Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(5): 11953–11972.
DOI: 10.3934/math.2023604
Received: 09 October 2022
Revised: 13 March 2023
Accepted: 15 March 2023
Published: 20 March 2023

Research article

Optical applications of a generalized fractional integro-differential equation with periodicity

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Abstract: Impulsive is the affinity to do something without thinking. In this effort, we model a mathematical formula types integro-differential equation (I-DE) to describe this behavior. We investigate periodic boundary value issues in Banach spaces for fractional a class of I-DEs with non-quick impulses. We provide numerous sufficient conditions of the existence of mild outcomes for I-DE utilizing the measure of non-compactness, the method of resolving domestic, and the fixed point result. Lastly, we illustrate a set of examples, which is given to demonstrate the investigations key findings. Our findings are generated some recent works in this direction.

Keywords: fractional calculus; fractional differential equation; fractional integral operator; fractional differential operator
Mathematics Subject Classification: 34A37, 26A33

1. Introduction

Many scholars have been drawn to fractional differential equations in recent decades, and many good results have been obtained. This class of the differential equations with instantaneous impulses is utilized to represent sudden events such as shocks and natural disasters, were explored by the many researchers and investigators. Some dynamics difficulties in the evolution process cannot be explained by differential equations with instantaneous impulse. Drug transport through the bloodstream, for example, is a gradual and ongoing process. Non-instantaneous impulse models, on the other hand, can explain these issues. Mathematically, researchers modeled this situation in different types of differential, integral and integro-differential styles. I-DEs can be used to model a variety of circumstances in science and engineering [1], such as circuit analysis. It is, in essence, a form of energy conservation. These types of I-DEs have been used in epidemiology and epidemic mathematical modeling, especially when the models include age-structure or depict spatial epidemics [2, 3].

Newly, many researchers investigate the impulsive problem by using the I-DEs. For example, Wang and Zhu modeled the BVP design of [4]

$$\begin{cases} {}^{C}\Delta^{\nu}\sigma(\tau) = \Lambda(\tau)\sigma(\tau) + \Phi\left(\tau, \sigma(\tau), \int_0^{\tau} \varphi(\tau, \varsigma, \sigma(\varsigma))d\varsigma\right) & \tau \in (\varsigma_{i+1}, \tau_{i+1}], i = 0, 1, ..., n \\ \sigma(\tau) = \rho_i(\tau) + \varrho_{\nu}(\tau, \tau_i) \int_{\tau_i}^{\tau} \varphi(\varsigma, \sigma(\varsigma))d\varsigma & \tau \in (\tau_i, \varsigma_i], i = 0, 1, ..., n \\ \sigma(0) = \sigma(\top), \end{cases}$$

where ${}^{C}\Delta^{\nu}$ indicates the Caputo' s fractional derivative of order $\nu \in (0, 1]$, Λ is a linear operator, ϱ_{ν} fractional supported functions and the numbers τ_i and ς_i satisfy

$$0 = \varsigma_0 < \tau_1 \leq \varsigma_1 < \tau_2 \leq ... < \tau_{n+1} = \top.$$

Also, they defined continuous functions $\varphi_i : (\tau_i, \varsigma_i] \times \Xi \to \Xi$, where Ξ indicates a Banach space, ϱ_{ν} is the resolvent operator generated by Λ and ρ_i are nonlinear functions in Ξ.

This BVP involved many recent designs, that can be seen in the efforts of Ibrahim [5], Malik and Kumar [6], Pierri et al. [7], Agarwal et al. [8], Ahmed et al. [9], Sitho et al. [10], Saadati et al. [11], Lu et al. [12], Chaudhary and Reich [13], Zhu and Liu [14], Hemant et al. [15] and Hadid and Ibrahim [16].

We investigate the periodicity of fractional multi-evolution equations (FME) via non-instantaneous impulses, which is created by the previous work:

$$\begin{cases} {}^{C}\Delta^{\nu}\sigma(\tau) = \Lambda(\tau)\sigma(\tau) + \phi(\tau, \varsigma) & \\ \qquad\qquad + \sum_{i=0}^{n} \int_0^{\tau} \varphi_i(\tau - \varsigma)\Phi_i(\varsigma, \sigma(\varsigma))d\varsigma & \tau \in (\varsigma_{i+1}, \tau_{i+1}], i = 0, 1, ..., n \\ \sigma(\tau) = \rho_i(\tau, \sigma(\tau))\varrho_{\nu}(\tau, \tau_i) & \tau \in (\tau_i, \varsigma_i], i = 1, ..., n \\ \sigma(0) = \sigma(\top), \end{cases} \tag{1.1}$$

The existence of mild outcomes for the FDEs (1.1), via the criteria of the non-compact semigroups is investigated in this study; nonetheless, the linear operator Λ is τ-dependent. Moreover, we clime the FME, where $n = 1$ is studied in [14]. As a result of utilizing a different strategy, the outcomes provided in this effort improve and extend the primary conclusions in many researches.

Iterative enhancement is based on the principle of incrementally developing a engineering system, letting the designer to benefit from what was learnt during the enlargement of previous, incremental, deliverable varieties of the system. Wherever possible, knowledge comes from both the improvement and usage of the system. Starting with a rudimentary implementation of a subset of the system requirements and alliteratively improving the evolving sequence of versions until the whole system was developed was a key phase in the process. Every iteration includes strategy changes as well as the addition of new functional features [17].

The article is devoted into the following sections: Section 2 deals with all preliminaries that we request in our investigation, such as the definitions of the fractional operators and the theory of fixed points; Section 3 pretenses, our outcomes, which are grouped into two parts depending on the compactness of $\varrho_\gamma(\tau,\tau_i)$; Section 4 offers an example and Section 5 involves the conclusion.

2. Preliminaries

We have two sets of information, as follows:

2.1. Theory of fixed points

- $\jmath = [0,\top]$;

- $C[\jmath,\Xi] = \{\sigma : \sigma : \jmath \to \Xi\}$ the space of all continuous functions on Ξ, where Ξ is a Banach space;
- $C_p[\jmath,\Xi] = \{\sigma : \sigma : (\varsigma_i,\tau_{i+1}] \to \Xi\}$ such that there occurs $\sigma(\tau_i^-)$ and $\sigma(\tau_i^+)$ satisfying $\sigma(\tau_i^-) = \sigma(\tau_i)$, $i = 1,...,n$ with the sup-norm

$$\|\sigma\|_{C_p} = \sup\{\|\sigma(\tau)\| : \tau \in \jmath\}.$$

2.1.1. Lemmas

Lemma 2.1. *[18] Suppose that $U \subset \Xi$ is a bounded closed and convex set, and Ξ is a Banach space. In addition, suppose that the mapping $F : U \to U$ is in the strict set contraction. Then F in U must have at least one fixed point.*

Lemma 2.2. *[19] Suppose that Ξ is a Banach space and $Q \subset C[\jmath,\Xi]$ is equicontinuous and bounded, then the closed convex hull of Q $\bar{C}oQ \subset C[\jmath,E]$ is equicontinuous and bounded.*

Lemma 2.3. *[20] Suppose that Ξ is a Banach space, and $U \subset \Xi$ is bounded, then there occurs a countable set $U_0 \subset U$ such that $\gamma(U) \leq 2\gamma(U_0)$, where $\gamma(U)$ is known as the Kuratowski measure of non-compactness of the bounded set $U \subset \Xi$. Clearly, $0 < \gamma(U) < \infty$ and*

$$\gamma(U) = \inf\{\epsilon > 0 : \bigcup_{i=0}^{n} u_i, \quad diam(u_i) \leq \epsilon\}.$$

Lemma 2.4. *[21] Suppose that Ξ is a Banach space, and let $U \subset C[\jmath,\Xi]$ is equicontinuous and bounded, then $\gamma(U(\tau))$ is continuous on $\jmath$, and*

$$\gamma\left(\int_{\jmath} F(\tau)d\varsigma\right) \leq \int_{\jmath} \gamma\left(F(\tau)\right)d\tau, \quad \gamma(F(\tau)) = \max\gamma(F(\tau)).$$

