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Research article

Solving an integral equation via orthogonal generalized α-ψ-Geraghty contractions

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Abstract: In this paper, we introduce orthogonal generalized $\mathbf{O}$-α-ψ-Geraghty contractive type mappings and prove some fixed point theorems in $\mathbf{O}$-complete $\mathbf{O}$-$\mathfrak{b}$-metric spaces. We also provide an illustrative example to support our theorem. The results proved here will be utilized to show the existence of a solution to an integral equation as an application.

Keywords: $\mathbf{O}$-$\mathfrak{b}$-metric space; fixed point; $\mathbf{O}$-α-admissible; orthogonal generalized $\mathbf{O}$-α-ψ-Geraghty contraction
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1. Introduction and preliminaries

Banach [1] introduced one of the most essential Banach Contraction Principles. In 1993, Czerwik [2] initiated the notion of a $\mathfrak{b}$-metric space and proved the fixed point theorem (FPT) in this space. Aydi et al. [3] proved the common FPT for weak ϕ-contractions in $\mathfrak{b}$-metric space. The existence and uniqueness of a fixed point of ϕ-contractions was proved by Pacurar [4]. In 2018, Zada et al. [5] elaborated a FPT of a rational contraction. Geraghty [6] expanded the Banach contraction principle in 1973 by factoring an auxiliary function of complete metric space.

One of the interesting results was given by Samet et al. [7] by defining α-ψ-contractive maps via α-admissible mappings. After that, Cho et al. [8] introduced the α-GC (Geraghty contraction) type maps in metric space and proved some FPT's of these functions. Popescu [9] developed $\boldsymbol{\alpha}$-GC maps and proved the fixed point theorem in complete metric space. From the above work, Karapınar [11] introduced generalized $\boldsymbol{\alpha}$-$\boldsymbol{\psi}$-GC type maps, and see also [10, 12]. Furthermore, several authors proved the common fixed point theorem in many metric spaces, see [13–19].

An orthogonality notion in metric spaces was introduced by Gordji et al. [20]. The theory of an orthogonal set has general application in a number of mathematical areas, and there are many types of orthogonality, see [21–29].

In this paper, we prove a FPT in an **O**-complete **O**-$\mathfrak{b}$-metric space ($\mathbf{O}_{\mathfrak{b}}$-MS) with **O**-generalized $\boldsymbol{\alpha}$-$\boldsymbol{\psi}$-GC type maps. Moreover, an example and application to an integral equation are given to strengthen our main results.

2. Preliminaries

Throughout this paper, the standard letter $\mathbb{R}$ is the set of all real numbers, $\mathbb{R}^+$ represents the set of all positive real numbers, $\mathbb{N}$ denotes the set of all natural numbers, $\mathbb{N}_0$ denotes the set of all positive natural numbers, and $\mathbb{Z}$ denotes the set of all integers. First, we recall some standard definitions and other results that will be needed in the sequel.

In 1993, Czerwik [2] introduced a $\mathfrak{b}$-metric space as follows:

Definition 2.1. [2] Let Q be a non-void set. Let $\tau\colon Q \times Q \to \mathbb{R}^+$ be called $\mathfrak{b}$-metric on Q if for all $\theta, \mu, \nu \in Q$ the below conditions hold:

(i) $0 \leq \tau(\theta,\mu)$ and $\tau(\theta,\mu) = 0$ iff $\theta = \mu$;

(ii) $\tau(\theta,\mu) = \tau(\mu,\theta)$;

(iii) $\tau(\theta,\mu) \leq \mathfrak{g}[\tau(\theta,\nu) + \tau(\nu,\mu)]$.

Then, (Q,τ) is said to be a $\mathfrak{b}$-metric space (with constant $\mathfrak{g} \geq 1$). In 2014, Karapınar [11] introduced the concept of $\boldsymbol{\alpha}$-regularity as follows:

Definition 2.2. [11] Let (Q,τ) be a $\mathfrak{b}$-metric space and $\boldsymbol{\alpha} : Q \times Q \to Q$ be a map. Q is called $\boldsymbol{\alpha}$-regular if for each sequence $\{\theta_\eta\} \in Q$ such that $\boldsymbol{\alpha}(\theta_\eta, \theta_{\eta+1}) \geq 1, \ \forall \ \eta \in \mathbb{N}$ and $\theta_\eta \to \theta \in Q$ as $\eta \to \infty, \ \exists$ a sub-sequence $\{\theta_{\eta(\gamma)}\}$ of $\{\theta_\eta\}$ with $\boldsymbol{\alpha}(\theta_{\eta(\gamma)}, \theta) \geq 1$, for all $\gamma \in \mathbb{N}$.

The notion of an orthogonal set was presented by Gordji et al. [21].

Definition 2.3. [21] Let Q be a non empty set and $\perp \subseteq Q \times Q$ be a binary relation. If $\perp$ holds with the constraint

$$\exists \ \theta_0 \in Q : (\forall \ \theta \in Q, \ \theta \perp \theta_0) \quad \text{or} \quad (\forall \ \theta \in Q, \ \theta_0 \perp \theta),$$

then $(Q,\perp)$ is said to be an orthogonal set.

Gordji et al. [21] presented the definition of an orthogonal sequence in 2017 as follows.

Definition 2.4. [21] Let $(Q,\perp)$ be an orthogonal set. A sequence $\{\theta_\eta\}_{\eta\in\mathbb{N}}$ is called an orthogonal sequence (**O**-sequence) if

$$(\forall \eta, \theta_\eta \perp \theta_{\eta+1}) \ \text{ or } \ (\forall \eta, \theta_{\eta+1} \perp \theta_\eta).$$

Definition 2.5. A tripled $(Q, \perp, \tau)$ is called an $\mathbf{O}_{\mathfrak{b}}$-MS if $(Q, \perp)$ is an orthogonal set and (Q, τ) is a $\mathfrak{b}$-metric space.

Definition 2.6. Let $(Q, \perp, \tau)$ be an $\mathbf{O}_{\mathfrak{b}}$-MS.

(1) $\{\mu_\eta\}$, an orthogonal sequence in Q, converges at a point μ if

$$\lim_{\eta\to\infty}(E(\mu_\eta, \mu)) = 0.$$

(2) $\{\mu_\eta\}, \{\mu_{\mathfrak{m}}\}$ are orthogonal sequences in Q and are said to be orthogonal-Cauchy sequences if

$$\lim_{\eta,\mathfrak{m}\to\infty}(E(\mu_\eta, \mu_m)) < \infty.$$

Definition 2.7. [20] Let $(Q, \perp, \tau)$ be an $\mathbf{O}_{\mathfrak{b}}$-MS. Then, $E\colon Q \to Q$ is said to be orthogonally continuous in $\mu \in Q$ if, for each $\mathbf{O}$-sequence $\{\mu_\eta\}_{\eta\in\mathbb{N}}$ in Q with $\mu_\eta \to \mu$, we have $E(\mu_\eta) \to E(\mu)$. Also, E is said to be orthogonal continuous on Q if E is orthogonal continuous in each $\mu \in Q$.

Example 2.8. [20] Let $Q = \mathbb{R}$ and suppose $\mu \perp \theta$ if

$$\theta, \mu \in \left(\eta + \frac{1}{3}, \eta + \frac{2}{3}\right)$$

for some $\eta \in \mathbb{Z}$ or $\theta = 0$.

It is clear that $(Q, \perp)$ is an orthogonal set. A map $E : Q \to Q$ is defined as $E(\theta) = [\theta]$. Then, E is orthogonal-continuous on Q, because if $\{\theta_{\mathfrak{r}}\}$ is an arbitrary $\mathbf{O}$-sequence in Q such that $\{\theta_{\mathfrak{r}}\}$ converges to $\theta \in Q$, then the below cases hold:

Case 1: If $\theta_{\mathfrak{r}} = 0 \ \forall \ \mathfrak{r}$, then $\theta = 0$ and $E(\theta_{\mathfrak{r}}) = 0 = E(\theta)$.

Case 2: If $\theta_{\mathfrak{r}_0} \neq 0$ for some $\mathfrak{r}_0$, then there exists $m \in \mathbb{Z}$ such that $\theta_{\mathfrak{r}} \in (m + \frac{1}{3}, m + \frac{2}{3})$, for all $\mathfrak{r} \geq \mathfrak{r}_0$. Thus, $\theta \in [m + \frac{1}{3}, m + \frac{2}{3}]$, and $E(\theta_{\mathfrak{r}}) = m = E(\theta)$.

This means that E is orthogonal-continuous on Q, but it is not continuous on Q.

The concept of orthogonal completeness in metric spaces is defined by Gordji et al. [21] as follows.

Definition 2.9. [21] Let $(Q, \perp, \tau)$ be an orthogonal metric space. Then, Q is said to be $\mathbf{O}$-complete if every orthogonal Cauchy sequence is convergent.

