho}_{0^+}\right)S_\alpha(t)x &= \left({}^c\mathcal{D}^{\alpha,\rho}_{0^+}\right)\sum_{n=0}^{\infty}\frac{\left[\left(\frac{t^\rho}{\rho}\right)^\alpha \mathrm{A}\right]^n}{\Gamma(1+\alpha n)}x\\
&= \left({}^c\mathcal{D}^{\alpha,\rho}_{0^+}\right)x + \sum_{n=1}^{\infty}\frac{\mathrm{A}^n}{\Gamma(1+\alpha n)}\left({}^c\mathcal{D}^{\alpha,\rho}_{0^+}\right)\left(\frac{t^\rho}{\rho}\right)^{\alpha n}x\\
&= \sum_{n=1}^{\infty}\frac{\mathrm{A}^n}{\Gamma(1+\alpha n)}\frac{\rho^{\alpha-\alpha n}\Gamma(1+\alpha n)}{\Gamma(1-\alpha+\alpha n)}\left(t^\rho\right)^{\alpha n-\alpha}x\\
&= \sum_{n=1}^{\infty}\frac{\mathrm{A}^n}{\Gamma(1+\alpha(n-1))}\left(\frac{t^\rho}{\rho}\right)^{\alpha(n-1)}x\\
&= \sum_{n=0}^{\infty}\frac{\mathrm{A}^{n+1}}{\Gamma(1+\alpha n)}\left(\frac{t^\rho}{\rho}\right)^{\alpha n}x\\
&= \mathrm{A}\sum_{n=0}^{\infty}\frac{\mathrm{A}^n}{\Gamma(1+\alpha n)}\left(\frac{t^\rho}{\rho}\right)^{\alpha n}x = \mathrm{A}S_\alpha(t)x.
\end{aligned}$$

The proof is complete. □

4. The mild solutions of abstract Cauchy problem (1.5-1.6)

4.1. Existence and uniqueness of mild solutions of nonlinear problem (1.5-1.6)

In order to study the existence and uniqueness of a mild solutions for a class of fractional abstract Cauchy problem of the form (1.5-1.6), we consider the following auxiliary abstract problem

$$\begin{cases} \left({}^C\mathcal{D}^{\alpha,\rho}_{0^+}\right)u(t) = \mathrm{A}u(t) + \left(\mathcal{J}^{1-\alpha,\rho}_{0^+}\right)F(t,u(t)),\ t>0,\\ u(0) = u_0, \end{cases} \tag{4.1}$$

where the auxiliary function

$$F(t, u(t)) = \left(\mathcal{D}_{0^+}^{1-\alpha,\rho}\right) f(t, u(t)),\ 0 < \alpha \leq 1,\ t \in [0, T], \tag{4.2}$$

where $\left(\mathcal{D}_{0^+}^{1-\alpha,\rho}\right) f$ the generalized Riemann−type fractional derivative of f.

It is reasonable to define the mild solution of (4.1) as follows.

Definition 4.1. A function $u \in \mathcal{C}([0, T]; \mathbb{X})$ is called a mild solution of (4.1) if

$$u(t) = u_0 + \mathrm{A}\left(\mathcal{J}_{0^+}^{\alpha,\rho}\right) u(t) + \int_0^t F\left(\tau, u\left(\tau\right)\right) \frac{d\tau}{\tau^{1-\rho}}. \tag{4.3}$$

By Lemma 2.1 and (4.1), we obtain (4.3).

Lemma 4.2. *Let* A *be the infinitesimal generator of a solution operator $S_\alpha(t)$, and let $F \in \mathcal{C}([0, T] \times \mathbb{X}, \mathbb{X})$ is a mild solution of (4.1), then*

$$u(t) = S_\alpha(t)u_0 + \int_0^t S_\alpha\left(\left(\rho\left(\frac{t^\rho}{\rho} - \frac{\tau^\rho}{\rho}\right)\right)^{\frac{1}{\rho}}\right) F\left(\tau, u\left(\tau\right)\right) \frac{d\tau}{\tau^{1-\rho}}. \tag{4.4}$$

Proof. Assume $u(t)$ satisfies (4.1). By applying the ρ-Laplace transform to (4.3), we get

$$\begin{aligned} \mathcal{L}_\rho\left\{u(t)\right\}(\tau) &= \mathcal{L}_\rho\left\{u_0\right\}(\tau) + \mathcal{L}_\rho\left\{\mathrm{A}\mathcal{J}_{0^+}^{\alpha,\rho}u(t)\right\}(\tau) + \mathcal{L}_\rho\left\{\mathcal{J}_{0^+}^{1,\rho}F(t, u(t))\right\}(\tau) \\ &= u_0\mathcal{L}_\rho\left\{1\right\}(\tau) + \lambda^{-\alpha}\mathrm{A}\mathcal{L}_\rho\left\{u(t)\right\}(\tau) + \lambda^{-1}\mathcal{L}_\rho\left\{F(t, u(t))\right\}(\tau). \end{aligned}$$

This implies that

$$\lambda^\alpha\mathcal{L}_\rho\left\{u(t)\right\}(\tau) = u_0\lambda^\alpha\mathcal{L}_\rho\left\{1\right\}(\tau) + \mathrm{A}\mathcal{L}_\rho\left\{u(t)\right\}(\tau) + \lambda^{\alpha-1}\mathcal{L}_\rho\left\{F(t, u(t))\right\}(\tau), \tag{4.5}$$

which implies that

$$\left(\lambda^\alpha - A\right)\mathcal{L}_\rho\left\{u(t)\right\}(\tau) = u_0\lambda^\alpha\mathcal{L}_\rho\left\{1\right\}(\tau) + \lambda^{\alpha-1}\mathcal{L}_\rho\left\{F(t, u(t))\right\}(\tau),$$

and consequently

$$\mathcal{L}_\rho\left\{u(t)\right\}(\tau) = u_0\lambda^{\alpha-1}\left(\lambda^\alpha - A\right)^{-1} + \lambda^{\alpha-1}\left(\lambda^\alpha - A\right)^{-1}\mathcal{L}_\rho\left\{F(t, u(t))\right\}(\tau). \tag{4.6}$$

On the other hand, using Theorem 3.5 and (4.6) we deduce that

$$\mathcal{L}_\rho\left\{u(t)\right\}(\tau) = \mathcal{L}_\rho\left\{S_\alpha(t)u_0\right\}(\tau) + \mathcal{L}_\rho\left\{S_\alpha(t)\right\}(\tau)\mathcal{L}_\rho\left\{F(t, u(t))\right\}(\tau).$$

By Definition 2.5, we obtain

$$\mathcal{L}_\rho\left\{u(t)\right\}(\tau) = \mathcal{L}_\rho\left\{S_\alpha(t)u_0\right\}(\tau) + \mathcal{L}_\rho\left\{S_\alpha(t) *_\rho F(t, u(t))\right\}(\tau).$$

Taking inverse ρ- Laplace transform into account, we arrived at (4.4). □

By virtue of Lemma 4.2, we get the following:

Theorem 4.3. *The mild solution operators of (1.5-1.6) is in fact a generalization of α-order fractional uniformly continuous solution operator $S_\alpha(t)$. The resolvent equation, we given*

$$u(t) = S_\alpha(t)u_0 + \int_0^t S_\alpha\left(\left(\rho\left(\frac{t^\rho}{\rho} - \frac{\tau^\rho}{\rho}\right)\right)^{\frac{1}{\rho}}\right) \mathcal{D}_{0^+}^{1-\alpha,\rho} f(\tau, u(\tau)) \frac{\mathrm{d}\tau}{\tau^{1-\rho}}, \tag{4.7}$$

where $\left(\mathcal{D}_{0^+}^{1-\alpha,\rho}\right) f$ the generalized Riemann−type fractional derivative of f and $S_\alpha(t)$ is a solution operator generated by A.

Proof. It can be easily shown by direct computation that the integral equation (4.7) satisfies the boundary value problem (1.5-1.6).