2.1.2. Definition

If $\sigma \in C_p(\jmath,\Xi)$ fulfills the resulting equations, it is supposed to be a mild solution to problematic (1.1)

$$\sigma(\tau) = \varrho_\gamma(\tau,0)\Big[\varrho_\gamma(\top,\tau_n)\rho_n(\varsigma_n,\sigma(\varsigma_n))$$

$$
\begin{aligned}
&+\int_{\varsigma_n}^{\top}\varrho_\nu(\tau,\varsigma)\left(\phi(\varsigma,\sigma(\varsigma))+\sum_{i=0}^{n}\int_0^{\varsigma}\varphi_i(\tau-\varsigma)\Phi_i(\varsigma,\sigma(\varsigma))d\varsigma\right)\Big]\\
&+\int_{0}^{\tau}\varrho_\nu(\tau,\varsigma)\left(\phi(\varsigma,\sigma(\varsigma))+\sum_{i=0}^{n}\int_0^{\varsigma}\varphi_i(\tau-\varsigma)\Phi_i(\varsigma,\sigma(\varsigma))d\varsigma\right),\quad \tau\in[0,\tau_1]\\
&\sigma(\tau)=\varrho_\nu(\tau,\tau_i)\rho_i(\tau,\sigma(\tau)),\quad \tau\in(\tau_i,\varsigma_i],\ i=1,...,n\\
&\sigma(\tau)=\varrho_\nu(\tau,\tau_i)\rho_i(\varsigma_i,\sigma(\varsigma_i))+\int_{\varsigma_i}^{\tau}\varrho_\nu(\tau,\varsigma)\left(\phi(\varsigma,\sigma(\varsigma))+\sum_{i=0}^{n}\int_0^{\varsigma}\varphi_i(\tau-\varsigma)\Phi_i(\varsigma,\sigma(\varsigma))d\varsigma\right)\\
&\tau\in(\varsigma_i,\tau_{i+1}],\ i=1,...,n.
\end{aligned}
$$

2.2. *Fractional calculus*

The Riemann-Liouville fractional order integral is given by the following formula [22]

$$I^\nu\sigma(\tau)=\frac{1}{\Gamma(\nu)}\int_0^\tau\sigma(\varsigma)(\tau-\varsigma)^{\nu-1}\,d\varsigma,\quad \nu>0,$$

where Γ indicates the gamma function. Note that

$$I^{\nu+\mu}\sigma(\tau)=I^\nu I^\mu\sigma(\tau)=I^\mu I^\nu\sigma(\tau).$$

For a function $\sigma\in C^n[0,\infty)$, the fractional derivative operator in the form of the Caputo formula of order $\nu\in(n,n+1]$ can be expressed as

$${}^{C}\Delta^\nu\sigma(\tau)=\frac{1}{\Gamma(n-\nu)}\int_0^\tau\sigma^{(n)}(\varsigma)(\tau-\varsigma)^{n-\nu-1}\,d\varsigma,\quad \tau>0,\ n\in\mathbb{N}.$$

The Caputo fractional differential operator has many applications in science, computer science and engineer.

3. Results

We have the following cases:
Define an operator $\mathbb{O}:C_p[\jmath,\Xi]\to C_p[\jmath,\Xi]$, as follows:

$$
\begin{aligned}
(\mathbb{O}\sigma)(\tau)&=\varrho_\nu(\tau,0)\Big[\varrho_\nu(\top,\tau_n)\rho_n(\varsigma_n,\sigma(\varsigma_n))\\
&+\int_{\varsigma_n}^{\top}\varrho_\nu(\tau,\varsigma)\left(\phi(\varsigma,\sigma(\varsigma))+\sum_{i=0}^{n}\int_0^{\varsigma}\varphi_i(\tau-\varsigma)\Phi_i(\varsigma,\sigma(\varsigma))d\varsigma\right)\Big]\\
&+\int_{0}^{\tau}\varrho_\nu(\tau,\varsigma)\left(\phi(\varsigma,\sigma(\varsigma))+\sum_{i=0}^{n}\int_0^{\varsigma}\varphi_i(\tau-\varsigma)\Phi_i(\varsigma,\sigma(\varsigma))d\varsigma\right),\quad \tau\in[0,\tau_1]\\
\sigma(\tau)&=\varrho_\nu(\tau,\tau_i)\rho_i(\tau,\sigma(\tau)),\quad \tau\in(\tau_i,\varsigma_i],\ i=1,...,n\\
\sigma(\tau)&=\varrho_\nu(\tau,\tau_i)\rho_i(\varsigma_i,\sigma(\varsigma_i))+\int_{\varsigma_i}^{\tau}\varrho_\nu(\tau,\varsigma)\left(\phi(\varsigma,\sigma(\varsigma))+\sum_{i=0}^{n}\int_0^{\varsigma}\varphi_i(\tau-\varsigma)\Phi_i(\varsigma,\sigma(\varsigma))d\varsigma\right)\\
&\tau\in(\varsigma_i,\tau_{i+1}],\ i=1,...,n.
\end{aligned}
\tag{3.1}
$$

We have the following result:

Theorem 3.1. *Consider the following hypotheses:*

(H1) The functions $\phi, \Phi_i, \rho_i\,(i = 1, ..., n) : \jmath \times \Xi \to \Xi$ *are bounded and continuous in* $\jmath \times \top_\ell$ *and*

$$\limsup_{\ell \to \infty} \frac{\chi(\ell)}{\ell} < \frac{1}{\Omega}, \tag{3.2}$$

where

$$\Omega = \max\Big\{\chi^2\Big(1 + (\top - \varsigma_n) + \int_{\varsigma_n}^{\top}\int_0^{\varsigma}\sum_{i=0}^{n}\varphi_i(\varsigma - t)dtd\varsigma\Big) + \chi\Big(\tau_1 + \int_0^{\tau}\int_0^{\varsigma}\sum_{i=0}^{n}\varphi_i(\varsigma - t)dtd\varsigma\Big), \chi\Big(1 + (\varsigma_{i+1} - \varsigma_i) + \int_{\varsigma_i}^{\tau}\int_0^{\varsigma}\sum_{i=0}^{n}\varphi_i(\varsigma - t)dtd\varsigma, i = 1, ..., n\Big),\ \chi\Big\},$$

where

$$\chi(\ell) = \sup\{\|\phi(\tau, \sigma)\|, \|\Phi_i(\tau, \sigma)\|, \|\rho_i(\tau, \sigma)\|, i = 1, ..., n : (\tau, \sigma) \in \jmath \times \top_\ell\}$$

and

$$\top_\ell = \{\sigma \in \Xi : \|\sigma\| \leq \ell\}.$$

The resolvent operator $\varrho_\nu(\tau, \varsigma)$ *is non-compact for* $\tau, \varsigma > 0$, *where*

$$\chi = \max_{0 \leq \varsigma < \tau \leq \top} \|\varrho_\nu(\tau, \varsigma)\| < \infty.$$

(H2) There occur non-negative Lebesgue integrable functions $L_\phi, L_{\Phi_i}, L_{\rho_i} \in L^1(\jmath, \mathbb{R}^+)(i = 1, 2, ..., n)$ *satisfying the following inequalities*

$$\gamma\,(\phi(\tau, \delta)) \leq L_\phi(\tau)\gamma(\delta)$$
$$\gamma\,(\Phi_i(\tau, \delta)) \leq L_{\Phi_i}(\tau)\gamma(\delta)$$
$$\gamma\,(\rho_i(\tau, \delta)) \leq L_{\rho_i}(\tau)\gamma(\delta),$$

where $\delta \subset \Xi$ *is equicontinuous and countable set. Define two sets as follows:*

$$\top_\ell = \{\sigma \in \Xi : \|\sigma\| \leq \ell, \quad \ell > 0\}$$

and

$$\omega = \max\Big\{\chi^2 L_{\rho_n}(\tau) + \chi^2\int_{\varsigma_n}^{\top} L_\phi(\varsigma)d\varsigma + \chi\int_0^{\tau} L_\phi(\varsigma)d\varsigma + (\chi + \chi^2)\int_0^{\tau}\int_0^{\varsigma}\sum_{i=0}^{n}\varphi_i(\varsigma - t)L_{\Phi_i}(\tau)dtd\varsigma, \chi L_{\rho_i}(\tau),\ L_{\rho_i}(\tau) + \int_{\varsigma_i}^{\tau} L_\phi(\varsigma)d\varsigma + \int_{\varsigma_i}^{\tau}\int_0^{\varsigma}\sum_{i=0}^{n}\varphi_i(\varsigma - t)L_{\Phi_i}(t)dtd\varsigma,\ i = 1, ..., n\Big\} < 1.$$

Then the BVP (1.1) *admits fully one mild outcome* $\sigma \in C_p[\jmath, \Xi]$.