Definition 2.10. [21] Let $(Q, \perp)$ be an orthogonal set. A function $E : Q \to Q$ is called orthogonal-preserving if $E\theta \perp E\mu$ whenever $\theta \perp \mu$.

Ramezani [26] introduced the notion of being orthogonal α-admissible as follows.

Definition 2.11. [26] Let $E : Q \to Q$ be a map and let $\boldsymbol{\alpha} : Q \times Q \to \mathbb{R}^+$ be a function. Then, E is said to be $\mathbf{O}$-$\boldsymbol{\alpha}$-admissible if $\forall\ \theta, \mu \in Q$ with $\theta \perp \mu$

$$\boldsymbol{\alpha}(\theta, \mu) \geq 1 \implies \boldsymbol{\alpha}(E\theta, E\mu) \geq 1.$$

In 2021, Gnanaprakasam et al. [25] introduced the notion of being orthogonal triangular $\boldsymbol{\alpha}$-admissible, defined as below:

Definition 2.12. [25] A self-map $E : Q \to Q$ is called **O**-triangular α-admissible if the following holds:

($E1$) E is **O**-α-admissible,

($E2$) $\alpha(\theta,\nu) \geq 1,\ \ \alpha(\nu,\mu) \geq 1$ *with* $\theta \perp \nu$ *and* $\nu \perp \mu \implies \alpha(\theta,\mu) \geq 1, \forall\ \theta,\mu,\nu \in Q$.

Recently, Popescu [9] has developed the notion of a triangular α-orbital admissible mappings, and we extend it to orthogonal in $\mathbf{O}_\flat$-MS. The following definitions will be needed in the main result.

Definition 2.13. Let $E : Q \to Q$ be a map, and let there be a function $\alpha : Q \times Q \to \mathbb{R}^+$. Then, E is called **O**-α-orbital admissible if the below constraint holds:

($E3$) $\quad \alpha(\theta, E\theta) \geq 1 \quad \implies \alpha(E\theta, E^2\theta) \geq 1, \forall\ \theta \in Q$.

Example 2.14. Let $Q = \{0,1,2,3\}$, $\tau\colon Q \times Q \to \mathbb{R}, \tau(\theta,\mu) = |\theta-\mu|^2, E\colon Q \to Q$ such that $E0 = 0$, $E1 = 2$, $E2 = 1$, $E3 = 3$. $\alpha\colon Q \times Q \to \mathbb{R}$, $\alpha(\theta,\mu) = 1$ if $(\theta,\mu) \in \{(0,1),(0,2),(1,1),(2,2),(1,2),(2,1),(1,3),(2,3)\}$ with $\theta \perp \mu$, and $\alpha(\theta,\mu) = 0$ otherwise.

Since $\alpha(1,E1) = \alpha(1,2) = 1$ and $\alpha(2,E2) = \alpha(2,1) = 1$, we have that E is orthogonal α-orbital admissible.

Definition 2.15. Let $E : Q \to Q$ and $\alpha : Q \times Q \to \mathbb{R}^+$ be maps. Then, E is called **O**-triangular-α-orbital admissible if E is **O**-α-orbital admissible and the following property holds:

($E4$) $\forall\ \theta,\mu \in Q$ with $\theta \perp \mu$, $\quad \alpha(\theta,\mu) \geq 1$ and $\alpha(\mu,E\mu) \geq 1 \implies \alpha(\theta,E\mu) \geq 1$.

Example 2.16. Let $Q = \mathbb{R}, E\mu = \mu^3 + \sqrt[7]{\mu}$, and $\alpha(\mu,\theta) = \mu^5 - \theta^5 + 1$, for all $\mu,\theta \in Q$ with $\mu \perp \theta$. Then, E is an **O**-triangular α-admissible mapping.

Lemma 2.17. *Let $E : Q \to Q$ be an* **O**-*triangular-α-orbital admissible map. Consider that $\exists\ \theta_0 \in Q$ such that $\theta_0 \perp E\theta_0$ and $\alpha(\theta_0,E\theta_0) \geq 1$. An* **O**-*sequence $\{\theta_\eta\}$ is defined by $\theta_{\eta+1} = E\theta_\eta$, $\forall\ \eta \in \mathbb{N}$ with $\theta_\eta \perp E\theta_\eta$ or $E\theta_\eta \perp \theta_\eta$. Then, we get $\alpha(\theta_\eta,\theta_\nu) \geq 1$, $\forall\ \eta,\nu \in \mathbb{N}$ with $\eta < \nu$.*

In this section, inspired by the concept of generalized α-ψ-GC type maps defined by Afshari et al. [28], we introduce a new orthogonal generalized α-ψ-GC type mapping and prove some FPT's for these contraction mappings in an **O**-complete $\mathbf{O}_\flat$-MS.

3. Main results

Let Λ be a set of all increasing and continuous functions, and $\psi \in \Lambda$ is defined as $\psi : \mathbb{R}^+ \to \mathbb{R}^+$, with $\psi^{-1}(\{0\}) = \{0\}$.

Let Γ be the family of all non-decreasing functions $\lambda : [0,\infty) \to [0, \frac{1}{\mathfrak{g}})$ which satisfy the condition

$$\lim_{\eta\to\infty} \lambda(\zeta_\eta) = \frac{1}{\mathfrak{g}} \implies \lim_{\eta\to\infty} \zeta_\eta = 0 \quad \text{for some } \mathfrak{g} \geq 1.$$

First, we explain the definition of an **O**-generalized α-ψ-GC type(A) map in an **O**-complete $\mathbf{O}_\flat$-MS.

Definition 3.1. Let $(Q,\perp,\tau)$ be an **O**-complete $\mathbf{O}_\flat$-MS, and let there be a map $E : Q \to Q$. E is called an **O**-generalized α-ψ-GC type(A) map whenever $\exists\ \alpha : Q \times Q \to \mathbb{R}^+$, and, for $\mathfrak{L} \geq 0$ such that

$$\mathbf{M}(\theta,\mu) = max\left\{\tau(\theta,\mu), \tau(\theta,E\theta), \tau(\mu,E\mu), \frac{\tau(\theta,E\mu)+\tau(\mu,E\theta)}{2\mathfrak{g}}\right\},$$

$$\mathcal{N}(\theta,\mu) = min\{\tau(\theta,E\theta),\tau(\mu,E\theta)\},$$

we have

$$\boldsymbol{\alpha}(\theta,\mu)\boldsymbol{\psi}(\mathfrak{g}^3\tau(E\theta,E\mu)) \leq \lambda(\boldsymbol{\psi}(\mathbf{M}(\theta,\mu)))\boldsymbol{\psi}(\mathbf{M}(\theta,\mu)) + \mathfrak{L}\vartheta(\mathcal{N}(\theta,\mu)), \tag{3.1}$$

for all $\theta,\mu \in \mathcal{Q}$ with $\theta \perp \mu$, where $\lambda \in \Gamma$ and $\boldsymbol{\psi},\vartheta \in \Lambda$.

$$\lambda(\boldsymbol{\psi}(\mathbf{M}(\theta,\mu))) < \frac{1}{\mathfrak{g}} \text{ for all } \theta,\mu \in \mathcal{Q}, \text{ with } \theta \perp \mu \text{ and } \theta \neq \mu.$$

Now, we generalize and improve our FPT from Afshari et al. [28] by introducing the notion of an $\mathbf{O}$-generalized $\boldsymbol{\alpha}$-$\boldsymbol{\psi}$-GC type(A) map in $\mathbf{O}$-complete $\mathbf{O}_\mathfrak{b}$-MS.

Theorem 3.2. *Let $(\mathcal{Q},\perp,\tau)$ be an $\mathbf{O}$-complete $\mathbf{O}_\mathfrak{b}$-MS, and $E:\mathcal{Q}\to\mathcal{Q}$ satisfies the below properties:*

(i) E is orthogonal-preserving,

(ii) E is an $\mathbf{O}$-generalized $\boldsymbol{\alpha}$-$\boldsymbol{\psi}$-GC type(A) map,

(iii) E is $\mathbf{O}$-triangular $\boldsymbol{\alpha}$-orbital admissible,

(iv) $\exists\ \theta_0 \in \mathcal{Q}$ such that $\theta_0 \perp E\theta_0$ and $\boldsymbol{\alpha}(\theta_0,E\theta_0) \geq 1$,

(v) E is $\mathbf{O}$-continuous.

Then, E has a UFP (unique fixed point).