Indeed, by virtue of the definition of F, let $u(t)$ be given by formula (4.7) and by Lemma 2.1, operating on both sides of it by $\left({}^{c}\mathcal{D}_{0^+}^{\alpha,\rho}\right)$, we get

$$\begin{aligned}
\left({}^{c}\mathcal{D}_{0^+}^{\alpha,\rho}\right) u(t) &= \left({}^{c}\mathcal{D}_{0^+}^{\alpha,\rho}\right)\left[S_\alpha(t)u_0 + \int_0^t S_{\alpha,\rho}(t,\tau)\,\mathcal{D}_{0^+}^{1-\alpha,\rho} f(\tau,u))\frac{\mathrm{d}\tau}{\tau^{1-\rho}}\right] \\
&= \mathrm{A}S_\alpha(t)u_0 + \sum_{n=0}^{\infty}\left({}^{c}\mathcal{D}_{0^+}^{\alpha,\rho}\right)\mathcal{J}_{0^+}^{\alpha+\alpha n,\rho}\mathrm{A}^n f(t,u(t)) \\
&= \mathrm{A}S_\alpha(t)u_0 + \sum_{n=0}^{\infty}\mathcal{J}_{0^+}^{\alpha n,\rho}\mathrm{A}^n f(t,u(t)) \\
&= \mathrm{A}S_\alpha(t)u_0 + f(t,u(t)) + \mathcal{J}_{0^+}^{\alpha,\rho}\mathrm{A}\sum_{n=0}^{\infty}\left({}^{c}\mathcal{D}_{0^+}^{\alpha,\rho}\right)\mathcal{J}_{0^+}^{\alpha+\alpha n,\rho}\mathrm{A}^n f(t,u(t)) \\
&= \mathrm{A}S_\alpha(t)u_0 + f(t,u(t)) + \mathcal{J}_{0^+}^{\alpha,\rho}\mathrm{A}\left({}^{c}\mathcal{D}_{0^+}^{\alpha,\rho}u(t) - \left({}^{c}\mathcal{D}_{0^+}^{\alpha,\rho}\right)S_\alpha(t)u_0\right) \\
&= \mathrm{A}S_\alpha(t)u_0 + f(t,u(t)) + \mathrm{A}u(t) - \mathrm{A}u(0) - \mathrm{A}S_\alpha(t)u_0 + \mathrm{A}S_\alpha(0)u_0 \\
&= \mathrm{A}u(t) + f(t,u(t)).
\end{aligned}$$

Since the initial condition is trivially satisfied, it follows that representation (4.7) does indeed solve the fractional abstract Cauchy problem (1.5-1.6). The proof is complete. □

The following theorem is regarding the existence of mild solution of the problem (1.5-1.6).

Theorem 4.4. *Assume $f \in \mathcal{C}([0,T],\mathbb{X})$ is continuous and (A_2) holds. Then the problem (1.5-1.6) has at least one mild solution defined on $\mathbb{X}$.*

Proof. Consider the operator $\Psi : \mathcal{C}([0,T],\mathbb{X}) \longrightarrow \mathcal{C}([0,T],\mathbb{X})$ defined by

$$(\Psi u)(t) = S_\alpha(t)u_0 + \int_0^t S_{\alpha,\rho}(t,\tau)\,\mathcal{D}_{0^+}^{1-\alpha,\rho} f(\tau,u(\tau))\frac{\mathrm{d}\tau}{\tau^{1-\rho}}. \tag{4.8}$$

Let

$$B_r := \left\{u \in \mathcal{C}([0,T),\mathbb{X}) : \|u\| \le r\right\},$$

where $r > 0$ is to be determined.

Clearly, the fixed points of the operator Ψ are solutions of the problem (1.5-1.6).

For any $u \in \mathcal{C}([0,T],\mathbb{X})$, and each $t \in [0,T]$, we have

$$\begin{aligned}
\|(\Psi u)\| &\le \|S_\alpha(t)u_0\| + \left\|\int_0^t S_{\alpha,\rho}(t,\tau)\,\mathcal{D}_{0^+}^{1-\alpha,\rho} f(\tau,u(\tau)))\frac{\mathrm{d}\tau}{\tau^{1-\rho}}\right\| \\
&\le \|S_\alpha(t)u_0\| + \|S_{\alpha,\rho}(t,\tau)\|\left\|\int_0^t \mathcal{D}_{0^+}^{1-\alpha,\rho} f(\tau,u(\tau)))\frac{\mathrm{d}\tau}{\tau^{1-\rho}}\right\| \\
&\le \|S_\alpha(t)u_0\| + \|S_{\alpha,\rho}(t,\tau)\|\,p^*\left\|\int_0^t \mathcal{D}_{0^+}^{1-\alpha,\rho}[1]\,\frac{\mathrm{d}\tau}{\tau^{1-\rho}}\right\| \\
&\le Me^{w^{\frac{1}{\alpha}}\frac{T^\rho}{\rho}}\left[u_0 + \frac{p^*}{\Gamma(\alpha)}\left(\frac{T^\rho}{\rho}\right)^\alpha\right] := r < \infty,
\end{aligned}$$

which implies that the $\Psi u \in \mathcal{C}([0,T],\mathbb{X})$,where $p^* = \sup\{p(t) : t \in [0,T]\}$.

Moreover, for $u, v \in \mathcal{C}([0,T],\mathbb{X})$ and $t \in [0,T]$, we get

$$\begin{aligned}
\|(\Psi u)-(\Psi v)\| &\leq \|S_{\alpha,\rho}(t,\tau)\| \left\| \int_0^t \mathcal{D}_{0^+}^{1-\alpha,\rho}(f(\tau,u)-f(\tau,v)) \frac{\mathrm{d}\tau}{\tau^{1-\rho}} \right\| \\
&\leq \|S_{\alpha,\rho}(t,\tau)\| \left\| \int_0^t \mathcal{D}_{0^+}^{1-\alpha,\rho} \|p(t) \frac{|u(t)|}{1+|u(t)|} - p(t) \frac{|v(t)|}{1+|v(t)|} \frac{\mathrm{d}\tau}{\tau^{1-\rho}} \right\| \\
&\leq \|S_{\alpha,\rho}(t,\tau)\| p^* \left\| \left(\frac{|u(t)|}{1+|u(t)|} - \frac{|v(t)|}{1+|v(t)|} \right) \right\| \int_0^t \mathcal{D}_{0^+}^{1-\alpha,\rho}[1] \frac{\mathrm{d}\tau}{\tau^{1-\rho}} \\
&\leq \|S_{\alpha,\rho}(t,\tau)\| p^* \int_0^t \mathcal{D}_{0^+}^{1-\alpha,\rho}[1] \frac{\mathrm{d}\tau}{\tau^{1-\rho}} \\
&\leq M e^{w^{\frac{1}{\alpha}} \frac{T^\rho}{\rho}} \frac{p^*}{\Gamma(\alpha)} \left(\frac{T^\rho}{\rho} \right)^\alpha .
\end{aligned}$$

This proves that Ψ transforms the ball B_r into itself. We shall show that the operator $\Psi : B_r \longrightarrow B_r$ satisfies all the conditions of Schauder fixed point theorem. The proof will be given in several steps.

Step 1: $\Psi : B_r \longrightarrow B_r$ is continuous. Let u_n be a sequence such that $u_n \to u$ in $\mathcal{C}([0,T),\mathbb{X})$. Then for each $t \in [0,T]$

$$\begin{aligned}
\|(\Psi u_n)-(\Psi u)\| &\leq \|S_{\alpha,\rho}(t,\tau)\| \left\| \int_0^t \mathcal{D}_{0^+}^{1-\alpha,\rho}(f(\tau,u_n)-f(\tau,u)) \frac{\mathrm{d}\tau}{\tau^{1-\rho}} \right\| \\
&\leq \|S_{\alpha,\rho}(t,\tau)\| \left\| \int_0^t \mathcal{D}_{0^+}^{1-\alpha,\rho} \|p(t) \frac{|u_n(t)|}{1+|u_n(t)|} - p(t) \frac{|u(t)|}{1+|u(t)|} \frac{\mathrm{d}\tau}{\tau^{1-\rho}} \right\| \\
&\leq \|S_{\alpha,\rho}(t,\tau)\| p^* \left\| \left(\frac{|u_n(t)|}{1+|u_n(t)|} - \frac{|u(t)|}{1+|u(t)|} \right) \right\| \int_0^t \mathcal{D}_{0^+}^{1-\alpha,\rho}[1] \frac{\mathrm{d}\tau}{\tau^{1-\rho}}.
\end{aligned}$$

Since $u_n \to u$ as $n \to \infty$ and f is continuous, then by the Lebesgue dominated convergence theorem, we have

$$\|(\Psi u_n)(t)-(\Psi u)(t)\| \longrightarrow 0 \text{ as } n \to \infty.$$

Step 2: $\Psi(B_r)$ is uniformly bounded. This is clear since $\Psi(B_r) \subset B_r$ and B_r is bounded.