Proof. From (3.2) indicates there is a positive number $\flat \in (0, \frac{1}{\Omega})$ and an initial number $\ell_0 > 0$ with $\ell > \ell_0$ fulfilling the inequality $\chi(\ell) < \flat\ell$. Also, for the initial value ℓ_0, there is a number ℓ^* satisfying the inequality $\ell^* \geq \ell_0$. Define a ball of radius ℓ^*, as follows:

$$B_{\ell^*} = \{\sigma \in C_p[\jmath, \Xi] : \|\sigma\| \leq \ell^*\}.$$

We aim to show that $(\mathbb{O}\sigma) \in B_{\ell^*}$. We follow the next steps.

Step 1 **Boundednees**

For $\tau \in [0, \tau_1]$, a computation implies that

$$\begin{aligned}
\|(\mathbb{O}\sigma)\| &\leq \|\varrho_\nu(\tau, 0)\|\|\varrho_\nu(\top, \tau_n)\rho_n(\varsigma_n, \sigma(\varsigma_n))\| \\
&+ \|\varrho_\nu(\tau, 0)\|\left\| \int_{\varsigma_n}^{\top} \varrho_\nu(\tau, \varsigma)\left(\phi(\varsigma, \sigma(\varsigma)) + \sum_{i=0}^{n} \int_0^{\varsigma} \varphi_i(\tau - \varsigma)\Phi_i(\varsigma, \sigma(\varsigma))d\varsigma\right)\right\| \\
&+ \|\varrho_\nu(\tau, 0)\|\left\| \int_0^{\tau} \varrho_\nu(\tau, \varsigma)\left(\phi(\varsigma, \sigma(\varsigma)) + \sum_{i=0}^{n} \int_0^{\varsigma} \varphi_i(\tau - \varsigma)\Phi_i(\varsigma, \sigma(\varsigma))d\varsigma\right)\right\| \\
&\leq \chi^2\flat\ell^*\left(1 + (\top - \varsigma_n) + \int_{\varsigma_n}^{\top} \int_0^{\varsigma} \sum_{i=0}^{n} \varphi_i(\varsigma - t)dtd\varsigma\right) \\
&+ \chi\flat\ell^*\left(\tau_1 + \int_0^{\tau} \int_0^{\varsigma} \sum_{i=0}^{n} \varphi_i(\varsigma - t)dtd\varsigma\right) \\
&\leq \ell^*.
\end{aligned}$$

In addition, we have for $\tau \in (\tau_i, \varsigma_i]$ the following inequality

$$\|(\mathbb{O}\sigma)\| \leq \|\varrho_\nu(\top, \tau_n)\rho_i(\varsigma_n, \sigma(\varsigma_n))\| \leq \chi\flat\ell^* \leq \ell^*.$$

Now for the interval $\tau \in (\varsigma_i, \tau_{i+1}]$, we obtain

$$\begin{aligned}
\|(\mathbb{O}\sigma)\| &\leq \|\varrho_\nu(\top, \tau_n)\rho_n(\varsigma_n, \sigma(\varsigma_n))\| \\
&+ \left\| \int_{\varsigma_n}^{\top} \varrho_\nu(\tau, \varsigma)\left(\phi(\varsigma, \sigma(\varsigma)) + \sum_{i=0}^{n} \int_0^{\varsigma} \varphi_i(\tau - \varsigma)\Phi_i(\varsigma, \sigma(\varsigma))d\varsigma\right)\right\| \\
&\leq \chi\flat\ell^*\left(1 + (\tau_{i+1} - \varsigma_i) + \int_{\varsigma_i}^{\tau} \int_0^{\varsigma} \sum_{i=0}^{n} \int_0^{\varsigma} \varphi_i(\tau - \varsigma)\right) \\
&\leq \ell^*.
\end{aligned}$$

Step 2 **Continuity**

We aim to show that $(\mathbb{O}\sigma)B_{\ell^*} \to B_{\ell^*}$ is continuous. By the continuity of ϕ, Φ_i and ρ_i, we get

$$\lim_{n\to\infty} \sup_{\tau\in\jmath} \|\phi(\tau, \sigma_n(\tau)) - \phi(\tau, \sigma(\tau))\| = 0$$

$$\lim_{n\to\infty}\sup_{\tau\in J}\|\Phi_i(\tau,\sigma_n(\tau))-\Phi_i(\tau,\sigma(\tau))\|=0$$
$$\lim_{n\to\infty}\sup_{\tau\in J}\|\rho_i(\tau,\sigma_n(\tau))-\rho_i(\tau,\sigma(\tau))\|=0.$$

Now, when $\tau\in[0,\tau_1]$, this implies

$$\begin{aligned}\|(\mathbb{O}\sigma_n)(\tau)-(\mathbb{O}\sigma)(\tau)\| \leq{}& \chi^2\sup_{\tau\in J}\|\rho_n(\tau,\sigma_n(\tau))-\rho_n(\tau,\sigma(\tau))\|\\ &+\chi^2\int_{\varsigma_n}^{\top}\|\phi(\tau,\sigma_n(\tau))-\phi(\tau,\sigma(\tau))\|d\varsigma\\ &+\chi^2\int_{\varsigma_n}^{\top}\int_0^{\varsigma}\sum_{i=0}^{n}(\|\Phi_i(\tau,\sigma_n(\tau))-\Phi_i(\tau,\sigma(\tau))\|\varphi_i(\varsigma-t))\,dtd\varsigma\\ &+\chi\tau_1\sup_{\tau\in J}(\|\phi(\tau,\sigma_n(\tau))-\phi(\tau,\sigma(\tau))\|)\\ &+\chi\int_0^{\top}\int_0^{\varsigma}\sum_{i=0}^{n}(\|\Phi_i(\tau,\sigma_n(\tau))-\Phi_i(\tau,\sigma(\tau))\|\varphi_i(\varsigma-t))\,dtd\varsigma.\end{aligned}$$

We proceed to determine the upper bound when $\tau\in(\tau_i,\varsigma_i], i=1,...,n$

$$\|(\mathbb{O}\sigma_n)(\tau)-(\mathbb{O}\sigma)(\tau)\|\leq\chi\sup_{\tau\in J}\|\rho_i(\tau,\sigma_n(\tau))-\rho_i(\tau,\sigma(\tau))\|.$$

And for $\tau\in(\varsigma_i,\tau_{i+1}], i=1,...,n$, we obtain

$$\begin{aligned}\|(\mathbb{O}\sigma_n)(\tau)-(\mathbb{O}\sigma)(\tau)\| \leq{}& \chi\sup_{\tau\in J}\|\rho_i(\tau,\sigma_n(\tau))-\rho_i(\tau,\sigma(\tau))\|\\ &+\chi\int_{\varsigma_i}^{\top}\|\phi(\tau,\sigma_n(\tau))-\phi(\tau,\sigma(\tau))\|d\varsigma\\ &+\chi\int_{\varsigma_i}^{\top}\int_0^{\varsigma}\sum_{i=0}^{n}(\|\Phi_i(\tau,\sigma_n(\tau))-\Phi_i(\tau,\sigma(\tau))\|\varphi_i(\varsigma-t))\,dtd\varsigma.\end{aligned}$$

As a conclusion, we receive the main result of this step, the continuity of $(\mathbb{O}\sigma)$, where

$$\lim_{n\to\infty}\|(\mathbb{O}\sigma_n)(\tau)-(\mathbb{O}\sigma)(\tau)\|_{C_p}=0.$$

Step 3 **Equi-continuity**

We have three cases. The first case, $\eta_1,\eta_2\in[0,\tau_1]$, where $\eta_1<\eta_2$.

$$\begin{aligned}\|(\mathbb{O}\sigma_n)(\eta_2)-(\mathbb{O}\sigma)(\eta_1)\| \leq{}& \|\varrho_\nu(\eta_2,0)-\varrho_\nu(\eta_1,0)\|\|\varrho_\nu(\top,\tau_i)\rho_i(\varsigma_i,\sigma(\varsigma_i))\|\\ &+\|(\mathbb{O}\sigma_n)(\eta_2)-(\mathbb{O}\sigma)(\eta_1)\|\\ &\times\left\|\int_{\varsigma_n}^{\top}\varrho_\nu(\top,\varsigma)\left(\phi(\varsigma,\sigma(\varsigma))+\int_0^{\varsigma}\sum_{i=1}^{n}\varphi_i(\varsigma-t)\Phi_i(t,\sigma(t))dt\right)d\varsigma\right\|\\ &+\sup_{\varsigma\in[0,\tau_1]}\|\varrho_\nu(\eta_2,0)-\varrho_\nu(\eta_1,0)\|\end{aligned}$$

$$\times \left\| \int_0^{\eta_1} \left(\phi(\varsigma, \sigma(\varsigma)) + \int_0^{\varsigma} \sum_{i=1}^{n} \varphi_i(\varsigma - t)\Phi_i(t, \sigma(t))dt \right) d\varsigma \right\|$$
$$+ \left\| \int_{\eta_1}^{\eta_2} \varrho_\nu(\eta_2, \varsigma) \left(\phi(\varsigma, \sigma(\varsigma)) + \int_0^{\varsigma} \sum_{i=1}^{n} \varphi_i(\varsigma - t)\Phi_i(t, \sigma(t))dt \right) d\varsigma \right\|.$$