Proof. By the condition (iv), there exists $\theta_0 \in \mathcal{Q}$ such that

$$\theta_0 \perp E(\theta_0) \text{ or } E(\theta_0) \perp \theta_0 \quad \text{and} \quad \boldsymbol{\alpha}(\theta_0 \perp E\theta_0) \geq 1.$$

Let

$$\theta_1 = E(\theta_0), \theta_2 = E(\theta_1) = E^2(\theta_0), \ldots\ldots., \theta_\eta = E(\theta_{\eta-1}) = E^\eta(\theta_0), \theta_{\eta+1} = E(\theta_\eta) = E^{\eta+1}(\theta_0), \ \ \forall\ \eta \in \mathbb{N}\cup\{0\}.$$

If $\theta_{\eta^*} = \theta_{\eta^*+1}$ for $\eta^* \in \mathbb{N}\cup\{0\}$, then θ_{η^*} is a fixed point of E. Therefore, the proof is complete. So, we consider $\theta_\eta \neq \theta_{\eta+1}$. Thus, we have $\tau(E\theta_\eta, E\theta_{\eta+1}) > 0$. Since E is $\mathbf{O}$-preserving, we get

$$\theta_\eta \perp \theta_{\eta+1} \ \text{ or } \ \theta_{\eta+1} \perp \theta_\eta, \ \ \forall\ \eta \in \mathbb{N}\cup\{0\}.$$

We construct an $\mathbf{O}$-sequence $\{\theta_\eta\}$. Since E is an $\mathbf{O}$-generalized $\boldsymbol{\alpha}$-$\boldsymbol{\psi}$ -GC type(A) map, we get

$$\theta_{\eta+1} = E\theta_\eta, \quad \forall\, \eta \in \mathbb{N}_0.$$

Since the map E is an orthogonal triangular α-orbital admissible, by Lemma 2.17, we get

$$\boldsymbol{\alpha}(\theta_\eta,\theta_{\eta+1}) \geq 1, \quad \forall\, \eta \in \mathbb{N}_0. \tag{3.2}$$

By letting $\theta = \theta_{\eta-1}$ and $\mu = \theta_\eta$ in the inequality (3.1), using (3.2) and that $\boldsymbol{\psi}$ is an ascending map, we get

$$\begin{aligned}\boldsymbol{\psi}(\tau(\theta_\eta,\theta_{\eta+1})) &= \boldsymbol{\psi}(\tau(E\theta_{\eta-1},E\theta_\eta))\\ &\le \boldsymbol{\alpha}(\theta_{\eta-1},\theta_\eta)\boldsymbol{\psi}(\mathfrak{g}^3(\tau(E\theta_{\eta-1},E\theta_\eta))\\ &\le \lambda(\boldsymbol{\psi}(\mathbf{M}(\theta_{\eta-1},\theta_\eta)))\boldsymbol{\psi}(\mathbf{M}(\theta_{\eta-1},\theta_\eta)) + \mathfrak{L}\vartheta(\mathcal{N}(\theta_{\eta-1},\theta_\eta)), \end{aligned} \tag{3.3}$$

for all $\eta\in\mathbb{N}$, where

$$\begin{aligned}\mathbf{M}(\theta_{\eta-1},\theta_\eta) &= \max\left\{\tau(\theta_{\eta-1},\theta_\eta),\tau(\theta_{\eta-1},E\theta_{\eta-1}),\tau(\theta_\eta,E\theta_\eta),\frac{\tau(\theta_{\eta-1},E\theta_\eta)+\tau(\theta_\eta,E\theta_{\eta-1})}{2\mathfrak{g}}\right\}\\ &= \max\left\{\tau(\theta_{\eta-1},\theta_\eta),\tau(\theta_{\eta-1},\theta_\eta),\tau(\theta_\eta,\theta_{\eta+1}),\frac{\tau(\theta_{\eta-1},\theta_{\eta+1})+\tau(\theta_\eta,\theta_\eta)}{2\mathfrak{g}}\right\}\\ &= \max\left\{\tau(\theta_{\eta-1},\theta_\eta),\tau(\theta_\eta,\theta_{\eta+1}),\frac{\tau(\theta_{\eta-1},\theta_{\eta+1})}{2\mathfrak{g}}\right\}\end{aligned}$$

and

$$\mathcal{N}(\theta_{\eta-1},\theta_\eta) = \min\{\tau(\theta_{\eta-1},E\theta_{\eta-1}),\tau(\theta_\eta,E\theta_{\eta-1})\} = \min\{\tau(\theta_{\eta-1},\theta_\eta),\tau(\theta_\eta,\theta_\eta)\} = 0. \tag{3.4}$$

Since

$$\frac{\tau(\theta_{\eta-1},\theta_{\eta+1})}{2\mathfrak{g}} \le \frac{\mathfrak{g}[\tau(\theta_{\eta-1},\theta_\eta)+\tau(\theta_\eta,\theta_{\eta+1})]}{2\mathfrak{g}} \le \max\{\tau(\theta_{\eta-1},\theta_\eta)+\tau(\theta_\eta,\theta_{\eta+1})\},$$

we get

$$\mathbf{M}(\theta_{\eta-1},\theta_\eta) \le max\{\tau(\theta_{\eta-1},\theta_\eta),\tau(\theta_\eta,\theta_{\eta+1})\}. \tag{3.5}$$

Taking (3.5) and (3.4) into account, (3.3) yields

$$\begin{aligned}\boldsymbol{\psi}(\tau(\theta_\eta,\theta_{\eta+1})) &\le \boldsymbol{\psi}(\mathfrak{g}^3\tau(\theta_\eta,\theta_{\eta+1}))\\ &\le \boldsymbol{\alpha}(\theta_{\eta-1},\theta_\eta)\boldsymbol{\psi}(\mathfrak{g}^3\tau(\theta_\eta,\theta_{\eta+1}))\\ &\le \lambda(\boldsymbol{\psi}(\mathbf{M}(\theta_{\eta-1},\theta_\eta)))\boldsymbol{\psi}(\max\{\tau(\theta_{\eta-1},\theta_\eta),\tau(\theta_\eta,\theta_{\eta+1})\}).\end{aligned} \tag{3.6}$$

If $\eta\in\mathbb{N}$, we get $\max\{\tau(\theta_{\eta-1},\theta_\eta),\tau(\theta_\eta,\theta_{\eta+1})\} = \tau(\theta_\eta,\theta_{\eta+1})$, and then by (3.6), we get

$$\boldsymbol{\psi}(\tau(\theta_\eta,\theta_{\eta+1})) \le \lambda(\boldsymbol{\psi}(\mathbf{M}(\theta_{\eta-1},\theta_\eta)))\boldsymbol{\psi}(\tau(\theta_\eta,\theta_{\eta+1})) < \frac{1}{\mathfrak{g}}\boldsymbol{\psi}(\tau(\theta_\eta,\theta_{\eta+1})) < \boldsymbol{\psi}(\tau(\theta_\eta,\theta_{\eta+1})),$$

which is a contradiction. Thus, from (3.6) we conclude that

$$\begin{aligned}\boldsymbol{\psi}(\tau(\theta_\eta,\theta_{\eta+1})) &\le \lambda(\boldsymbol{\psi}(\mathbf{M}(\theta_{\eta-1},\theta_\eta)))\boldsymbol{\psi}(\tau(\theta_{\eta-1},\theta_\eta))\\ &< \frac{1}{\mathfrak{g}}\boldsymbol{\psi}(\tau(\theta_{\eta-1},\theta_\eta)) < \boldsymbol{\psi}(\tau(\theta_{\eta-1},\theta_\eta)),\ \forall\ \eta\in\mathbb{N}.\end{aligned} \tag{3.7}$$

Hence, $\{\boldsymbol{\psi}(\tau(\theta_\eta,\theta_{\eta+1}))\}$ is a positive non-increasing sequence. Since $\boldsymbol{\psi}$ is ascending, the sequence $\{\tau(\theta_\eta,\theta_{\eta+1})\}$ is decreasing. Consequently, $\exists\ \epsilon\ge 0$ such that $\lim_{\eta\to\infty}\tau(\theta_\eta,\theta_{\eta+1}) = \epsilon$. We claim that $\epsilon = 0$. Suppose, on the contrary, that

$$\lim_{\eta\to\infty}\tau(\theta_\eta,\theta_{\eta+1}) = \epsilon \succ 0.$$

Since $\mathfrak{g} \geq 1$, (3.7) can be approximated as

$$\frac{1}{\mathfrak{g}}\boldsymbol{\psi}(\tau(\theta_\eta, \theta_{\eta+1})) \leq \boldsymbol{\psi}(\tau(\theta_\eta, \theta_{\eta+1})) \leq \lambda(\boldsymbol{\psi}(\mathbf{M}(\theta_{\eta-1}, \theta_\eta)))\boldsymbol{\psi}(\tau(\theta_{\eta-1}, \theta_\eta)). \tag{3.8}$$

With regard to (3.2), in (3.8) we get

$$\frac{1}{\mathfrak{g}}\frac{\boldsymbol{\psi}(\tau(\theta_\eta, \theta_{\eta+1}))}{\boldsymbol{\psi}(\tau(\theta_{\eta-1}, \theta_\eta))} \leq \lambda(\boldsymbol{\psi}(\mathbf{M}(\theta_{\eta-1}, \theta_\eta))) < \frac{1}{\mathfrak{g}}.$$

This yields $\lim_{\eta\to\infty} \lambda(\boldsymbol{\psi}(\mathbf{M}(\theta_{\eta-1}, \theta_\eta))) = \frac{1}{\mathfrak{g}}$. Since $\lambda \in \varGamma$, we get $\lim_{\eta\to\infty} \boldsymbol{\psi}(\mathbf{M}(\theta_{\eta-1}, \theta_\eta)) = 0$. We simplify that:

$$\lim_{\eta\to\infty} \boldsymbol{\psi}(\tau(\theta_\eta, \theta_{\eta+1})) = 0.$$

Thus, by the fact that $\tau(\theta_\eta, \theta_{\eta+1}) \to \epsilon$ and the continuity of $\boldsymbol{\psi}$, we get $\boldsymbol{\psi}(\epsilon) = 0$. Since $\boldsymbol{\psi}^{-1}(\{0\}) = \{0\}$, we have $\epsilon = 0$, and this is a contradiction. Thus, we get

$$\lim_{\eta\to\infty} \tau(\theta_\eta, \theta_{\eta+1}) = 0. \tag{3.9}$$

Now, we claim that $\lim_{\pi,\eta\to\infty} \tau(\theta_\eta, \theta_\pi) = 0$.