Step 3: $\Psi(B_r)$ is equicontinuous. Let $t_1, t_2 \in [0,T), t_1 < t_2$ and let $u \in B_r$. Thus, we have

$$\begin{aligned}
\|(\Psi u)(t_2)-(\Psi u)(t_1)\| &\leq \|S_\alpha(t_2)-S_\alpha(t_1)\| u_0 \\
&\quad + \left\| \int_0^{t_2} S_{\alpha,\rho}(t_2,\tau) \mathcal{D}_{0^+,\tau}^{1-\alpha,\rho}[f] \frac{\mathrm{d}\tau}{\tau^{1-\rho}} - \int_0^{t_1} S_{\alpha,\rho}(t_1,\tau) \mathcal{D}_{0^+,\tau}^{1-\alpha,\rho}[f] \frac{\mathrm{d}\tau}{\tau^{1-\rho}} \right\| \\
&\leq \|S_\alpha(t_2)-S_\alpha(t_1)\| u_0 \\
&\quad + \left\| \int_0^{t_1} [S_{\alpha,\rho}(t_2,\tau) - S_{\alpha,\rho}(t_1,\tau)] \mathcal{D}_{0^+,\tau}^{1-\alpha,\rho} f(\tau,u) \frac{\mathrm{d}\tau}{\tau^{1-\rho}} \right\| \\
&\quad + \left\| \int_{t_1}^{t_2} S_{\alpha,\rho}(t_2,\tau) \mathcal{D}_{0^+,\tau}^{1-\alpha,\rho} f(\tau,u) \frac{\mathrm{d}\tau}{\tau^{1-\rho}} \right\|.
\end{aligned}$$

From the property (3.17), we have

$$\begin{aligned}
\|(\Psi u)(t_2)-(\Psi u)(t_1)\| &\leq \|S_{\alpha,\rho}(t_2,\tau)\| p^* \int_{t_1}^{t_2} \mathcal{D}_{0^+}^{1-\alpha,\rho}[1] \frac{\mathrm{d}\tau}{\tau^{1-\rho}} \\
&\leq M e^{w^{\frac{1}{\alpha}} \frac{T^\rho}{\rho}} \frac{p^*}{\Gamma(\alpha)} \left(\frac{T^\rho}{\rho} \right)^{\alpha-1} \left| \frac{t_2^\rho - t_1^\rho}{\rho} \right|.
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero.

As a consequence of Step 1-3 together with the Arzela-Ascoli theorem, we can conclude that Ψ is continuous and compact. From an application of Theorem 2.12, we deduce that Ψ has a fixed point u which is a solution of the problem (1.5-1.6). □

Theorem 4.5. *Assume that the hypotheses (A_1) and (3.3) hold. If*

$$Me^{w^{\frac{1}{\alpha}}\frac{T^\rho}{\rho}}\frac{L}{\Gamma(\alpha)}\left(\frac{T^\rho}{\rho}\right)^\alpha < 1; M \geq 1, w \geq 0. \tag{4.9}$$

Then there exists a unique mild solution of (1.5-1.6).

Proof. Let $\sup_{t\in[0,T)}|f(t,0)| = N$. Let $u \in \mathcal{C}([0,T),\mathbb{X})$, by (A_1) and (4.8), we see that

$$|f(\tau,u)| = |f(\tau,u) - f(\tau,0) + f(\tau,0)| \leq |f(\tau,u) - f(\tau,0)| + |f(\tau,0)| \leq L\|u\| + N \tag{4.10}$$

and

$$\begin{aligned}
\|\Psi u\| &\leq \|S_\alpha(t)u_0\| + \left\|\int_0^t S_{\alpha,\rho}(t,\tau)\mathcal{D}_{0^+}^{1-\alpha,\rho}f(\tau,u(\tau)))\frac{\mathrm{d}\tau}{\tau^{1-\rho}}\right\| \\
&\leq \|S_\alpha(t)u_0\| + \|S_{\alpha,\rho}(t,\tau)\|\left\|\int_0^t \mathcal{D}_{0^+}^{1-\alpha,\rho}f(\tau,u(\tau)))\frac{\mathrm{d}\tau}{\tau^{1-\rho}}\right\| \\
&\leq \|S_\alpha(t)u_0\| + \|S_{\alpha,\rho}(t,\tau)\|\|f\|\left\|\int_0^t \mathcal{D}_{0^+}^{1-\alpha,\rho}[1]\frac{\mathrm{d}\tau}{\tau^{1-\rho}}\right\| \\
&\leq Me^{w^{\frac{1}{\alpha}}\frac{T^\rho}{\rho}}\left[u_0 + \frac{L\|u\|+N}{\Gamma(\alpha)}\left(\frac{T^\rho}{\rho}\right)^\alpha\right] < \infty, \text{ for all } T < \infty,
\end{aligned}$$

which implies that the $\Psi u \in \mathcal{C}([0,T];\mathbb{X})$, for all $t \in [0,T)$.

Moreover, for $u, v \in \mathcal{C}([0,T),\mathbb{X})$ and $t \in [0,T]$ we get

$$\begin{aligned}
\|(\Psi u) - (\Psi v)\| &\leq \int_0^t S_{\alpha,\rho}(t,\tau)\|\mathcal{D}_{0^+}^{1-\alpha,\rho}\|f(\tau,u) - f(\tau,v)\|\frac{\mathrm{d}\tau}{\tau^{1-\rho}} \\
&\leq Me^{w^{\frac{1}{\alpha}}\frac{T^\rho}{\rho}}\frac{L}{\Gamma(\alpha)}\left(\frac{T^\rho}{\rho}\right)^\alpha\|u(t) - v(t)\|.
\end{aligned}$$

If

$$Me^{w^{\frac{1}{\alpha}}\frac{T^\rho}{\rho}}\frac{L}{\Gamma(\alpha)}\left(\frac{T^\rho}{\rho}\right)^\alpha < 1, \tag{4.11}$$

then

$$T^* < \left(\rho^\alpha\frac{\Gamma(\alpha)}{ML}\right)^{\frac{1}{\alpha\rho}}. \tag{4.12}$$

In this way, we have actually shown that the mapping Ψ is a contraction in space $\mathcal{C}([0,T^*),\mathbb{X})$, and it follows from Banach′s contraction principle that the mapping Ψ has a unique fixed point in that space.

Since the choice of T^* given by (4.12) does not depend on the initial condition, it follows that the solution obtained can be extended to arbitrary $T > 0$, repeating the previous procedure and choosing new initial conditions. □

4.2. Mild solutions and Mittag–Leffler function

By other method, representation of the mild solution of problem (1.5-1.6), obtained by applying the fractional analog of the generalized Duhamel principle, is given in the following statement:

Lemma 4.6. *The mild solution of the fractional abstract Cauchy problem (1.5-1.6) is given by*

$$u(t) = E_\alpha\left(\left(\frac{t^\rho}{\rho}\right)^\alpha \mathrm{A}\right)u_0 + \int_0^t E_\alpha\left(\left(\frac{t^\rho}{\rho} - \frac{\tau^\rho}{\rho}\right)^\alpha \mathrm{A}\right)\mathcal{D}_{0^+}^{1-\alpha,\rho}f(\tau,u(\tau)))\frac{\mathrm{d}\tau}{\tau^{1-\rho}}. \tag{4.13}$$

where $\mathcal{D}_{0^+}^{1-\alpha,\rho}f$ the generalized Riemann– type fractional derivative of f.