The second case is in the interval $(\tau_i, \varsigma_i]$, we have

$$\begin{aligned} \|(\mathbb{O}\sigma_n)(\eta_2) - (\mathbb{O}\sigma)(\eta_1)\| &\leq \|\varrho_\nu(\eta_2, \tau_i)\rho_i(\eta_2, \sigma(\eta_2)) - \varrho_\nu(\eta_1, \tau_i)\rho_i(\eta_1, \sigma(\eta_1))\| \\ &\leq \|\varrho_\nu(\eta_2, \eta_1)\varrho_\nu(\eta_1, \tau_i)\rho_i(\eta_2, \sigma(\eta_2)) - \varrho_\nu(\eta_1, \tau_i)\rho_i(\eta_1, \sigma(\eta_1))\| \\ &\leq \chi\|\varrho_\nu(\eta_2, \eta_1)\rho_i(\eta_2, \sigma(\eta_2)) - \rho_i(\eta_1, \sigma(\eta_1))\|. \end{aligned}$$

The third case is obtained in the interval $(\varsigma_i, \tau_{i+1}]$, which yields

$$\begin{aligned} \|(\mathbb{O}\sigma_n)(\eta_2) - (\mathbb{O}\sigma)(\eta_1)\| \leq & \|\varrho_\nu(\eta_2, \tau_i) - \varrho_\nu(\eta_1, \tau_i)\|\|\rho_i(\varsigma_i, \sigma(\varsigma_i))\| \\ & + \sup_{\varsigma \in [0,\tau_1]} \|\varrho_\nu(\eta_2, \varsigma) - \varrho_\nu(\eta_1, \varsigma)\| \\ & \times \left\| \int_{\varsigma_i}^{\eta_1} \left(\phi(\varsigma, \sigma(\varsigma)) + \int_0^{\varsigma} \sum_{i=1}^{n} \varphi_i(\varsigma - t)\Phi_i(t, \sigma(t))dt \right) d\varsigma \right\| \\ & + \left\| \int_{\eta_1}^{\eta_2} \varrho_\nu(\eta_2, \varsigma) \left(\phi(\varsigma, \sigma(\varsigma)) + \int_0^{\varsigma} \sum_{i=1}^{n} \varphi_i(\varsigma - t)\Phi_i(t, \sigma(t))dt \right) d\varsigma \right\|. \end{aligned}$$

Clearly, when $\eta_2 \to \eta_1$ we have $\|(\mathbb{O}\sigma_n)(\eta_2) - (\mathbb{O}\sigma)(\eta_1)\| = 0$. Hence, $(\mathbb{O}\sigma)$ is equicontinuous in B_{ℓ^*}. Consequently, in view of Lemma 2.2, we obtain that $\bar{Co}\mathbb{O}(B_{\ell^*}) \subset B_{\ell^*}$ is bounded and equicontinuous.

Step 4 **Condensity**

We aim to show that $\mathbb{O} : B_{\ell^*} \to B_{\ell^*}$ is a condensing operator. In view of Lemma 2.3, there occurs a countable set $\Theta_0 = \{\sigma_n\} \subset \Theta \subset \bar{Co}\mathbb{O}(B_{\ell^*})$ satisfying the inequality

$$\gamma\left(\mathbb{O}(\Theta)\right) \leq 2\gamma\left(\mathbb{O}(\Theta_0)\right).$$

We have three cases, as follows: for the interval $[0, \tau_1]$ there is a set $\Theta_0 \subset \Theta \subset \bar{Co}\mathbb{O}(B_{\ell^*})$ such that

$$\begin{aligned} \gamma\left(\mathbb{O}(\Theta_0)(\tau)\right) \leq & \chi^2\ \gamma\left(\rho_n(\varsigma_n, (\Theta_0)(\varsigma_n))\right) \\ & + \chi^2\ \gamma\left(\int_{\varsigma_n}^{\top} \left(\phi(\varsigma, \Theta_0(\varsigma)) + \int_0^{\varsigma} \sum_{i=1}^{n} \varphi_i(\varsigma - t)\Phi_i(t, \Theta_0(t))dt \right) d\varsigma \right) \\ & + \chi\ \gamma\left(\int_0^{\tau} \left(\phi(\varsigma, \Theta_0(\varsigma)) + \int_0^{\varsigma} \sum_{i=1}^{n} \varphi_i(\varsigma - t)\Phi_i(t, \Theta_0(t))dt \right) d\varsigma \right) \\ \leq & \chi^2 L_{\rho_n}\gamma(\Theta) \\ & + \chi^2 \left(\int_{\varsigma_n}^{\top} \left(L_\phi(\varsigma)\gamma(\Theta_0(\varsigma)) + \int_0^{\varsigma} \sum_{i=1}^{n} \varphi_i(\varsigma - t)L_{\Phi_i}(t)\gamma(\Theta_0(t))dt \right) d\varsigma \right) \\ & + \chi \left(\int_0^{\tau} \left(L_\phi(\varsigma)\gamma(\Theta_0(\varsigma)) + \int_0^{\varsigma} \sum_{i=1}^{n} \varphi_i(\varsigma - t)L_{\Phi_i}(t)\gamma(\Theta_0(t))dt \right) d\varsigma \right) \end{aligned}$$

$$\leq \gamma(\Theta)\Big[\chi^2\left(L_{\rho_n} + \int_{\varsigma_n}^{\top} L_\phi(\varsigma)d\varsigma + \int_0^{\tau}\int_0^{\varsigma}\sum_{i=1}^{n}\varphi_i(\varsigma - t)L_{\Phi_i}(t)dtd\varsigma\right)$$
$$+\chi\left(\int_0^{\top} L_\phi(\varsigma)d\varsigma + \int_0^{\tau}\int_0^{\varsigma}\sum_{i=1}^{n}\varphi_i(\varsigma - t)L_{\Phi_i}(t)dtd\varsigma\right)\Big].$$

The second case is obtained in the interval $(\tau_i, \varsigma_i], i = 1, ..., n$ yields

$$\gamma\left(\mathbb{O}(\Theta_0)(\tau)\right) \leq \chi L_{\rho_i}(\tau)\gamma(\Theta).$$

While, for the third case in the interval $(\varsigma_i, \tau_{i+1}], i = 1, ..., n$, we get

$$\begin{aligned}\gamma\left(\mathbb{O}(\Theta_0)(\tau)\right) &\leq \chi L_{\rho_i}(\tau)\gamma(\Theta)\\ &\quad+\chi\left(\int_{\varsigma_i}^{\top}\left(L_\phi(\varsigma)\gamma(\Theta_0(\varsigma)) + \int_0^{\varsigma}\sum_{i=1}^{n}\varphi_i(\varsigma - t)L_{\Phi_i}(t)\gamma(\Theta_0(t))dt\right)d\varsigma\right)\\ &\leq \chi\gamma(\Theta)\left(L_{\rho_i}(\tau) + \int_{\varsigma_i}^{\top} L_\phi(\varsigma) + \int_{\varsigma_i}^{\top}\int_0^{\varsigma}\sum_{i=1}^{n}\varphi_i(\varsigma - t)L_{\Phi_i}(t)dtd\varsigma\right).\end{aligned}$$

According to Lemma 2.4, we obtain

$$\gamma\left(\Theta_0\right) \leq \max_{\tau\in J}\gamma\left(\Theta_0(t)\right),$$

which leads to

$$\gamma\left(\Theta\right) \leq \omega\gamma\left(\Theta\right), \quad 0 < \omega < 1.$$

As a conclusion, we confirm that $\mathbb{O}$ is a strict contraction mapping in $\bar{Co}\mathbb{O}(B_{\ell^*})$. As a result, according to Lemma 2.1, $\mathbb{O}$ has a fully fixed point in $\bar{Co}\mathbb{O}(B_{\ell^*}) \subset C_p[J, \Xi]$. Hence, Eq (1.1) has a fully mild solution in $C[J, \Xi]$.