Consider, on the contrary, that $\exists\ \delta \succ 0$, and orthogonal subsequences $\{\theta_{\pi_\mathfrak{j}}\}$, and $\{\theta_{\eta_\mathfrak{j}}\}$ of $\{\theta_\eta\}$, with $\eta_\mathfrak{j} \succ \pi_\mathfrak{j} \geq \mathfrak{j}$, such that

$$\tau(\theta_{\pi_\mathfrak{j}}, \theta_{\eta_\mathfrak{j}}) \geq \delta. \tag{3.10}$$

Additionally, for every $\pi_\mathfrak{j}$, we choose the smallest integer $\eta_\mathfrak{j}$ to fulfill (3.10), and $\eta_\mathfrak{j} \succ \pi_\mathfrak{j} \geq \mathfrak{j}$. Then, we get

$$\tau(\theta_{\pi_\mathfrak{j}}, \theta_{\eta_{\mathfrak{j}-1}}) < \delta. \tag{3.11}$$

From (3.10) and the condition (iii) in Definition 2.1, we get

$$\begin{aligned}\delta \leq \tau(\theta_{\eta_\mathfrak{j}}, \theta_{\pi_\mathfrak{j}}) &\leq \mathfrak{g}\tau(\theta_{\eta_\mathfrak{j}}, \theta_{\eta_{\mathfrak{j}+1}}) + \mathfrak{g}\tau(\theta_{\eta_{\mathfrak{j}+1}}, \theta_{\pi_\mathfrak{j}})\\ &\leq \mathfrak{g}\tau(\theta_{\eta_\mathfrak{j}}, \theta_{\eta_{\mathfrak{j}+1}}) + \mathfrak{g}^2\tau(\theta_{\eta_{\mathfrak{j}+1}}, \theta_{\pi_{\mathfrak{j}+1}}) + \mathfrak{g}^2\tau(\theta_{\pi_{\mathfrak{j}+1}}, \theta_{\pi_\mathfrak{j}}).\end{aligned} \tag{3.12}$$

Taking $\mathfrak{j} \to \infty$ and (3.9) into account, Eq (3.12) yields

$$\begin{aligned}\delta &\leq \mathfrak{g}\tau(\theta_{\eta_\mathfrak{j}}, \theta_{\eta_{\mathfrak{j}+1}}) + \mathfrak{g}^2\tau(\theta_{\eta_{\mathfrak{j}+1}}, \theta_{\pi_{\mathfrak{j}+1}}) + \mathfrak{g}^2\tau(\theta_{\pi_{\mathfrak{j}+1}}, \theta_{\pi_\mathfrak{j}})\\ &\leq \lim_{\mathfrak{j}\to\infty}(\mathfrak{g}\tau(\theta_{\eta_\mathfrak{j}}, \theta_{\eta_{\mathfrak{j}+1}}) + \mathfrak{g}^2\tau(\theta_{\eta_{\mathfrak{j}+1}}, \theta_{\pi_{\mathfrak{j}+1}}) + \mathfrak{g}^2\tau(\theta_{\pi_{\mathfrak{j}+1}}, \theta_{\pi_\mathfrak{j}}))\\ &\leq \mathfrak{g}^2\lim_{\mathfrak{j}\to\infty}(\tau(\theta_{\eta_{\mathfrak{j}+1}}, \theta_{\pi_{\mathfrak{j}+1}})),\\ \frac{\delta}{\mathfrak{g}^2} &\leq \limsup_{\mathfrak{j}\to\infty} \tau(\theta_{\eta_{\mathfrak{j}+1}}, \theta_{\pi_{\mathfrak{j}+1}}),\end{aligned} \tag{3.13}$$

where $\lim_{j\to\infty}\tau(\theta_{\eta_j},\theta_{\eta_{j+1}})=\lim_{j\to\infty}\tau(\theta_{\pi_{j+1}},\theta_{\pi_j})=0$. By Lemma 2.17, $\boldsymbol{\alpha}(\theta_{\pi_j},\theta_{\eta_j})\geq 1$. Followed by (3.1), we get

$$\begin{aligned}\boldsymbol{\psi}(\tau(\theta_{\eta_{j+1}},\theta_{\pi_{j+1}}))&=\boldsymbol{\psi}(\tau(E\theta_{\eta_j},E\theta_{\pi_j}))\\&\leq\boldsymbol{\psi}(\mathfrak{g}^3\tau(E\theta_{\eta_j},E\theta_{\pi_j}))\\&\leq\boldsymbol{\alpha}(\theta_{\pi_j},\theta_{\eta_j})\boldsymbol{\psi}(\mathfrak{g}^3\tau(E\theta_{\eta_j},E\theta_{\pi_j}))\\&\leq\lambda(\boldsymbol{\psi}(\mathbf{M}(\theta_{\eta_j},\theta_{\pi_j})))\boldsymbol{\psi}(\mathbf{M}(\theta_{\eta_j},\theta_{\pi_j}))+\mathfrak{L}\vartheta(\tau(\theta_{\pi_j},E\theta_{\eta_j})),\end{aligned}\tag{3.14}$$

where

$$\begin{aligned}\mathbf{M}(\theta_{\eta_j},\theta_{\pi_j})&=\max\left\{\tau(\theta_{\eta_j},\theta_{\pi_j}),\tau(\theta_{\eta_j},E\theta_{\eta_j}),\tau(\theta_{\pi_j},E\theta_{\pi_j}),\frac{\tau(\theta_{\eta_j},E\theta_{\pi_j})+\tau(\theta_{\pi_j},E\theta_{\eta_j})}{2\mathfrak{g}}\right\}\\&=\max\left\{\tau(\theta_{\eta_j},\theta_{\pi_j}),\tau(\theta_{\eta_j},\theta_{\eta_{j+1}}),\tau(\theta_{\pi_j},\theta_{\pi_{j+1}}),\frac{\tau(\theta_{\eta_j},\theta_{\pi_{j+1}})+\tau(\theta_{\pi_j},\theta_{\eta_{j+1}})}{2\mathfrak{g}}\right\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{N}(\theta_{\eta_j},\theta_{\pi_j})&=\min\{\tau(\theta_{\eta_j},E\theta_{\eta_j}),\tau(\theta_{\pi_j},E\theta_{\eta_j})\}\\&=\min\{\tau(\theta_{\eta_j},\theta_{\eta_{j+1}}),\tau(\theta_{\pi_j},\theta_{\eta_{j+1}})\}.\end{aligned}$$

Notice that

$$\frac{\tau(\theta_{\eta_j},\theta_{\pi_{j+1}})+\tau(\theta_{\pi_j},\theta_{\eta_{j+1}})}{2\mathfrak{g}}\leq\frac{\mathfrak{g}[\tau(\theta_{\eta_j},\theta_{\pi_j})+\tau(\theta_{\pi_j},\theta_{\pi_{j+1}})]+\mathfrak{g}[\tau(\theta_{\pi_j},\theta_{\eta_j})+\tau(\theta_{\eta_j},\theta_{\eta_{j+1}})]}{2\mathfrak{g}}\tag{3.15}$$

and

$$\tau(\theta_{\eta_j},\theta_{\pi_j})\leq\mathfrak{g}[\tau(\theta_{\eta_j},\theta_{\eta_{j-1}})+\tau(\theta_{\eta_{j-1}},\theta_{\pi_j})]<\mathfrak{g}\tau(\theta_{\eta_j},\theta_{\eta_{j-1}})+\mathfrak{g}\delta.\tag{3.16}$$