Proof. Let $u(t)$ be given by formula (4.7) then

$$\begin{aligned}
u(t) &= S_\alpha(t)u_0 + \int_0^t S_{\alpha,\rho}(t,\tau)\mathcal{D}_{0^+}^{1-\alpha,\rho} f(\tau,u(\tau)))\frac{\mathrm{d}\tau}{\tau^{1-\rho}}\\
&= E_\alpha\left(\left(\frac{t^\rho}{\rho}\right)^\alpha \mathrm{A}\right)u_0 + \int_0^t \sum_{n=0}^{\infty}\frac{1}{\Gamma(1+\alpha n)}\left[\mathrm{A}\left(\frac{1}{\rho}\left(\left(\rho\left(\frac{t^\rho}{\rho}-\frac{\tau^\rho}{\rho}\right)\right)^{\frac{1}{\rho}}\right)^\rho\right)^\alpha\right]^n \mathcal{D}_{0^+}^{1-\alpha,\rho} f(u)\frac{\mathrm{d}\tau}{\tau^{1-\rho}}\\
&= E_\alpha\left(\left(\frac{t^\rho}{\rho}\right)^\alpha \mathrm{A}\right)u_0 + \int_0^t \sum_{n=0}^{\infty}\frac{\mathrm{A}^n}{\Gamma(1+\alpha n)}\left(\frac{t^\rho}{\rho}-\frac{\tau^\rho}{\rho}\right)^{\alpha n}\mathcal{D}_{0^+}^{1-\alpha,\rho} f(u)\frac{\mathrm{d}\tau}{\tau^{1-\rho}}\\
&= E_\alpha\left(\left(\frac{t^\rho}{\rho}\right)^\alpha \mathrm{A}\right)u_0 + \int_0^t E_\alpha\left(\left(\frac{t^\rho}{\rho}-\frac{\tau^\rho}{\rho}\right)^\alpha \mathrm{A}\right)\mathcal{D}_{0^+}^{1-\alpha,\rho} f(u)\frac{\mathrm{d}\tau}{\tau^{1-\rho}}.
\end{aligned}$$

□

Integral representation stated in the next Lemma will often be used in proving some auxiliary results as well as in proving our main result.

Lemma 4.7. *Let $0<\alpha<1$ and let $S_\alpha(t)$ be a solution operator generated by* A *then*

$$\int_0^t S_{\alpha,\rho}(t,\tau)D_{0^+}^{1-\alpha,\rho} f(\tau,u(\tau)))\frac{\mathrm{d}\tau}{\tau^{1-\rho}}=\int_0^t \frac{E_{\alpha,\alpha}\left(\mathrm{A}\left(\frac{t^\rho}{\rho}-\frac{\tau^\rho}{\rho}\right)^\alpha\right)}{\left(\frac{t^\rho}{\rho}-\frac{\tau^\rho}{\rho}\right)^{1-\alpha}} f(\tau,u))\frac{\mathrm{d}\tau}{\tau^{1-\rho}}. \tag{4.14}$$

Proof. By Theorem 3.8-i and Lemma 2.1-iii, we have

$$\begin{aligned}
\int_0^t S_{\alpha,\rho}(t,\tau)\mathcal{D}_{0^+}^{1-\alpha,\rho} f(u)\frac{\mathrm{d}\tau}{\tau^{1-\rho}} &= \int_0^t \sum_{n=0}^{\infty}\frac{1}{\Gamma(1+\alpha n)}\left[\left(\frac{1}{\rho}\left(\rho\left(\frac{t^\rho}{\rho}-\frac{\tau^\rho}{\rho}\right)^{\frac{1}{\rho}}\right)^\rho\right)^\alpha \mathrm{A}\right]^n \mathcal{D}_{0^+}^{1-\alpha,\rho} f(u)\frac{\mathrm{d}\tau}{\tau^{1-\rho}}\\
&= \int_0^t \sum_{n=0}^{\infty}\frac{\mathrm{A}^n}{\Gamma(1+\alpha n)}\left(\frac{t^\rho}{\rho}-\frac{\tau^\rho}{\rho}\right)^{\alpha n}\mathcal{D}_{0^+}^{1-\alpha,\rho} f\tau,u(\tau))\frac{\mathrm{d}\tau}{\tau^{1-\rho}}\\
&= \sum_{n=0}^{\infty}\frac{1}{\Gamma(1+\alpha n)}\int_0^t \left(\frac{t^\rho}{\rho}-\frac{\tau^\rho}{\rho}\right)^{\alpha n}\mathrm{A}^n\mathcal{D}_{0^+}^{1-\alpha,\rho} f\tau,u(\tau))\frac{\mathrm{d}\tau}{\tau^{1-\rho}}\\
&= \sum_{n=0}^{\infty}\mathcal{J}_{0^+}^{1+\alpha n,\rho}\mathrm{A}^n\mathcal{D}_{0^+}^{1-\alpha,\rho} f(t,u(t) = \sum_{n=0}^{\infty}\mathcal{J}_{0^+}^{\alpha+\alpha n,\rho}\mathrm{A}^n f(t,u(t))\\
&= \sum_{n=0}^{\infty}\frac{1}{\Gamma(\alpha+\alpha n)}\int_0^t \left(\frac{t^\rho}{\rho}-\frac{\tau^\rho}{\rho}\right)^{\alpha+\alpha n-1}\mathrm{A}^n f(\tau,u(\tau))\frac{\mathrm{d}\tau}{\tau^{1-\rho}}\\
&= \int_0^t \left(\frac{t^\rho}{\rho}-\frac{\tau^\rho}{\rho}\right)^{\alpha-1}\sum_{n=0}^{\infty}\frac{\left[\mathrm{A}\left(\frac{t^\rho}{\rho}-\frac{\tau^\rho}{\rho}\right)^\alpha\right]^n}{\Gamma(\alpha+\alpha n)}\mathrm{A}^n f\tau,u(\tau))\frac{\mathrm{d}\tau}{\tau^{1-\rho}}\\
&= \int_0^t \left(\frac{t^\rho}{\rho}-\frac{\tau^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\left(\frac{t^\rho}{\rho}-\frac{\tau^\rho}{\rho}\right)^\alpha \mathrm{A}\right) f(\tau,u(\tau)))\frac{\mathrm{d}\tau}{\tau^{1-\rho}}.
\end{aligned}$$

□

Theorem 4.8. *The solution operators is in fact a generalization of α -order fractional uniformly continuous solution operator of (1.5-1.6). The resolvent equation, we given*

$$u(t) = E_\alpha\left(\left(\frac{t^\rho}{\rho}\right)^\alpha \mathrm{A}\right)u_0 + \int_0^t \left(\frac{t^\rho}{\rho}-\frac{\tau^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\left(\frac{t^\rho}{\rho}-\frac{\tau^\rho}{\rho}\right)^\alpha \mathrm{A}\right) f\tau,u(\tau))\frac{\mathrm{d}\tau}{\tau^{1-\rho}}. \tag{4.15}$$

4.3. Dependence of mild solution on the parameters

For f Lipschitz, the mild solution's dependence on the order of the differential operator, the boundary values, and the nonlinear term f are also discussed.

4.3.1. The dependence on parameters of the left-hand side of (1.5)

We show that the mild solutions of two equations with neighbouring orders will (under suitable conditions on their right hand sides f) lie close to one another.

Theorem 4.9. *Suppose that the conditions of Theorem 4.5 hold. Let $u(t), u_\epsilon(t)$ be the mild solutions, respectively, of the problems (1.5-1.6) and*

$$ {}^{c}\mathcal{D}_{0^+}^{\alpha-\epsilon,\rho}u(t) = \mathrm{A}u(t) + f(t,u(t)),\ 0 < t \leq T, \tag{4.16}$$

with the boundary conditions (1.6), where $0 < \alpha - \epsilon < \alpha \leq 1$. Then $\|u - u_\epsilon\| = \mathcal{O}(\epsilon)$, for ϵ sufficiently small.

Proof. By the above theorems, we can obtain the following results. Let

$$u_\epsilon(t) = S_{\alpha-\epsilon}(t)\, u_0 + \int_0^t S_{\alpha-\epsilon,\rho}(t,\tau)\, \mathcal{D}_{0^+}^{1-\alpha+\epsilon,\rho} f(\tau, u_\epsilon(\tau)) \frac{\mathrm{d}\tau}{\tau^{1-\rho}}, \tag{4.17}$$

be the mild solution of (4.161.6).

