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Sawi transform and Hyers-Ulam stability of n^{th} order linear differential equations

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Abstract

The use of the Sawi transform has increased in the light of recent events in different approaches. The Sawi transform is also seen as the easiest and most effective way among the other transforms. In line with this, the research deals with the Hyers-Ulam stability of n^{th} order differential equations using the Sawi transform. The study aims at deriving a generalised Hyers-Ulam stability result for linear homogeneous and non-homogeneous differential equations.

Keywords: Hyers-Ulam stability (HUS), Hyers-Ulam σ-stability (σHUS), differential equation (DE), Sawi transform (ST).

2020 MSC: 26D10, 34A40, 34K20, 39A30, 39B82, 44A10.

1. Introduction

Integral transforms are the best option for analysts to find the solution to the fundamental issues of the hypothesis of versatility, fluid mechanics, mechanics, stargazing, radar, and sign handling because these changes give us the specific solution to the issues and all around reported in writing. Some important applications of differential equations in engineering and basic science can be found in [5–14, 16–20, 23].

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The mathematicians [1, 2] focused primarily on the Aboodh, Mahgoub, and Sumudu transforms, as well as the natives of constant coefficients of linear DEs. Laplace transforms have been applied to solve different problems in [11, 21, 22, 24]. Singh [29] tackled the popular issue of development by fostering its numerical model as far as first order linear DE using ST. Gupta [3] introduced connection among Sawi and other essential transforms. The applications of differential equations in various fields, for example, the environment, financial matters, science and engineering can be displayed as [25–32]. Recently Viglialoro [10, 15, 30] investigated the properties of solutions to porous medium problems with different sources and boundary conditions, boundedness in a chemotaxis system with consumed chemoattractant and bounded solutions to a parabolic-elliptic chemotaxis system with nonlinear diffusion and signal-dependent sensitivity.

Aggarwal [4] used ST to solve volterra integro-DE convolutions of the first kind. Higazy [13] used Sawi decomposition method for the Volterra integral equation, along with an application. Khan [14] investigated Hyers-Ulam stability and existence criteria for coupled fractional differential equations involving the p-Laplacian operator. For this, the creators present the meaning of Sawi transform, a few helpful qualities of ST, ST of derivatives, and inverse ST in sections two to four. In segments five to six, we examined the ST for the arrangement of standard DEs, using HUS, σHUS, Mittag-Leffler HUS, and Mittag-Leffler σHUS stability of n^{th} order with constant co-efficient by using the ST method.

2. Properties of Sawi transformation

In this section, we discuss some useful characteristics of the Sawi transform of the function $\Lambda(\rho), \rho \geqslant 0$, which is defined by [23]

$$S\{\Lambda(\rho)\} = \frac{1}{\sigma^2}\int_0^\infty \Lambda(\rho)e^{-\left(\frac{1}{\sigma}\right)\rho}d\rho = T(\sigma), \sigma > 0.$$

Some properties of ST are as follows:

1. (Linearity) If $S\{\Lambda_1(\rho)\} = T_1(\sigma)$ and $S\{\Lambda_2(\rho)\} = T_2(\sigma)$, then $S\{l\Lambda_1(\rho) + m\Lambda_2(\rho)\} = [lT_1(\sigma) + mT_2(\sigma)]$.
2. (Scaling) If $T(\sigma)$ is the ST of $\Lambda(\rho)$, then $kT(k\sigma)$ is the ST of $\Lambda(k\rho)$.
3. (Translation) If $T(\sigma)$ is the ST of $\Lambda(\rho)$, then $\frac{1}{(1-k\sigma)^2}T\left(\frac{\sigma}{1-k\sigma}\right)$ is the ST of $e^{k\rho}\Lambda(\rho)$.

2.1. *Relation between Sawi transform and laplace Transform*

If $S\{\Lambda(\rho)\} = T(\sigma)$ and $L\{\Lambda(\rho)\} = M(\sigma) = \int_0^\infty \Lambda(\rho)e^{-\sigma\rho}d\rho$ is the Laplace transform of $\Lambda(\rho)$, then $M(\sigma) = \frac{1}{\sigma^2}T\left(\frac{1}{\sigma}\right)$.

Proof. Letting $L\{\Lambda(\rho)\} = M(\sigma)$, then

$$M(\sigma) = \int_0^\infty \Lambda(\rho)e^{-\sigma t}d\rho = \frac{1}{\sigma^2}\left(\sigma^2\int_0^\infty \Lambda(\rho)e^{-\sigma t}d\rho\right) = \frac{1}{\sigma^2}T\left(\frac{1}{\sigma}\right).$$

□

3. Sawi Transform of derivatives

In this section, we discuss the ST of derivatives of the functions as follows. If $T(\sigma)$ is the ST of $\Lambda(\rho)$, then

(a) $\frac{1}{\sigma}T(\sigma) - \frac{1}{\sigma^2}\Lambda(0)$ is the ST of $\Lambda'(\rho)$.
(b) $\frac{1}{\sigma}T(\sigma) - \frac{1}{\sigma^2}\Lambda(0) - \frac{1}{\sigma^2}\Lambda'(0)$ is the ST of $\Lambda''(\rho)$.
(c) $\frac{1}{\sigma^n}T(\sigma) - \sum_{k=0}^{(n-1)}\left(\frac{1}{\sigma}\right)^{n-(k-1)}\Lambda^{(k)}(0)$ is the ST of $\Lambda^{(n)}(\rho)$.

Proof.

(a) From the definition of ST, we have

$$S\{\Lambda(\rho)\} = \frac{1}{\sigma^2}\int_0^\infty \Lambda(\rho)e^{-\left(\frac{1}{\sigma}\right)\rho}d\rho, \qquad S\{\Lambda'(\rho)\} = \frac{1}{\sigma^2}\int_0^\infty \Lambda'(\rho)e^{-\left(\frac{1}{\sigma}\right)\rho}d\rho.$$

Using the integration by parts rule, then we have

$$= \frac{1}{\sigma^2}\left[\Lambda(\rho)e^{-\left(\frac{1}{\sigma}\right)\rho}\right]_0^\infty - \frac{1}{\sigma^2}\int_0^\infty -\left(\frac{1}{\sigma}\right)\Lambda(\rho)e^{-\left(\frac{1}{\sigma}\right)\rho}d\rho, \qquad S\{\Lambda'(\rho)\} = \frac{1}{\sigma}T(\sigma) - \frac{1}{\sigma^2}\Lambda(0).$$

(b) From (a), we know that $S\{\Lambda'(\rho)\} = \frac{1}{\sigma}T(\sigma) - \frac{1}{\sigma^2}\Lambda(0) = \frac{1}{\sigma}S\{\Lambda(\rho)\} - \frac{1}{\sigma^2}\Lambda(0)$, which implies

$$S\{\Lambda''(\rho)\} = \frac{1}{\sigma}S\{\Lambda'(\rho)\} - \frac{1}{\sigma^2}\Lambda'(0) = \frac{1}{\sigma}\left[\frac{1}{\sigma}S\{\Lambda(\rho)\} - \frac{1}{\sigma^2}\Lambda(0)\right] - \frac{1}{\sigma^2}\Lambda'(0),$$
$$S\{\Lambda''(\rho)\} = \frac{1}{\sigma^2}T(\Lambda) - \frac{1}{\sigma^3}\Lambda(0) - \frac{1}{\sigma^2}\Lambda'(0).$$

(c) From (b), we know that $S\{\Lambda''(\rho)\} = \frac{1}{\sigma^2}T(\Lambda) - \frac{1}{\sigma^3}\Lambda(0) - \frac{1}{\sigma^2}\Lambda'(0)$, which implies

$$S\{\Lambda''(\rho)\} = \frac{1}{\sigma^2}S\{\Lambda(\rho)\} - \frac{1}{\sigma^3}\Lambda(0) - \frac{1}{\sigma^2}\Lambda'(0),$$
$$S\{\Lambda'''(\rho)\} = \frac{1}{\sigma^2}S\{\Lambda'(\rho)\} - \frac{1}{\sigma^3}\Lambda'(0) - \frac{1}{\sigma^2}\Lambda''(0),$$
$$S\{\Lambda'''(\rho)\} = \frac{1}{\sigma^3}S\{\Lambda(\rho)\} - \frac{1}{\sigma^4}\Lambda(0) - \frac{1}{\sigma^3}\Lambda'(0) - \frac{1}{\sigma^2}\Lambda''(0),$$
$$S\{\Lambda'''(\rho)\} = \frac{1}{\sigma^3}T(\sigma) - \frac{1}{\sigma^4}\Lambda(0) - \frac{1}{\sigma^3}\Lambda'(0) - \frac{1}{\sigma^2}\Lambda''(0).$$

Generalization gives,

$$S\{\Lambda^{(n)}(\rho)\} = \frac{1}{\sigma^{(n)}}S\{\Lambda(\rho)\} - \frac{1}{\sigma^{n+1}}\Lambda(0) - \frac{1}{\sigma^{(n)}}\Lambda'(0) - \cdots - \frac{1}{\sigma^{(n-1)}}\Lambda^{(n-1)}(0),$$
$$S\{\Lambda^{(n)}(\rho)\} = \frac{1}{\sigma^{(n)}}T(\sigma) - \frac{1}{\sigma^{n+1}}\Lambda(0) - \frac{1}{\sigma^{(n)}}\Lambda'(0) - \cdots - \frac{1}{\sigma^{(n-1)}}\Lambda^{(n-1)}(0),$$
$$S\{\Lambda^{(n)}(\rho)\} = \frac{1}{\sigma^{(n)}}T(\sigma) - \sum_{k=0}^{(n-1)}\left(\frac{1}{\Lambda}\right)^{n-(k-1)}\Lambda^{k}(0).$$

□

3.1. *Inverse Sawi transform*

The function $\Lambda(\rho)$ is called the inverse ST of $T(\sigma)$, if $S^{-1}\{T(\sigma)\} = \Lambda(\rho)$.