□

The next result indicates the maximum value of φ_i and Φ_i. The proof is quite similar to Theorem 3.1.

Theorem 3.2. *Consider the following hypotheses:*

(H3) The functions $\phi, \Phi_i, \rho_i\ (i = 1, ..., n) : J \times \Xi \to \Xi$ are bounded and continuous in $J \times \top_\ell$ and

$$\limsup_{\ell\to\infty}\frac{\chi(\ell)}{\ell} < \frac{1}{\overline{\Omega}_n}, \tag{3.3}$$

where

$$\overline{\Omega}_n = \max\Big\{\chi^2\left(1 + (\top - \varsigma_n) + n\int_{\varsigma_n}^{\top}\int_0^{\varsigma}\varphi(\varsigma - t)dtd\varsigma\right)$$
$$+\chi\left(\tau_1 + n\int_0^{\tau}\int_0^{\varsigma}\varphi(\varsigma - t)dtd\varsigma\right),$$

$$\chi\left(1 + (\varsigma_{i+1} - \varsigma_i) + n\int_{\varsigma_i}^{\tau}\int_0^{\varsigma}\varphi(\varsigma - t)dtd\varsigma\right), i = 1, ..., n,\ \chi\Big\},$$

where $\varphi := \max(\varphi_i(\varsigma - t))$ *and* $\Phi := \max(\Phi_i(\tau, \sigma))$,

$$\chi(\ell) = \sup\{\|\phi(\tau,\sigma)\|, \|\Phi(\tau,\sigma)\|, \|\rho_i(\tau,\sigma)\|, i = 1, ..., n : (\tau,\sigma) \in \jmath \times \top_\ell\}$$

and

$$\top_\ell = \{\sigma \in \Xi : \|\sigma\| \leq \ell\}.$$

The resolvent operator $\varrho_\nu(\tau,\varsigma)$ *is non-compact for* $\tau,\varsigma > 0$, *where*

$$\chi = \max_{0\leq\varsigma<\tau\leq\top} \|\varrho_\nu(\tau,\varsigma)\| < \infty.$$

(H4) There occur non-negative Lebesgue integrable functions $L_\phi, L_\Phi, L_{\rho_i} \in L^1(\jmath, \mathbb{R}^+)(i = 1, 2, ..., n)$ *satisfying the following inequalities*

$$\gamma(\phi(\tau,\delta)) \leq L_\phi(\tau)\gamma(\delta)$$
$$\gamma(\Phi(\tau,\delta)) \leq L_\Phi(\tau)\gamma(\delta)$$
$$\gamma(\rho_i(\tau,\delta)) \leq L_{\rho_i}(\tau)\gamma(\delta),$$

where $\delta \subset \Xi$ *is equicontinuous and countable set. Define two sets as follows:*

$$\top_\ell = \{\sigma \in \Xi : \|\sigma\| \leq \ell, \quad \ell > 0\}$$

and

$$\begin{aligned}\overline{\omega}_n = \max\Big\{&\chi^2 L_{\rho_n}(\tau) + \chi^2\int_{\varsigma_n}^{\top} L_\phi(\varsigma)d\varsigma + \chi\int_0^{\tau} L_\phi(\varsigma)d\varsigma\\ &+ n(\chi + \chi^2)\int_0^{\tau}\int_0^{\varsigma}\varphi(\varsigma - t)L_\Phi(\tau)dtd\varsigma,\\ &\chi L_{\rho_i}(\tau),\ L_{\rho_i}(\tau) + \int_{\varsigma_i}^{\tau} L_\phi(\varsigma)d\varsigma + n\int_{\varsigma_i}^{\tau}\int_0^{\varsigma}\varphi(\varsigma - t)L_\Phi(t)dtd\varsigma,\ i = 1, ..., n.\Big\}\\ &< 1.\end{aligned}$$

Then the BVP (1.1) *admits a fully mild outcome* $\sigma \in C_p[\jmath, \Xi]$.

The next consequence can be found in [14]

Corollary 3.3. *Consider the following hypotheses:*

(H5) The functions $\phi, \Phi, \rho_i\ (i = 1, ..., n) : \jmath \times \Xi \to \Xi$ *are bounded and continuous in* $\jmath \times \top_\ell$ *and*

$$\limsup_{\ell\to\infty} \frac{\chi(\ell)}{\ell} < \frac{1}{\overline{\Omega}}, \tag{3.4}$$

where

$$\overline{\Omega} = \max\left\{\chi^2\left(1 + (\top - \varsigma_n) + \int_{\varsigma_n}^{\top}\int_0^{\varsigma}\varphi(\varsigma - t)dtd\varsigma\right)\right.$$

$$+\chi\left(\tau_1+\int_0^\tau\int_0^\varsigma \varphi(\varsigma-t)dtd\varsigma\right),$$
$$\chi\left(1+(\varsigma_{i+1}-\varsigma_i)+\int_{\varsigma_i}^\tau\int_0^\varsigma \varphi(\varsigma-t)dtd\varsigma\right), i=1,...,n,\ \chi\Big\},$$

$$\chi(\ell)=\sup\{\|\phi(\tau,\sigma)\|,\|\Phi(\tau,\sigma)\|,\|\rho_i(\tau,\sigma)\|, i=1,...,n:(\tau,\sigma)\in \jmath\times\top_\ell\}$$

and

$$\top_\ell=\{\sigma\in\Xi:\|\sigma\|\le\ell\}.$$

The resolvent operator $\varrho_\nu(\tau,\varsigma)$ is non-compact for $\tau,\varsigma>0$, where

$$\chi=\max_{0\le\varsigma<\tau\le\top}\|\varrho_\nu(\tau,\varsigma)\|<\infty.$$

(H6) *There occur non-negative Lebesgue integrable functions $L_\phi, L_\Phi, L_{\rho_i}\in L^1(\jmath,\mathbb{R}^+)(i=1,2,...,n)$ satisfying the following inequalities*

$$\gamma(\phi(\tau,\delta))\le L_\phi(\tau)\gamma(\delta)$$
$$\gamma(\Phi(\tau,\delta))\le L_\Phi(\tau)\gamma(\delta)$$
$$\gamma(\rho_i(\tau,\delta))\le L_{\rho_i}(\tau)\gamma(\delta),$$

where $\delta\subset\Xi$ is equicontinuous and countable set. Define two sets as follows:

$$\top_\ell=\{\sigma\in\Xi:\|\sigma\|\le\ell,\quad \ell>0\}$$

and

$$\begin{aligned}\overline{\omega}=\max\Big\{&\chi^2L_{\rho_n}(\tau)+\chi^2\int_{\varsigma_n}^\top L_\phi(\varsigma)d\varsigma+\chi\int_0^\tau L_\phi(\varsigma)d\varsigma\\&+(\chi+\chi^2)\int_0^\tau\int_0^\varsigma\varphi(\varsigma-t)L_\Phi(\tau)dtd\varsigma,\\&\chi L_{\rho_i}(\tau),\ L_{\rho_i}(\tau)+\int_{\varsigma_i}^\tau L_\phi(\varsigma)d\varsigma+\int_{\varsigma_i}^\tau\int_0^\varsigma\varphi(\varsigma-t)L_\Phi(t)dtd\varsigma,\ i=1,...,n.\Big\}\\&<1.\end{aligned}$$

Then the BVP (1.1) *admits a fully mild outcome $\sigma\in C_p[\jmath,\Xi]$.*

Corollary 3.4. *Let the assumptions of Theorem 3.2 be hold. Then the maximum mild solution $\sigma\in C_p(\jmath,\Xi)$ of Eq* (1.1) *can be formulated by*

$$\begin{aligned}\sigma(\tau)=&\varrho_\nu(\tau,0)\Big[\varrho_\nu(\top,\tau_n)\rho_n(\varsigma_n,\sigma(\varsigma_n))\\&+\int_{\varsigma_n}^\top\varrho_\nu(\tau,\varsigma)\left(\phi(\varsigma,\sigma(\varsigma))+n\int_0^\varsigma\varphi(\tau-\varsigma)\Phi(\varsigma,\sigma(\varsigma))d\varsigma\right)\Big]\\&+\int_0^\tau\varrho_\nu(\tau,\varsigma)\left(\phi(\varsigma,\sigma(\varsigma))+n\int_0^\varsigma\varphi(\tau-\varsigma)\Phi(\varsigma,\sigma(\varsigma))d\varsigma\right),\quad\tau\in[0,\tau_1]\end{aligned}$$

$$\sigma(\tau)=\varrho_{\nu}(\tau,\tau_i)\rho_i(\tau,\sigma(\tau)),\quad \tau\in(\tau_i,\varsigma_i],\ i=1,...,n$$
$$\sigma(\tau)=\varrho_{\nu}(\tau,\tau_i)\rho_i(\varsigma_i,\sigma(\varsigma_i))+\int_{\varsigma_i}^{\tau}\varrho_{\nu}(\tau,\varsigma)\left(\phi(\varsigma,\sigma(\varsigma))+n\int_0^{\varsigma}\varphi(\tau-\varsigma)\Phi(\varsigma,\sigma(\varsigma))d\varsigma\right)$$
$$\tau\in(\varsigma_i,\tau_{i+1}],\ i=1,...,n,$$

where φ and Φ have the same sign.