On one hand, from (4.7) and (4.16) yields

$$\begin{aligned}
\|u_\epsilon - u\| &= \|(S_{\alpha-\epsilon}(t) - S_\alpha(t))\, u_0\| \\
&\quad + \left\|\int_0^t \left[S_{\alpha-\epsilon,\rho}(t,\tau)\, \mathcal{D}_{0^+}^{1-\alpha+\epsilon,\rho} f(u_\epsilon)) - S_{\alpha,\rho}(t,\tau)\, \mathcal{D}_{0^+}^{1-\alpha,\rho}(f(u_\epsilon) - f(u))\right] \frac{\mathrm{d}\tau}{\tau^{1-\rho}}\right\| \\
&\leq \|(S_{\alpha-\epsilon}(t) - S_\alpha(t))\, u_0\| \\
&\quad + \left\|\int_0^t S_{\alpha,\rho}(t,\tau) \left(\mathcal{D}_{0^+}^{1-\alpha+\epsilon,\rho} f(\tau,u)) - \mathcal{D}_{0^+}^{1-\alpha,\rho} f(\tau,u))\right) \frac{\mathrm{d}\tau}{\tau^{1-\rho}}\right\| \\
&\quad + \left\|\int_0^t \left[S_{\alpha-\epsilon,\rho}(t,\tau) - S_{\alpha,\rho}(t,\tau)\right] \mathcal{D}_{0^+}^{1-\alpha+\epsilon,\rho}(f(\tau,u_\epsilon))) \frac{\mathrm{d}\tau}{\tau^{1-\rho}}\right\| \\
&\quad + \left\|\int_0^t S_{\alpha,\rho}(t,\tau) \left(\mathcal{D}_{0^+}^{1-\alpha+\epsilon,\rho} f(\tau,u_\epsilon)) - \mathcal{D}_{0^+}^{1-\alpha+\epsilon,\rho} f(\tau,u))\right) \frac{\mathrm{d}\tau}{\tau^{1-\rho}}\right\|.
\end{aligned}$$

By Theorem 3.8-i, we have

$$\lim_{\epsilon\to 0} \|(S_\alpha(t) - S_{\alpha-\epsilon}(t))\, u_0\| = 0$$

and

$$\begin{aligned}
\|u_\epsilon - u\| &\leq M e^{w^{\frac{1}{\alpha}} \frac{T^\rho}{\rho}} \|f(\tau, u(\tau))\| \left\|\left(\mathcal{D}_{0^+}^{1-\alpha+\epsilon,\rho}[1] - \mathcal{D}_{0^+}^{1-\alpha,\rho}[1]\right)\right\| \left(\frac{T^\rho}{\rho}\right) \\
&\quad + M e^{w^{\frac{1}{\alpha}} \frac{T^\rho}{\rho}} L \|u_\epsilon - u\| \left\|\mathcal{D}_{0^+}^{1-\alpha+\epsilon,\rho}[1]\right\| \left(\frac{T^\rho}{\rho}\right).
\end{aligned}$$

From Lemma 2.2, that

$$\|u_\epsilon - u\| \leq \frac{1}{k_\epsilon} M e^{w^{\frac{1}{\alpha}} \frac{T^\rho}{\rho}} \|f(\tau, u(\tau))\| \left[\frac{1}{\Gamma(\alpha-\epsilon)} \left(\frac{T^\rho}{\rho}\right)^{\alpha-\epsilon} - \frac{1}{\Gamma(\alpha)} \left(\frac{T^\rho}{\rho}\right)^{\alpha}\right], \tag{4.18}$$

where

$$k_\epsilon = 1 - M e^{w^{\frac{1}{\alpha}} \frac{T^\rho}{\rho}} L \frac{1}{\Gamma(\alpha-\epsilon)} \left(\frac{T^\rho}{\rho}\right)^{\alpha-\epsilon} > 0. \tag{4.19}$$

Thus, in accordance with (4.18) and (4.19) we obtain $\|u - u_\epsilon\| = O(\epsilon)$. □

4.3.2. The dependence on parameters of initial conditions

Let us introduce small changes in the initial conditions of (1.6) and consider (1.5) with boundary conditions

$$u(0) = u_0 + \epsilon. \tag{4.20}$$

Theorem 4.10. *Assume the conditions of Theorem 4.5 hold. Let $u(t), u_\epsilon(t)$ be respective solutions, of the problems (1.5-1.6) and the boundary conditions (1.5-4.20). Then $\|u_\epsilon - u\| = O(\epsilon)$.*

Proof. Let

$$u_\epsilon(t) = S_\alpha(t)(u_0+\epsilon) + \int_0^t S_{\alpha,\rho}(t,\tau)\,\mathcal{D}_{0^+}^{1-\alpha,\rho} f(\tau, u(\tau)))\frac{\mathrm{d}\tau}{\tau^{1-\rho}}, \tag{4.21}$$

solutions of the problem (1.5-4.20), Then

$$\begin{aligned}
\|u_\epsilon - u\| &= \left| S_\alpha(t)\,\epsilon + \int_0^t S_{\alpha,\rho}(t,\tau)\,\mathcal{D}_{0^+}^{1-\alpha,\rho}\left(f(\tau,u_\epsilon)) - f(\tau,u))\right)\frac{\mathrm{d}\tau}{\tau^{1-\rho}}\right| \\
&\leq \|S_\alpha(t)\,\epsilon\| + \|S_{\alpha,\rho}(t,\tau)\|\|f(\tau,u_\epsilon) - f(\tau,u))\| \left\| \int_0^t \mathcal{D}_{0^+}^{1-\alpha,\rho}[1]\frac{\mathrm{d}\tau}{\tau^{1-\rho}}\right\| \\
&\leq \varepsilon M e^{w^{\frac{1}{\alpha}}\frac{T^\rho}{\rho}} + M e^{w^{\frac{1}{\alpha}}\frac{T^\rho}{\rho}}\left[\frac{L}{\Gamma(\alpha)}\left(\frac{T^\rho}{\rho}\right)^\alpha\right]\|u_\epsilon - u\|.
\end{aligned}$$

Consequently, we obtain

$$\|u_\epsilon - u\| \leq \frac{1}{k_0}\epsilon M e^{w^{\frac{1}{\alpha}}\frac{T^\rho}{\rho}}, \tag{4.22}$$

where k_0 is defined by (4.19). It is easy to see that $\|u_\epsilon - u\| = O(\epsilon)$. □

4.3.3. The dependence on parameters of the right-hand side of (1.6)

Theorem 4.11. *Suppose that the conditions of Theorem 4.5 hold. Let $u(t), u_\epsilon(t)$ be the mild solutions, respectively, of the problems (1.5-1.6) and*

$${}^{c}\mathcal{D}_{0^+}^{\alpha,\rho}u(t) = \mathrm{A}u(t) + f(t,u(t)) + \epsilon h(t,u(t)),\ 0 < t \leq T,\ \alpha \in (0,1), \tag{4.23}$$

with boundary conditions (1.6) where $h \in C[0,T]$. Then $\|u - u_\epsilon\| = O(\epsilon)$.