Taking (3.11), (3.15) and (3.16) into account, we find that

$$\limsup_{j\to\infty}\mathbf{M}(\theta_{\eta_j},\theta_{\pi_j})\leq\mathfrak{g}\delta,\tag{3.17}$$

$$\limsup_{j\to\infty}\mathcal{N}(\theta_{\eta_j},\theta_{\pi_j})=0.\tag{3.18}$$

By allowing the upper limit to be $j\to\infty$ and using constraint ($E4$), (3.13), (3.17) and (3.18), inequality (3.14) becomes

$$\begin{aligned}\frac{1}{\mathfrak{g}}\boldsymbol{\psi}(\mathfrak{g}\delta)\leq\boldsymbol{\psi}(\mathfrak{g}\delta)&\leq\limsup_{j\to\infty}\boldsymbol{\psi}(\mathfrak{g}^3\tau(\theta_{\eta_{j+1}},\theta_{\pi_{j+1}}))\\&\leq\limsup_{j\to\infty}\boldsymbol{\alpha}(\theta_{\pi_j},\theta_{\eta_j})\boldsymbol{\psi}(\mathfrak{g}^3\tau(\theta_{\eta_{j+1}},\theta_{\pi_{j+1}}))\\&\leq\limsup_{j\to\infty}\boldsymbol{\alpha}(\theta_{\pi_j},\theta_{\eta_j})\boldsymbol{\psi}(\mathfrak{g}^3\tau(E\theta_{\eta_j},E\theta_{\pi_j}))\\&\leq\limsup_{j\to\infty}[\lambda(\boldsymbol{\psi}(\mathbf{M}(\theta_{\eta_j},\theta_{\pi_j})))\boldsymbol{\psi}(\mathbf{M}(\theta_{\eta_j},\theta_{\pi_j}))+\mathfrak{L}\vartheta(\mathcal{N}(\tau(\theta_{\eta_j},\theta_{\pi_j})))]\end{aligned}$$

$$\leq \psi(\mathfrak{g}\delta) \limsup_{\mathrm{j}\to\infty} \lambda(\psi(\mathbf{M}(\theta_{\eta_{\mathrm{j}}}, \theta_{\pi_{\mathrm{j}}})))$$
$$\leq \frac{1}{\mathfrak{g}}\psi(\mathfrak{g}\delta).$$

Then,

$$\limsup_{\mathrm{j}\to\infty} \lambda(\psi(\mathbf{M}(\theta_{\eta_{\mathrm{j}}}, \theta_{\pi_{\mathrm{j}}}))) = \frac{1}{\mathfrak{g}}.$$

Due to the fact that $\lambda \in \Gamma$, we get

$$\limsup_{\mathrm{j}\to\infty} \lambda(\psi(\mathbf{M}(\theta_{\eta_{\mathrm{j}}}, \theta_{\pi_{\mathrm{j}}}))) = 0.$$

Thus, we assume that

$$\lim_{\mathrm{j}\to\infty} \lambda(\psi(\tau(\theta_{\eta_{\mathrm{j}}}, \theta_{\pi_{\mathrm{j}}}))) = 0.$$

Consequently, due to the continuity of ψ and $\psi^{-1}(\{0\}) = \{0\}$, we obtain

$$\lim_{\mathrm{j}\to\infty} \tau(\theta_{\eta_{\mathrm{j}}}, \theta_{\pi_{\mathrm{j}}}) = 0,$$

which contradicts (3.10). Therefore, $\{\theta_\eta\}$ is a $\mathbf{O}$-Cauchy sequence in Q. Since Q is an $\mathbf{O}$-complete $\mathbf{O}_{\mathfrak{b}}$-MS, $\exists\ \theta^* \in Q$ such that $\lim_{\eta\to\infty} \theta_\eta = \theta^*$. The map E is $\mathbf{O}$-continuous, and it is obvious that $E\theta^* = \theta^*$. Hence, θ^* is a fixed point of E.

Now, we prove θ^* is a UFP of E. Suppose μ^* is another fixed point of E. If $\theta_\eta \to \mu^*$ as $\eta \to \infty$, we get $\theta^* = \mu^*$.

If $\lim_{\eta\to\infty}\{\theta_\eta\} \nrightarrow \mu^*$, there is an orthogonal sub-sequence $\{\theta_{\eta_\gamma}\}$ such that $E\theta_{\eta_\gamma} \neq \mu^*$, $\forall\ \gamma \in \mathbb{N}$. By the choice of θ_0, we get

$$\theta_0 \perp \mu^* \ \text{(or)}\ \mu^* \perp \theta_0.$$

Since E is $\mathbf{O}$-preserving and $E^\eta\mu^* = \mu^*$, for all $\eta \in \mathbb{N}$, we get

$$E^\eta\theta_0 \perp \mu^* \ \text{(or)}\ \mu^* \perp E^\eta\theta_0, \ \forall\ \eta \in \mathbb{N}.$$

Since E is an $\mathbf{O}$-generalized α-ψ-GC type(A) map, we get

$$\psi(\tau(E^{\eta_\gamma}\theta_0, \mu^*)) = \psi(\tau(E^{\eta_\gamma}\theta_0, E^{\eta_\gamma}\mu^*)) \leq \psi(\tau(\theta_0, \mu^*)), \quad \gamma \in \mathbb{N}.$$

This implies $\psi(\tau(E^{\eta_\gamma}\theta_0, \mu^*)) \to -\infty$ as $\gamma \to \infty$. This yields that $\theta_\eta \to \mu^*$ as $\eta \to \infty$, which is a contradiction. Hence, E has a UFP. □

We replace the continuity of map E in the above theorem by a suitable condition on Q.

Theorem 3.3. *Let $(Q, \perp, \tau)$ be an $\mathbf{O}$-complete $\mathbf{O}$-$\mathfrak{b}$-metric space, and $E : Q \to Q$ fulfills the following properties:*

(i) E is orthogonal preserving,

(ii) E is an $\mathbf{O}$*-generalized* $\boldsymbol{\alpha}$*-*$\boldsymbol{\psi}$ *-GC type(A) map,*

(iii) E is $\mathbf{O}$*-triangular-*α *orbital admissible,*

(iv) $\exists\ \theta_0 \in Q$ *such that* $\theta_0 \perp E\theta_0$ *and* $\boldsymbol{\alpha}(\theta_0, E\theta_0) \geq 1$,

(v) E is $\mathbf{O}$*-*$\boldsymbol{\alpha}$*- regular.*

Then, E has a UFP.

Proof. From the proof of Theorem 3.2, we conclude that $\lim\limits_{\eta\to\infty} \theta_\eta = \theta^*$. If Q is $\mathbf{O}$-$\boldsymbol{\alpha}$-regular, then $\boldsymbol{\alpha}(\theta_\eta, \theta_{\eta+1}) \geq 1$, $\exists$ a subsequence $\{\theta_{\eta_\gamma}\}$ of $\{\theta_\eta\}$ such that

$$\boldsymbol{\alpha}(\theta_{\eta_\gamma}, \theta^*) \geq 1, \quad \forall\ \gamma \in \mathbb{N}. \tag{3.19}$$

By the (iii) in Definition 2.1, we get

$$\tau(\theta^*, E\theta^*) \leq \mathfrak{g}\tau(\theta^*, \theta_{\eta_{\gamma+1}}) + \mathfrak{g}\tau(\theta_{\eta_{\gamma+1}}, E\theta^*) = \mathfrak{g}\tau(\theta^*, \theta_{\eta_{\gamma+1}}) + \mathfrak{g}\tau(E\theta_{\eta_\gamma}, E\theta^*).$$

Letting $\gamma \to \infty$, yields

$$\tau(\theta^*, E\theta^*) \leq \liminf_{\gamma\to\infty} \mathfrak{g}\tau(E\theta_{\eta_\gamma}, E\theta^*). \tag{3.20}$$

Using that $\boldsymbol{\psi} \in \Lambda$, (3.19) and (3.20), we have

$$\begin{aligned}
\boldsymbol{\psi}(\mathfrak{g}^2\tau(\theta^*, E\theta^*)) &\leq \lim_{\gamma\to\infty} \boldsymbol{\psi}(\mathfrak{g}^3\tau(E\theta_{\eta_\gamma}, E\theta^*)) \\
&\leq \lim_{\gamma\to\infty} \boldsymbol{\alpha}(\theta_{\eta_{\gamma+1}}, \theta^*)\boldsymbol{\psi}(\mathfrak{g}^3\tau(E\theta_{\eta_\gamma}, E\theta^*)) \\
&\leq \lim_{\gamma\to\infty} [\lambda(\boldsymbol{\psi}(\mathbf{M}(\theta_{\eta_\gamma}, \theta^*)))\boldsymbol{\psi}(\mathbf{M}(\theta_{\eta_\gamma}, \theta^*)) + \mathfrak{L}\vartheta(\mathcal{N}(\theta_{\eta_\gamma}, \theta^*))].
\end{aligned} \tag{3.21}$$

