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*Research article*

## On Hilbert-Pachpatte type inequalities within $\psi$ -Hilfer fractional generalized derivatives

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**Abstract:** In this manuscript, we discussed various new Hilbert-Pachpatte type inequalities implying the left sided  $\psi$ -Hilfer fractional derivatives with the general kernel. Our results are a generalization of the inequalities of Pečarić and Vuković [1]. Furthermore, using the specific cases of the  $\psi$ -Hilfer fractional derivative, we proceed with wide class of fractional derivatives by selecting  $\psi$ ,  $a_1$ ,  $b_1$  and considering the limit of the parameters  $\alpha$  and  $\beta$ .

**Keywords:** Hilbert-Pachpatte type inequality;  $\psi$ -Hilfer fractional derivative; Riemann-Liouville derivative; homogeneous kernel; fractional derivative

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### 1. Introduction

Starting from 1695 [2–4], non-integer calculus has been applied in various fields of science and engineering [5–18]. Introducing new fractional integrals and derivatives leads to an increasing rate of growth and cleanses its numerous applications [12, 19–26]. Various researchers reported integral inequalities involving the different definitions of fractional derivatives [27–34].

We notice that Hilbert [35] presented the following integral inequality:

**Theorem 1.1.** [35] (Chapter IX, Theorem 316) If  $g \in L^p(0, \infty)$ ,  $h \in L^q(0, \infty)$ ,  $g, h \geq 0$ ,  $p > 1$  and

$q = \frac{p}{(p-1)}$ , then

$$\int_0^\infty \int_0^\infty \frac{g(s)h(t)}{s+t} ds dt \leq \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left( \int_0^\infty g^p(s) ds \right)^{1/p} \times \left( \int_0^\infty h^q(t) dt \right)^{1/q}, \quad (1.1)$$

such that  $\pi / \sin\left(\frac{\pi}{p}\right)$  is the best value.

We recall (1.1) as Hilbert's inequality. It is a fact that Hilbert's inequalities has an important place in analysis. In recent years, various mathematicians investigated Hilbert's inequalities, Hilbert's type inequalities and their several generalizations, see [1, 22, 36–66] and the references therein.

Pachpatte [47] reported the following integral inequality in a similar way as Hilbert's inequality:

**Theorem 1.2.** [47] Let  $0 \leq r \leq m - 1$  and  $m \geq 1$  be integers. Besides, let  $u_1 \in C^m([0, x])$  and  $v_1 \in C^m([0, y])$  such that  $x > 0$ ,  $y > 0$  and  $u_1^{(i)}(0) = v_1^{(i)}(0) = 0$  for  $i \in \{0, 1, \dots, m - 1\}$ . As a results we give

$$\begin{aligned} & t_0^x \int_0^y \frac{|u_1^{(r)}(s)| |v_1^{(r)}(t)|}{s^{2m-2r-1} + t^{2m-2r-1}} ds dt \\ & \leq M(m, r, x, y) \times \left( \int_0^x (x-s) (u_1^{(m)}(s))^2 ds \right)^{1/2} \left( \int_0^y (y-t) (v_1^{(m)}(t))^2 dt \right)^{1/2}, \end{aligned} \quad (1.2)$$

where

$$M(m, r, x, y) := \frac{\sqrt{xy}}{2((m-r-1)!)^2(2m-2r-1)}.$$

Handly et al. [40] derived new type Hilbert-Pachpatte integral inequalities from the results of Pachpatte in [47, 48].

Lü [46] gave some new inequalities related to the Hilbert-Pachpatte inequalities. Same year, Dragomir and Kim [38] extended and proved the results obtained by Handly et al. [40, 41]. So, they obtained new inequalities like Hilbert-Pachpatte-type inequalities.

He and Li [22] established new inequalities related to the Hilbert-Pachpatte inequalities. Also, they presented some new generalizations of Hilbert-Pachpatte inequality.

Anastassiou [36] gave very general weighted Hilbert-Pachpatte type integral inequalities involving Caputo and Riemann-Liouville fractional derivatives as well as fractional partial derivatives of the above mentioned types. Also, in 2021, Anastassiou [37] we presented Hilfer-Polya,  $\psi$ -Hilfer Ostrowski as well as  $\psi$ -Hilfer-Hilbert-Pachpatte types fractional inequalities implying left and right Hilfer and  $\psi$ -Hilfer fractional derivatives.

Zhao et al. [64] obtained some multiple integral Hilbert-Pachpatte type inequalities. One year later, Zhao and Cheung [65] established Hilbert-Pachpatte-type integral inequalities.

Pečarić and Vuković [1] studied of some generalizations of Hilbert-Pachpatte type inequality involving the fractional derivative utilizing the Taylor series of function and refinement of Arithmetic-Geometric Inequality (A.G.I.) from [67]. Their results based on the results of Krnić and Pečarić [45].

A new kind of fractional derivative presented by Sousa and Oliveira [68]. They gave  $\psi$ -Hilfer fractional derivative with respect to (w.r.t.) another function, having in mind to combine many fractional derivatives into a single fractional operator, and as a result they open a path for new

applications. Later, their authors studied Gronwall inequality and the Cauchy-type problem by using of  $\psi$ -Hilfer operator in [69].

Our work is organized as given below. In Section 1 and Section 2, we present the introduction and preliminaries, respectively. Motivated by [1], we developed several new Hilbert-Pachpatte type inequalities for the left sided  $\psi$ -Hilfer fractional derivatives with the general kernel. Furthermore, using the particular cases of the  $\psi$ -Hilfer fractional derivative, we proceed with the wide class of fractional derivatives by taking into account  $\psi$ ,  $a_1$ ,  $b_1$  and taking the limit of the parameters  $\alpha$  and  $\beta$ .

## 2. Basic tools

In the following some basic tools required in this manuscript are given.

**Definition 2.1.** [68] Let  $(a_1, b_1)$  ( $-\infty \leq a_1 < x < b_1 \leq \infty$ ) be a finite or infinite interval and  $\alpha > 0$ . Also, let  $g$  be an integrable function on  $[a_1, b_1]$ , and  $\psi$  be positive monotone and an increasing function on  $(a_1, b_1]$  such that  $\psi'(x)$  is a continuous derivative on  $(a_1, b_1)$ . The left-sided fractional integral of a function  $g$  w.r.t. another function  $\psi$  becomes

$$I_{a_1^+}^{\alpha; \psi} g(x) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} g(t) dt. \quad (2.1)$$

**Definition 2.2.** [68] Let  $\psi'(x) \neq 0$  in  $-\infty \leq a_1 < x < b_1 \leq \infty$ ,  $m \in \mathbb{N}$  and  $\alpha > 0$ . The Riemann-Liouville derivatives of a function  $g$  w.r.t.  $\psi$  of order  $\alpha$  correspondent to the Riemann-Liouville is given by

$$\begin{aligned} D_{a_1^+}^{\alpha; \psi} g(x) &= \left( \frac{1}{\psi'(x)} \frac{d}{dx} \right)^m I_{a_1^+}^{m-\alpha; \psi} g(x) \\ &= \frac{1}{\Gamma(m-\alpha)} \left( \frac{1}{\psi'(x)} \frac{d}{dx} \right)^m \int_{a_1}^x \psi'(t) (\psi(x) - \psi(t))^{m-\alpha-1} g(t) dt, \end{aligned} \quad (2.2)$$

such that  $m = [\alpha] + 1$ .