3.2. *Linearity property*

If $S^{-1}\{T_1(\sigma)\} = \Lambda_1(\rho)$ and $S^{-1}\{T_2(\sigma)\} = \Lambda_2(\rho)$, then

$$S^{-1}\{lT_1(\sigma) + mT_2(\sigma)\} = lS^{-1}\{T_1(\sigma)\} + mS^{-1}\{T_2(\sigma)\}, \qquad S^{-1}\{lT_1(\sigma) + mT_2(\sigma)\} = l\Lambda_1(\rho) + m\Lambda_2(\rho),$$

where l, m are arbitrary constants.

3.3. Sawi transform of some functions

In this section, we find the ST of simple functions. We consider a function $\Lambda(\rho),\ \rho \geqslant 0$ of exponential order is piecewise continuous.
(i) Let $\Lambda(\rho) = 1$. Then

$$S\{\Lambda(\rho)\} = \frac{1}{\sigma^2}\int_0^\infty \Lambda(\rho)e^{-(\frac{1}{\sigma})\rho}d\rho, \qquad S\{1\} = \frac{1}{\sigma^2}\int_0^\infty 1e^{-(\frac{1}{\sigma})\rho}d\rho = \frac{1}{\sigma}.$$

(ii) Let $\Lambda(\rho) = \rho$. Then

$$S\{\Lambda(\rho)\} = \frac{1}{\sigma^2}\int_0^\infty \Lambda(\rho)e^{-(\frac{1}{\sigma})\rho}d\rho, \qquad S\{\rho\} = \frac{1}{\sigma^2}\int_0^\infty \rho e^{-(\frac{1}{\sigma})\rho}d\rho, \qquad S\{\rho\} = 1.$$

In the general case if $n = 0, 1, 2, \ldots$ is integer number, then $S\{\rho^{(n)}\} = \rho^{(n-1)}n!$.
(iii) Let $\Lambda(\rho) = e^{l\rho}$. Then

$$S\{\Lambda(\rho)\} = \frac{1}{\sigma^2}\int_0^\infty \Lambda(\rho)e^{-(\frac{1}{\sigma})\rho}d\rho, \qquad S\{e^{l\rho}\} = \frac{1}{\sigma^2}\int_0^\infty e^{l\rho}e^{-(\frac{1}{\sigma})\rho}d\rho.$$

Using the integration by parts rule, then we have

$$S\{e^{l\rho}\} = \frac{1}{\sigma^2}\left[\left(e^{l\rho}(-\sigma)e^{-(\frac{1}{\sigma}t}\right)_0^\infty - \int_0^\infty(-\sigma)e^{-(\frac{1}{\sigma}t}le^{l\rho}d\rho\right] = \frac{1}{\sigma^2}\left[\sigma + \sigma l\int_0^\infty e^{(l-\frac{1}{\sigma})\rho}d\rho\right] = \frac{1}{\sigma^2}\left[\sigma + \frac{l\sigma^2(-1)}{l\sigma - 1}\right],$$
$$S\{e^{l\rho}\} = \frac{1}{\sigma(1-l\sigma)}.$$

Based on the above results, we prove the HUS of n^{th} order homogeneous linear DE

$$\frac{d^{(n)}\Lambda}{d\rho^{(n)}} + a_1(\rho)\frac{d^{(n-1)}\Lambda}{d\rho^{(n-1)}} + \cdots + a_{n-1}(\rho)\frac{d\Lambda}{d\rho} + a_n(\rho)\Lambda(\rho) = 0 \tag{3.1}$$

and non-homogeneous linear DE

$$\frac{d^{(n)}\Lambda}{d\rho^{(n)}} + a_1(\rho)\frac{d^{(n-1)}\Lambda}{d\rho^{(n-1)}} + \cdots + a_{n-1}(\rho)\frac{d\Lambda}{d\rho} + a_n(\rho)\Lambda(\rho) = m(\rho) \tag{3.2}$$

by using the ST method, where $a_1(\rho), a_2(\rho), \ldots, a_{n-1}(\rho), a_n(\rho)$ is a scalar and $\Lambda(\rho)$ is a continuously differentiable function of exponential order.

4. Preliminaries

In this section, we present a few standard documents and definitions that will be valuable to demonstrate our principal results.

Definition 4.1. The conversion is to change the form of a value on an expression without a change in the value. The conversion of two functions $\Lambda(\rho)$ and $\chi(\rho)$ are defined by

$$\Lambda(\rho) \star \chi(\rho) = (\Lambda \star \chi)\rho = \int_0^\rho \Lambda(s)\chi(\rho - s)ds = \int_0^\rho \Lambda(\rho - s)\chi(s)ds.$$

Theorem 4.2. *The convolution of two functions, $\Lambda(\rho)$ and $\chi(\rho)$ is defined by*

$$(\Lambda \star \chi)\rho = \int_0^\rho \Lambda(s)\chi(\rho - s)ds = \int_0^\rho \Lambda(\rho - s)\chi(s)ds.$$

The convolution theorem for the ST of two functions Λ and χ is given by,

$$s(\Lambda \star \chi)\rho = \int_0^\infty (\Lambda \star \chi)\rho e^{-\frac{\rho}{\sigma}} d\rho = \int_0^\infty \int_0^\infty \Lambda(s)\chi(\rho - s)e^{-\frac{\rho}{\sigma}} d\rho = \int_0^\infty \Lambda(s)ds \int_0^\infty \chi(\rho - s)e^{-\frac{\rho}{\sigma}} d\rho.$$

Substituting $\rho - s = y$ in above equation, we have

$$s(\Lambda \star \chi)\rho = \int_0^\infty \Lambda(s)ds \int_0^\infty \chi(y)e^{-\frac{(y+s)}{\sigma}} dy,$$
$$s(\Lambda \star \chi)\rho = \frac{\sigma^2}{\sigma^2}\left[\int_0^\infty \Lambda(s)e^{-\frac{s}{\sigma}} ds \int_0^\infty \chi(y)e^{-\frac{y}{\sigma}} dy\right],$$
$$s(\Lambda \star \chi)\rho = \sigma^2 T(\sigma)\Phi(\sigma).$$

Definition 4.3. The Mittag-Leffler function of one parameter is defined by,

$$E_\beta(\rho) = \sum_{k=0}^{\infty} \frac{\rho^k}{\Gamma(\beta k + 1)}, \quad \rho, \beta \in C \text{ and } R(\beta) > 0.$$

If we put $\beta = 1$, then

$$E_1(\rho) = \sum_{k=0}^{\infty} \frac{\rho^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{\rho^k}{k!}, \qquad E_1(\rho) = e^\rho.$$

In this section, we consider

$$T = \{\Lambda : [0, \infty) \to K/\Lambda \text{ is a continuously differentiable function of exponential order}\},$$

where K is a subset of either real or complex field and $k > 0$ be a stability constant.

Definition 4.4.

(i) The DE (3.1) is said to have the HUS if there exists $k > 0$ such that, if a function $\Lambda \in T$ satisfying

$$|\Lambda^{(n)}(\rho) + a_1(\rho)\Lambda^{(n-1)}(\rho) + \cdots + a_{n-1}(\rho)\Lambda'(\rho) + a_n(\rho)\Lambda(\rho)| \leqslant \in, \quad \in > 0, \text{ for all } \rho \geqslant 0,$$

then there exists a solution $\chi : [0, \infty) \to K$ of the DE (3.1) such that $\chi \in T$ and $|\Lambda(\rho) - \chi(\rho)| \leqslant k$.

(ii) The DE (3.2) has the HUS if there exists $k > 0$ such that for each $\in > 0$, if a function $\Lambda \in T$ satisfying

$$|\Lambda^{(n)}(\rho) + a_1(\rho)\Lambda^{(n-1)}(\rho) + \cdots + a_{n-1}(\rho)\Lambda'(\rho) + a_n(\rho)\Lambda(\rho) - m(\rho)| \leqslant \in, \tag{4.1}$$

then there exists a solution $\chi : [0, \infty) \to K$ of the DE (3.2) such that $\chi \in T$ and

$$|\Lambda(\rho) - \chi(\rho)| \leqslant k, \quad \text{for all } \rho \geqslant 0,$$

where the constant k is called as stability constant.

Definition 4.5. Let $\sigma : [0, \infty) \to (0, \infty)$ be a function.

(i) The homogeneous DE (3.1) has the σHUS (for the class T) if there exists $k > 0$ such that for every $\epsilon > 0$ and the function $\Lambda \in T$ satisfying

$$|\Lambda^{(n)}(\rho) + a_1(\rho)\Lambda^{(n-1)}(\rho) + \cdots + a_{n-1}(\rho)\Lambda'(\rho) + a_n(\rho)\Lambda(\rho)| \leqslant \sigma(\rho)\epsilon, \text{ for all } \rho \geqslant 0, \tag{4.2}$$

there exists a solution $\chi : [0, \infty) \to K$ of the DE (3.1) such that $\chi \in T$ and

$$|\Lambda(\rho) - \chi(\rho)| \leqslant k\sigma(\rho)\epsilon, \quad \text{for all } \rho \geqslant 0.$$

(ii) The non-homogeneous DE (3.2) has the HUS if there exists $k > 0$ such that for each $\epsilon > 0$, if a function $\chi \in T$ satisfying

$$|\Lambda^{(n)}(\rho) + a_1(\rho)\Lambda^{(n-1)}(\rho) + \cdots + a_{n-1}(\rho)\Lambda'(\rho) + a_n(\rho)\Lambda(\rho) - m(\rho)| \leqslant \sigma(\rho)\epsilon, \quad \text{for all } \rho \geqslant 0,$$

where the constant k is called as stability constant.

Definition 4.6. Let $E_\beta(\rho)$ be the Mittag-Leffler function.