Proof. Assume that $u(t)$ and $u_\epsilon(t)$ on $[0,T]$ are the mild solutions of the initial value problems (1.5-1.6) and (4.23-1.6), respectively. In accordance with Theorem 4.3, we have

$$u_\epsilon(t) = S_\alpha(t)u_0 + \int_0^t S_{\alpha,\rho}(t,\tau)\,\mathcal{D}_{0^+}^{1-\alpha,\rho}\left[f(\tau,u_\epsilon) + \epsilon h(\tau,u_\epsilon)\right]\frac{\mathrm{d}\tau}{\tau^{1-\rho}}. \tag{4.24}$$

Then

$$\begin{aligned}
\|u_\epsilon - u\| &= \left|\int_0^t S_{\alpha,\rho}(t,\tau)\,\mathcal{D}_{0^+}^{1-\alpha,\rho}\left(f(\tau,u_\epsilon(\tau))) - f(\tau,u)) + \varepsilon h(\tau,u_\epsilon)\right)\frac{\mathrm{d}\tau}{\tau^{1-\rho}}\right| \\
&\leq \|S_{\alpha,\rho}(t,\tau)\|\|f(\tau,u_\epsilon) - f(\tau,u) + \varepsilon h(\tau,u_\epsilon)\| \left\|\int_0^t \mathcal{D}_{0^+}^{1-\alpha,\rho}[1]\frac{\mathrm{d}\tau}{\tau^{1-\rho}}\right\| \\
&\leq M e^{w^{\frac{1}{\alpha}}\frac{T^\rho}{\rho}}\left[\epsilon\frac{\|h\|}{\Gamma(\alpha)}\left(\frac{T^\rho}{\rho}\right)^\alpha\right] + M e^{w^{\frac{1}{\alpha}}\frac{T^\rho}{\rho}}\left[\frac{L}{\Gamma(\alpha)}\left(\frac{T^\rho}{\rho}\right)^\alpha\right]\|u_\epsilon - u\|.
\end{aligned}$$

We get

$$\|u_\epsilon - u\| \leq \frac{1}{k_0} M e^{w^{\frac{1}{\alpha}}\frac{T^\rho}{\rho}}\left[\epsilon\|h\|\left(\frac{T^\rho}{\rho}\right)^\alpha\right], \tag{4.25}$$

where k_0 is defined by (4.19). It is easy to see that $\|u_\epsilon - u\| = O(\epsilon)$. □

Finally, we state and prove the Ulam–Hyers stability result.

4.4. Ulam–Hyers stability of mild solution

Next, we are going to discuss stability in Ulam–Hyers sense of the fractional equation (1.5) on $[0,T]$. In this subsection, we study Hyers-Ulam stability for the solutions of our proposed system. Thanks to Definitions 2.13-2.15 and Theorem 4.3, the respective results are received. In the proofs of Theorems 4.13-4.14, we use one of the most important techniques of classical calculus: Gronwall type inequality and integration by parts in the settings of ρ-fractional operators.

Lemma 4.12. *if $\tilde{u} \in \mathcal{C}$ is a mild solution of the inequation (2.2) then $\tilde{u}$ is a mild solution of the following integral inequation*

$$\left|\tilde{u}(t) - S_\alpha(t)u_0 - \int_0^t S_{\alpha,\rho}(t,\tau)\,\mathcal{D}_{0^+}^{1-\alpha,\rho} f(\tau,\tilde{u}(\tau))\frac{\mathrm{d}\tau}{\tau^{1-\rho}}\right| \leq \epsilon c_f, \tag{4.26}$$

with

$$c_f = \sup\left\{\int_0^t S_{\alpha,\rho}(t,\tau)\,\mathcal{D}_{0^+}^{1-\alpha,\rho}[1]\,\frac{\mathrm{d}\tau}{\tau^{1-\rho}} : t \in (0,T]\right\}, \tag{4.27}$$

where c_f is independent of $\tilde{u}(t)$ and f.

Proof. Let $\tilde{u} \in \mathcal{C}$ is a mild solution of the inequation (2.2), by Remark 2.16-ii, we have that

$$\tilde{u}(t) = S_\alpha(t)u_0 - \int_0^t S_{\alpha,\rho}(t,\tau)\left(\mathcal{D}_{0^+}^{1-\alpha,\rho}\right)(f(\tau,\tilde{u}(\tau)) + g(\tau))\,\frac{\mathrm{d}\tau}{\tau^{1-\rho}}. \tag{4.28}$$

In view of (3.3) and (4.28), we obtain

$$\begin{aligned}
&\left|\tilde{u}(t) - S_\alpha(t)u_0 - \int_0^t S_{\alpha,\rho}(t,\tau)\left(\mathcal{D}_{0^+}^{1-\alpha,\rho}\right)f(\tau,\tilde{u}(\tau))\frac{\mathrm{d}\tau}{\tau^{1-\rho}}\right| \\
\leq\ & \int_0^t S_{\alpha,\rho}(t,\tau)\left(\mathcal{D}_{0^+}^{1-\alpha,\rho}\right)g(\tau)\,\frac{\mathrm{d}\tau}{\tau^{1-\rho}} \\
\leq\ & \epsilon\left\|S_{\alpha,\rho}(t,\tau)\right\|\left\|\left(\mathcal{D}_{0^+}^{1-\alpha,\rho}\right)[1]\right\|\left(\frac{T^\rho}{\rho}\right) \leq \varepsilon c_f.
\end{aligned}$$

where c_f is defined below in (4.27). □

Theorem 4.13. *Assume that the assumptions (A_1) and (4.26) hold. Then the equation (1.5) is Ulam-Hyers stable with*

$$|\tilde{u}(0) - u(0)| = 0. \tag{4.29}$$

Proof. By Lemmas 2.4 , integrating by parts one shows that

$$\begin{aligned}
|u(t) - \tilde{u}(t)| &\leq \varepsilon c_f + \int_0^t S_{\alpha,\rho}(t,\tau)\left(\mathcal{D}_{0^+}^{1-\alpha,\rho}\right)(f(\tau,u(\tau)) - f(\tau,\tilde{u}(\tau)))\,\frac{\mathrm{d}\tau}{\tau^{1-\rho}} \\
&\leq \varepsilon c_f + L\int_0^t S_{\alpha,\rho}(t,\tau)\left(\mathcal{D}_{0^+,\tau}^{1-\alpha,\rho}\right)(|u(\tau) - \tilde{u}(\tau)|)\,\frac{\mathrm{d}\tau}{\tau^{1-\rho}} \\
&\leq \varepsilon c_f + L\int_0^t \left(\mathcal{D}_{\tau,T}^{1-\alpha,\rho}\right)S_{\alpha,\rho}(t,\tau)\,|u(\tau) - \tilde{u}(\tau)|\,\frac{\mathrm{d}\tau}{\tau^{1-\rho}}.
\end{aligned}$$

Then Gronwall's inequality implies that

$$v(t) = \varepsilon c_f \text{ and } v(\tau) = L\left(\mathcal{D}_{t,T}^{1-\alpha,\rho}\right)\left[S_{\alpha,\rho}(t,\tau)\,\tau^{\rho-1}\right].$$

It follows that

$$\mid \tilde{u}(t) - u(t) \mid \leq v(t)\exp(\mathit{\Lambda}(t)),\ \ \mathit{\Lambda}(t) = \int_0^t b(\tau)\mathrm{d}\tau. \tag{4.30}$$

It is immediately to find that $v(t) \longrightarrow 0$ in the presence of $\epsilon \longrightarrow 0$. Therefore, we conclude that $\tilde{u}(t) \longrightarrow u(t)$.

This proves that the problem (1.5-1.6) is Hyers-Ulam stable.Thus we complete this proof. □

Theorem 4.14. *Assume that the assumptions (A_1) and (A_3) hold. If a continuously differentiable function $\tilde{u} : [0,T] \longrightarrow \mathbb{R}$ satisfies (2.3), where $\Phi : [0,T] \longrightarrow \mathbb{R}^+$ is a continuous function with (A_3), then there exists a mild solution $u : [0,T] \longrightarrow \mathbb{R}$ of problem (1.5-1.6) such that*

$$|\tilde{u}(t) - u(t)| \leq \epsilon c_f \Phi(t), \tag{4.31}$$

with (4.29).