**Definition 2.3.** [68] Let  $m - 1 < \alpha < m$  ( $m \in \mathbb{N}$ ),  $I = [a_1, b_1]$  be the interval such that  $-\infty \leq a_1 < b_1 \leq \infty$  and  $g, \psi \in C^m([a_1, b_1], \mathbb{R})$  two functions such that  $\psi'(x) > 0$  and  $\psi'(x) \neq 0$ , for all  $x \in I$ . The  $\psi$ -Hilfer fractional derivative (left-sided)  ${}^H D_{a_1^+}^{\alpha, \beta; \psi}(\cdot)$  of function of order  $\alpha$  and type  $0 \leq \beta \leq 1$ , is defined as

$${}^H D_{a_1^+}^{\alpha, \beta; \psi} g(x) = I_{a_1^+}^{\beta(m-\alpha); \psi} \left( \frac{1}{\psi'(x)} \frac{d}{dx} \right)^m I_{a_1^+}^{(1-\beta)(m-\alpha); \psi} g(x). \quad (2.3)$$

Also, (2.3) becomes

$$\begin{aligned} {}^H D_{a_1^+}^{\alpha, \beta; \psi} g(x) &= I_{a_1^+}^{\gamma-\alpha; \psi} D_{a_1^+}^{\gamma; \psi} g(x) \\ &= \frac{1}{\Gamma(\gamma-\alpha)} \int_{a_1}^x \psi'(t) (\psi(x) - \psi(t))^{\gamma-\alpha-1} D_{a_1^+}^{\gamma; \psi} g(t) dt \end{aligned} \quad (2.4)$$

with  $\gamma = \alpha + \beta(m - \alpha)$ . Here,  $I_{a_1^+}^{\gamma; \psi}(\cdot)$  and  $D_{a_1^+}^{\gamma; \psi}(\cdot)$  are defined in (2.1) and (2.2), respectively.

**Theorem 2.1.** [37] Let  $\psi, g \in C^m([a_1, b_1])$  such that  $\psi$  is increasing and  $\psi'(x) \neq 0$  over  $[a_1, b_1]$ , where  $m - 1 < \alpha < m$ ,  $0 \leq \beta \leq 1$  and  $\gamma = \alpha + \beta(m - \alpha)$ ,  $x \in [a_1, b_1]$ . Then,

$$g(x) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} D_{a_1^+}^{\alpha, \beta; \psi} g(t) dt + \sum_{k=1}^{m-1} \frac{(\psi(x) - \psi(a_1))^{\gamma-k}}{\Gamma(\gamma-k+1)} g^{[m-k]} \left( I_{a_1^+}^{(1-\beta)(m-\alpha); \psi} g \right) (a_1). \quad (2.5)$$

Here notice that

$$I_{a_1^+}^{(1-\beta)(m-\alpha); \psi} g(a_1) = 0.$$

If

$$g^{[m-k]} \left( I_{a_1^+}^{(1-\beta)(m-\alpha); \psi} g \right) (a_1) = 0, \quad (k = 1, 2, \dots, m-1),$$

then (2.5) becomes

$$g(x) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} D_{a_1^+}^{\alpha, \beta; \psi} g(t) dt. \quad (2.6)$$

Krnić and Pecarić reported the following inequalities in [45]:

$$\int_{\Omega \times \Omega} K(x, y) g(x) h(y) d\mu_1(x) d\mu_2(y) \leq \left[ \int_{\Omega} \theta^p(x) G(x) g^p(x) d\mu_1(x) \right]^{1/p} \left[ \int_{\Omega} \phi^q(y) H(y) h^q(y) d\mu_2(y) \right]^{1/q} \quad (2.7)$$

and

$$\int_{\Omega} H^{1-p}(y) \phi^{-p}(y) \left[ \int_{\Omega} K(x, y) g(x) d\mu_1(x) \right]^p d\mu_2(y) \leq \int_{\Omega} \theta^p(x) G(x) g^p(x) d\mu_1(x). \quad (2.8)$$

Here  $p > 1$ ,  $\mu_1$  and  $\mu_2$  are positive  $\sigma$ -finite measures,  $K : \Omega \times \Omega \rightarrow \mathbb{R}$ ,  $g, h, \theta, \phi : \Omega \rightarrow \mathbb{R}$  are measurable nonnegative functions and

$$G(x) = \int_{\Omega} \frac{K(x, y)}{\phi^p(y)} d\mu_2(y), \quad (2.9)$$

$$H(y) = \int_{\Omega} \frac{K(x, y)}{\theta^q(x)} d\mu_1(x). \quad (2.10)$$

On the other hand, we give some definitions of Yang et al. [55]. That is, let  $p_1, \dots, p_l > 1$  be real parameters such that  $\sum_{i=1}^l \frac{1}{p_i} = 1$ . Also, let  $K : \Omega^l \rightarrow \mathbb{R}$  and  $\omega_{ij} : \Omega \rightarrow \mathbb{R}$  ( $i, j = 1, \dots, l$ ) be nonnegative measurable functions. If  $\prod_{i,j=1}^l \omega_{ij}(x_j) = 1$ , then the following inequality gives for all measurable nonnegative functions  $g_1, \dots, g_l : \Omega \rightarrow \mathbb{R}$ :

$$\int_{\Omega^l} K(x_1, \dots, x_l) \prod_{i=1}^l g_i(x_i) dx_1 \dots dx_l \leq \prod_{i=1}^l \left( \int_{\Omega} G_i(x_i) (\omega_{ij} g_i)^{p_i}(x_i) dx_i \right)^{\frac{1}{p_i}}, \quad (2.11)$$

where

$$G_i(x_i) = \int_{\Omega^{l-1}} K(x_1, \dots, x_l) \prod_{j=1, j \neq i}^l \omega_{ij}^{p_i}(x_j) dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_l, \quad (2.12)$$

for  $i = 1, \dots, l$ .

In [67], Krnić et al. showed the following refinements and converses of Young's inequality in quotient and difference form. If  $x = (x_1, x_2, \dots, x_m)$  and  $p = (p_1, p_2, \dots, p_m)$ , we denote  $P_m = \sum_{i=1}^m p_i$ ,

$$A_m(x) = \frac{\sum_{i=1}^m x_i}{m}, H_m(x) = \left( \prod_{i=1}^m x_i \right)^{1/m},$$

and

$$M_r(x, p) = \begin{cases} \left( \frac{1}{P_m} \sum_{i=1}^m p_i x_i^r \right)^{1/r}, & r \neq 0. \\ \left( \prod_{i=1}^m x_i^{p_i} \right)^{1/P_m}, & r = 0. \end{cases}$$

**Lemma 2.1.** [52] Let  $x = (x_1, x_2, \dots, x_m)$  and  $p = (p_1, p_2, \dots, p_m)$  be positive  $m$ -tuples, such that  $\sum_{i=1}^m \frac{1}{p_i} = 1$ , and

$$x^p = (x_1^{p_1}, x_2^{p_2}, \dots, x_m^{p_m}), \quad \frac{1}{p} = \left( \frac{1}{p_1}, \frac{1}{p_2}, \dots, \frac{1}{p_m} \right).$$

Then, the followings hold:

(i)

$$\left[ \frac{A_m(x^p)}{H_m(x^p)} \right]^{m \min_{1 \leq i \leq m} \left\{ \frac{1}{p_i} \right\}} \leq \frac{M_1\left(x^p, \frac{1}{p}\right)}{M_0\left(x^p, \frac{1}{p}\right)} \leq \left[ \frac{A_m(x^p)}{H_m(x^p)} \right]^{m \max_{1 \leq i \leq m} \left\{ \frac{1}{p_i} \right\}},$$

(ii)

$$\begin{aligned} m \min_{1 \leq i \leq m} \left\{ \frac{1}{p_i} \right\} [A_m(x^p) - H_m(x^p)] &\leq M_1\left(x^p, \frac{1}{p}\right) - M_0\left(x^p, \frac{1}{p}\right) \\ &\leq m \max_{1 \leq i \leq m} \left\{ \frac{1}{p_i} \right\} [A_m(x^p) - H_m(x^p)]. \end{aligned}$$

Now, define

$$k(\xi) := \int_0^\infty K(1, u) u^{-\xi} ds,$$

such that  $K(x, y)$  represents a nonnegative homogeneous function of degree  $-s$  ( $s > 0$ ), that is,  $K(tx, ty) = t^{-s} K(x, y)$  is satisfied. Suppose that  $k(\xi) < \infty$  for  $1 - s < \xi < 1$ .

