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Research article

Approximation of solutions for nonlinear functional integral equations

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Abstract: In this article, we consider a class of nonlinear functional integral equations, motivated by an equation that offers increasing evidence to the extant literature through replication studies. We investigate the existence of solution for nonlinear functional integral equations on Banach space $C[0, 1]$. We use the technique of the generalized Darbo's fixed-point theorem associated with the measure of noncompactness (MNC) to prove our existence result. Also, we have given two examples of the applicability of established existence result in the theory of functional integral equations. Further, we construct an efficient iterative algorithm to compute the solution of the first example, by employing the modified homotopy perturbation (MHP) method associated with Adomian decomposition. Moreover, the condition of convergence and an upper bound of errors are presented.

Keywords: measure of noncompactness; nonlinear functional integral equation; fixed point theorem; modified homotopy perturbation
Mathematics Subject Classification : 45G10, 45M99, 47H08, 47H10

1. Introduction

Integral equations play a notable role in applied mathematics. The practical importance of nonlinear integral equations is increasingly evident from studies that incorporate the same in distinct areas of knowledge that include biology, traffic theory, the theory of optimal control, economics, acoustic scattering, etc. [1,2,9,39]. Precisely, extensive studies on these equations are focused on their solutions by employing the MNC [3, 29, 36, 47]. Besides, other concepts like quasi linearization [35] and pseudo-spectral methods [52] have also been used in similar studies. In such studies, the existence of solutions is proved with the theory of fixed points. Verifying the existence of solutions, their behavioral properties are also extensively studied. For instance, Hu et al. [20] discussed the global attractivity and asymptotic stability of the solutions, whereas Wang et al. [51] discussed the local attractivity and local stability, and Aghajani et al. [4] and Alvarez et al. [5] gave globally and uniformly locally attractive solutions. Notably, Banaś et al. [6, 8, 10] and Dhage et al. [12, 13, 16, 17] also discussed the attractivity of solutions. Xu et al. [53] renders radially symmetric solutions and their asymptotic estimates. Furthermore, systems of equations have been studied and numerical methods to find solutions have also been proposed [19, 22, 23, 25, 38, 40, 48, 49].

In recent times, the fixed point theory (FPT) is applicable in various scientific fields suggested by Banach [11, 21, 26, 37, 50]. Also, FPT can be applied to seeking solutions of functional integral equations. Functional integral equations of variety of forms chair as a extraordinary and prestigious branch of non-linear analysis and seek various invocations in demonstrating numerous real-life together with real-world problems (cf. [14, 27, 28, 30–34, 41]).

Recently, several research articles have been published in connection:

In 2020, El-Sayeda and Ebeada [18] have studied the solvability of self-reference functional and quadratic functional integral equations:

$$x(t) = f\Big(t, \int_0^t g(s, x(x(s)))ds\Big),$$

$$\text{and } x(t) = f\Big(t, \int_0^t f_1(s, x(x(s)))ds \int_0^t f_2(s, x(x(s)))ds\Big), \text{ respectively,}$$

where $x \in C[0, T],\ t \in [0, T]$, and $g,\ f_1,\ f_2$ satisfy Carathéodory condition. To realize the existence of a solution to those integral equations, they used Schauder's fixed point theorem in the Banach space $C[0, T]$.

In 2020, Deep et al. [15] established the existence of solutions of some non-linear functional integral equations in Banach algebra with applications:

$$\begin{aligned} y(t) = &\Big(f(t, y(t), y(\theta(t))) + F\Big(t, \int_0^t r(t, s, y(\theta(s)))ds, \int_0^t u(t, s, y(a(s)))ds, y(d(t))\Big)\Big) \\ &\times L\Big(t, \int_0^b p(t, s, y(c(s)))ds, \int_0^b q(t, s, y(\chi(s)))ds, y(\eta(t))\Big), \end{aligned}$$

for $t \in [0, b]$. Existence result is obtained through the techniques of MNC and Darbo's fixed point theorem in $[0, b]$.

In 2021, Rabbani et al. [46] established some generalized non-linear functional integral equations of two variables via measures of noncompactness and numerical methods to solve it,

$$\begin{aligned} y(\zeta) = &\Big(f(\zeta) + A(\zeta, y(\zeta), Y(\alpha(\zeta))) + P\Big(\zeta, y(\zeta), \int_0^\zeta F(\zeta), r, y(\beta(r)))dr, y(\gamma(\zeta)))\Big)\Big) \\ &\times \Big(U(\zeta, y(\zeta), Y(\theta(\zeta))) + Q\Big(\zeta, y(\zeta), \int_0^b G(\zeta, r, y(\delta(r))), y(\phi(\zeta)))\Big)\Big), \end{aligned}$$

where $\zeta \in [0, b]$. To realize the existence of the solution of those integral equations, they have used the concept of MNC and Petryshyn fixed point theorem for the operators in a Banach algebra $C([0, b] \times$

$[0, b], \mathbb{R})$, for $b > 0$ in the form of two operators. They have also discussed an iterative algorithm which was constructed by modified homotopy perturbation method and Adomian polynomials to compute the solution of the example.