Proof. By Lemma 2.4 and Remark 2.16, one can obtain

$$\begin{aligned}
|u(t) - \tilde{u}(t)| &= \int_0^t S_{\alpha,\rho}(t,\tau)\left(\mathcal{D}_{0^+}^{1-\alpha,\rho}\right)(f(\tau,u) - f(\tau,\tilde{u}) + g(\tau))\frac{\mathrm{d}\tau}{\tau^{1-\rho}} \\
&\leq \epsilon \int_0^t S_{\alpha,\rho}(t,\tau)\left(\mathcal{D}_{0^+}^{1-\alpha,\rho}\right)\Phi(\tau)\frac{\mathrm{d}\tau}{\tau^{1-\rho}} \\
&\quad + \int_0^t S_{\alpha,\rho}(t,\tau)\left(\mathcal{D}_{0^+}^{1-\alpha,\rho}\right)(f(\tau,u) - f(\tau,\tilde{u}))\frac{\mathrm{d}\tau}{\tau^{1-\rho}} \\
&\leq \epsilon \int_0^t S_{\alpha,\rho}(t,\tau)\left(\mathcal{D}_{0^+}^{1-\alpha,\rho}\right)\Phi(\tau)\frac{\mathrm{d}\tau}{\tau^{1-\rho}} \\
&\quad + L\int_0^t S_{\alpha,\rho}(t,\tau)\left(\mathcal{D}_{0^+,\tau}^{1-\alpha,\rho}\right)|u(\tau) - \tilde{u}(\tau)|\frac{\mathrm{d}\tau}{\tau^{1-\rho}} \\
&\leq \epsilon \int_0^t S_{\alpha,\rho}(t,\tau)\left(\mathcal{D}_{0^+}^{1-\alpha,\rho}\right)\Phi(\tau)\frac{\mathrm{d}\tau}{\tau^{1-\rho}} \\
&\quad + L\int_0^t \left(\mathcal{D}_{\tau,T}^{1-\alpha,\rho}\right)S_{\alpha,\rho}(t,\tau)|u(\tau) - \tilde{u}(\tau)|\frac{\mathrm{d}\tau}{\tau^{1-\rho}}.
\end{aligned}$$

For the Lemma 2.10, we have

$$\begin{aligned}
v(t) &= \epsilon \int_0^t S_{\alpha,\rho}(t,\tau)\left(\mathcal{D}_{0^+}^{1-\alpha,\rho}\right)\Phi(\tau)\frac{\mathrm{d}\tau}{\tau^{1-\rho}} \\
&\leq \epsilon \left\|\left(\mathcal{D}_{\tau,T}^{1-\alpha,\rho}\right)[1]\right\| \left(\int_0^t \Phi(\tau)\frac{\mathrm{d}\tau}{\tau^{1-\rho}}\right) \\
&\leq \epsilon \left\|\left(\mathcal{D}_{\tau,T}^{1-\alpha,\rho}\right)[1]\right\| \left|\left(\mathcal{J}_{0^+}^{1,\rho}\right)[\Phi]\right| \\
&\leq \epsilon \left\|\left(\mathcal{D}_{\tau,T}^{1-\alpha,\rho}\right)[1]\right\| l_\Phi \Phi(t).
\end{aligned}$$

and

$$b(\tau) = L\left(\mathcal{D}_{t,T}^{1-\alpha,\rho}\right)\left[S_{\alpha,\rho}(t,\tau)\,\tau^{\rho-1}\right].$$

So

$$|\ \tilde{u}(t) - u(t)\ | \leq v(t)\exp(\mathit{\Lambda}(t)),\ \ \mathit{\Lambda}(t) = \int_0^t b(\tau)\mathrm{d}\tau. \tag{4.32}$$

It is immediately to find that $v(t) \longrightarrow 0$ in the presence of $\epsilon \longrightarrow 0$. Therefore, we conclude that $\tilde{u}(t) \longrightarrow u(t)$.

This proves that the problem (1.5-1.6) is Hyers-Ulam-Rassias stable.Thus we complete this proof. □

Theorem 4.15. *Assume that the assumptions (A_1), (A_3) and (A_4). If a continuously differentiable function $\tilde{u} : [0,T] \longrightarrow \mathbb{R}$ satisfies (2.4), where $\Phi : [0,T] \longrightarrow \mathbb{R}^+$ is a continuous function with (A_4), then there exists a mild solution $u : [0,T] \longrightarrow \mathbb{R}$ of problem (1.5-1.6) such that*

$$|\tilde{u}\,(t) - u\,(t)| \leq c_f \Phi\,(t)\,, \tag{4.33}$$

with (4.29), where

$$c_f = \sup \left\{ (1+2q^*) \left\| S_{\alpha,\rho}\,(t,\tau) \right\| \left\| \left(\mathcal{D}^{1-\alpha,\rho}_{\tau,T} \right) [1] \right\| l_\Phi : t \in [0,T] \right\}. \tag{4.34}$$

Proof. By Remark 4.16-ii, we have that

$$\begin{aligned} &\left| \tilde{u}(t) - S_\alpha(t) u_0 - \int_0^t S_{\alpha,\rho}\,(t,\tau) \left(\mathcal{D}^{1-\alpha,\rho}_{0^+} \right) f(\tau,\tilde{u}) \frac{\mathrm{d}\tau}{\tau^{1-\rho}} \right| \\ \leq &\int_0^t S_{\alpha,\rho}\,(t,\tau) \left(\mathcal{D}^{1-\alpha,\rho}_{0^+} \right) (\Phi\,(\tau)) \frac{\mathrm{d}\tau}{\tau^{1-\rho}}. \end{aligned} \tag{4.35}$$

In view of Lemma 2.4, (A_3) and A_4), we obtain

$$\begin{aligned} |u\,(t) - \tilde{u}\,(t)| &\leq \int_0^t S_{\alpha,\rho}\,(t,\tau) \left(\mathcal{D}^{1-\alpha,\rho}_{0^+} \right) (\Phi\,(\tau)) \frac{\mathrm{d}\tau}{\tau^{1-\rho}} \\ &\quad + \int_0^t S_{\alpha,\rho}\,(t,\tau) \left(\mathcal{D}^{1-\alpha,\rho}_{0^+} \right) (f(u) - f(\tilde{u}) \frac{\mathrm{d}\tau}{\tau^{1-\rho}} \\ &\leq \int_0^t S_{\alpha,\rho}\,(t,\tau) \left(\mathcal{D}^{1-\alpha,\rho}_{0^+} \right) (\Phi\,(\tau)) \frac{\mathrm{d}\tau}{\tau^{1-\rho}} \\ &\quad + 2 \int_0^t S_{\alpha,\rho}\,(t,\tau) \left(\mathcal{D}^{1-\alpha,\rho}_{0^+,\tau} \right) (q\,(\tau)\,\Phi\,(\tau)) \frac{\mathrm{d}\tau}{\tau^{1-\rho}} \\ &\leq (1+2q^*) \int_0^t S_{\alpha,\rho}\,(t,\tau) \left(\mathcal{D}^{1-\alpha,\rho}_{0^+,\tau} \right) (\Phi\,(\tau)) \frac{\mathrm{d}\tau}{\tau^{1-\rho}} \\ &\leq (1+2q^*) \left\| S_{\alpha,\rho}\,(t,\tau) \right\| \left\| \left(\mathcal{D}^{1-\alpha,\rho}_{\tau,T} \right) [1] \right\| \left| \left(\mathcal{J}^{1,\rho}_{0^+} \right) [\Phi] \right|. \end{aligned}$$

This proves that the problem (1.5-1.6) is generalized Ulam-Hyers-Rassias stable.Thus we complete this proof. □

Remark 4.16.

- **(i)** Under the assumptions of Theorem 4.4, we consider (1.5-1.6) and the inequality (2.2), one can renew the same procedure to confirm that (1.5-1.6) is Ulam-Hyers stable.
- **(ii)** Other stability results for the equation (1.5-1.6) can be discussed similarly.