**Lemma 2.2.** [1] If  $\lambda > 0$ ,  $1 - \lambda < \xi < 1$  and  $K : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$  is a nonnegative homogeneous function of degree  $-\lambda$ , then

$$\int_0^\infty K(x, y) \left( \frac{x}{y} \right)^\xi dy = x^{1-\lambda} k(\xi),$$

and

$$\int_0^\infty K(x, y) \left( \frac{y}{x} \right)^\xi dx = y^{1-\lambda} k(2 - \lambda - \xi).$$

### 3. Main results

Below, we will prove various new Hilbert-Pachpatte type inequalities including the left sided  $\psi$ -Hilfer fractional derivatives.

**Theorem 3.1.** Let  $\frac{1}{p} + \frac{1}{q} = 1$  with  $p, q > 1$ ,  $-\infty \leq a_1 < b_1 \leq \infty$  and  $\gamma \geq \alpha + 1$ . Assume that  $K : [a_1, b_1] \times [a_1, b_1] \rightarrow \mathbb{R}$  is nonnegative function,  $\theta(x), \phi(y)$  are nonnegative functions on  $[a_1, b_1]$ , and  $g, h, \psi \in C^n([a_1, b_1], \mathbb{R})$  are functions such that  $\psi$  is an increasing function with  $\psi'(x) \neq 0$ , for all  $x \in [a_1, b_1]$ . Also, let  $D_{a_1^+}^{\gamma; \psi}$  for  $\alpha = \gamma$  and  ${}^H D_{a_1^+}^{\alpha, \beta; \psi}$  be defined by (2.2) and (2.4), respectively. Then, we have

$$\begin{aligned} & \int_{a_1}^{b_1} \int_{a_1}^{b_1} \frac{K(x, y) \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} g(x) \right| \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} h(y) \right|}{\left[ (x - a_1)^{\frac{1}{q(M_1 - m_1)}} + (y - b_1)^{\frac{1}{p(M_1 - m_1)}} \right]^{2(M_1 - m_1)}} dx dy \\ & \leq \frac{1}{4^{M_1 - m_1}} \int_{a_1}^{b_1} \int_{a_1}^{b_1} \frac{K(x, y) \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} g(x) \right| \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} h(y) \right|}{(x - a_1)^{\frac{1}{q}} (y - b_1)^{\frac{1}{p}}} dx dy \\ & \leq \frac{1}{4^{M_1 - m_1} (\Gamma(\gamma - \alpha))^2} \left( \int_{a_1}^{b_1} \int_{a_1}^x (\psi'(t))^p (\psi(x) - \psi(t))^{p(\gamma - \alpha - 1)} \right. \\ & \quad \times \theta^p(x) G(x) \left| D_{a_1^+}^{\gamma; \psi} g(t) \right|^p dt dx \Big)^{1/p} \\ & \quad \times \left( \int_{a_1}^{b_1} \int_{a_1}^y (\psi'(t))^q (\psi(y) - \psi(t))^{q(\gamma - \alpha - 1)} \phi^q(y) H(y) \left| D_{a_1^+}^{\gamma; \psi} h(t) \right|^q dt dy \right)^{1/q} \end{aligned} \quad (3.1)$$

and

$$\begin{aligned} & \int_{a_1}^{b_1} H^{1-p}(y) \phi^{-p}(y) \left( \int_{a_1}^{b_1} K(x, y) \left( \int_{a_1}^x (\psi'(t))^p (\psi(x) - \psi(t))^{p(\gamma - \alpha - 1)} \left| D_{a_1^+}^{\gamma; \psi} g(t) \right|^p dt \right)^{1/p} dx \right)^p dy \\ & \leq \int_{a_1}^{b_1} \int_{a_1}^x (\psi'(t))^p (\psi(x) - \psi(t))^{p(\gamma - \alpha - 1)} \theta^p(x) G(x) \left| D_{a_1^+}^{\gamma; \psi} g(t) \right|^p dt dx, \end{aligned} \quad (3.2)$$

where  $m_1 = \min\{\frac{1}{p}, \frac{1}{q}\}$ ,  $M_1 = \max\{\frac{1}{p}, \frac{1}{q}\}$ , and  $G(x)$  and  $H(y)$  are defined as in (2.9) and (2.10), respectively.

*Proof.* Applying Hölder's inequality in (2.4) and using an increasing of the function  $\psi(x)$ , for  $x \in [a_1, b_1]$  and  $t \in [a_1, x]$  we obtain

$$\begin{aligned} \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} g(x) \right| &= \frac{1}{\Gamma(\gamma - \alpha)} \left| \int_{a_1}^x \psi'(t) (\psi(x) - \psi(t))^{\gamma - \alpha - 1} D_{a_1^+}^{\gamma; \psi} g(t) dt \right| \\ &\leq \frac{1}{\Gamma(\gamma - \alpha)} \int_{a_1}^x (\psi'(t)) (\psi(x) - \psi(t))^{\gamma - \alpha - 1} \left| D_{a_1^+}^{\gamma; \psi} g(t) \right| dt \\ &\leq \frac{1}{\Gamma(\gamma - \alpha)} \left( \int_{a_1}^x (\psi'(t))^p (\psi(x) - \psi(t))^{p(\gamma - \alpha - 1)} \left| D_{a_1^+}^{\gamma; \psi} g(t) \right|^p dt \right)^{1/p} \left( \int_{a_1}^x 1 dt \right)^{1/q}. \end{aligned}$$

From here, we can write

$$\left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} g(x) \right| \leq \frac{(x - a_1)^{1/q}}{\Gamma(\gamma - \alpha)} \left( \int_{a_1}^x (\psi'(t))^p (\psi(x) - \psi(t))^{p(\gamma - \alpha - 1)} \left| D_{a_1^+}^{\gamma; \psi} g(t) \right|^p dt \right)^{1/p}. \quad (3.3)$$

Similarly, for  $y \in [a_1, b_1]$  and  $t \in [a_1, y]$  we can derive

$$\left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} h(y) \right| \leq \frac{(y - a_1)^{1/p}}{\Gamma(\gamma - \alpha)} \left( \int_{a_1}^y (\psi'(t))^q (\psi(y) - \psi(t))^{q(\gamma - \alpha - 1)} \times \left| D_{a_1^+}^{\gamma; \psi} h(t) \right|^q dt \right)^{1/q}. \quad (3.4)$$

Multiplying (3.3) by (3.4), we get

$$\begin{aligned} \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} g(x) \right| \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} h(y) \right| &\leq \frac{(x - a_1)^{1/q} (y - a_1)^{1/p}}{[\Gamma(\gamma - \alpha)]^2} \\ &\times \left( \int_{a_1}^x (\psi'(t))^p (\psi(x) - \psi(t))^{p(\gamma - \alpha - 1)} \left| D_{a_1^+}^{\gamma; \psi} g(t) \right|^p dt \right)^{1/p} \\ &\times \left( \int_{a_1}^y (\psi'(t))^q (\psi(y) - \psi(t))^{q(\gamma - \alpha - 1)} \left| D_{a_1^+}^{\gamma; \psi} h(t) \right|^q dt \right)^{1/q}. \end{aligned} \quad (3.5)$$