In 2022, Karmakar et al. [24] have studied the existence of solutions to non-linear quadratic integral equations via measure of noncompactness:

$$x(t) = g(t, x(t)) + \lambda \int_0^t \mu_1(t, s)\varsigma_1(s, x(s))ds \int_0^t \mu_2(t, s)\varsigma_2(s, x(s))ds,$$

for $t > 0$, where g, μ_1, μ_2, ς_1, ς_2 are real valued continuous functions defined on $\mathbb{R}^+ \times \mathbb{R}$ and λ is a positive constant.

In our work, we study an existence result for the solution of the following nonlinear functional integral equation:

$$\varpi(\varrho) = f\left(\varrho,\ \varpi(\varrho),\ g(\varrho, \varpi(\varrho)) \int_0^1 v(\varrho, \eta, \varpi(\eta))d\eta,\ \varpi(\varrho) \int_0^\varrho u(\varrho, \eta, \varpi(\eta))d\eta\right), \tag{1.1}$$

where $\varrho \in I = [0, 1]$. In this work, our main work aims to obtain the existence result of Eq (1.1) on Banach space $C[0, 1]$ by applying the technique of the generalized Darbo's fixed-point theorem associated with the MNC, and also, work to obtain the analytic solution of it by applying the semi-analytic method. Now, we describe the importance of why we study Eq (1.1) and what is the perfection of our findings. The first one is that the conditions estimated in several research articles will be analyzed and the second one is that this manuscript affiliate the relevant work in this area. The third one is the bounded condition implies that the "sublinear condition" that has been identified in various literature works does not have a relevant appearance here. Our findings generalized, extended, and complement several results existing in the literature.

The estimate of our work is organized as follows: In Section 2, some notations, definitions and auxiliary facts are given. In Section 3, we prove the existence of solution by using the generalized Darbo fixed point theorem associated with the MNC on $C[0, 1]$. Also, we present two examples to illustrate our theorem. In Section 4, we state an algorithm to find the solution by using MHP and Adomian decomposition method. Correspondingly, we apply the algorithm to one of the examples for finding an approximate solution and tabulate the errors. Also, we show the graphs of $v(\varrho)$ and $\varpi(\varrho)$. In Section 5, we analyze the errors and provide an upper bound of the errors. Our conclusion is presented in Section 6.

2. Preliminaries

In this section, we organize some notations, definitions and auxiliary facts which we need throughout the paper.

Let A be a nonempty subset of a Banach space X. We use $\bar{A}$ and $Conv\,A$ to denote the closure and convex closure of A respectively. Also, we use P_X to denote the class of nonempty bounded subsets of X and Q_X to denote the subclass of relatively compact sets of P_X. Let $C(I)$ denotes the set of all continuous real valued functions on $[0, 1]$, a classical Banach space.

Definition 2.1. *[7] A function $\mu : P_X \to \mathbb{R}^+$ is called a MNC on X if the following conditions are satisfied:*

(i) *The kernel of the function,* $\ker \mu = \{A \in P_X : \mu(A) = 0\} \neq \emptyset \quad \& \quad \ker \mu \subseteq Q_X$.

(ii) $A \subseteq B \Rightarrow \mu(A) \leq \mu(B), \forall A, B \in P_X$.

(iii) $\mu(\bar{A}) = \mu(Conv\, A) = \mu(A), \forall A \in P_X$.

(iv) $\mu(\lambda A + (1-\lambda)B) \leq \lambda\mu(A) + (1-\lambda)\mu(B)$, *for* $\lambda \in [0, 1]$.

(v) *If* $\{A_n\}_{n=1}^{\infty}$ *is a sequence of closed sets in* P_X *such that* $A_{n+1} \subseteq A_n, \forall n \in \mathbb{N}$ *and* $\mu(A_n) \to 0$, *then* $\cap_{n=1}^{\infty} A_n = A_\infty \neq \emptyset$.

Definition 2.2. *Define* $\Psi = \{\psi' : \mathbb{R}^+ \to \mathbb{R}^+\}$ *such that each* ψ' *satisfies the following:*

(i) ψ' *is an upper semi-continuous function from the right.*

(ii) $\psi'(\varrho) < \varrho, \forall \varrho \in [0, 1]$.