We have

$$\begin{aligned}
\mathbf{M}(\theta_{\eta_\gamma}, \theta^*) &= \max\left\{\tau(\theta_{\eta_\gamma}, \theta^*), \tau(\theta_{\eta_\gamma}, E\theta_{\eta_\gamma}), \tau(\theta^*, E\theta^*), \frac{\tau(\theta_{\eta_\gamma}, E\theta^*) + \tau(\theta^*, E\theta_{\eta_\gamma})}{2\mathfrak{g}}\right\} \\
&= \max\left\{\tau(\theta_{\eta_\gamma}, \theta^*), \tau(\theta_{\eta_\gamma}, \theta_{\eta_{\gamma+1}}), \tau(\theta^*, E\theta^*), \frac{\tau(\theta_{\eta_\gamma}, E\theta^*) + \tau(\theta^*, \theta_{\eta_{\gamma+1}})}{2\mathfrak{g}}\right\}
\end{aligned}$$

and

$$\mathcal{N}(\theta_{\eta_\gamma}, \theta^*) = \min\{\tau(\theta_{\eta_\gamma}, E\theta_{\eta_\gamma}), \tau(\theta^*, E\theta_{\eta_\gamma})\} = \min\{\tau(\theta_{\eta_\gamma}, \theta_{\eta_{\gamma+1}}), \tau(\theta^*, \theta_{\eta_{\gamma+1}})\}.$$

Recall that

$$\frac{\tau(\theta_{\eta_\gamma}, E\theta^*) + \tau(\theta^*, \theta_{\eta_{\gamma+1}})}{2\mathfrak{g}} \leq \frac{\mathfrak{g}\tau(\theta_{\eta_\gamma}, \theta^*) + \mathfrak{g}\tau(\theta^*, E\theta^*) + \tau(\theta^*, \theta_{\eta_{\gamma+1}})}{2\mathfrak{g}}.$$

Then, by (3.9), we get

$$\limsup_{\gamma\to\infty} \frac{\tau(\theta_{\eta_\gamma}, E\theta^*) + \tau(\theta^*, \theta_{\eta_{\gamma+1}})}{2\mathfrak{g}} \le \frac{\tau(\theta^*, E\theta^*)}{2}.$$

When $\gamma \to \infty$, we deduce

$$\lim_{\gamma\to\infty} \mathbf{M}(\theta_{\eta_\gamma}, \theta^*) = \tau(\theta^*, E\theta^*),$$
$$\lim_{\gamma\to\infty} \mathcal{N}(\theta_{\eta_\gamma}, \theta^*) = 0.$$

Since $\lambda(\boldsymbol{\psi}(\mathbf{M}(\theta_{\eta_\gamma}, \theta^*))) \le \frac{1}{\mathfrak{g}}$, for all $\gamma \in \mathbb{N}$, from (3.21), we obtain

$$\boldsymbol{\psi}(\mathfrak{g}^2\tau(\theta^*, E\theta^*)) \le \frac{1}{\mathfrak{g}}\boldsymbol{\psi}(\tau(\theta^*, E\theta^*)) \le \boldsymbol{\psi}(\tau(\theta^*, E\theta^*)).$$

Hence, $\tau(\theta^*, E\theta^*) = 0$, that is $E\theta^* = \theta^*$. Therefore, E has a fixed point.

Now, we prove θ^* is a UFP of E. Suppose μ^* is another fixed point of E. If $\theta_\eta \to \mu^*$ as $\eta \to \infty$, we get $\theta^* = \mu^*$.

If $\lim_{\eta\to\infty}\{\theta_\eta\} \nrightarrow \mu^*$, there is a subsequence $\{\theta_{\eta_\gamma}\}$ such that $E\theta_{\eta_\gamma} \neq \mu^*$, $\forall\, \gamma \in \mathbb{N}$. By the choice θ_0, we get

$$\theta_0 \perp \mu^* \ \text{ or } \ \mu^* \perp \theta_0.$$

Since E is $\mathbf{O}$-preserving and $E^\eta\mu^* = \mu^*$, $\forall\, \eta \in \mathbb{N}$, we get

$$E^\eta\theta_0 \perp \mu^* \ \text{ or } \ \mu^* \perp E^\eta\theta_0, \ \forall \ \eta \in \mathbb{N}.$$

Since E is an $\mathbf{O}$-generalized $\boldsymbol{\alpha}$-$\boldsymbol{\psi}$ -GC type(A) map, we get

$$\boldsymbol{\psi}(\tau(E^{\eta_\gamma}\theta_0, \mu^*)) = \boldsymbol{\psi}(\tau(E^{\eta_\gamma}\theta_0, E^{\eta_\gamma}\mu^*)) \le \boldsymbol{\psi}(\tau(\theta_0, \mu^*)), \quad \gamma \in \mathbb{N}.$$

This implies $\boldsymbol{\psi}(\tau(E^{\eta_\gamma}\theta_0, \mu^*)) \to -\infty$ as $\gamma \to \infty$. This yields that $\theta_\eta \to \mu^*$ as $\eta \to \infty$, and this is contradiction. Hence, E has a UFP. □

We initiate the definition of an $\mathbf{O}$-generalized $\boldsymbol{\alpha}$-$\boldsymbol{\psi}$-GC type(B) map as follows:

Definition 3.4. Let $(Q, \perp, \tau)$ be an $\mathbf{O}$-complete $\mathbf{O}_\mathfrak{b}$-MS and let $E : Q \to Q$ be a map. E is called $\mathbf{O}$-generalized $\boldsymbol{\alpha}$-$\boldsymbol{\psi}$-GC map of type(B) whenever $\exists\ \boldsymbol{\alpha} : Q \times Q \to \mathbb{R}^+$ such that for all $\theta, \mu \in Q$ with $\theta \perp \mu$ or $\mu \perp \theta$,

$$\boldsymbol{\alpha}(\theta, \mu)\boldsymbol{\psi}(\mathfrak{g}^3\tau(E\theta, E\mu)) \le \lambda(\boldsymbol{\psi}(\mathbf{M}(\theta, \mu)))\boldsymbol{\psi}(\mathbf{M}(\theta, \mu)), \tag{3.22}$$

where

$$\mathbf{M}(\theta, \mu) = \max\left\{\tau(\theta, \mu), \tau(\theta, E\theta), \tau(\mu, E\mu), \frac{\tau(\theta, E\mu) + \tau(\mu, E\theta)}{2\mathfrak{g}}\right\},$$

$\lambda \in \Gamma$ and $\boldsymbol{\psi} \in \Lambda$.

Now, we generalize and improve our fixed point theorem from Afshari et al. [28] by introducing the notion of an **O**-generalized $\boldsymbol{\alpha}$-$\boldsymbol{\psi}$-GC type(B) map in **O**-complete $\mathbf{O}_{\mathfrak{b}}$-MS.

Theorem 3.5. *Let $(Q, \perp, \tau)$ be an* **O***-complete* $\mathbf{O}_{\mathfrak{b}}$*-MS, and let $E : Q \to Q$ satisfy the below properties:*

(i) E is orthogonal preserving,

(ii) E is an **O***-generalized* $\boldsymbol{\alpha}$-$\boldsymbol{\psi}$ *-GC map of type(B),*

(iii) E is **O***-triangular-α orbital admissible,*

(iv) $\exists\ \theta_0 \in Q$ *such that* $\theta_0 \perp E\theta_0$ *and* $\boldsymbol{\alpha}(\theta_0, E\theta_0) \geq 1$,

(v) E is **O***-continuous or Q is* **O**-$\boldsymbol{\alpha}$*-regular.*

Then, E has a UFP.

Now, we provide the example for Theorem 3.3.

Example 3.6. Let Q be a set of Lebesgue measurable functions [0,1] such that $\int_0^1 |\theta(\zeta)|\tau\zeta < 1$. Define a relation $\perp$ on Q by

$$\theta \perp \mu \ \ if \ \ \theta(\zeta)\mu(\zeta) \leq \theta(\zeta) \vee \mu(\zeta),$$

where $\theta(\zeta) \vee \mu(\zeta) = \theta(\zeta)$ or $\mu(\zeta)$. Define $\tau\colon Q \times Q \to \mathbb{R}^+$ by

$$\tau(\theta, \mu) = \left(\int_0^1 |\theta(\zeta) - \mu(\zeta)|\tau\zeta\right)^2.$$

Then, (Q, τ) is an **O**-complete $\mathbf{O}_{\mathfrak{b}}$-MS with $\mathfrak{g} = 2$. The operator $E\colon Q \times Q \to \mathbb{R}^+$ is defined by

$$E\theta(\zeta) = \frac{1}{4}In(1 + |\theta(\zeta)|).$$

Consider the map $\boldsymbol{\alpha}\colon Q \times Q \to \mathbb{R}^+$, with $\boldsymbol{\lambda}\colon \mathbb{R}^+ \to [0, \frac{1}{2})$ and $\boldsymbol{\psi}\colon \mathbb{R}^+ \to \mathbb{R}^+$, defined by

$$\boldsymbol{\alpha}(\theta, \mu) = \begin{cases} 1, & if\ \theta(\zeta) \geq \mu(\zeta),\ \forall\ \zeta \in [0, 1], \\ 0, & otherwise. \end{cases}$$

$$\lambda(\zeta) = \frac{(In(1 + \sqrt{\zeta}))^2}{2\zeta} \text{ and } \psi(\zeta) = \zeta.$$

Obviously, $\psi \in \Lambda$, and $\lambda \in \Gamma$. Moreover, E is an **O**-triangular-α orbital admissible map, and $\boldsymbol{\alpha}(1, E1) \geq 1$.