Applying Lemma 2.1(i), we obtain

$$4^{M_1 - m_1} (x^p y^q)^{M_1 - m_1} \leq (x^p + y^q)^{2(M_1 - m_1)}, \quad x \geq 0, \quad y \geq 0, \quad (3.6)$$

where  $\frac{1}{p} + \frac{1}{q} = 1$  with  $p, q > 1$ , and  $m_1 = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$ ,  $M_1 = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$ . Replacing  $x$  by  $x^{\frac{1}{pq(M_1 - m_1)}}$  and  $y$  by  $y^{\frac{1}{pq(M_1 - m_1)}}$  in (3.6), taking it into (3.5), one has

$$\begin{aligned} &\frac{4^{M_1 - m_1} \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} g(x) \right| \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} h(y) \right|}{\left[ (x - a_1)^{\frac{1}{q(M_1 - m_1)}} + (y - a_1)^{\frac{1}{p(M_1 - m_1)}} \right]^{2(M_1 - m_1)}} \\ &\leq \frac{\left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} g(x) \right| \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} h(y) \right|}{(x - a_1)^{\frac{1}{q}} (y - a_1)^{\frac{1}{p}}} \\ &\leq \frac{1}{(\Gamma(\gamma - \alpha))^2} \left( \int_{a_1}^x (\psi'(t))^p (\psi(x) - \psi(t))^{p(\gamma - \alpha - 1)} \left| D_{a_1^+}^{\gamma; \psi} g(t) \right|^p dt \right)^{1/p} \\ &\times \left( \int_{a_1}^y (\psi'(t))^q (\psi(y) - \psi(t))^{q(\gamma - \alpha - 1)} \left| D_{a_1^+}^{\gamma; \psi} h(t) \right|^q dt \right)^{1/q}. \end{aligned} \quad (3.7)$$

Multiplying (3.7) by the kernel  $K(x, y)$  and integrating  $x$  and  $y$  on domain  $[a_1, b_1] \times [a_1, b_1]$ , we obtain

$$\begin{aligned} &4^{M_1 - m_1} \int_{a_1}^{b_1} \int_{a_1}^{b_1} \frac{K(x, y) \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} g(x) \right| \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} h(y) \right|}{\left[ (x - a_1)^{\frac{1}{q(M_1 - m_1)}} + (y - a_1)^{\frac{1}{p(M_1 - m_1)}} \right]^{2(M_1 - m_1)}} dx dy \\ &\leq \int_{a_1}^{b_1} \int_{a_1}^{b_1} \frac{K(x, y) \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} g(x) \right| \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} h(y) \right|}{(x - a_1)^{\frac{1}{q}} (y - a_1)^{\frac{1}{p}}} dx dy \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{(\Gamma(\gamma - \alpha))^2} \int_{a_1}^{b_1} \int_{a_1}^{b_1} K(x, y) \left( \int_{a_1}^x (\psi'(t))^p (\psi(x) - \psi(t))^{p(\gamma - \alpha - 1)} \left| D_{a_1^+}^{\gamma; \psi} g(t) \right|^p dt \right)^{1/p} \\ &\times \left( \int_{a_1}^y (\psi'(t))^q (\psi(y) - \psi(t))^{q(\gamma - \alpha - 1)} \left| D_{a_1^+}^{\gamma; \psi} h(t) \right|^q dt \right)^{1/q} dx dy. \end{aligned} \quad (3.8)$$

Taking

$$g_1(x) = \left( \int_{a_1}^x (\psi'(t))^p (\psi(x) - \psi(t))^{p(\gamma - \alpha - 1)} \left| D_{a_1^+}^{\gamma; \psi} g(t) \right|^p dt \right)^{1/p}$$

and

$$h_1(y) = \left( \int_{a_1}^y (\psi'(t))^q (\psi(y) - \psi(t))^{q(\gamma - \alpha - 1)} \left| D_{a_1^+}^{\gamma; \psi} h(t) \right|^q dt \right)^{1/q}.$$

Replacing  $g(t)$  by  $g_1(t)$  and  $h(t)$  by  $h_1(t)$  in (2.7), we have

$$\begin{aligned} &\int_{a_1}^{b_1} \int_{a_1}^{b_1} K(x, y) g_1(x) h_1(y) dx dy \\ &\leq \left( \int_{a_1}^{b_1} \theta^p(x) G(x) g_1^p(x) dx \right)^{1/p} \left( \int_{a_1}^{b_1} \phi^q(y) H(y) h_1^q(y) dy \right)^{1/q} \\ &= \left( \int_{a_1}^{b_1} \int_{a_1}^x (\psi'(t))^p (\psi(y) - \psi(t))^{p(\gamma - \alpha - 1)} \theta^p(x) G(x) \left| D_{a_1^+}^{\gamma; \psi} g(t) \right|^p dt dx \right)^{1/p} \\ &\times \left( \int_{a_1}^{b_1} \int_{a_1}^y (\psi'(t))^q (\psi(y) - \psi(t))^{q(\gamma - \alpha - 1)} \phi^q(y) H(y) \left| D_{a_1^+}^{\gamma; \psi} h(t) \right|^q dt dy \right)^{1/q}. \end{aligned} \quad (3.9)$$

Using (3.8) and (3.9), we obtain (3.1). Also, the inequality (3.2) can be proved by applying (2.8).  $\square$

**Corollary 3.1.** Let  $\frac{1}{p} + \frac{1}{q} = 1$  with  $p, q > 1$  and  $\gamma \geq \alpha + 1$ . Assume that  $K : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$  is a nonnegative homogeneous function for degree  $-\lambda$ ,  $\lambda > 0$ , and  $g, h, \psi \in C_0^n([0, \infty])$  are functions such that  $\psi$  is an increasing function with  $\psi'(x) \neq 0$ , for all  $x \in ([0, \infty), \mathbb{R})$ . Also, let  ${}^H D_{a_1^+}^{\alpha, \beta; \psi}$  and  $D_{a_1^+}^{\gamma; \psi}$  be defined by (2.2) and (2.4), respectively. Then the inequalities hold:

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{K(x, y) \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} g(x) \right| \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} h(y) \right|}{\left[ x^{\frac{1}{q(M_1 - m_1)}} + y^{\frac{1}{p(M_1 - m_1)}} \right]^{2(M_1 - m_1)}} dx dy \\ &\leq \frac{pq}{4^{M_1 - m_1}} \int_0^\infty \int_0^\infty K(x, y) \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} g(x) \right| \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} h(y) \right| d(x^{1/p}) d(y^{1/q}) \\ &\leq \frac{L}{4^{M_1 - m_1} (\Gamma(\gamma - \alpha))^2} \left( \int_0^\infty \int_0^x (\psi'(t))^p (\psi(x) - \psi(t))^{p(\gamma - \alpha - 1)} x^{p(A_1 - A_2) + 1 - \lambda} \left| D_{a_1^+}^{\gamma; \psi} g(t) \right|^p dt dx \right)^{1/p} \\ &\times \left( \int_0^\infty \int_0^y (\psi'(t))^q (\psi(y) - \psi(t))^{q(\gamma - \alpha - 1)} y^{q(A_2 - A_1) + 1 - \lambda} \left| D_{a_1^+}^{\gamma; \psi} h(t) \right|^q dt dy \right)^{1/q}, \end{aligned} \quad (3.10)$$



and

$$\begin{aligned} & \int_0^\infty y^{(p-1)(\lambda-1)+p(A_1-A_2)} \left( \int_0^\infty K(x,y) \left( \int_0^x (\psi'(t))^p (\psi(x)-\psi(t))^{p(\gamma-\alpha-1)} \left| D_{a_1^+}^{\gamma;\psi} g(t) \right|^p dt \right)^{1/p} dx \right)^p dy \\ & \leq L^p \int_0^\infty \int_0^x (\psi'(t))^p (\psi(x))^{p(\gamma-\alpha-1)} x^{p(A_1-A_2)+1-\lambda} \left| D_{a_1^+}^{\gamma;\psi} g(t) \right|^p dt dx, \end{aligned} \quad (3.11)$$

where  $M_1$  and  $m_1$  are defined as Theorem 3.1, and

$$A_1 = \left( \frac{1-\lambda}{q}, \frac{1}{q} \right), \quad A_2 = \left( \frac{1-\lambda}{p}, \frac{1}{p} \right), \quad L = [k(pA_2)]^{1/p} [k(2-\lambda-qA_1)]^{1/q}.$$