Theorem 3.1 can be extended into 2D(n,m)-parametric designing as follows, with similar proof:

Theorem 3.5. *Consider the following hypotheses:*

(H7) The functions $\phi,\Phi_j,\rho_i\,(i=1,...,n,\ j=0,...,m):\jmath\times\Xi\to\Xi$ are bounded and continuous in $\jmath\times\top_\ell$ and

$$\limsup_{\ell\to\infty}\frac{\chi(\ell)}{\ell}<\frac{1}{\Omega_m},\tag{3.5}$$

where

$$\Omega_m=\max\Big\{\chi^2\left(1+(\top-\varsigma_n)+\int_{\varsigma_n}^{\top}\int_0^{\varsigma}\sum_{j=0}^{m}\varphi_j(\varsigma-t)dtd\varsigma\right)$$
$$+\chi\left(\tau_1+\int_0^{\tau}\int_0^{\varsigma}\sum_{j=0}^{m}\varphi_j(\varsigma-t)dtd\varsigma\right),$$
$$\chi\left(1+(\varsigma_{i+1}-\varsigma_i)+\int_{\varsigma_i}^{\tau}\int_0^{\varsigma}\sum_{j=0}^{m}\varphi_j(\varsigma-t)dtd\varsigma,i=1,...,n\right),\ \chi\Big\},$$

where

$$\chi(\ell)=\sup\{\|\phi(\tau,\sigma)\|,\|\Phi_j(\tau,\sigma)\|,\|\rho_i(\tau,\sigma)\|,i=1,...,n,\ j=0,...,m:(\tau,\sigma)\in\jmath\times\top_\ell\}$$

and

$$\top_\ell=\{\sigma\in\Xi:\|\sigma\|\le\ell\}.$$

The resolvent operator $\varrho_\nu(\tau,\varsigma)$ is non-compact for $\tau,\varsigma>0$, where

$$\chi=\max_{0\le\varsigma<\tau\le\top}\|\varrho_\nu(\tau,\varsigma)\|<\infty.$$

(H8) There occur non-negative Lebesgue integrable functions $L_\phi,L_{\Phi_j},L_{\rho_i}\in L^1(\jmath,\mathbb{R}^+)(i=1,2,...,n,\ j=0,...,m)$ satisfying the following inequalities

$$\gamma\left(\phi(\tau,\delta)\right)\le L_\phi(\tau)\gamma(\delta)$$
$$\gamma\left(\Phi_j(\tau,\delta)\right)\le L_{\Phi_j}(\tau)\gamma(\delta)$$
$$\gamma\left(\rho_i(\tau,\delta)\right)\le L_{\rho_i}(\tau)\gamma(\delta),$$

where $\delta\subset\Xi$ is equicontinuous and countable set. Define two sets as follows:

$$\top_\ell=\{\sigma\in\Xi:\|\sigma\|\le\ell,\quad \ell>0\}$$

and

$$\omega_m = \max\Big\{\chi^2 L_{\rho_n}(\tau) + \chi^2 \int_{\varsigma_n}^{\top} L_\phi(\varsigma)d\varsigma + \chi \int_0^\tau L_\phi(\varsigma)d\varsigma$$
$$+ (\chi+\chi^2)\int_0^\tau \int_0^\varsigma \sum_{j=0}^{m} \varphi_j(\varsigma - t)L_{\Phi_j}(\tau)dtd\varsigma,$$
$$\chi L_{\rho_i}(\tau),\ L_{\rho_i}(\tau) + \int_{\varsigma_i}^{\tau} L_\phi(\varsigma)d\varsigma + \int_{\varsigma_i}^{\tau}\int_0^\varsigma \sum_{j=0}^{m} \varphi_j(\varsigma - t)L_{\Phi_j}(t)dtd\varsigma\Big\} < 1.$$

Then the following BVP

$$\begin{cases} {}^C\Delta^\nu \sigma(\tau) = \Lambda(\tau)\sigma(\tau) + \phi(\tau,\varsigma) & \\ \qquad\qquad + \sum_{j=0}^{m}\int_0^\tau \varphi_j(\tau-\varsigma)\Phi_j(\varsigma,\sigma(\varsigma))d\,\varsigma & \tau \in (\varsigma_{i+1}, \tau_{i+1}], i = 0,1,...,n \\ \sigma(\tau) = \rho_i(\tau,\sigma(\tau))\varrho_\nu(\tau,\tau_i) & \tau \in (\tau_i, \varsigma_i], i = 1,...,n \\ \sigma(0) = \sigma(\top), & \end{cases} \tag{3.6}$$

admits at least one mild solution $\sigma \in C_p[\jmath, \Xi]$ *formulating by*

$$\sigma(\tau) = \varrho_\nu(\tau,0)\Big[\varrho_\nu(\top,\tau_n)\rho_n(\varsigma_n,\sigma(\varsigma_n))$$
$$+ \int_{\varsigma_n}^{\top} \varrho_\nu(\tau,\varsigma)\left(\phi(\varsigma,\sigma(\varsigma)) + \sum_{j=0}^{m}\int_0^\varsigma \varphi_j(\tau-\varsigma)\Phi_j(\varsigma,\sigma(\varsigma))d\varsigma\right)\Big]$$
$$+ \int_0^\tau \varrho_\nu(\tau,\varsigma)\left(\phi(\varsigma,\sigma(\varsigma)) + \sum_{j=0}^{m}\int_0^\varsigma \varphi_j(\tau-\varsigma)\Phi_j(\varsigma,\sigma(\varsigma))d\varsigma\right), \quad \tau\in[0,\tau_1]$$
$$\sigma(\tau) = \varrho_\nu(\tau,\tau_i)\rho_i(\tau,\sigma(\tau)), \quad \tau\in(\tau_i,\varsigma_i],\ i = 1,...,n$$
$$\sigma(\tau) = \varrho_\nu(\tau,\tau_i)\rho_i(\varsigma_i,\sigma(\varsigma_i)) + \int_{\varsigma_i}^{\tau} \varrho_\nu(\tau,\varsigma)\left(\phi(\varsigma,\sigma(\varsigma)) + \sum_{j=0}^{m}\int_0^\varsigma \varphi_j(\tau-\varsigma)\Phi_j(\varsigma,\sigma(\varsigma))d\varsigma\right)$$
$$\tau\in(\varsigma_i,\tau_{i+1}],\ i = 1,...,n.$$

The following outcome indicates the maximum value of φ_j and $\Phi_j,\ j = 0,...,m$. The proof is quite similar to Theorem 3.5.