Now, we prove E is an **O**-generalized $\boldsymbol{\alpha}$-$\boldsymbol{\psi}$-GC type(A) map. Certainly, $\forall\ \zeta \in [0, 1]$, we get

$$\begin{aligned}\sqrt{\boldsymbol{\alpha}(\theta(\zeta), \mu(\zeta))\boldsymbol{\psi}(\mathfrak{g}^3\tau(E\theta(\zeta), E\mu(\zeta))} &\leq \sqrt{2^3\Big(\int_0^1 |E\theta(\zeta) - E\mu(\zeta|\tau\zeta\Big)^2} \\ &\leq 2\sqrt{2}\int_0^1 \Big|\frac{1}{4}In(1 + |\theta(\zeta)|) - \frac{1}{4}In(1 + |\mu(\zeta)|)\Big|\tau\zeta\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2}} \int_0^1 \Big| In\Big(\frac{1 + |\theta(\zeta)|}{1 + |\mu(\zeta)|}\Big)\Big| \tau\zeta \\
&= \frac{1}{\sqrt{2}} \int_0^1 \Big| In\Big(1 + \frac{|\theta(\zeta)| - |\mu(\zeta)|}{1 + |\mu(\zeta)|}\Big)\Big| \tau\zeta \\
&\leq \frac{1}{\sqrt{2}} \int_0^1 |In(1 + |\theta(\zeta)| - |\mu(\zeta)|)| \tau\zeta,
\end{aligned}
$$

and we have

$$
\begin{aligned}
\int_0^1 |In(1 + |\theta(\zeta)| - |\mu(\zeta)|)|\tau\zeta &\leq In\Big(\int_0^1 |(1 + |\theta(\zeta)| - |\mu(\zeta)|)|\tau\zeta\Big) \\
&= In\Big(1 + \int_0^1 |\theta(\zeta) - \mu(\zeta)|\tau\zeta\Big).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sqrt{\boldsymbol{\alpha}(\theta(\zeta), \mu(\zeta))\boldsymbol{\psi}(\mathfrak{g}^3\tau(E\theta(\zeta), E\mu(\zeta))} &\leq \frac{1}{\sqrt{2}} In\Big(1 + \int_0^1 |\theta(\zeta) - \mu(\zeta)|\tau\zeta\Big) \\
&\leq \frac{1}{\sqrt{2}} In(1 + \sqrt{\tau(\theta, \mu)}).
\end{aligned}
$$

So, we get

$$
\begin{aligned}
\boldsymbol{\alpha}(\theta(\zeta), \mu(\zeta))\boldsymbol{\psi}(\mathfrak{g}^3\tau(E\theta(\zeta), E\mu(\zeta)) &\leq \frac{1}{2}(In(1 + \sqrt{\tau(\theta, \mu)}))^2 \\
&\leq \frac{1}{2}(In(1 + \sqrt{\mathbf{M}(\theta, \mu)}))^2 \\
&\leq \frac{(In(1 + \sqrt{\mathbf{M}(\theta, \mu)}))^2}{2\mathbf{M}(\theta, \mu)}\mathbf{M}(\theta, \mu) \\
&= \lambda(\boldsymbol{\psi}(\mathbf{M}(\theta, \mu)))\boldsymbol{\psi}(\mathbf{M}(\theta, \mu)).
\end{aligned}
$$

Hence by Theorem 3.3, we get that E has a UFP.

4. Application

As an application of Theorem 3.2, we find an existence and uniqueness result of the following type of integral equation:

$$
\omega(\theta) = \lambda(\theta) + \int_0^{\mathfrak{a}} E(\theta, \mathfrak{s})\mathcal{H}(\theta, \mathfrak{s}, \omega(\mathfrak{s}))d\mathfrak{s}, \quad \theta \in [0, \mathfrak{a}], \mathfrak{a} > 0. \tag{4.1}
$$

Consider $Q = C([0, \mathfrak{a}], \mathbb{R})$ to be the real continuous functions on $[0, \mathfrak{a}]$, and a mapping $\mathcal{D} : Q \to Q$ is defined by

$$
\tau(\omega, \mu) = \max_{0 \leq \theta \leq \mathfrak{a}} |\omega(\theta) - \mu(\theta)|^2, \quad \omega, \mu \in Q. \tag{4.2}
$$

Obviously, (Q, τ) is a complete b-metric space, and $\omega(\theta)$ is a solution of integral equation (4.1) iff $\omega(\theta)$ is a fixed point of $\mathcal{D}$.

Theorem 4.1. *Suppose the following.*

(1) The mappings $E : [0, \mathfrak{a}] \times \mathbb{R} \to \mathbb{R}^+$, $\mathcal{H} : [0, \mathfrak{a}] \times \mathbb{R} \to \mathbb{R}$, *and* $\lambda : [0, \mathfrak{a}] \to \mathbb{R}$ *are O-continuous functions.*

(2) There exists $\mathcal{K} > 0$, *such that, for all* $\theta, \mathfrak{s} \in [0, \mathfrak{a}]$ *and* $\omega, \mu \in \mathcal{Q}$,

$$|\mathcal{H}(\theta, \mathfrak{s}, \omega(\mathfrak{s})) - \mathcal{H}(\theta, \mathfrak{s}, \mu(\mathfrak{s}))| \leq \sqrt{\frac{e^{-\mathcal{K}(\omega,\mu)}\mathcal{K}(\omega, \mu)}{2}}. \tag{4.3}$$

(3) For all $\theta, \mathfrak{s} \in [0, \mathfrak{a}]$, *we have*

$$\max \int_0^{\mathfrak{a}} E(\theta, \mathfrak{s})^2 d\mathfrak{s} \leq \frac{1}{\mathfrak{a}}.$$

Then, (4.1) *has a unique solution in* $\mathcal{Q}$.

Proof. Define the O-relation $\perp$ on $\mathcal{Q}$ by

$$\omega \perp \mu \iff \omega(\theta)\mu(\theta) \geq \omega(\theta) \quad or \quad \omega(\theta)\mu(\theta) \geq \mu(\theta), \ \forall\ \theta \in [0, \mathfrak{a}].$$

Define $\tau : \mathcal{Q} \times \mathcal{Q} \to \mathbb{R}^+$, given by

$$\tau(\omega, \mu) = \max_{0 \leq \theta \leq \mathfrak{a}} |\omega(\theta) - \mu(\theta)|^2,$$

for all $\omega, \mu \in \mathcal{Q}$. It is easy to see that, $(\mathcal{Q}, \perp, \tau)$ is an **O**-complete $\mathbf{O}_{\mathfrak{b}}$-MS. For all $\omega, \mu \in \mathcal{Q}$ with $\omega \perp \mu$ and $\theta \in [0, \mathfrak{a}]$, we have

$$\mathcal{D}(\omega(\theta)) = \lambda(\theta) + \int_0^{\mathfrak{a}} E(\theta, \mathfrak{s})\mathcal{H}(\theta, \mathfrak{s}, \omega(\mathfrak{s}))d\mathfrak{s} \geq 1. \tag{4.4}$$

Accordingly, $[(\mathcal{D}\omega)(\theta)][(\mathcal{D}\mu)(\theta)] \geq (\mathcal{D}\mu)(\theta)$, and so $(\mathcal{D}\omega)(\theta) \perp (\mathcal{D}\mu)(\theta)$. Then, $\mathcal{D}$ is $\perp$-preserving.