*Proof.* Let  $G(x)$  and  $H(y)$  be defined by (2.9) and (2.10), respectively. In (3.9), taking  $\theta(x) = x^{A_1}$  and  $\phi(y) = y^{A_2}$ , and from  $\psi(x)$  is an increasing function and  $\gamma \geq \alpha + 1$ , we write

$$(\psi(x) - \psi(t))^{p(\gamma-\alpha-1)} \leq \psi^{p(\gamma-\alpha-1)}(x), \text{ for } x \in [0, \infty] \text{ and } t \in [0, x].$$

By using Lemma 2.2, we obtain

$$\begin{aligned} & \int_0^\infty \int_0^x (\psi'(t))^p (\psi(x) - \psi(t))^{p(\gamma-\alpha-1)} \theta^p(x) G(x) \left| D_{a_1^+}^{\gamma;\psi} g(t) \right|^p dt dx \\ & \leq \int_0^\infty \int_0^x (\psi'(t))^p \psi^{p(\gamma-\alpha-1)}(x) x^{p(A_1-A_2)} \left( \int_0^\infty K(x,y) \left( \frac{x}{y} \right)^{pA_2} dy \right) \left| D_{a_1^+}^{\gamma;\psi} g(t) \right|^p dt dx \\ & = k(pA_2) \int_0^\infty \int_0^x (\psi'(t))^p \psi^{p(\gamma-\alpha-1)}(x) x^{p(A_1-A_2)+1-\lambda} \left| D_{a_1^+}^{\gamma;\psi} g(t) \right|^p dt dx. \end{aligned} \quad (3.12)$$

Similarly, we can derive

$$\begin{aligned} & \int_0^\infty \int_0^y (\psi'(t))^q (\psi(y) - \psi(t))^{q(\gamma-\alpha-1)} \phi^q(y) H(y) \left| D_{a_1^+}^{\gamma;\psi} h(t) \right|^q dt dy \\ & \leq k(2-\lambda-qA_1) \int_0^\infty \int_0^y (\psi'(t))^q \psi^{q(\gamma-\alpha-1)}(y) y^{q(A_2-A_1)+1-\lambda} \left| D_{a_1^+}^{\gamma;\psi} h(t) \right|^q dt dy. \end{aligned} \quad (3.13)$$

Therefore, from (3.1), (3.12) and (3.13), we obtain (3.10). Also, the inequality (3.11) can be proved applying (3.2). So, we complete the proof of Corollary 3.1.  $\square$

Let's choose the special homogeneous function  $K(x, y)$ . Taking  $K(x, y) = \frac{\ln \frac{y}{x}}{y-x}$  in Corollary 3.1, then we get the following Corollary 3.2:

**Corollary 3.2.** Let  $\frac{1}{p} + \frac{1}{q} = 1$  for  $p, q > 1$  and  $\gamma \geq \alpha + 1$ . Also, let  $M_1, m_1, g, h$  as in Corollary 3.1. Then we have

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{\ln \frac{y}{x} \left| {}^H D_{a_1^+}^{\alpha,\beta;\psi} g(x) \right| \left| {}^H D_{a_1^+}^{\alpha,\beta;\psi} h(y) \right|}{(y-x) \left[ x^{\frac{1}{q(M_1-m_1)}} + y^{\frac{1}{p(M_1-m_1)}} \right]^{2(M_1-m_1)}} dx dy \\ & \leq \frac{pq}{4^{M_1-m_1}} \int_0^\infty \int_0^\infty \frac{\ln \frac{y}{x}}{y-x} \left| {}^H D_{a_1^+}^{\alpha,\beta;\psi} g(x) \right| \left| {}^H D_{a_1^+}^{\alpha,\beta;\psi} h(y) \right| d(x^{1/p}) d(y^{1/q}) \end{aligned}$$

$$\begin{aligned} &\leq \frac{L_1}{4^{M_1-m_1} (\Gamma(\gamma-\alpha))^2} \left( \int_0^\infty \int_0^x (\psi'(t))^p (\psi(x)-\psi(t))^{p(\gamma-\alpha-1)} x^{p(A_1-A_2)} \left| D_{a_1^+}^{\gamma;\psi} g(t) \right|^p dt dx \right)^{1/p} \\ &\times \left( \int_0^\infty \int_0^y (\psi'(t))^q (\psi(y)-\psi(t))^{q(\gamma-\alpha-1)} y^{q(A_2-A_1)} \left| D_{a_1^+}^{\gamma;\psi} h(t) \right|^q dt dy \right)^{1/q}, \end{aligned} \quad (3.14)$$

and

$$\begin{aligned} &\int_0^\infty y^{p(A_1-A_2)} \left( \int_0^\infty \frac{\ln \frac{y}{x}}{y-x} \left( \int_0^x (\psi'(t))^p (\psi(x)-\psi(t))^{p(\gamma-\alpha-1)} \left| D_{a_1^+}^{\gamma;\psi} g(t) \right|^p dt \right)^{1/p} dx \right)^p dy \\ &\leq L_1^p \int_0^\infty \int_0^x (\psi'(t))^p (\psi(x))^{p(\gamma-\alpha-1)} x^{p(A_1-A_2)} \left| D_{a_1^+}^{\gamma;\psi} g(t) \right|^p dt dx, \end{aligned} \quad (3.15)$$

where  $A_1 = (0, \frac{1}{q})$ ,  $A_2 = (0, \frac{1}{p})$  and  $L_1 = \pi^2 (\sin(pA_2\pi))^{-2/p} (\sin(qA_1\pi))^{-2/q}$ .

Furthermore, for the homogeneous function of degree  $-\lambda$ ,  $\lambda > 0$ , taking  $K(x, y) = (\max\{x, y\})^{-\lambda}$ ,  $A_1 = A_2 = \frac{2-\lambda}{pq}$  with the condition  $\lambda > 2 - \min\{p, q\}$  in Corollary 3.1, we can give the following result:

**Corollary 3.3.** Let  $\frac{1}{p} + \frac{1}{q} = 1$  for  $p, q > 1$  and  $\gamma \geq \alpha + 1$ . Also, let  $M_1, m_1, g, h$  as in Corollary 3.1. Then inequalities hold:

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{(\max\{x, y\})^{-\lambda} \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} g(x) \right| \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} h(y) \right|}{\left[ x^{\frac{1}{q(M_1-m_1)}} + y^{\frac{1}{p(M_1-m_1)}} \right]^{2(M_1-m_1)}} dx dy \\ &\leq \frac{pq}{4^{M_1-m_1}} \int_0^\infty \int_0^\infty \frac{\left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} g(x) \right| \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} h(y) \right|}{(\max\{x, y\})^\lambda} d(x^{1/p}) d(y^{1/q}) \\ &\leq \frac{L_2}{4^{M_1-m_1} (\Gamma(\gamma-\alpha))^2} \left( \int_0^\infty \int_0^x (\psi'(t))^p (\psi(x)-\psi(t))^{p(\gamma-\alpha-1)} \right. \\ &\times \left. x^{\lambda-1} \left| D_{a_1^+}^{\gamma;\psi} g(t) \right|^p dt dx \right)^{1/p} \left( \int_0^\infty \int_0^y (\psi'(t))^q (\psi(y)-\psi(t))^{q(\gamma-\alpha-1)} y^{\lambda-1} \left| D_{a_1^+}^{\gamma;\psi} h(t) \right|^q dt dy \right)^{1/q}, \end{aligned} \quad (3.16)$$

and

$$\begin{aligned} &\int_0^\infty y^{(p-1)(\lambda-1)} \left( \int_0^\infty (\max\{x, y\})^{-\lambda} \left( \int_0^x (\psi'(t))^p (\psi(x)-\psi(t))^{p(\gamma-\alpha-1)} \left| D_{a_1^+}^{\gamma;\psi} g(t) \right|^p dt \right)^{1/p} dx \right)^p dy \\ &\leq L_2^p \int_0^\infty \int_0^x (\psi'(t))^p (\psi(x))^{p(\gamma-\alpha-1)} x^{\lambda-1} \left| D_{a_1^+}^{\gamma;\psi} g(t) \right|^p dt dx, \end{aligned} \quad (3.17)$$

where  $L_2 = k\left(\frac{2-\lambda}{q}\right)$  and  $k(\alpha) = \frac{\lambda}{(1-\alpha)(\lambda+\alpha-1)}$ .

If we are applying the second refinement of A.G.I. (see Lemma 2.1 (ii)), then we report the related theorem:

**Theorem 3.2.** Assume that the conditions of Theorem 3.1 provided. Then, the following inequalities hold:

$$\begin{aligned}
& \int_{a_1}^{b_1} \int_{a_1}^{b_1} \frac{K(x, y) \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} g(x) \right| \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} h(y) \right|}{\left( \frac{1}{2} \left[ (x - a_1)^{\frac{1}{q}} + (y - b_1)^{\frac{1}{p}} \right] + \frac{1}{M_1 - m_1} \right)^2} dx dy \\
& \leq \int_{a_1}^{b_1} \int_{a_1}^{b_1} \frac{K(x, y) \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} g(x) \right| \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} h(y) \right|}{(x - a_1)^{\frac{1}{q}} (y - b_1)^{\frac{1}{p}}} dx dy \\
& \leq \frac{1}{(\Gamma(\gamma - \alpha))^2} \left( \int_{a_1}^{b_1} \int_{a_1}^x (\psi'(t))^p (\psi(x) - \psi(t))^{p(\gamma - \alpha - 1)} \theta^p(x) G(x) \left| D_{a_1^+}^{\gamma; \psi} g(t) \right|^p dt dx \right)^{1/p} \\
& \times \left( \int_{a_1}^{b_1} \int_{a_1}^y (\psi'(t))^q (\psi(y) - \psi(t))^{q(\gamma - \alpha - 1)} \phi^q(y) H(y) \left| D_{a_1^+}^{\gamma; \psi} h(t) \right|^q dt dy \right)^{1/q}. \tag{3.18}
\end{aligned}$$

*Proof.* Using Lemma 2.1 (ii), we get

$$x^p y^q \leq \left( \frac{x^p + y^q}{2} + \frac{1}{M_1 - m_1} \right)^2, \quad x \geq 0 \text{ and } y \geq 0, \tag{3.19}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$  with  $p > 1$ , and  $m_1 = \min \left\{ \frac{1}{p}, \frac{1}{q} \right\}$ ,  $M_1 = \max \left\{ \frac{1}{p}, \frac{1}{q} \right\}$ . If we take (3.19) and proceed as in the proof of Theorem 3.1, then we obtain (3.18).  $\square$

Now, we can derive the following theorem, which is the generalization of Theorem 3.1. In the proof of the Theorem 3.3, we used a general Hilbert-type inequality (5) of Yang et al. [51].

**Theorem 3.3.** Let  $n, l \in \mathbb{R}$ ,  $l \geq 2$ ,  $\sum_{i=1}^l \frac{1}{p_i} = 1$  with  $p_1, \dots, p_l > 1$  and  $\gamma \geq \alpha + 1$ . Also, let  $\alpha_i$  ( $i = 1, \dots, l$ ) be defined by  $\alpha_i = \prod_{j=1, j \neq i}^l p_j$ . Assume that  $K : [a_1, b_1]^l \rightarrow \mathbb{R}$  is nonnegative function,  $\omega_{ij}(x_j)$  are nonnegative functions on  $[a_1, b_1]$  for  $i, j = 1, \dots, l$  such that  $\prod_{i,j=1}^l \omega_{ij}(x_j) = 1$ , and  $g_i, \psi \in C^n([a_1, b_1])$  are functions such that  $\psi$  is an increasing function with  $\psi'(x) \neq 0$ , for all  $x \in [a_1, b_1]$ . Then we report the following:

$$\begin{aligned}
& \int_{(a_1, b_1)^l} \frac{K(x_1, \dots, x_l) \prod_{i=1}^l \left| {}^H D_{a_1^+}^{\alpha_i, \beta; \psi} g_i(x_i) \right|}{\left( \sum_{i=1}^l (x_i - a_1)^{\frac{1}{\alpha_i(M_1 - m_1)}} \right)^{l(M_1 - m_1)}} dx_1 \dots dx_l \\
& \leq \frac{1}{l^{(M_1 - m_1)l}} \int_{(a_1, b_1)^l} \frac{K(x_1, \dots, x_l) \prod_{i=1}^l \left| {}^H D_{a_1^+}^{\alpha_i, \beta; \psi} g_i(x_i) \right|}{\prod_{i=1}^l (x_i - a_1)^{\frac{1}{\alpha_i}}} dx_1 \dots dx_l \\
& \leq \frac{1}{l^{(M_1 - m_1)l} [\Gamma(\gamma - \alpha)]^l} \prod_{i=1}^l \left( \int_{a_1}^{b_1} \int_a^{x_i} (\psi'(t))^{p_i} (\psi(x_i) - \psi(t))^{p_i(\gamma - \alpha_i - 1)} \omega_{ij}^{p_i}(x_j) \right. \\
& \quad \left. \times G_i(x_i) \left| D_{a_1^+}^{\gamma; \psi} g_i(t) \right|^{p_i} dt dx_i \right)^{1/p_i}, \tag{3.20}
\end{aligned}$$

where  $m_1 = \min_{1 \leq i \leq l} \left\{ \frac{1}{p_i} \right\}$ ,  $M_1 = \max_{1 \leq i \leq l} \left\{ \frac{1}{p_i} \right\}$ , and  $G_i(x_i)$  is defined by (2.12) for  $i = 1, \dots, l$ .