(i) The homogeneous DE 3.1 has the Mittag-Leffler HUS if there exists a constant $k > 0$ such that for every $\epsilon > 0$, if a function $\Lambda \in T$ satisfying

$$|\Lambda^{(n)}(\rho) + a_1(\rho)\Lambda^{(n-1)}(\rho) + \cdots + a_{n-1}(\rho)\Lambda'(\rho) + a_n(\rho)\Lambda(\rho)| \leqslant E_\beta(\rho)\epsilon, \quad \text{for all } \rho \geqslant 0,$$

then there exists a solution $\chi : [0, \infty) \to K$ of the DE (3.1) such that $\chi \in T$ and

$$|\Lambda(\rho) - \chi(\rho)| \leqslant E_\beta(\rho)\epsilon, \quad \text{for all } \rho \geqslant 0.$$

(ii) The non-homogeneous DE (3.2) has the Mittag-Leffler HUS if there exists a constant $k > 0$ such that for each $\epsilon > 0$ if a function $\Lambda \in T$ satisfying

$$|\Lambda^{(n)}(\rho) + a_1(\rho)\Lambda^{(n-1)}(\rho) + \cdots + a_{n-1}(\rho)\Lambda'(\rho) + a_n(\rho)\Lambda(\rho) - m(\rho)| \leqslant E_\beta(\rho)\epsilon,$$

$\forall\ \rho \geqslant 0$, then there exists a solution $\chi : [0, \infty) \to K$ of the DE 3.2 such that $\chi \in T$ and

$$|\Lambda(\rho) - \chi(\rho)| \leqslant kE_\beta(\rho)\epsilon,$$

$\forall\ \rho \geqslant 0$, where the constant k is called as Mittag-Leffler HUS constant.

Definition 4.7. Let $E_\beta(\rho)$ be the Mittag-Leffler function.

(i) The homogeneous DE (3.1) has the Mittag-Leffler σHUS, if there exists $k > 0$ such that for every $\epsilon > 0$, if a function $\Lambda \in T$ satisfying

$$|\Lambda^{(n)}(\rho) + a_1(\rho)\Lambda^{(n-1)}(\rho) + \cdots + a_{n-1}(\rho)\Lambda'(\rho) + a_n(\rho)\Lambda(\rho)| \leqslant \sigma(\rho)E_\beta(\rho)\epsilon, \quad \text{for all } \rho \geqslant 0,$$

then there exists a solution $\chi : [0, \infty) \to K$ of the DE (3.1) such that $\chi \in T$ and

$$|\Lambda(\rho) - \chi(\rho)| \leqslant k\sigma(\rho)E_\beta(\rho)\epsilon, \ \ \forall\ \ \rho \geqslant 0.$$

(ii) The non-homogeneous DE (3.2) has the Mittag-Leffler σHUS if $\exists\ k > 0$ such that tor each $\epsilon > 0$ if a function $\Lambda \in T$ satisfying

$$|\Lambda^{(n)}(\rho) + a_1(\rho)\Lambda^{(n-1)}(\rho) + \cdots + a_{n-1}(\rho)\Lambda'(\rho) + a_n(\rho)\Lambda(\rho) - m(\rho)| \leqslant \sigma(\rho)E_\beta(\rho)\epsilon,$$

for all $\rho \geqslant 0$, then there exists a solution $\chi : [0, \infty) \to K$ of the DE (3.2) such that $\chi \in T$ and

$$|\Lambda(\rho) - \chi(\rho)| \leqslant k\sigma(\rho)E_\beta(\rho)\epsilon \quad \text{for all } \rho \geqslant 0,$$

where the constant k is called as Mittag-Leffler σ HUS constant.

5. Hyers-Ulam stability of (3.1)

In this section, we prove several types of HUS of n^{th} order DE (3.1) by using ST method. Let $\chi : [0,\infty) \to K$ be the solution of the DE (3.1). For any $a \in K$, we denote the real part of a by $R(a)$.

Theorem 5.1. *Assume that $a_1 + a_2 + \cdots + a_{n-1} + a_n$ is a constant with $R(a_1 + a_2 + \cdots + a_{n-1} + a_n) > 0$. Then the DE (3.1) is Hyers-Ulam stable in the class T.*

Proof. Assume that $\Lambda \in T$ satisfying (4.1) and $i : [0,\infty) \to K$ by

$$i(\rho) = \Lambda^{(n)}(\rho) + a_1(\rho)\Lambda^{(n-1)}(\rho) + \cdots + a_{n-1}(\rho)\Lambda'(\rho) + a_n(\rho)\Lambda(\rho), \quad \text{for all } \rho \geqslant 0. \tag{5.1}$$

Now, the equation (5.1) becomes

$$P(\sigma) = s\{i(\rho)\} = s\{\Lambda^{(n)}(\rho) + a_1(\rho)\Lambda^{(n-1)}(\rho) + \cdots + a_{n-1}(\rho)\Lambda'(\rho) + a_n(\rho)\Lambda(\rho)\},$$
$$P(\sigma) = s\{\Lambda^{(n)}(\rho)\} + a_1(\rho)s\{\Lambda^{(n-1)}(\rho)\} + \cdots + a_{n-1}(\rho)s\{\Lambda'(\rho)\} + a_n(\rho)s\{\Lambda(\rho)\},$$

where $s\{\Lambda(\rho)\} = T(\sigma)$. Since

$$\begin{aligned}
S\{\Lambda'(\rho)\} &= \frac{1}{\sigma}T(\sigma) - \frac{1}{\sigma^2}\Lambda(0),\\
S\{\Lambda''(\rho)\} &= \frac{1}{\sigma^2}T(\sigma) - \frac{1}{\sigma^2}\Lambda(0) - \frac{1}{\sigma^2}\Lambda'(0),\\
&\vdots\\
S\{\Lambda^{(n-1)}(\rho)\} &= \frac{1}{\sigma^{(n-1)}}T(\sigma) - \sum_{k=0}^{n-2}\left(\frac{1}{\sigma}\right)^{(n-1)-(k-1)}\Lambda^k(0),\\
S\{\Lambda^n(\rho)\} &= \frac{1}{\sigma^{(n-1)}}T(\sigma) - \sum_{k=0}^{(n-1)}\left(\frac{1}{\sigma}\right)^{n-(k-1)}\Lambda^k(0).
\end{aligned}$$

Clearly,

$$\begin{aligned}
P(\sigma) = &\frac{1}{\sigma^{(n)}}T(\sigma) - \sum_{k=0}\left(\frac{1}{\sigma}\right)^{n-(k-1)}\Lambda^{(k)}(0)\\
&+ a_1(\rho)\left[\frac{1}{\sigma^{(n-1)}}T(\sigma) - \sum_{k=0}^{n-2}\left(\frac{1}{\sigma}\right)^{(n-1)-(k-1)}\Lambda^k(0)\right]\\
&\vdots\\
&+ a_{n-1}(\rho)\left[\frac{1}{\sigma}T(\sigma) - \frac{1}{\sigma^2}\Lambda(0)\right] + a_n(\rho)T(\sigma).
\end{aligned}$$

Set

$$T(\sigma) = s\{\Lambda(\rho)\}$$
$$= \frac{P(\sigma) + \sum_{k=0}^{(n-1)}\left(\frac{1}{\sigma}\right)^{n-(k-1)}\Lambda^{(k)}(0) + a_1(\rho)\sum_{k=0}^{n-2}\left(\frac{1}{\sigma}\right)^{(n-1)-(k-1)}\Lambda^k(0) + \cdots + a_{n-1}(\rho)\frac{1}{\sigma^2}\Lambda(0)}{\left(\frac{1}{\sigma^{(n)}} + a_1(\rho)\frac{1}{\sigma^{(n-1)}} + \cdots + a_{n-1}(\rho)\frac{1}{\sigma} + a_n(\rho)\right)}. \tag{5.2}$$

If we put $\rho = 0$ in $\chi(\rho) = e^{-(a_1+\cdots+a_{n-1}+a_n)\rho}\Lambda(\rho)$, then $\chi(0) = \Lambda(0)$ and $\chi \in T$, then the ST of $\chi(\rho)$ gives the following result

$$\Phi(\sigma) = S\{\chi(\rho)\}$$
$$= \frac{\sum_{k=0}^{(n-1)} \left(\frac{1}{\sigma}\right)^{n-(k-1)} \chi^k(0) + a_1(\rho) \sum_{k=0}^{n-2} \left(\frac{1}{\sigma}\right)^{(n-1)-(k-1)} \chi^k(0) + \cdots + a_{n-1}(\rho)\frac{1}{\sigma^2}\chi(0)}{\left(\frac{1}{\sigma^{(n)}} + a_1(\rho)\frac{1}{\sigma^{(n-1)}} + \cdots + a_{n-1}(\rho)\frac{1}{\sigma} + a_n(\rho)\right)}. \tag{5.3}$$

Therefore,

$$\begin{aligned} &S\left[\chi^{(n)}(\rho) + a_1(\rho)\chi^{(n-1)}(\rho) + \cdots + a_{n-1}(\rho)\chi'(\rho) + a_n(\rho)\chi(\rho)\right] \\ &= \frac{1}{\sigma^{(n)}}\Phi(\sigma) - \sum_{k=0}^{(n-1)} \left(\frac{1}{\sigma}\right)^{n-(k-1)} \chi^k(0) + a_1(\rho)\left[\frac{1}{\sigma^{(n-1)}}\Phi(\sigma) - \sum_{k=0}^{n-2} \left(\frac{1}{\sigma}\right)^{(n-1)-(k-1)} \chi^k(0)\right] \\ &\quad + \cdots + a_{n-1}(\rho)\left[\frac{1}{\sigma}\Phi(\sigma) - \frac{1}{\sigma^2}\chi(0)\right] + a_n(\rho)\Phi(\sigma) \\ &= \frac{1}{\sigma^{(n)}}\Phi(\sigma) - \sum_{k=0}^{(n-1)} \left(\frac{1}{\sigma}\right)^{n-(k-1)} \chi^k(0) - a_1(\rho) \sum_{k=0}^{n-2} \left(\frac{1}{\sigma}\right)^{(n-1)-(k-1)} \chi^k(0) \\ &\quad + \cdots - a_{n-1}(\rho)\frac{1}{\sigma}\Phi(\sigma) - a_{n-1}(\rho)\frac{1}{\sigma^2}\chi(0) + a_n(\rho)\Phi(\sigma) = 0. \end{aligned} \tag{5.4}$$