5. Examples

In this section we present two examples to illustrate our results. We take

$$\mathbb{X} = C_0(\mathbb{R}) = \{u \in C(\mathbb{R}) : \lim_{|s|\to\infty} u(s) = 0\}$$

and for a fixed constant $\beta > 0$, we define an operator A_β by the difference quotients

$$(\mathrm{A}_\beta u)\,(s) = \frac{1}{\beta} \left(u\,(s+\beta) - u(s) \right), u \in \mathbb{X}.$$

It is well known that $\mathrm{A}_\beta \in LB(X)$ is the infinitesimal generator of an semigroup $S_\alpha(t)\,(t \geqslant 0)$ with $w := \|\mathrm{A}_\beta\| = 2/\beta$, and hence one has the estimate

$$\|S_\alpha(t)u\| = \left\|E_\alpha\left(\left(\frac{t^\rho}{\rho}\right)^\alpha(\mathrm{A}_\beta)\right)u\right\| \le Me^{2/\beta\left(\frac{t^\rho}{\rho}\right)^\alpha}\|u\|\,,\ M \ge 1.$$

However, with multiplication of the product of two series, we can rewrite $E_\alpha\left(\left(\frac{t^\rho}{\rho}\right)^\alpha(\mathrm{A}_\beta)\right)$ as

$$E_\alpha\left(\left(\frac{t^\rho}{\rho}\right)^\alpha(\mathrm{A}_\beta)\right)u\,(s) = E_\alpha\left(-\left(\frac{t^\rho}{\rho}\right)^\alpha\left(\frac{1}{\beta}\right)\right)\sum_{n=0}^{\infty}\frac{\left(\left(\frac{t^\rho}{\rho}\right)^\alpha\frac{1}{\beta}\right)^n}{\Gamma(1+\alpha n)}u\,(s+n\beta)\,.$$

From this it follows that $S_\alpha(t)\,(t \geqslant 0)$ is a uniformly continuous semigroup of bounded linear operators $\mathrm{A}_\beta \in \mathcal{SG}_\alpha(M,\omega)$ on $\mathbb{X} = \mathrm{D}\,(\mathrm{A}_\beta)\,.$

By Theorem 3.8, we have that

$$(\mathrm{A}_\beta S_\alpha(t))\,u\,(s) = \mathrm{A}_\beta\,(S_\alpha(t)u)\,(s) = \frac{1}{\beta}\,(S_\alpha(t)u\,(s+\beta) - S_\alpha(t)u(s)) = S_\alpha(t)\,(\mathrm{A}_\beta u)\,(s)$$

and

$$\left({}^c\mathcal{D}_{0^+}^{\alpha,\rho}\right)S_\alpha(t)u\,(s) = \left({}^c\mathcal{D}_{0^+}^{\alpha,\rho}\right)\left(E_\alpha\left(\left(\frac{t^\rho}{\rho}\right)^\alpha\mathrm{A}_\beta\right)u(s)\right) = \mathrm{A}_\beta\left(E_\alpha\left(\left(\frac{t^\rho}{\rho}\right)^\alpha\mathrm{A}_\beta\right)u\right)(s) = (\mathrm{A}_\beta S_\alpha(t))\,u\,(s)\,.$$

Example 5.1. Let us first consider the fractional abstract Cauchy problem (1.5-1.6) with the following parameters $T = 1,\ \alpha = 1/2, \rho = 3,\ u(0) = -3$. Here

$$f(t,u) = \frac{1}{3}u(t) + E_{1/2}(t^{1/2}).$$

It is easy to see that the function f satisfy condition (A_1) and $L = 1/3$. Furthermore, the inequality (4.9) holds with

$$\frac{ML}{\Gamma(\alpha)}\left(\frac{T^\rho}{\rho}\right)^\alpha e^{w\frac{1}{\alpha}\frac{T^\rho}{\rho}} = \frac{1}{9}\frac{\sqrt{3}}{\sqrt{\pi}}Me^{3\sqrt{\frac{2}{\beta}}} < 1,$$

which implies that

$$1 \le M < \frac{9\sqrt{\pi}}{\sqrt{3}}e^{-3\sqrt{\frac{2}{\beta}}},\ \beta > 0.$$

It follows from Theorem 4.5, the problem (1.5-1.6) has a unique mild solution.

In the view of Theorem 4.13 the problem (1.5-1.6) is Ulam-Hyers stable if the condition (4.27) is satisfied with

$$c_f \backsimeq \|S_{\alpha,\rho}\,(t,\tau)\|\left\|\left(\mathcal{D}_{0^+}^{1-\alpha,\rho}\right)[1]\right\|\left(\frac{T^\rho}{\rho}\right) \backsimeq \frac{1}{3}\frac{\sqrt{3}}{\sqrt{\pi}}Me^{3\sqrt{\frac{2}{\beta}}} < \infty.$$

Next, in the view of Theorem 4.14 the problem (1.5-1.6) is Ulam-Hyers-Rassias stable with $\phi(t) = E_{1/2}(t^{1/2})$, hence

$$\mid \mathcal{J}_{0^+}^{1/2,3}\phi(t) \mid \le l_\phi\phi(t)\ \text{ with } l_\phi = 1.$$

From the condition (A_3), which implies that

$$c_f \backsimeq \frac{Ml_\phi}{\Gamma(\alpha)}\left(\frac{T^\rho}{\rho}\right)^\alpha e^{w\frac{1}{\alpha}\frac{T^\rho}{\rho}} \backsimeq \frac{1}{3}\frac{\sqrt{3}}{\sqrt{\pi}}Me^{3\sqrt{\frac{2}{\beta}}} < \infty.$$

Example 5.2. Consider the following fractional abstract Cauchy problem (1.5-1.6) with $T = 1,\ \alpha = 1/2,\ \rho = 1/2\ u(0) = -3$

$$f(t,u) = \frac{\kappa t^{1/2}}{20} E_{1,2}(t^{1/2}) \frac{\mid u(t) \mid}{1 + \mid u(t) \mid},\ \kappa \in [0,\infty),\ t \in [0,1],\ u \in \mathbb{R}.$$

From the conditions (A_2) and (A_4), $p(t) = \dfrac{\kappa t^{\frac{1}{2}}}{20} E_{1,2}(t^{1/2})$ and $q(t) = \dfrac{1}{2} p(t)$

$$p^* = \sup\left\{ \frac{\kappa t^{1/2}}{20} E_{1,2}(t^{1/2}) : t \in [0,1] \right\} = \frac{\kappa}{20}(e-1),\ \kappa > 0,$$

Before, Theorem 4.4, the problem (1.5-1.6) has at least one mild solution, when $M \geq 1$ and

$$r = M\left(u(0) + \frac{p^*}{\Gamma(\alpha)} \left(\frac{T^\rho}{\rho}\right)^\alpha \right) e^{w^{\frac{1}{\alpha}} \frac{T^\rho}{\rho}} = M\left(\frac{1}{20} \frac{\sqrt{2}}{\sqrt{\pi}} \kappa (e-1) - 3 \right) e^{3\sqrt{\frac{2}{\beta}}} < \infty.$$

In view of Theorem 4.15, the problem (1.5-1.6) is generalized Ulam-Hyers-Rassias stable with $\phi(t) = 2\sqrt{t}$, hence

$$\mid \mathcal{J}_{0^+}^{1,1/2} \phi(t) \mid \leq l_\phi \phi(t) = 2\sqrt{t} \text{ with } l_\phi = 1.$$

We derive that

$$c_f = M(1 + 2q^*) \frac{1}{\Gamma(\alpha)} \left(\frac{T^\rho}{\rho}\right)^\alpha e^{w^{\frac{1}{\alpha}} \frac{T^\rho}{\rho}} = M\left(1 + \frac{\kappa}{20}(e-1)\right) \frac{\sqrt{2}}{\sqrt{\pi}} e^{3\sqrt{\frac{2}{\beta}}} < \infty.$$

6. Conclusions

We have presented some results dealing with the existence and uniqueness of mild solutions for abstract Cauchy problem involving generalized fractional order The mild solutions are obtained by using the theory of uniformly continuous semigroups of operators, we also analyze the continuous dependence of solutions all on its right side function, initial value condition and the fractional order for abstract Cauchy problem and We also presented a discuss on the Ulam-Hyers stability of the mild solution of the proposed problem. The concerned theory has been enriched by providing suitable examples.

Reported results in this paper can be considered as a promising contribution to the theory of fractional abstract Cauchy problems. These results can be used to study and develop further quantitative and qualitative properties of generalized fractional abstract differential equations. Furthermore, the form of a fundamental mild solution obtained in this work is a foundation result for further investigation such as the problem with perturbation, delay and a nonlocal term. For problem (1.5-1.6), we can also consider its approximation controllability, which will be our future work.

However, some questions remain open and are motivations for a future work among them, the main one is the possibility to investigate the existence and uniqueness of classical, mild and strong solutions to a class of nonlocal term Cauchy problem with weighted φ-Caputo fractional mixed generalized Volterra-Fredholm-type integrodifferential equations of order $\alpha \in (1,2)$ of integrodifferential and differential fractional equations in the sense of the ψ-Caputo fractional derivative, we will develop approximate controllability results for Sobolev type fractional delay evolution inclusions of order $\alpha \in (1,2)$. We claim that the results of this paper is new and generalize some earlier results.

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