Let $\omega, \mu \in \mathcal{Q}$ with $\omega \perp \mu$. Suppose that $\mathcal{D}(\omega) \neq \mathcal{D}(\mu)$. For each $\theta \in [0, \mathfrak{a}]$, we have

$$\begin{aligned}
\tau(\mathcal{D}\omega, \mathcal{D}\mu) &= \max_{\theta \in [0,\mathfrak{a}]} |\mathcal{D}\omega(\theta) - \mathcal{D}\mu(\theta)|^2 \\
&= \max_{\theta \in [0,\mathfrak{a}]} \left\{ \left| \lambda(\theta) + \int_0^{\mathfrak{a}} E(\theta, \mathfrak{s})\mathcal{H}(\theta, \mathfrak{s}, \omega(\mathfrak{s}))d\mathfrak{s} - \lambda(\theta) - \int_0^{\mathfrak{a}} E(\theta, \mathfrak{s})\mathcal{H}(\theta, \mathfrak{s}, \mu(\mathfrak{s}))d\mathfrak{s} \right|^2 \right\} \\
&= \max_{\theta \in [0,\mathfrak{a}]} \left\{ \left| \int_0^{\mathfrak{a}} E(\theta, \mathfrak{s})(\mathcal{H}(\theta, \mathfrak{s}, \omega(\mathfrak{s})) - \mathcal{H}(\theta, \mathfrak{s}, \mu(\mathfrak{s})))d\mathfrak{s} \right|^2 \right\} \\
&\leq \max_{\theta \in [0,\mathfrak{a}]} \left\{ \int_0^{\mathfrak{a}} E(\theta, \mathfrak{s})^2 d\mathfrak{s} \int_0^{\mathfrak{a}} |\mathcal{H}(\theta, \mathfrak{s}, \omega(\mathfrak{s})) - \mathcal{H}(\theta, \mathfrak{s}, \mu(\mathfrak{s}))|^2 d\mathfrak{s} \right\} \\
&\leq \frac{1}{\mathfrak{a}} \int_0^{\mathfrak{a}} \left| \sqrt{\frac{e^{-\mathcal{K}(\omega,\mu)}\mathcal{K}(\omega, \mu)}{2}} \right|^2 d\mathfrak{s} \\
&\leq \frac{e^{-\mathcal{K}(\omega,\mu)}}{2}\mathcal{K}(\omega, \mu).
\end{aligned}$$

Thus,

$$\tau(\mathcal{D}\omega, \mathcal{D}\mu) \leq \gamma(\mathcal{K}(\omega, \mu))\mathcal{K}(\omega, \mu),$$

for all $\omega, \mu \in \mathcal{Q}$. Therefore, all the conditions of Theorem (3.2) are satisfied. Hence, (4.1) has a unique solution. □

Example 4.2. Consider the integral equation as follows:

$$\mathfrak{g}(\theta) = \sin(\pi\theta^2) - \frac{\theta^2}{\pi} + \int_0^x \theta^2\sigma\mathfrak{g}(\sigma)\delta\sigma, \;\; 0 \le x \le 1. \tag{4.5}$$

Clearly, the above Eq (4.5) satisfies the assumption of Theorem 4.1, that is, $\sin(\pi\theta^2) - \frac{\theta^2}{\pi}$ is an orthogonal continuous function on $[0,1]$. Kernel $\mathtt{K}(\theta,\sigma)$ is orthogonal continuous on $\mathbb{R} = \{(\theta,\sigma), 0 < \theta, \sigma < 1\}$.

The solution will be determined from Eq (4.5) by the fixed point iteration method:

$$\mathfrak{g}_{\vartheta+1}(\theta) = \sin(\pi\theta^2) - \frac{\theta^2}{\pi} + \int_0^x \theta^2\sigma\mathfrak{g}_{\vartheta}(\sigma)\delta\sigma, \;\; 0 \le x \le 1.$$

By choosing $\sin(\pi\theta^2) - \frac{\theta^2}{\pi}$ as the initial function, we can apply the fixed point iteration method to get a numerical solution:

$$\begin{aligned}
\mathfrak{g}_1(\theta) &= \sin(\pi\theta^2) - \frac{\theta^2}{\pi} + \int_0^x \theta^2\sigma\mathfrak{g}_0(\sigma)\delta\sigma\\
&= \sin(\pi\theta^2) - \frac{\theta^2}{\pi} + \int_0^x \theta^2\sigma\sin(\pi\sigma^2)\delta\sigma\\
&= \sin(\pi\theta^2) - \frac{\theta^2}{\pi} + \theta^2\frac{1}{2\pi}(1-\cos(\pi\theta^2)),\\
\mathfrak{g}_2(\theta) &= \sin(\pi\theta^2) - \frac{\theta^2}{\pi} + \int_0^x \theta^2\sigma\mathfrak{g}_1(\sigma)\delta\sigma\\
&= \sin(\pi\theta^2) - \frac{\theta^2}{\pi} + \int_0^x \theta^2\sigma\Big(\sin(\pi\sigma^2) - \frac{\sigma^2}{\pi} + \sigma^2\frac{1}{2\pi}(1-\cos(\pi\sigma^2))\Big)\delta\sigma\\
&= \sin(\pi\theta^2) - \frac{\theta^2}{\pi} + \frac{\theta^2}{8\pi^3}\Big(-4\pi^2 - 2 + 4\pi^2\cos(\pi\theta^2) + \pi^2\theta^4 + 2\theta^2\pi\sin(\pi\theta^2) + 2\cos(\pi\theta^2)\Big),\\
\mathfrak{g}_3(\theta) &= \sin(\pi\theta^2) - \frac{\theta^2}{\pi} + \int_0^x \theta^2\sigma\mathfrak{g}_2(\sigma)\delta\sigma\\
&= \sin(\pi\theta^2) - \frac{\theta^2}{\pi} + \int_0^x \theta^2\sigma\Big(\sin(\pi\sigma^2) - \frac{\sigma^2}{\pi} + \frac{\sigma^2}{8\pi^2}(-4\pi^2 - 2 + 4\pi^2\cos(\pi\sigma^2) + \sigma^4\pi^2\\
&\quad + 2\theta^2\pi\sin(\pi\theta^2) + 2\cos(\pi\theta^2))\Big)\delta\sigma\\
&= \sin(\pi\theta^2) - \frac{\theta^2}{\pi} + \theta^2\Bigg\{-\frac{1}{64\pi^6}\Bigg(-32\pi^5 - 16\pi^3 - 24\pi + 32\pi^5\cos(\pi\theta^2) + 8\theta^4\pi^5 - 4\theta^4\pi^3\\
&\quad + 16\theta^2\pi^4\Big(\sin(\pi\theta^2) + 16\pi^3\cos(\pi\theta^2) + \theta^8\pi^5 - 8\theta^4\pi^3\cos(\pi\theta^2) + 24\theta^2\pi^2\sin(\pi\theta^2)\\
&\quad + 24\pi\cos(\pi\theta^2)\Big)\Bigg)\Bigg\}.
\end{aligned}$$

Consider that for $|\theta| \le 1$, an O-sequence $\{\mathfrak{g}_{\vartheta}(\theta)\}$ will converge to $\mathfrak{g}(\theta) = \sin(\pi\theta^2) - \frac{\theta^2}{\pi}$.

Error calculation for an approximate solution compared to an exact solution is given in Figure 1. Table 1 shows that the error of an approximate solution compared to an exact solution is relatively small.

Table 1. Comparison of an approximate solution and an exact solution.

θ_j	approximate solution	exact solution	error
0.000	0.000	0.000	0.000
0.100	0.023	0.028	0.005
0.200	0.102	0.113	0.011
0.300	0.234	0.250	0.016
0.400	0.412	0.431	0.019
0.500	0.609	0.628	0.018
0.600	0.779	0.790	0.011
0.700	0.848	0.844	0.004
0.800	0.730	0.701	0.029
0.900	0.358	0.304	0.054
1.000	-0.251	-0.318	0.067

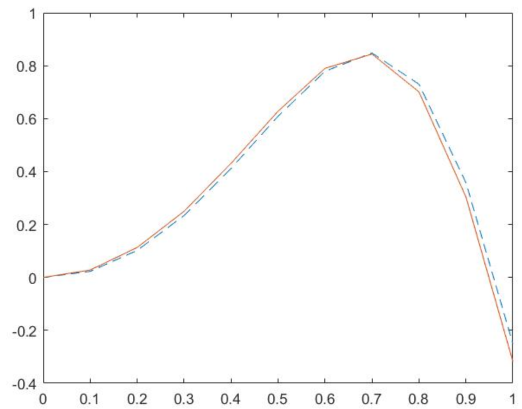

Figure 1. Graph of an approximate solution compare to an exact solution with h=0.1.

5. Conclusions

In this paper, we proved fixed point theorems for **O**-generalized α-ψ-GC type contraction mappings in an **O**-complete $\mathbf{O}_b$-MS. Furthermore, we presented some examples to strengthen our main results. Also, we provided an application to the existence of the solution of an integral equation and we have compared the approximate solution with the exact solution.

Khalehoghli et al. [30, 31] presented a real generalization of the mentioned Banach's contraction principle by introducing $\mathbb{R}$-metric spaces, where R is an arbitrary relation on L. We note that in a special case, $\mathbb{R}$ can be considered as $\mathbb{R} = \leq$[partially ordered relation], $\mathbb{R} = \perp$[orthogonal relation], etc. If one can find a suitable replacement for a Banach theorem that may determine the values of fixed points, then many problems can be solved in this $\mathbb{R}$-relation. This will provide a structural method for finding a value of a fixed point. It is an interesting open problem to study the fixed-point results on $\mathbb{R}$-complete $\mathbb{R}$-metric spaces.

Conflict of interest

The authors declare that they have no competing interests.

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