From (5.4), we have

$$\begin{aligned} &S\left[\chi^{(n)}(\rho) + a_1(\rho)\chi^{(n-1)}(\rho) + \cdots + a_{n-1}(\rho)\chi'(\rho) + a_n(\rho)\chi(\rho)\right] \\ &= \frac{1}{\sigma^{(n)}}\Phi(\sigma) + a_1(\rho)\frac{1}{\sigma^{(n-1)}}\Phi(\sigma) + \cdots + a_{n-1}(\rho)\frac{1}{\sigma^n}\Phi(\sigma) = \sum_{k=0}^{(n-1)} \left(\frac{1}{\sigma}\right)^{n-(k-1)} \chi^k(0) \\ &\quad + a_1(\rho) \sum_{k=0}^{n-2} \left(\frac{1}{\sigma}\right)^{(n-1)-(k-1)} \chi^k(0) + \cdots + a_{n-1}(\rho)\frac{1}{\sigma^2}\chi(0) \\ &= \Phi(\sigma)\left[\frac{1}{\sigma^{(n)}} + a_1(\rho)\frac{1}{\sigma^{(n-1)}} + \cdots + a_{n-1}(\rho)\frac{1}{\sigma} + a_n(\rho)\right] = \sum_{k=0}^{(n-1)} \left(\frac{1}{\sigma}\right)^{n-(k-1)} \chi^k(0) \\ &\quad + a_1(\rho) \sum_{k=0}^{n-2} \left(\frac{1}{\sigma}\right)^{(n-1)-(k-1)} \chi^k(0) + \cdots + a_{n-1}(\rho)\frac{1}{\sigma^2}\chi(0), \end{aligned}$$

$$\Phi(\sigma) = \frac{\sum_{k=0}^{(n-1)} \left(\frac{1}{\sigma}\right)^{n-(k-1)} \chi^k(0) + a_1(\rho) \sum_{k=0}^{n-2} \left(\frac{1}{\sigma}\right)^{(n-1)-(k-1)} \chi^k(0) + \cdots + a_{n-1}(\rho)\frac{1}{\sigma^2}\chi(0)}{\left(\frac{1}{\sigma^{(n)}} + a_1(\rho)\frac{1}{\sigma^{(n-1)}} + \cdots + a_{n-1}(\rho)\frac{1}{\sigma} + a_n(\rho)\right)}.$$

From (5.3), we have

$$S\{\chi^{(n)}(\rho) + a_1(\rho)\chi^{(n-1)}(\rho) + \cdots + a_{n-1}(\rho)\chi'(\rho) + a_n(\rho)\chi(\rho)\} = 0.$$

Since S is a one to one operator. Then we get

$$\chi^{(n)}(\rho) + a_1(\rho)\chi^{(n-1)}(\rho) + \cdots + a_{n-1}(\rho)\chi'(\rho) + a_n(\rho)\chi(\rho) = 0.$$

Here, $\chi(\rho)$ is a solution of the DE (3.1). Using (5.2) and (5.3), we obtain

$$\begin{aligned} S\{\Lambda(\rho)\}-S\{\chi(\rho)\} &= T(\sigma)-\Phi(\sigma) \\ &= \frac{P(\sigma)}{\frac{1}{\sigma^{(n)}}+a_1(\rho)\frac{1}{\sigma^{(n-1)}}+\cdots+a_{n-1}(\rho)\frac{1}{\sigma}+a_n(\rho)} \\ &= \frac{\sigma^2}{\sigma^2}\frac{P(\sigma)}{\left(\frac{1}{\sigma^{(n)}}+a_1(\rho)\frac{1}{\sigma^{(n-1)}}+\cdots+a_{n-1}(\rho)\frac{1}{\sigma}+a_n(\rho)\right)} \\ &= \sigma^2 P(\sigma)\frac{1}{\sigma^2\left(\frac{1}{\sigma^{(n)}}+a_1(\rho)\frac{1}{\sigma^{(n-1)}}+\cdots+a_{n-1}(\rho)\frac{1}{\sigma}+a_n(\rho)\right)}, \\ S\{\Lambda(\rho)\}-S\{\chi(\rho)\} &= \sigma^2 P(\sigma)Q(\sigma)=\sigma^2[i(\rho)\star j(\rho)]=S\{\Lambda\star\chi\}(\rho), \end{aligned}$$

where

$$Q(\sigma)=\frac{1}{\sigma^2\left(\frac{1}{\sigma^{(n)}}+a_1(\rho)\frac{1}{\sigma^{(n-1)}}+\cdots+a_{n-1}(\rho)\frac{1}{\sigma}+a_n(\rho)\right)}. \tag{5.5}$$

From (5.5), we get

$$\begin{aligned} S\{j(\rho)\} &= \frac{1}{\sigma^2\left(\frac{1}{\sigma^{(n)}}+a_1(\rho)\frac{1}{\sigma^{(n-1)}}+\cdots+a_{n-1}(\rho)\frac{1}{\sigma}+a_n(\rho)\right)}, \\ \{j(\rho)\} &= S^{-1}\left(\frac{1}{\sigma^2\left(\frac{1}{\sigma^{(n)}}+a_1(\rho)\frac{1}{\sigma^{(n-1)}}+\cdots+a_{n-1}(\rho)\frac{1}{\sigma}+a_n(\rho)\right)}\right), \\ j(\rho) &= S^{-1}\left(\frac{1}{\sigma^2\frac{1}{\sigma}\left(\frac{1}{\sigma^{(n-1)}}+a_1(\rho)\frac{1}{\sigma^{n-2}}+\cdots+a_{n-1}(\rho)+a_n(\rho)\sigma\right)}\right), \\ j(\rho) &= S^{-1}\left(\frac{1}{\sigma\left(\frac{1}{\sigma^{(n-1)}}+a_1(\rho)\frac{1}{\sigma^{n-2}}+\cdots+a_{n-1}(\rho)+a_n(\rho)\sigma\right)}\right), \\ j(\rho) &= e^{-(a_1(\rho)+\cdots+a_{n-1}(\rho)+a_n(\rho))\rho}. \end{aligned}$$

Consequently,

$$S\{\Lambda(\rho)-\chi(\rho)\}=S\{i(\rho)\star j(\rho)\}$$

and thus,

$$\Lambda(\rho)-\chi(\rho)=i(\rho)\star j(\rho).$$

Taking modulus on both sides, we have

$$\begin{aligned} |\Lambda(\rho)-\chi(\rho)| &= |i(\rho)\star j(\rho)|=\left|\int_0^\rho i(s)j(\rho-s)ds\right|\leqslant\int_0^\rho i(s)j(\rho-s)ds, \\ |\Lambda(\rho)-\chi(\rho)| &= \epsilon\int_0^\rho j(\rho-s)ds. \end{aligned}$$

Since $j(\rho)=e^{(-a_1(\rho)+\cdots+a_{n-1}(\rho)+a_n(\rho))\rho}$ or $e^{R(-a_1(\rho)+\cdots+a_{n-1}(\rho)+a_n(\rho))\rho}$. Then

$$|\Lambda(\rho)-\chi(\rho)|$$

$$
\begin{aligned}
&\leqslant \epsilon \int_0^\rho e^{-R(-a_1(\rho)+\cdots+a_{n-1}(\rho)+a_n(\rho))(\rho-s)} ds \\
&\leqslant \epsilon \int_0^\rho e^{-R(-a_1(\rho)+\cdots+a_{n-1}(\rho)+a_n(\rho))}\rho^{+R(a_1(\rho)+\cdots+a_{n-1}(\rho)+a_n(\rho))s} ds \\
&\leqslant \epsilon e^{-R(a_1(\rho)+\cdots+a_{n-1}(\rho)+a_n(\rho))\rho} \int_0^\rho e^{R(a_1(\rho)+\cdots+a_{n-1}(\rho)+a_n(\rho))s} ds \\
&\leqslant \epsilon e^{-R(a_1(\rho)+\cdots+a_{n-1}(\rho)+a_n(\rho))\rho} \left[\frac{e^{R(a_1(\rho)+\cdots+a_{n-1}(\rho)+a_n(\rho))\rho}-1}{R\left(a_1(\rho)+\cdots+a_{n-1}(\rho)+a_n(\rho)\right)}\right] \\
&\leqslant \epsilon \left[\frac{e^{R(a_1(\rho)+\cdots+a_{n-1}(\rho)+a_n(\rho))\rho}.e^{-R(a_1(\rho)+\cdots+a_{n-1}(\rho)+a_n(\rho))\rho}-e^{-R(a_1(\rho)+\cdots+a_{n-1}(\rho)+a_n(\rho))\rho}}{R\left(a_1(\rho)+\cdots+a_{n-1}(\rho)+a_n(\rho)\right)}\right] \\
&\leqslant \epsilon \left[\frac{1-e^{-R(a_1(\rho)+\cdots+a_{n-1}(\rho)+a_n(\rho))\rho}}{R\left(a_1(\rho)+\cdots+a_{n-1}(\rho)+a_n(\rho)\right)}\right] \\
&\leqslant k\epsilon, \quad \text{for all } \rho \geqslant 0,
\end{aligned}
$$

where $k = \frac{1}{R(a_1(\rho)+\cdots+a_{n-1}(\rho)+a_n(\rho))}$. Hence, (3.1) has HUS in the class T. □

Note 5.1. *If* $-R\left(a_1(\rho)+\cdots+a_{n-1}(\rho)+a_n(\rho)\right)<0$, *then*

$$
\frac{\epsilon}{R\left(a_1(\rho)+\cdots+a_{n-1}(\rho)+a_n(\rho)\right)}\left(1-e^{-R(a_1(\rho)+\cdots+a_{n-1}(\rho)+a_n(\rho))}\right)
$$

diverges to infinity as ρ *tends to infinity.*

Hence, in the case of $-R\left(a_1(\rho)+\cdots+a_{n-1}(\rho)+a_n(\rho)\right)<0$, *we notice that we cannot prove the HUS by applying the ST method.*

Theorem 5.2. *Assume that* $\sigma : [0,\infty) \to (0,\infty)$ *is an increasing function and* $a_1(\rho)+\cdots+a_{n-1}(\rho)+a_n(\rho)$ *is a constant with* $R\left(a_1(\rho)+\cdots+a_{n-1}(\rho)+a_n(\rho)\right)>0$. *Then the DE* (3.1) *has the* σ*HUS for the* T.