Theorem 3.6. *Consider the following hypotheses:*

(H9) *The functions* $\phi, \Phi_j, \rho_i\ (i = 1,...,n,\ j = 0,...,m) : \jmath\times\Xi \to \Xi$ *are bounded and continuous in* $\jmath\times\top_\ell$ *and*

$$\limsup_{\ell\to\infty}\frac{\chi(\ell)}{\ell} < \frac{1}{\overline{\Omega}_m}, \tag{3.7}$$

where

$$\overline{\Omega}_m = \max\Big\{\chi^2\left(1 + (\top - \varsigma_n) + m\int_{\varsigma_n}^{\top}\int_0^\varsigma \varphi(\varsigma - t)dtd\varsigma\right)$$

$$+\chi\left(\tau_1+m\int_0^{\tau}\int_0^{\varsigma}\varphi(\varsigma-t)dtd\varsigma\right),$$
$$\chi\left(1+(\varsigma_{i+1}-\varsigma_i)+m\int_{\varsigma_i}^{\tau}\int_0^{\varsigma}\varphi(\varsigma-t)dtd\varsigma, i=1,...,n\right),\ \chi\Big\},$$

where $\varphi := \max(\varphi_j(\varsigma-t))$ *and* $\Phi := \max(\Phi_j(\tau,\sigma))$

$$\chi(\ell)=\sup\{\|\phi(\tau,\sigma)\|,\|\Phi(\tau,\sigma)\|,\|\rho_i(\tau,\sigma)\|, i=1,...,n:(\tau,\sigma)\in \jmath\times\top_\ell\}$$

and

$$\top_\ell=\{\sigma\in\Xi:\|\sigma\|\le\ell\}.$$

The resolvent operator $\varrho_\nu(\tau,\varsigma)$ *is non-compact for* $\tau,\varsigma>0$, *where*

$$\chi=\max_{0\le\varsigma<\tau\le\top}\|\varrho_\nu(\tau,\varsigma)\|<\infty.$$

(H10) There occur non-negative Lebesgue integrable functions $L_\phi, L_\Phi, L_{\rho_i}\in L^1(\jmath,\mathbb{R}^+)(i=1,2,...,n)$ *satisfying the following inequalities*

$$\gamma(\phi(\tau,\delta))\le L_\phi(\tau)\gamma(\delta)$$
$$\gamma(\Phi(\tau,\delta))\le L_\Phi(\tau)\gamma(\delta)$$
$$\gamma(\rho_i(\tau,\delta))\le L_{\rho_i}(\tau)\gamma(\delta),$$

where $\delta\subset\Xi$ *is equicontinuous and countable set. Define two sets as follows:*

$$\top_\ell=\{\sigma\in\Xi:\|\sigma\|\le\ell,\quad \ell>0\}$$

and

$$\begin{aligned}\overline{\omega}_m=\max\Big\{&\chi^2L_{\rho_n}(\tau)+\chi^2\int_{\varsigma_n}^{\top}L_\phi(\varsigma)d\varsigma+\chi\int_0^{\tau}L_\phi(\varsigma)d\varsigma\\&+m(\chi+\chi^2)\int_0^{\tau}\int_0^{\varsigma}\varphi(\varsigma-t)L_\Phi(\tau)dtd\varsigma,\\&\chi L_{\rho_i}(\tau),\ L_{\rho_i}(\tau)+\int_{\varsigma_i}^{\tau}L_\phi(\varsigma)d\varsigma+m\int_{\varsigma_i}^{\tau}\int_0^{\varsigma}\varphi(\varsigma-t)L_\Phi(t)dtd\varsigma,\ i=1,...,n\Big\}\\&<1.\end{aligned}$$

Then the BVP (3.6) *admits a fully mild outcome* $\sigma\in C_p[\jmath,\Xi]$.

Corollary 3.7. *Let the assumptions of Theorem 3.6 be hold. Then the maximum mild solution* $\sigma\in C_p(\jmath,\Xi)$ *of Eq* (3.6) *can be formulated by*

$$\begin{aligned}\sigma(\tau)=\varrho_\nu(\tau,0)\Big[&\varrho_\nu(\top,\tau_n)\rho_n(\varsigma_n,\sigma(\varsigma_n))\\&+\int_{\varsigma_n}^{\top}\varrho_\nu(\tau,\varsigma)\left(\phi(\varsigma,\sigma(\varsigma))+m\int_0^{\varsigma}\varphi(\tau-\varsigma)\Phi(\varsigma,\sigma(\varsigma))d\varsigma\right)\Big]\end{aligned}$$

$$+\int_0^\tau \varrho_\nu(\tau,\varsigma)\left(\phi(\varsigma,\sigma(\varsigma))+m\int_0^\varsigma \varphi(\tau-\varsigma)\Phi(\varsigma,\sigma(\varsigma))d\varsigma\right),\quad \tau\in[0,\tau_1]$$
$$\sigma(\tau)=\varrho_\nu(\tau,\tau_i)\rho_i(\tau,\sigma(\tau)),\quad \tau\in(\tau_i,\varsigma_i],\ i=1,...,n$$
$$\sigma(\tau)=\varrho_\nu(\tau,\tau_i)\rho_i(\varsigma_i,\sigma(\varsigma_i))+\int_{\varsigma_i}^\tau \varrho_\nu(\tau,\varsigma)\left(\phi(\varsigma,\sigma(\varsigma))+m\int_0^\varsigma \varphi(\tau-\varsigma)\Phi(\varsigma,\sigma(\varsigma))d\varsigma\right)$$
$$\tau\in(\varsigma_i,\tau_{i+1}],\ i=1,...,n,$$

where φ and Φ have the same sign.

Remark 3.8. The kernel function $\varphi(\tau-\varsigma)$ can be replaced by the fractional kernel $\varphi_\alpha(\tau-\varsigma)$ of any types of fractional integral operators including the classic fractional integral operator (Riemann-Liouville integral operator, ABC-fractional integral operator [23], etc.) providing that $\varphi_\alpha(\tau-\varsigma)\le\varphi(\tau-\varsigma)$, where α is the fractional power of the fractional integral operator.

Applications in optical studies

In this section, we introduce an application of the theory results in optics studies. The best-focused point of light that a perfect lens with a circular aperture may produce is described by the Airy floppy and 2D-Airy function in the field of optics and is constrained by light diffraction. In the fields of physics, optics, and astronomy, the Airy floppy is significant.
We consider the following problem:

Example 3.9. Consider the BVP

$$\begin{cases} {}^C\Delta^\nu\sigma(\tau)=\tau\sigma(\tau)+\dfrac{\tau}{1-\tau}\cos^2\sigma(\tau) & \\ +\int_0^\tau \exp(\tau-\varsigma)(\dfrac{1}{2}+\dfrac{1}{3}\varsigma)d\,\varsigma & \tau\in[0,\dfrac{1}{2})\bigcup(\dfrac{3}{4},1] \\ \sigma(\tau)=\dfrac{1}{4}\varrho_\nu(\tau,1) & \tau\in(\dfrac{1}{2},\dfrac{3}{4}],i=1,...,n \\ \sigma(0)=\sigma(1)=0, & \end{cases}\tag{3.8}$$

where

$$\int_0^\tau \exp(\tau-\varsigma)(\frac{1}{2}+\frac{1}{3}\varsigma)d\,\varsigma=\frac{1}{6}(e^\tau-1)(2\tau+3).$$

Since $\Lambda(\tau)\sigma(\tau)=\tau\sigma(\tau)$ then Λ is generated the resolve operator ϱ_ν, $\Xi=[0,1],\nu\in(n,1+n]$. Define the set $\Theta:=\{\sigma:\sigma(0)=\sigma(1)\}\subset\Xi$. Assume that the functions

$$\phi(\tau)=\frac{\tau}{1-\tau}\cos^2\sigma(\tau),\ \Phi_1(\tau)=\frac{1}{2},\quad \Phi_2(\tau)=\frac{1}{3}\tau,\ \varphi_{1,2}(\tau-\varsigma)=\exp(\tau-\varsigma)$$

satisfy the hypotheses (H1) and (H2). Then in view of Theorem 3.1 has at least one mild solution of the form that given in Definition 2.1.2. The exact solution of the above BVP is formulated for different values of ν in terms of the Airy functions, which are represented as periodic functions. For instant, when $\nu = 2$, we obtain different solutions in the formula of Airy functions $Ai(\tau)$ and $Bi(\tau)$ (see Figure 1) and Figure 2 for the special solution