Also, using the Taylor series (see Theorem 2.1), we get the following the Hilbert-Pachpatte Type inequalities:

**Theorem 3.4.** Let  $\frac{1}{p} + \frac{1}{q} = 1$  with  $p, q > 1$ ,  $-\infty \leq a_1 < b_1 \leq \infty$  and  $\alpha \geq 1$ . Assume that  $K : [a_1, b_1] \times [a_1, b_1] \rightarrow \mathbb{R}$  is nonnegative function,  $\theta(x)$ ,  $\phi(y)$  are nonnegative functions on  $[a_1, b_1]$ , and  $g, h, \psi \in C^n([a_1, b_1])$  are functions such that  $\psi$  is an increasing with  $\psi'(x) \neq 0$ , for all  $x \in [a_1, b_1]$ . Also, for  $k = 1, 2, \dots, m-1$  suppose that

$$g^{[m-k]} \left( I_{a_1^+}^{(1-\beta)(m-\alpha); \psi} g \right) (a_1) = 0$$

and

$$h^{[m-k]} \left( I_{a_1^+}^{(1-\beta)(m-\alpha); \psi} h \right) (a_1) = 0.$$

Let  ${}^H D_{a_1^+}^{\alpha, \beta; \psi}$  be defined by (2.4). As a results we get

$$\begin{aligned} & \int_{a_1}^{b_1} \int_{a_1}^{b_1} \frac{K(x, y) |g(x)| |h(y)|}{\left[ (x - a_1)^{\frac{1}{q(M_1 - m_1)}} + (y - b_1)^{\frac{1}{p(M_1 - m_1)}} \right]^{2(M_1 - m_1)}} dx dy \\ & \leq \frac{1}{4^{M_1 - m_1}} \int_{a_1}^{b_1} \int_{a_1}^{b_1} \frac{K(x, y) |g(x)| |h(y)|}{(x - a_1)^{\frac{1}{q}} (y - b_1)^{\frac{1}{p}}} dx dy \\ & \leq \frac{1}{4^{M_1 - m_1} (\Gamma(\alpha))^2} \left( \int_{a_1}^{b_1} \int_{a_1}^x (\psi'(t))^p (\psi(x) - \psi(t))^{p(\alpha-1)} \theta^p(x) G(x) \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} g(t) \right|^p dt dx \right)^{1/p} \\ & \quad \times \left( \int_{a_1}^{b_1} \int_{a_1}^y (\psi'(t))^q (\psi(y) - \psi(t))^{q(\alpha-1)} \phi^q(y) H(y) \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} h(t) \right|^q dt dy \right)^{1/q} \end{aligned} \quad (3.21)$$

and

$$\begin{aligned} & \int_{a_1}^{b_1} H^{1-p}(y) \Phi^{-p}(y) \left( \int_{a_1}^{b_1} K(x, y) \left( \int_{a_1}^x (\psi'(t))^p (\psi(x) - \psi(t))^{p(\alpha-1)} \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} g(t) \right|^p dt \right)^{1/p} dx \right)^p dy \\ & \leq \int_{a_1}^{b_1} \int_{a_1}^x (\psi'(t))^p (\psi(x) - \psi(t))^{p(\alpha-1)} \theta^p(x) G(x) \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} g(t) \right|^p dt dx, \end{aligned} \quad (3.22)$$

where  $m_1 = \min\{\frac{1}{p}, \frac{1}{q}\}$ ,  $M_1 = \max\{\frac{1}{p}, \frac{1}{q}\}$ , and  $G(x)$  and  $H(y)$  are defined as in (2.9) and (2.10) respectively.

*Proof.* Applying the Hölder's inequality to (2.6), we obtain

$$\begin{aligned} |g(x)| &= \frac{1}{\Gamma(\alpha)} \left| \int_{a_1}^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} {}^H D_{a_1^+}^{\alpha, \beta; \psi} g(t) dt \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_{a_1}^x (\psi'(t)) (\psi(x) - \psi(t))^{\alpha-1} \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} g(t) \right| dt \\ &\leq \frac{1}{\Gamma(\alpha)} \left( \int_{a_1}^x (\psi'(t))^p (\psi(x) - \psi(t))^{p(\alpha-1)} \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} g(t) \right|^p dt \right)^{1/p} \left( \int_{a_1}^x 1 dt \right)^{1/q} \end{aligned}$$

$$= \frac{(x - a_1)^{1/q}}{\Gamma(\alpha)} \left( \int_{a_1}^x (\psi'(t))^p (\psi(x) - \psi(t))^{p(\alpha-1)} \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} g(t) \right|^p dt \right)^{1/p}. \quad (3.23)$$

Similarly, we have

$$|h(y)| = \frac{(y - a_1)^{1/p}}{\Gamma(\alpha)} \left( \int_{a_1}^y (\psi'(t))^q (\psi(y) - \psi(t))^{q(\alpha-1)} \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} h(t) \right|^q dt \right)^{1/q}. \quad (3.24)$$

Repeating the same procedure as in the proof of Theorem 3.1, only multiplying (3.23) by (3.24) and using (3.6), one gets Theorem 3.4.  $\square$

**Corollary 3.4.** Let  $\frac{1}{p} + \frac{1}{q} = 1$  with  $p, q > 1$  and  $\alpha \geq 1$ . Assume that  $K : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$  is a nonnegative homogeneous function for degree  $-\lambda$ ,  $\lambda > 0$ , and  $g, h, \psi \in C_0^n([0, \infty])$  are functions such that  $\psi$  is an increasing with  $\psi'(x) \neq 0$ , for all  $x \in [0, \infty)$ . Also, for  $k = 1, 2, \dots, m - 1$  suppose that

$$g^{[m-k]} \left( I_{a_1^+}^{(1-\beta)(m-\alpha); \psi} g \right) (a_1) = 0$$

and

$$h^{[m-k]} \left( I_{a_1^+}^{(1-\beta)(m-\alpha); \psi} h \right) (a_1) = 0.$$

Let  ${}^H D_{a_1^+}^{\alpha, \beta; \psi}$  be defined by (2.4). Then the inequalities hold:

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{K(x, y) |g(x)| |h(y)|}{\left[ x^{\frac{1}{q(M_1-m_1)}} + y^{\frac{1}{p(M_1-m_1)}} \right]^{2(M_1-m_1)}} dx dy \\ & \leq \frac{pq}{4^{M_1-m_1}} \int_0^\infty \int_0^\infty K(x, y) |g(x)| |h(y)| d(x^{1/p}) d(y^{1/q}) \\ & \leq \frac{L}{4^{M_1-m_1} [\Gamma(\alpha)]^2} \left( \int_0^\infty \int_0^x (\psi'(t))^p (\psi(x) - \psi(t))^{p(\alpha-1)} x^{p(A_1-A_2)+1-\lambda} \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} g(t) \right|^p dt dx \right)^{1/p} \\ & \quad \times \left( \int_0^\infty \int_0^y (\psi'(t))^q (\psi(y) - \psi(t))^{q(\alpha-1)} y^{q(A_2-A_1)+1-\lambda} \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} h(t) \right|^q dt dy \right)^{1/q}, \end{aligned} \quad (3.25)$$

and

$$\begin{aligned} & \int_0^\infty y^{(p-1)(\lambda-1)+p(A_1-A_2)} \left( \int_0^\infty K(x, y) \left( \int_0^x (\psi'(t))^p (\psi(x) - \psi(t))^{p(\alpha-1)} \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} g(t) \right|^p dt \right)^{1/p} dx \right)^p dy \\ & \leq L^p \int_0^\infty \int_0^x (\psi'(t))^p (\psi(x) - \psi(t))^{p(\alpha-1)} x^{p(A_1-A_2)+1-\lambda} \left| {}^H D_{a_1^+}^{\alpha, \beta; \psi} g(t) \right|^p dt dx, \end{aligned} \quad (3.26)$$

where  $M_1$  and  $m_1$  are defined as Theorem 3.2. Also,

$$A_1 = \left( \frac{1-\lambda}{q}, \frac{1}{q} \right), \quad A_2 = \left( \frac{1-\lambda}{p}, \frac{1}{p} \right), \quad L = [k(pA_2)]^{1/p} [k(2-\lambda-qA_1)]^{1/q}.$$

Now, we recall that in some particular cases of the  $\psi$ -Hilfer fractional derivative operator Eq (2.3), we proceed with wide class of fractional derivatives by taking some special values of  $\psi(x)$ ,  $a_1$ ,  $\alpha$  and  $\beta$  (see [12, 26, 68]).