Proof. Suppose that $\sigma : [0,\infty) \to (0,\infty)$ is an increasing function and the inequality 4.2 holds for all $\rho \geqslant 0$. If we define the function $i : [0,\infty) \to K$ by

$$
i(\rho) = \Lambda^{(n)}(\rho) + a_1(\rho)\Lambda^{(n-1)}(\rho) + \cdots + a_{n-1}(\rho)\Lambda'(\rho) + a_n(\rho)\Lambda(\rho), \quad \text{for all } \rho \geqslant 0,
$$

then

$$
|i(\rho)| \leqslant \sigma(\rho)\epsilon, \quad \text{for all } \rho \geqslant 0.
$$

By Theorem 5.1, we can prove that $\chi(\rho) = e^{-(a_1+\cdots+a_{n-1}+a_n)\rho}\Lambda(\rho)$ is a solution of (3.1) of course, $\chi \in T$. On the other hand

$$
Q(\sigma) = \frac{1}{\sigma\left(\frac{1}{\sigma^{(n-1)}} + a_1(\rho)\frac{1}{\sigma^{n-2}} + \cdots + a_{n-1}(\rho) + a_n(\rho)\sigma\right)},
$$

which gives

$$
\begin{aligned}
j(\rho) &= s^{-1}\left[\frac{1}{\sigma\left(\frac{1}{\sigma^{(n-1)}} + a_1(\rho)\frac{1}{\sigma^{n-2}} + \cdots + a_{n-1}(\rho) + a_n(\rho)\sigma\right)}\right], \\
j(\rho) &= e^{-(a_1(\rho)+\cdots+a_{n-1}(\rho)+a_n(\rho))\rho}.
\end{aligned}
$$

Moreover, it follows from (5.2) and (5.3), we obtain

$$\begin{aligned} S[\Lambda(\rho)] - S[\chi(\rho)] = T(\sigma) - \Phi(\sigma) &= \frac{P(\sigma)}{\left(\frac{1}{\sigma^n} + a_1(\rho)\frac{1}{\sigma^{(n-1)}} + \cdots + a_{n-1}(\rho) + a_n(\rho)\right)} \\ &= \sigma^2 P(\sigma)Q(\sigma) = \sigma^2\{i(\rho) \star j(\rho)\} = S\{i(\rho) \star j(\rho)\}. \end{aligned}$$

Which yields

$$S[\Lambda(\rho) - \chi(\rho)] = S[i(\rho) \star j(\rho)].$$

Therefore,

$$\Lambda(\rho) - \chi(\rho) = i(\rho) \star j(\rho) \qquad \text{and} \qquad \Lambda(\rho) - \chi(\rho) = i(\rho) \star e^{-(a_1+\cdots+a_{n-1}+a_n)\rho}.$$

By Theorem 5.1, we get

$$\begin{aligned} |\Lambda(\rho) - \chi(\rho)| &= |i(\rho) \star e^{-(a_1+\cdots+a_{n-1}+a_n)\rho}| \\ &= |\int_0^\rho i(s) e^{-(a_1+\cdots+a_{n-1}+a_n)(\rho-s)} ds| \\ &\leqslant \int_0^\rho |i(s)||e^{-(a_1+\cdots+a_{n-1}+a_n)(\rho-s)}|ds \\ &\leqslant \frac{\sigma(\rho)\epsilon\left(1 - e^{-R(a_1+\cdots+a_{n-1}+a_n)\rho}\right)}{R(a_1+\cdots+a_{n-1}+a_n)} \\ &\leqslant k\sigma(\rho)\epsilon, \quad \text{for all } \rho \geqslant 0, \end{aligned}$$

where $k = \frac{1}{R(a_1(\rho)+\cdots+a_{n-1}(\rho)+a_n(\rho))}$. This completes the proof. □

Theorem 5.3. *Let* $a_1(\rho) + \cdots + a_{n-1}(\rho) + a_n(\rho)$ *and* β *be constants satisfying* $R(a_1(\rho) + \cdots + a_{n-1}(\rho) + a_n(\rho)) > 0$ *and* $\beta > 0$. *Then the DE (3.1) has Mittag-Leffler HUS for the class* T.

Proof. Assume that $\Lambda \in T$ satisfying (4.5), for all $\rho \geqslant 0$. Let us define the function $i : [0, \infty) \to K$ by

$$i(\rho) = \Lambda^{(n)}(\rho) + a_1(\rho)\Lambda^{(n-1)}(\rho) + \cdots + a_{n-1}(\rho)\Lambda'(\rho) + a_n(\rho)\Lambda(\rho), \quad \text{for all} \rho \geqslant 0.$$

In view of (4.5), we have $|i(\rho)| \leqslant \epsilon, \forall \ \rho \geqslant 0$.

The ST of $i(\rho)$, which gives

$$\begin{aligned} P(\sigma) = S\{i(\rho)\} &= S\left[\Lambda^{(n)}(\rho) + a_1(\rho)\Lambda^{(n-1)}(\rho) + \cdots + a_{n-1}(\rho)\Lambda'(\rho) + a_n(\rho)\Lambda(\rho)\right] \\ P(\sigma) &= \frac{1}{\sigma^{(n)}}T(\sigma) - \sum_{k=0}^{(n-1)} \left(\frac{1}{\sigma}\right)^{n-(k-1)} \Lambda^k(0) \\ &\quad + a_1(\rho)\left[\frac{1}{\sigma^{(n-1)}}T(\sigma) - \sum_{k=0}^{n-2}\left(\frac{1}{\sigma}\right)^{(n-1)-(k-1)} \Lambda^k(0)\right] \\ &\quad + \cdots + a_{n-1}(\rho)\left[\frac{1}{\sigma}T(\sigma) - \frac{1}{\sigma^2}\Lambda(0)\right] + a_n(\rho)T(\sigma). \end{aligned} \tag{5.6}$$

From (5.6), we have

$$\begin{aligned} &T(\sigma) = s\{\Lambda(\rho)\} \\ &= \frac{P(\sigma) + \sum_{k=0}^{(n-1)}\left(\frac{1}{\sigma}\right)^{n-(k-1)} \Lambda^k(0) + a_1(\rho)\sum_{k=0}^{n-2}\left(\frac{1}{\sigma}\right)^{(n-1)-(k-1)} \Lambda^k(0) + \cdots a_{n-1}(\rho)\frac{1}{\sigma^2}\Lambda(0)}{\left(\frac{1}{\sigma^{(n)}} + a_1(\rho)\frac{1}{\sigma^{(n-1)}} + \cdots + a_{n-1}(\rho)\frac{1}{\sigma} + a_n(\rho)\right)}. \end{aligned}$$

If we put $\chi(\rho) = e^{-(a_1(\rho)+\cdots+a_{n-1}(\rho)+a_n(\rho))\rho}\Lambda(\rho)$, then $\chi(0) = \Lambda(0)$ and $\chi \in T$. The ST of $\chi(\rho)$ gives

$$S[\chi(\rho)] = \Phi(\sigma)$$
$$= \frac{P(\sigma) + \sum_{k=0}^{(n-1)} \left(\frac{1}{\sigma}\right)^{n-(k-1)} \chi^k(0) + a_1(\rho) \sum_{k=0}^{n-2} \left(\frac{1}{\sigma}\right)^{(n-1)-(k-1)} \chi^k(0) + \cdots a_{n-1}(\rho)\frac{1}{\sigma^2}\chi(0)}{\left(\frac{1}{\sigma^{(n)}} + a_1(\rho)\frac{1}{\sigma^{(n-1)}} + \cdots + a_{n-1}(\rho)\frac{1}{\sigma} + a_n(\rho)\right)}. \tag{5.7}$$

From (5.7), we get

$$\begin{aligned} S[\chi^{(n)}(\rho) &+ a_1(\rho)\chi^{(n-1)}(\rho) + \cdots + a_{n-1}(\rho)\chi'(\rho)a_n(\rho)\chi(\rho)] \\ &= \frac{1}{\sigma^{(n)}}\Phi(\sigma) - \sum_{k=0}^{(n-1)} \left(\frac{1}{\sigma}\right)^{n-(k-1)} \chi^k(0) \\ &+ a_1(\rho)\left[\frac{1}{\sigma^{(n-1)}}\Phi(\sigma) - \sum_{k=0}^{n-2} \left(\frac{1}{\sigma}\right)^{(n-1)-(k-1)} \chi^k(0)\right] \\ &+ \cdots + a_{n-1}(\rho)\left[\frac{1}{\sigma}\Phi(\sigma) - \frac{1}{\sigma^2}\chi(0)\right] + a_n(\rho)\Phi(\sigma). \end{aligned}$$

Since k is a one to one operator. Then

$$\chi^{(n)}(\rho) + a_1(\rho)\chi^{(n-1)}(\rho) + \cdots + a_{n-1}(\rho)\chi'(\rho)a_n(\rho)\chi(\rho) = 0.$$