$$\sigma(\tau) = \frac{aAi(\tau) * Bi(\tau)}{bAi(\tau) - cBi(\tau)}.$$

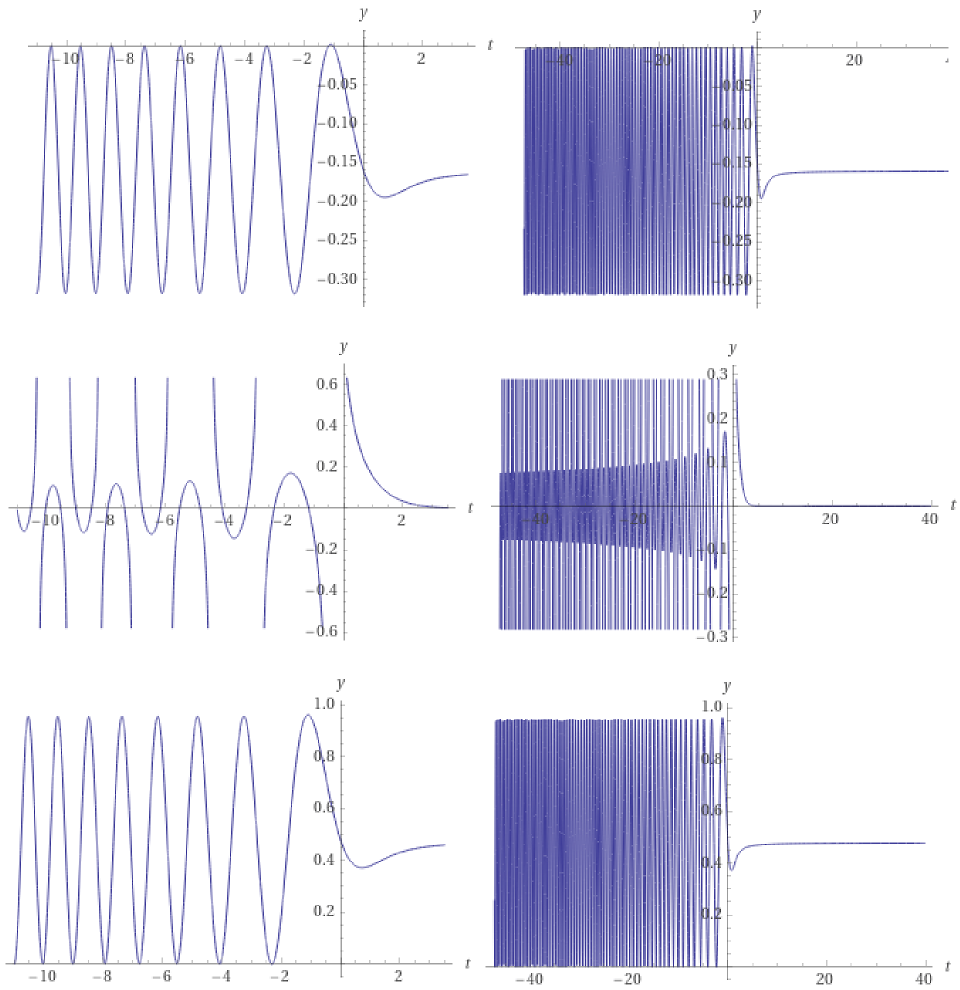

Figure 1. The exact solutions of the BVP from the top : $(Ai(\tau))' * Bi(\tau), -\dfrac{Ai(\tau) * Bi(\tau)}{Ai(\tau) - Bi(\tau)}, Ai(\tau) * (Bi(\tau))'$, where $'$ indicates the first derivative respectively.

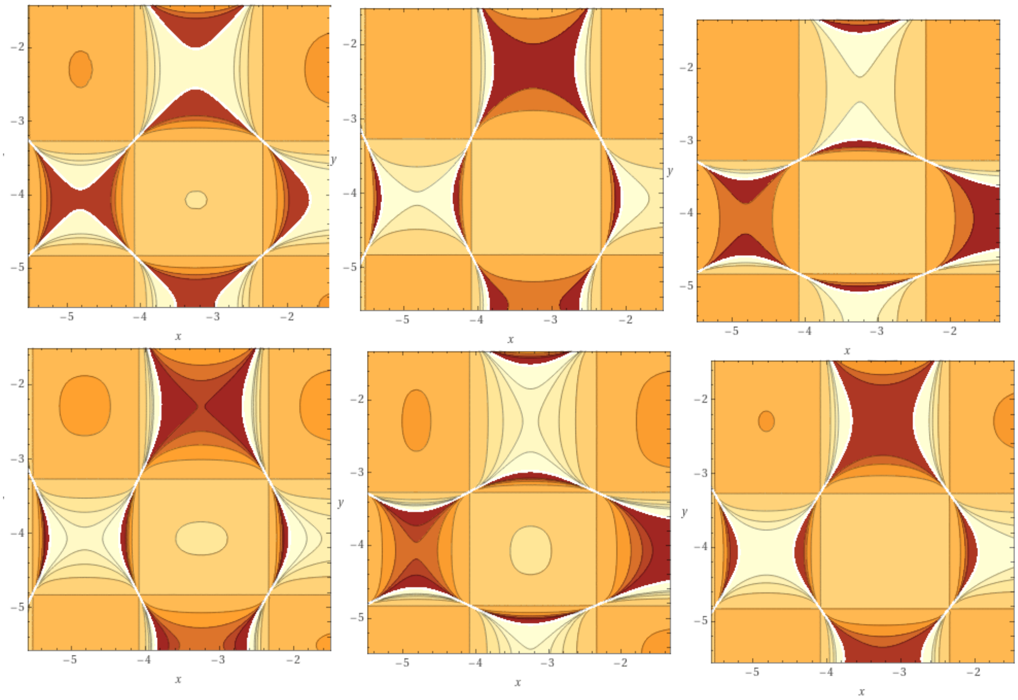

Figure 2. One of the exact solution BVP of the form $\dfrac{aAi(\tau) * Bi(\tau)}{bAi(\tau) - cBi(\tau)}$ for different values of the constants a, b and c. This formula captured the camera lens by an Airy disk. From the left $a = 1, b = 2, c = 1; a = 2, b = 1, c = -1/2; a = 2, b = 1/2, c = 1; a = 2, b = 1/2, c = 1/3; a = 1, b = 1/2, c = 1/3; a = 2, b = 1/2, c = 1/3$ respectively.

In the above example, ϕ is suggested to be a convex function in the unit interval. In the next example, we generalize it to starlike function.

Example 3.10. Consider the BVP

$$\begin{cases} {}^{C}\Delta^{\nu}\sigma(\tau) = \tau\sigma(\tau) + \dfrac{\tau}{(1-\tau)^2}\cos^2\sigma(\tau) & \\ + \int_0^\tau \exp(\tau - \varsigma)(1+\varsigma)d\varsigma & \tau \in [0, \frac{1}{2}) \bigcup (\frac{3}{4}, 1] \\ \sigma(\tau) = \dfrac{1}{4}\varrho_\nu(\tau, 1) & \tau \in (\frac{1}{2}, \frac{3}{4}], i = 1, ..., n \\ \sigma(0) = \sigma(1) = 0, & \end{cases} \tag{3.9}$$

where

$$\begin{aligned} \int_0^\tau \exp(\tau - \varsigma)(1+\varsigma)d\varsigma &= (e^\tau - 1)(\tau + 1) \\ &= \tau + \frac{(3\tau^2)}{2} + \frac{(2\tau^3)}{3} + \frac{(5\tau^4)}{24} + \frac{\tau^5}{20} + O(\tau^6). \end{aligned}$$

Since $\Lambda(\tau)\sigma(\tau) = \tau\sigma(\tau)$ then Λ is generated the resolve operator ϱ_ν, $\Xi = [0, 1], \nu \in (n, 1 + n]$. Define the set $\Theta := \{\sigma : \sigma(0) = \sigma(1)\} \subset \Xi$. Assume that the functions

$$\phi(\tau) = \frac{\tau}{(1-\tau)^2} \cos^2 \sigma(\tau),\ \ \Phi_1(\tau) = 1, \quad \Phi_2(\tau) = \tau,\ \ \varphi_{1,2}(\tau - \varsigma) = \exp(\tau - \varsigma)$$

achieve the assumptions (H1) and (H2). There is then at least one mild solution of the form given in Definition 2.1.2 in light of Theorem 3.1.

4. Conclusions

The benefit of using I-DEs is that they allow for the investigation of the complete diffusion process, including the start, intermediate, and long time scales of the process. As a result, this approach can tell the difference between an evolution detail for a system that exhibits the same behavior over a long period of time but distinct behaviors at the beginning and middle of the evolution. In order to describe the sub-diffusive and super-diffusive regimes, an I-DE for diffusion is also introduced. Additionally, techniques for resolving I-DEs are established, and for the instances of force-free and linear force, differential equations have analytically solutions.

We presented the necessary criteria for the existence of a mild periodic solution of fractional BVP in the previous inquiry, where the fractional resolve operator is non-compact. We proposed using a multi-evolution equation. There are some examples of both unique and generic instances. To demonstrate how the abstract theory mechanism works, a simple example is given. Our technique was based on the fixed point theory of measure of non-compactness. For the future work, one can use the compact case and formulate the sufficient conditions to get a mild periodic outcome.

Conflict of interest

The authors declare no conflict of interest.

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