(1) Using the limit  $\beta \rightarrow 1$  on both sides of the Eq (2.3), we obtain the following  $\psi$ -Caputo fractional operator w.r.t. another function:

$${}^H D_{a_1^+}^{\alpha,1;\psi} g(x) = I_{a_1^+}^{m-\alpha;\psi} \left( \frac{1}{\psi'(x)} \frac{d}{dx} \right)^m g(x) = {}^C D_{a_1^+}^{\alpha;\psi} g(x). \quad (3.27)$$

(2) Using the limit  $\beta \rightarrow 0$  on both sides of the Eq (2.3), we get the following  $\psi$ -Riemann-Liouville fractional operator w.r.t. another function, namely:

$${}^H D_{a_1^+}^{\alpha,0;\psi} g(x) = \left( \frac{1}{\psi'(x)} \frac{d}{dx} \right)^m I_{a_1^+}^{(m-\alpha);\psi} g(x) = D_{a_1^+}^{\alpha;\psi} g(x). \quad (3.28)$$

(3) For  $\psi(x) = x$ , considering the limit  $\beta \rightarrow 1$  on both sides of the Eq (2.3), we end up with the following Caputo fractional operator:

$${}^H D_{a_1^+}^{\alpha,1;x} g(x) = I_{a_1^+}^{(m-\alpha);x} \left( \frac{d}{dx} \right)^m g(x) = \frac{1}{\Gamma(m-\alpha)} \int_{a_1}^x (x-t)^{m-\alpha-1} \left( \frac{d}{dt} \right)^m g(t) dt = {}^C D_{a_1^+}^{\alpha} g(x). \quad (3.29)$$

(4) For  $\psi(x) = x^\rho$ , considering the limit  $\beta \rightarrow 0$  on both sides of the Eq (2.3), we obtain the below fractional operator (Katugampola):

$$\begin{aligned} \rho^\alpha {}^H D_{a_1^+}^{\alpha,0;x^\rho} g(x) &= \rho^\alpha \left( \frac{1}{\rho x^{\rho-1}} \frac{d}{dx} \right)^m I_{a_1^+}^{(m-\alpha);x^\rho} g(x) \\ &= \rho^{\alpha-m+1} \left( \frac{1}{\rho x^{\rho-1}} \frac{d}{dx} \right)^m \frac{1}{\Gamma(m-\alpha)} \int_{a_1}^x t^{\rho-1} (x^\rho - t^\rho)^{m-\alpha-1} g(t) dt \\ &= {}^\rho D_{a_1^+}^{\alpha} g(x). \end{aligned} \quad (3.30)$$

(5) When  $\psi(x) = x$ , considering the limit  $\beta \rightarrow 0$  on both sides of the Eq (2.3), we obtain the following Riemann-Liouville fractional operator:

$${}^H D_{a_1^+}^{\alpha,0;x} g(x) = \left( \frac{d}{dx} \right)^m I_{a_1^+}^{(m-\alpha);x} g(x) = \left( \frac{d}{dx} \right)^m \frac{1}{\Gamma(m-\alpha)} \int_{a_1}^x (x-t)^{m-\alpha-1} g(t) dt = D_{a_1^+}^{\alpha} g(x). \quad (3.31)$$

(6) When  $\psi(x) = \ln x$ , considering the limit  $\beta \rightarrow 0$  on both sides of the Eq (2.3), we get the below Hadamard fractional operator:

$${}^H D_{a_1^+}^{\alpha,0;\ln x} g(x) = \left( x \frac{d}{dx} \right)^m I_{a_1^+}^{(m-\alpha);\ln x} g(x) = \left( x \frac{d}{dx} \right)^m \frac{1}{\Gamma(m-\alpha)} \int_{a_1}^x \left( \ln \left( \frac{x}{t} \right) \right)^{m-\alpha-1} g(t) \frac{dt}{t} = {}^H D_{a_1^+}^{\alpha} g(x). \quad (3.32)$$

(7) When  $\psi(x) = \ln x$ , considering the limit  $\beta \rightarrow 1$  on both sides of the Eq (2.3), we obtain the Caputo-Hadamard fractional operator, namely:

$${}^H D_{a_1^+}^{\alpha,1;\ln x} g(x) = I_{a_1^+}^{(m-\alpha);\ln x} \left( x \frac{d}{dx} \right)^m g(x) = \frac{1}{\Gamma(m-\alpha)} \int_{a_1}^x \left( \ln \left( \frac{x}{t} \right) \right)^{m-\alpha-1} \left( t \frac{d}{dt} \right)^m g(t) \frac{dt}{t} = {}^{CH} D_{a_1^+}^{\alpha} g(x). \quad (3.33)$$

(8) For  $\psi(x) = x^\rho$ , considering the limit  $\beta \rightarrow 1$  on both sides of the Eq (2.3), we get the following fractional operator (Caputo-Katugampola):

$$\begin{aligned} \rho^\alpha {}^H D_{a_1^+}^{\alpha,1;x^\rho} g(x) &= \rho^\alpha I_{a_1^+}^{(m-\alpha);x^\rho} \left( \frac{1}{\rho x^{\rho-1}} \frac{d}{dx} \right)^m g(x) \\ &= \rho^{\alpha-m+1} \frac{1}{\Gamma(m-\alpha)} \int_{a_1}^x t^{\rho-1} (x^\rho - t^\rho)^{m-\alpha-1} \left( \frac{1}{t^{\rho-1}} \frac{d}{dt} \right)^m g(t) dt = {}^{CK} D_{a_1^+}^{\alpha,\rho} g(x). \end{aligned} \quad (3.34)$$

(9) For  $\psi(x) = x$  and  $a_1 = 0$ , considering the limit  $\beta \rightarrow 0$  on both sides of the Eq (2.3), we obtain the Riemann fractional operator:

$${}^H D_{0^+}^{\alpha,0;x} g(x) = \left( \frac{d}{dx} \right)^m I_{0^+}^{(m-\alpha);x} g(x) = {}^R D_{a_1^+}^\alpha g(x). \quad (3.35)$$

(10) For  $\psi(x) = x$  and  $a_1 = c$ , considering the limit  $\beta \rightarrow 0$  on both sides of the Eq (2.3), we report the following Chen fractional operator:

$${}^H D_{c^+}^{\alpha,0;x} g(x) = \left( \frac{d}{dx} \right)^m I_{c^+}^{(m-\alpha);x} g(x) = \left( \frac{d}{dx} \right)^m \frac{1}{\Gamma(m-\alpha)} \int_c^x (x-t)^{m-\alpha-1} g(t) dt = D_{c^+}^\alpha g(x). \quad (3.36)$$

**Remark 3.1.** For the above some particular cases of the  $\psi$ -Hilfer fractional derivative operator Eq (2.3), in Theorems 3.1–3.4 and Corollaries 3.1–3.4, we obtain new results for them.

## 4. Conclusions

Studies involving Hilbert's inequalities play an important place in analysis and application several. Recently, such inequalities were generalized and developed by mathematics. In this study, by generalizing of the inequalities in [1], we establish several new Hilbert-Pachpatte type for the left sided  $\psi$ -Hilfer fractional derivatives. Furthermore, using the particular cases of the  $\psi$ -Hilfer fractional derivative, we proceed with wide class of fractional derivatives by selecting  $\psi$ ,  $a_1$ ,  $b_1$  and considering the limit of the parameters  $\alpha$  and  $\beta$ .

## Conflict of interest

The authors declare no conflict of interests.

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