If we set

$$Q(\sigma) = \left[\frac{1}{\sigma\left(\frac{1}{\sigma^{(n)}} + a_1(\rho)\frac{1}{\sigma^{(n-1)}} + \cdots + a_{n-1}(\rho)\frac{1}{\sigma} + a_n(\rho)\sigma\right)}\right],$$

then we have

$$\begin{aligned} j(\rho) &= s^{-1}\left[\frac{1}{\sigma\left(\frac{1}{\sigma^{(n)}} + a_1(\rho)\frac{1}{\sigma^{(n-1)}} + \cdots + a_{n-1}(\rho)\frac{1}{\sigma} + a_n(\rho)\sigma\right)}\right], \\ j(\rho) &= e^{-(a_1(\rho)+\cdots+a_{n-1}(\rho)a_n(\rho))\rho}. \end{aligned} \tag{5.8}$$

By (5.7) and (5.8), we obtain

$$\begin{aligned} S[\Lambda(\rho)] - S[\chi(\rho)] &= T(\sigma) - \Phi(\sigma) \\ &= \frac{p(\sigma)}{\left(\frac{1}{\sigma^{(n)}} + a_1(\rho)\frac{1}{\sigma^{(n-1)}} + \cdots + a_{n-1}(\rho)\frac{1}{\sigma} + a_n(\rho)\right)} = \sigma^2 P(\sigma)Q(\sigma) = S\{i(\rho) \star j(\rho)\}. \end{aligned}$$

This gives

$$\Lambda(\rho) - \chi(\rho) = i(\rho) \star j(\rho) = i(\rho) \star e^{-(a_1+\ldots+a_{n-1}+a_n)\rho}.$$

Taking modulus on both sides and using $|i(\rho)| \leqslant \epsilon E_\beta(\rho)$ for $\rho \geqslant 0$ and $E_\beta(\rho)$ is increasing for $\rho \geqslant 0$, we have

$$\begin{aligned} |\Lambda(\rho) - \chi(\rho)| &= |i(\rho) \star e^{-(a_1+\cdots+a_{n-1}+a_n)\rho}| \\ &= |\int_0^\rho i(s) \star e^{-(a_1+\cdots+a_{n-1}+a_n)(\rho-s)} ds| \end{aligned}$$

$$\begin{aligned}
&\leqslant \int_0^\rho |i(s)||e^{-(a_1+\cdots+a_{n-1}+a_n)(\rho-s)}ds| \\
&\leqslant \frac{E_\beta(\rho)\epsilon}{R(a_1+\cdots+a_{n-1}+a_n)}\left(1-e^{-R(a_1+\cdots+a_{n-1}+a_n)}\right) \\
&\leqslant kE_\beta(\rho)\epsilon \quad \text{for all } \rho \geqslant 0,
\end{aligned}$$

where $k = \frac{1}{R(a_1+\cdots+a_{n-1}+a_n)}$. This completes the proof. □

Theorem 5.4. *Assume that* $\sigma : [,\infty) \to (0,\infty)$ *is an increasing function and* $a_1 + \cdots + a_{n-1} + a_n$ *and* β *are constants which satisfy* $R(a_1 + \cdots + a_{n-1} + a_n) > 0$. *Then the DE* (3.1) *has the Mittag-Leffler σHUS for the class* T.

Proof. Assume that $\Lambda \in T$ and $\sigma : [0,\infty) \to (0,\infty)$ is a function and $\Lambda(\rho)$ and $\chi(\rho)$ satisfy the inequality (4.7), for all $\rho \geqslant 0$. There exists a $k > 0$ and a solution $\chi : [0,\infty) \to K$ of the DE (3.1) such that $\chi \in T$ and

$$|\Lambda(\rho) - \chi(\rho)| \leqslant k\sigma(\rho)\epsilon E_\beta(\rho), \quad \text{for all} \rho \geqslant 0.$$

Define a function $i : [0,\infty) \to K$ by

$$i(\rho) = \Lambda^n(\rho) + a_1(\rho)\Lambda^{(n-1)}(\rho) + \cdots + a_{n-1}(\rho)\Lambda'(\rho) + a_n(\rho)\Lambda(\rho) \quad \text{for all } \rho \geqslant 0.$$

Then we have $|i(\rho)| \leqslant \sigma(\rho) \in E_\beta(\rho)$, for all $\rho \geqslant 0$. By Theorem 5.3, we get

$$\begin{aligned}
|\Lambda(\rho) - \chi(\rho)| &= |i(\rho) \star e^{-(a_1+\cdots+a_{n-1}+a_n)\rho}| \\
&= |\int_0^\rho i(s)e^{-(a_1+\cdots+a_{n-1}+a_n)(\rho-s)}ds| \\
&\leqslant \int_0^\rho |i(s)||e^{-(a_1+\cdots+a_{n-1}+a_n)(\rho-s)}|ds \\
&\leqslant \frac{\sigma(\rho)E_\beta(\rho)\epsilon\left(1-e^{-R(a_1+\cdots+a_{n-1}+a_n)\rho}\right)}{R(a_1+\cdots+a_{n-1}+a_n)} \\
&\leqslant k\sigma(\rho)E_\beta(\rho)\epsilon, \quad \text{for all } \rho \geqslant 0,
\end{aligned}$$

where $k = \frac{1}{R(a_1+\cdots+a_{n-1}+a_n)}$. This completes the proof. □

6. Hyers-Ulam stability of (3.2)

In this section, we prove several types of the HUS of n^{th} order DE (3.2) by using ST method.

Theorem 6.1. *Assume that* $m : [0,\infty) \to \infty$ *is a continuous function of exponential order and* $a_1 + \cdots a_{n-1} + a_n$ *is a constant with* $R(a_1 + \cdots + a_{n-1} + a_n) > 0$. *Then the DE* (3.2) *has the HUS for the class* T.

Proof. Suppose that $\Lambda \in T$ satisfying (4.1) and the function $i : [0,\infty) \to K$ is defined by

$$i(\rho) = \Lambda^{(n)}(\rho) + a_1(\rho)\Lambda^{(n-1)}(\rho) + \cdots + a_{n-1}(\rho)\Lambda'(\rho) + a_n(\rho)\Lambda(\rho) - m(\rho), \quad \text{for all} \rho \geqslant 0.$$

Then $|i(\rho)| \leqslant \epsilon$ holds, $\forall\ \rho \geqslant 0$. The ST of $i(\rho)$ gives

$$P(\sigma) = S\{i(\rho)\} = S\left[\Lambda^{(n)}(\rho) + a_1(\rho)\Lambda^{(n-1)}(\rho) + \cdots + a_{n-1}(\rho)\Lambda'(\rho) + a_n(\rho)\Lambda(\rho) - m(\rho)\right].$$

This implies that

$$\begin{aligned}
&T(\sigma) = s\{\Lambda(\rho)\} \\
&= \frac{P(\sigma) + \sum_{k=0}^{(n-1)}\left(\frac{1}{\sigma}\right)^{n-(k-1)}\Lambda^k(0) + a_1(\rho)\sum_{k=0}^{n-2}\left(\frac{1}{\sigma}\right)^{(n-1)-(k-1)}\Lambda^k(0) + \cdots + a_{n-1}(\rho)\frac{1}{\sigma^2}\Lambda(0)}{\left(\frac{1}{\sigma^{(n)}} + a_1(\rho)\frac{1}{\sigma^{(n-1)}} + \cdots + a_{n-1}(\rho)\frac{1}{\sigma} + a_n(\rho)\right)}. \quad (6.1)
\end{aligned}$$

If we set $\chi(\rho) = e^{-(a_1+\cdots+a_{n-1}+a_n)\rho}\Lambda(\rho) + \left(m(\rho) \star e^{-(a_1+\cdots+a_{n-1}+a_n)\rho}\right)$, then $\chi(0) = \Lambda(0)$ and $\chi \in T$.

The ST of $\chi(\rho)$ gives the following:

$$\begin{aligned} S[\chi(\rho)] &= \Phi(\sigma) \\ &= \frac{\sum_{k=0}^{(n-1)}\left(\frac{1}{\sigma}\right)^{n-(k-1)}\chi^k(0) + a_1(\rho)\sum_{k=0}^{n-2}\left(\frac{1}{\sigma}\right)^{(n-1)-(k-1)}\chi^k(0) + \cdots a_{n-1}(\rho)\frac{1}{\sigma^2}\chi(0)}{\left(\frac{1}{\sigma^{(n)}} + a_1(\rho)\frac{1}{\sigma^{(n-1)}} + \cdots + a_{n-1}(\rho)\frac{1}{\sigma} + a_n(\rho)\right)}. \end{aligned} \tag{6.2}$$

On the other hand

$$\begin{aligned} &S[\chi^{(n)}(\rho) + a_1(\rho)\chi^{(n-1)}(\rho) + \cdots + a_{n-1}(\rho)\chi'(\rho)a_n(\rho)\chi(\rho)] \\ &= \frac{1}{\sigma^{(n)}}\Phi(\sigma) - \sum_{k=0}^{(n-1)}\left(\frac{1}{\sigma}\right)^{n-(k-1)}\chi^k(0) + a_1(\rho)\left[\frac{1}{\sigma^{(n-1)}}\Phi(\sigma) - \sum_{k=0}^{n-2}\left(\frac{1}{\sigma}\right)^{(n-1)-(k-1)}\chi^k(0)\right] \\ &\quad + \cdots + a_{n-1}(\rho)\left[\frac{1}{\sigma}\Phi(\sigma) - \frac{1}{\sigma^2}\chi(0)\right] + a_n(\rho)\Phi(\sigma). \end{aligned}$$

By using (6.2), we have

$$S[\chi^{(n)}(\rho) + a_1(\rho)\chi^{(n-1)}(\rho) + \cdots + a_{n-1}(\rho)\chi'(\rho)a_n(\rho)\chi(\rho)] = \mathcal{N}(\sigma) = S\{m(\rho)\}.$$

Hence, $\chi(\rho)$ is a solution of the DE (3.1). In addition by using (6.1) and (6.2), we obtain

$$\begin{aligned} S[\Lambda(\rho)] - S[\chi(\rho)] &= T(\sigma) - \Phi(\sigma) \\ &= \frac{P(\sigma)}{\left(\frac{1}{\sigma^{(n)}} + a_1(\rho)\frac{1}{\sigma^{(n-1)}} + \cdots + a_{n-1}(\rho)\frac{1}{\sigma} + a_n(\rho)\right)} = \sigma^2 P(\sigma)Q(\sigma) = S\{i(\rho) \star j(\rho)\}, \end{aligned}$$

where we set

$$Q(\sigma) = \frac{1}{\sigma\left(\frac{1}{\sigma^{(n)}} + a_1(\rho)\frac{1}{\sigma^{(n-1)}} + \cdots + a_{n-1}(\rho)\frac{1}{\sigma} + a_n(\rho)\sigma\right)},$$

which gives,

$$\begin{aligned} j(\rho) &= s^{-1}\left[\frac{1}{\sigma\left(\frac{1}{\sigma^{(n)}} + a_1(\rho)\frac{1}{\sigma^{(n-1)}} + \cdots + a_{n-1}(\rho)\frac{1}{\sigma} + a_n(\rho)\sigma\right)}\right], \\ j(\rho) &= e^{-(a_1(\rho)+\cdots+a_{n-1}(\rho)a_n(\rho))\rho}. \end{aligned}$$

Therefore

$$S[\Lambda(\rho)] - S[\chi(\rho)] = S[i(\rho) \star j(\rho)] = S\left[i(\rho) \star e^{-(a_1+\cdots a_{n-1}+a_n)\rho}\right],$$

which yields

$$\Lambda(\rho) - \chi(\rho) = i(\rho) \star e^{-(a_1+\cdots a_{n-1}+a_n)\rho}.$$

Furthermore,

$$\begin{aligned} |\Lambda(\rho) - \chi(\rho)| &= |i(\rho) \star e^{-(a_1+\cdots+a_{n-1}+a_n)\rho}| \\ &= |\int_0^\rho i(s) \star e^{-(a_1+\cdots+a_{n-1}+a_n)(\rho-s)}ds| \end{aligned}$$

$$\leqslant \int_0^\rho |i(s)||e^{-(a_1+\cdots+a_{n-1}+a_n)(\rho-s)}ds|$$
$$\leqslant E_\beta(\rho)\epsilon e^{-R(a_1+\cdots+a_{n-1}+a_n)\rho}\int_0^\rho e^{R(a_1+\cdots+a_{n-1}+a_n)s}ds$$
$$\leqslant \frac{E_\beta(\rho)\epsilon}{R(a_1+\cdots+a_{n-1}+a_n)}\left(1-e^{-R(a_1+\cdots+a_{n-1}+a_n)}\right)$$
$$\leqslant k\epsilon, \quad \text{for all } \rho \geqslant 0,$$

where $k = \frac{1}{R(a_1+\cdots+a_{n-1}+a_n)}$. This completes the proof. □

Theorem 6.2. *Assume that* $m : [0,\infty) \to K$ *is a continuous function of exponential order,* $\sigma : [0,\infty) \to (0,\infty)$ *is an increasing function and* $a_1+\cdots a_{n-1}+a_n$ *is a constant with* $R(a_1+\cdots+a_{n-1}+a_n) > 0$. *Then the DE* (3.2) *has the σHUS for the class* T.

Proof. Let $\Lambda \in T$ satisfies (4.4), for all $\rho \geqslant 0$. Define the function $i : [0,\infty) \to K$ by

$$i(\rho) = \Lambda^{(n)}(\rho) + a_1(\rho)\Lambda^{(n-1)}(\rho) + \cdots + a_{n-1}(\rho)\Lambda'(\rho) + a_n(\rho)\Lambda(\rho) - m(\rho),$$

for each $\rho \geqslant 0$. Then $|i(\rho)| \leqslant \sigma(\rho)\epsilon$, for all $\rho \geqslant 0$. It is not difficult to check

$$S[\Lambda(\rho)] = T(\sigma)$$
$$= \frac{P(\sigma)+\sum_{k=0}^{(n-1)}\left(\frac{1}{\sigma}\right)^{n-(k-1)}\Lambda^k(0)+a_1(\rho)\sum_{k=0}^{n-2}\left(\frac{1}{\sigma}\right)^{(n-1)-(k-1)}\Lambda^k(0)+\cdots+a_{n-1}(\rho)\frac{1}{\sigma^2}\Lambda(0)}{\left(\frac{1}{\sigma^{(n)}}+a_1(\rho)\frac{1}{\sigma^{(n-1)}}+\cdots+a_{n-1}(\rho)\frac{1}{\sigma}+a_n(\rho)\right)}. \tag{6.3}$$

If we set $\chi(\rho) = e^{-(a_1+\cdots+a_{n-1}+a_n)\rho}\Lambda(\rho) + \left(m(\rho) \star e^{-(a_1+\cdots+a_{n-1}+a_n)\rho}\right)$, then $\chi = T$. Further, we apply the ST on both sides to get

$$S[\chi(\rho)] = \Phi(\sigma)$$
$$= \frac{\sum_{k=0}^{(n-1)}\left(\frac{1}{\sigma}\right)^{n-(k-1)}\chi^k(0)+a_1(\rho)\sum_{k=0}^{n-2}\left(\frac{1}{\sigma}\right)^{(n-1)-(k-1)}\chi^k(0)+\cdots a_{n-1}(\rho)\frac{1}{\sigma^2}\chi(0)}{\left(\frac{1}{\sigma^{(n)}}+a_1(\rho)\frac{1}{\sigma^{(n-1)}}+\cdots+a_{n-1}(\rho)\frac{1}{\sigma}+a_n(\rho)\right)}. \tag{6.4}$$

On the other hand

$$S[\chi^{(n)}(\rho)+a_1(\rho)\chi^{(n-1)}(\rho)+\cdots+a_{n-1}(\rho)\chi'(\rho)a_n(\rho)\chi(\rho)]$$
$$= \frac{1}{\sigma^{(n)}}\Phi(\sigma) - \sum_{k=0}^{(n-1)}\left(\frac{1}{\sigma}\right)^{n-(k-1)}\chi^k(0) + a_1(\rho)\left[\frac{1}{\sigma^{(n-1)}}\Phi(\sigma) - \sum_{k=0}^{n-2}\left(\frac{1}{\sigma}\right)^{(n-1)-(k-1)}\chi^k(0)\right]$$
$$+\cdots+a_{n-1}(\rho)\left[\frac{1}{\sigma}\Phi(\sigma)-\frac{1}{\sigma^2}\chi(0)\right]+a_n(\rho)\Phi(\sigma).$$

The relation (6.4) implies that

$$S[\chi^{(n)}(\rho)+a_1(\rho)\chi^{(n-1)}(\rho)+\cdots+a_{n-1}(\rho)\chi'(\rho)a_n(\rho)\chi(\rho)] = \mathcal{N}(\sigma) = S\{m(\rho)\}$$

and thus

$$\chi^{(n)}(\rho)+a_1(\rho)\chi^{(n-1)}(\rho)+\cdots+a_{n-1}(\rho)\chi'(\rho)a_n(\rho)\chi(\rho) = m(\rho).$$

That is $\chi(\rho)$ is a solution of (3.2). Using (6.3) and (6.4), we get

$$S[\Lambda(\rho)] - S[\chi(\rho)] = T(\sigma) - \Phi(\sigma)$$

$$= \frac{P(\sigma)}{\left(\frac{1}{\sigma^{(n)}} + a_1(\rho)\frac{1}{\sigma^{(n-1)}} + \cdots + a_{n-1}(\rho)\frac{1}{\sigma} + a_n(\rho)\right)} = \sigma^2 P(\sigma)Q(\sigma) = S\{i(\rho) \star j(\rho)\},$$

where

$$Q(\sigma) = \frac{1}{\sigma\left(\frac{1}{\sigma^{(n)}} + a_1(\rho)\frac{1}{\sigma^{(n-1)}} + \cdots + a_{n-1}(\rho)\frac{1}{\sigma} + a_n(\rho)\sigma\right)}.$$

This gives

$$j(\rho) = s^{-1}\left[\frac{1}{\sigma\left(\frac{1}{\sigma^{(n)}} + a_1(\rho)\frac{1}{\sigma^{(n-1)}} + \cdots + a_{n-1}(\rho)\frac{1}{\sigma} + a_n(\rho)\sigma\right)}\right], \quad j(\rho) = e^{-(a_1(\rho)+\cdots+a_{n-1}(\rho)a_n(\rho))\rho}.$$

Therefore, we have

$$S[\Lambda(\rho)] - S[\chi(\rho)] = S[i(\rho) \star j(\rho)],$$

which gives

$$\Lambda(\rho) - \chi(\rho) = i(\rho) \star j(\rho).$$

By Theorem (5.2), we have

$$|\Lambda(\rho) - \chi(\rho)| = |i(\rho) \star j(\rho)| = |\int_0^\rho i(s) \star j(\rho - s)ds|$$
$$\leqslant \int_0^\rho |i(s)||j(\rho - s)ds| \leqslant \sigma(\rho)\epsilon \int_0^\rho |i(\rho - s)ds| \leqslant k\sigma(\rho)\epsilon \quad \text{for all } \rho \geqslant 0,$$

where $k = \frac{1}{R(a_1+\cdots+a_{n-1}+a_n)}$. This completes the proof. □

Theorem 6.3. *Assume that* $m : [0, \infty) \to (0, \infty)$ *is a continuous function of exponential order and that* $a_1 + \cdots + a_{n-1} + a_n$ *and* β *are constants satisfying* $R(a_1 + \cdots + a_{n-1} + a_n) > 0$ *and* $\beta > 0$. *Then the DE* (3.2) *has Mittag-Leffler HUS for the class* T.

Proof. Suppose that $\Lambda(\rho) \in T$ satisfying (4.6), $\forall\ \rho \geqslant 0$. Consider a function $i : [0, \infty) \to K$ which is defined by

$$i(\rho) = \Lambda^{(n)}(\rho) + a_1(\rho)\Lambda^{(n-1)}(\rho) + \cdots + a_{n-1}(\rho)\Lambda'(\rho) + a_n(\rho)\Lambda(\rho) - m(\rho), \quad \text{for all } \rho \geqslant 0.$$

From (4.6) that $|i(\rho)| \leqslant E_\beta(\rho)\epsilon$, $\forall\ \rho \geqslant 0$. The ST of $i(\rho)$ gives

$$P(\sigma) = S\{i(\rho)\} = S\{\Lambda^{(n)}(\rho) + a_1(\rho)\Lambda^{(n-1)}(\rho) + \cdots + a_{n-1}(\rho)\Lambda'(\rho) + a_n(\rho)\Lambda(\rho) - m(\rho)\}.$$

That is,

$$T(\sigma) = s\{\Lambda(\rho)\}$$
$$= \frac{P(\sigma) + \sum_{k=0}^{(n-1)}\left(\frac{1}{\sigma}\right)^{n-(k-1)}\Lambda^k(0) + a_1(\rho)\sum_{k=0}^{n-2}\left(\frac{1}{\sigma}\right)^{(n-1)-(k-1)}\Lambda^k(0) + \cdots + a_{n-1}(\rho)\frac{1}{\sigma^2}\Lambda(0)}{\left(\frac{1}{\sigma^{(n)}} + a_1(\rho)\frac{1}{\sigma^{(n-1)}} + \cdots + a_{n-1}(\rho)\frac{1}{\sigma} + a_n(\rho)\right)}. \quad (6.5)$$

If we set $\chi(\rho) = e^{-(a_1+\cdots+a_{n-1}+a_n)\rho}\Lambda(\rho) + \left(m(\rho) \star e^{-(a_1+\cdots+a_{n-1}+a_n)\rho}\right)$, then $\chi(0) = \Lambda(0)$ and $\chi \in T$. We apply ST on both sides, we get

$$S[\chi(\rho)] = \Phi(\sigma)$$
$$= \frac{\sum_{k=0}^{(n-1)}\left(\frac{1}{\sigma}\right)^{n-(k-1)}\chi^k(0) + a_1(\rho)\sum_{k=0}^{n-2}\left(\frac{1}{\sigma}\right)^{(n-1)-(k-1)}\chi^k(0) + \cdots a_{n-1}(\rho)\frac{1}{\sigma^2}\chi(0)}{\left(\frac{1}{\sigma^{(n)}} + a_1(\rho)\frac{1}{\sigma^{(n-1)}} + \cdots + a_{n-1}(\rho)\frac{1}{\sigma} + a_n(\rho)\right)}. \quad (6.6)$$

On the other hand

$$\begin{aligned}
&S\{\chi^{(n)}(\rho)+a_1(\rho)\chi^{(n-1)}(\rho)+\cdots+a_{n-1}(\rho)\chi'(\rho)a_n(\rho)\chi(\rho)\}\\
&=\frac{1}{\sigma^{(n)}}\Phi(\sigma)-\sum_{k=0}^{(n-1)}\left(\frac{1}{\sigma}\right)^{n-(k-1)}\chi^{k}(0)+a_1(\rho)\left[\frac{1}{\sigma^{(n-1)}}\Phi(\sigma)-\sum_{k=0}^{n-2}\left(\frac{1}{\sigma}\right)^{(n-1)-(k-1)}\chi^{k}(0)\right]\\
&\quad+\cdots+a_{n-1}(\rho)\left[\frac{1}{\sigma}\Phi(\sigma)-\frac{1}{\sigma^2}\chi(0)\right]+a_n(\rho)\Phi(\sigma).
\end{aligned}$$

Then by (6.6), we have

$$S[\chi^{(n)}(\rho)+a_1(\rho)\chi^{(n-1)}(\rho)+\cdots+a_{n-1}(\rho)\chi'(\rho)a_n(\rho)\chi(\rho)]=\mathcal{N}(\sigma)=S\{m(\rho)\}$$

and thus

$$\chi^{(n)}(\rho)+a_1(\rho)\chi^{(n-1)}(\rho)+\cdots+a_{n-1}(\rho)\chi'(\rho)a_n(\rho)\chi(\rho)=m(\rho).$$

Hence, $\chi(\rho)$ is a solution of the DE (3.2). In addition by using (6.5) and (6.6), we obtain

$$\begin{aligned}
S[\Lambda(\rho)]-S[\chi(\rho)]=T(\sigma)-\Phi(\sigma)&=\frac{P(\sigma)}{\left(\frac{1}{\sigma^{(n)}}+a_1(\rho)\frac{1}{\sigma^{(n-1)}}+\cdots+a_{n-1}(\rho)\frac{1}{\sigma}+a_n(\rho)\right)}\\
&=\sigma^2P(\sigma)\frac{1}{\sigma\left(\frac{1}{\sigma^{(n)}}+a_1(\rho)\frac{1}{\sigma^{(n-1)}}+\cdots+a_{n-1}(\rho)\frac{1}{\sigma}+a_n(\rho)\sigma\right)}\\
&=\sigma^2P(\sigma)Q(\sigma)=S\{i(\rho)\star j(\rho)\},
\end{aligned}$$

where

$$Q(\sigma)=\frac{1}{\sigma\left(\frac{1}{\sigma^{(n)}}+a_1(\rho)\frac{1}{\sigma^{(n-1)}}+\cdots+a_{n-1}(\rho)\frac{1}{\sigma}+a_n(\rho)\sigma\right)}.$$

This gives,

$$j(\rho)=s^{-1}\left[\frac{1}{\sigma\left(\frac{1}{\sigma^{(n)}}+a_1(\rho)\frac{1}{\sigma^{(n-1)}}+\cdots+a_{n-1}(\rho)\frac{1}{\sigma}+a_n(\rho)\sigma\right)}\right],\quad j(\rho)=e^{-(a_1(\rho)+\cdots+a_{n-1}(\rho)a_n(\rho))\rho}.$$

Therefore, $S[\Lambda(\rho)]-S[\chi(\rho)]=S[i(\rho)\star j(\rho)]$ which yields

$$\Lambda(\rho)-\chi(\rho)=i(\rho)\star j(\rho)\quad\text{for all }\rho\geqslant 0.$$

Furthermore,

$$\begin{aligned}
|\Lambda(\rho)-\chi(\rho)|&=|i(\rho)\star e^{-(a_1+\cdots+a_{n-1}+a_n)\rho}|\\
&=|\int_0^{\rho}i(s)\star e^{-(a_1+\cdots+a_{n-1}+a_n)(\rho-s)}ds|\\
&\leqslant\int_0^{\rho}|i(s)||e^{-(a_1+\cdots+a_{n-1}+a_n)(\rho-s)}ds|\\
&\leqslant\frac{E_\beta(\rho)\epsilon}{R(a_1+\cdots+a_{n-1}+a_n)}\left(1-e^{-R(a_1+\cdots+a_{n-1}+a_n)}\right)\leqslant kE_\beta(\rho)\epsilon,\quad\text{for all }\rho\geqslant 0,
\end{aligned}$$

where $k=\frac{1}{R(a_1+\cdots+a_{n-1}+a_n)}$. This completes the proof. □

Theorem 6.4. *Assume that* $m : [0,\infty) \to (0,\infty)$ *is an continuous function of exponential order and that* $\sigma : [0,\infty) \to (0,\infty)$ *is an increasing function and* $a_1 + \cdots + a_{n-1} + a_n$ *and* β *are constants which satisfy* $R(a_1 + \cdots + a_{n-1} + a_n) > 0$. *Then the DE* (3.2) *has the Mittag-Leffler* σ*HUS for the class* T.

Proof. Assume that $\Lambda \in T$ satisfying (4.7), $\forall\ \rho \geqslant 0$. It is easy to prove that $\exists\ k > 0$ and a solution $\chi : [0,\infty) \to K$ of (3.2) such that $\chi \in T$ and

$$|\Lambda(\rho) - \chi(\rho)| \leqslant k\sigma(\rho)\epsilon E_\beta(\rho),$$

$\forall\ \rho \geqslant 0$. Define a mapping $i : [0,\infty) \to K$ by

$$i(\rho) = \Lambda^{(n)}(\rho) + a_1(\rho)\Lambda^{(n-1)}(\rho) + \cdots\cdots + a_{n-1}(\rho)\Lambda'(\rho) + a_n(\rho)\Lambda(\rho) - m(\rho),$$

$\forall\ \rho \geqslant 0$. Then $|i(\rho)| \leqslant \sigma(\rho)\epsilon$, $\forall\ \rho \geqslant 0$. By applying of Theorem 6.3, there exists a solution $\chi : [0,\infty) \to K$ of (3.2) satisfying $\chi \in T$ and

$$\begin{aligned}
|\Lambda(\rho) - \chi(\rho)| &= |i(\rho) \star e^{-(a_1+\cdots+a_{n-1}+a_n)\rho}| \\
&= |\int_0^\rho i(s) \star e^{-(a_1+\cdots+a_{n-1}+a_n)(\rho-s)} ds| \\
&\leqslant \int_0^\rho |i(s)||e^{-(a_1+\cdots+a_{n-1}+a_n)(\rho-s)} ds| \\
&\leqslant \frac{\sigma(\rho)E_\beta(\rho)\epsilon}{R(a_1+\cdots+a_{n-1}+a_n)}\left(1 - e^{-R(a_1+\cdots+a_{n-1}+a_n)}\right) \leqslant k\sigma(\rho)E_\beta(\rho)\epsilon, \quad \text{for all } \rho \geqslant 0,
\end{aligned}$$

where $k = \frac{1}{R(a_1+\cdots+a_{n-1}+a_n)}$. This completes the proof. □

7. Conclusion

We successfully determined the solutions to the ST of various derivatives. This research has made an attempt to analyse the HUS of DE of n^{th} order using the ST method. This paper has also proved the general solution and generalised HUS of homogeneous and non-homogeneous linear DE. Hence, this study proved the stability in the sense of "Hyers-Ulam."

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