Definition 2.3. *We recall the definition of MNC in* $C(I)$ *as defined in [8]. Let* $A \subseteq C(I)$ *such that* A *is nonempty, bounded, and let* $\varpi \in A$ *together with* $\epsilon \geq 0$, *then the modulus of continuity of* ϖ *in* I *is defined as*

$$\omega(\varpi, \epsilon) = \sup_{\varrho, \eta \in I} \{|\varpi(\varrho) - \varpi(\eta)| : |\varrho - \eta| \leq \epsilon\}.$$

$$\text{Let} \quad \omega(A, \epsilon) = \sup_{\varpi \in A} \omega(\varpi, \epsilon) \quad \text{and} \quad \omega_0(A) = \lim_{\epsilon \to 0} \omega(A, \epsilon).$$

$$\text{Define} \quad i(\varpi) = \sup_{\varrho, \eta \in I} \{|\varpi(\varrho) - \varpi(\eta)| - [|\varpi(\varrho) - \varpi(\eta)|] : \eta \leq \varrho\} \quad \text{and} \quad i(A) = \sup_{\varpi \in A} i(\varpi).$$

Now, MNC is defined as

$$\mu(A) = \omega_0(A) + i(A).$$

Theorem 2.1. *[16] Let* A *be a nonempty, closed, bounded together with convex subset of a Banach space, and let* $T : A \to A$ *be a continuous function satisfying*

$$\mu(TB) \leq \psi'(\mu(B)), \quad \forall B \subseteq A (B \neq \emptyset), \tag{2.1}$$

for some MNC μ *and some* $\psi' \in \Psi$. *Then, there exist at least one fixed point in* T.

This theorem is known as the generalized Darbo fixed point theorem.

3. Main result

In this section, we prove the nonlinear equation (1.1) has a solution in the Banach space $C(I)$, with the supremum norm.

The nonlinear equation (1.1) is studied with the following assumptions:

(A1) The function $f : I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous such that $f : I \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$. Also, there exists a function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\psi(0) = 0$ and $\psi(\varrho) < \frac{\varrho}{2}$ together with $\psi(\varrho) + \psi(\eta) \leq \psi(\varrho + \eta)$ satisfying

$$|f(\varrho, u_1, v_1, w_1) - f(\varrho, u_2, v_2, w_2)| \leq \psi(|u_1 - u_2|) + |v_1 - v_2| + |w_1 - w_2|. \tag{3.1}$$

For $u, v, w \in \mathbb{R}^+$, $\varrho \mapsto f(\varrho, u, v, w)$ is increasing on I and for $\varrho \in I$ and $u \in \mathbb{R}^+$, $v \mapsto f(\varrho, u, v, w)$ and $w \mapsto f(\varrho, u, v, w)$ are increasing on I and for some $N > 0$, $f(\varrho, 0, 0, 0) \leq N$.

(A2) The function $g : I \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists $\kappa', \kappa \geq 0$ such that

$$|g(\varrho, 0)| \leq \kappa' \quad \text{and} \quad |g(\varrho, y_1) - g(\varrho, y_2)| \leq \kappa |y_1 - y_2|. \tag{3.2}$$

(A3) The functions $u : I \times I \times \mathbb{R} \to \mathbb{R}$ and $v : I \times I \times \mathbb{R} \to \mathbb{R}$ are continuous such that $u : I \times I \times \mathbb{R}^+ \to \mathbb{R}^+$ and $v : I \times I \times \mathbb{R}^+ \to \mathbb{R}^+$. For arbitrarily fixed $\eta \in I$ ($\Rightarrow \varpi(\eta)$ is also fixed), $\varrho \mapsto u(\varrho, \eta, \varpi(\eta))$ and $\varrho \mapsto v(\varrho, \eta, \varpi(\eta))$ are increasing in I. Also, there exists a constant $l \in [0, \frac{1}{4})$ such that

$$\int_0^1 v(\varrho, \eta, \varpi(\eta)) d\eta \leq l \quad \text{and} \quad \int_0^\varrho u(\varrho, \eta, \varpi(\eta)) d\eta \leq l. \tag{3.3}$$

(A4) There exists $r_0 > 0$ such that

$$\psi(r_0) + (r_0(\kappa + 1) + \kappa')l + N \leq r_0. \tag{3.4}$$

(A5) The constant $\kappa l < 1/4$.

Theorem 3.1. *Under the above assumptions, the nonlinear equation* (1.1) *has at least one solution in* $C(I)$.

Proof. Let $T : C(I) \to C(I)$ be an operator defined as

$$(T\varpi)(\varrho) = f\left(\varrho, \varpi(\varrho), g(\varrho, \varpi(\varrho)) \int_0^1 v(\varrho, \eta, \varpi(\eta)) d\eta, \varpi(\varrho) \int_0^\varrho u(\varrho, \eta, \varpi(\eta)) d\eta\right).$$

The operator T having a fixed point in $C(I)$ is equivalent to the nonlinear equation (1.1) having a solution in $C(I)$. Hence, we prove T has a fixed point by using Theorem 2.1.

By applying system of Eq (1.1) and imposed postulates (A1–A5), we estimate for every $\varrho \in I$ such that

$$\begin{aligned}
&|(T\varpi)(\varrho)| \\
\leq &\left| f\left(\varrho, \varpi(\varrho), g(\varrho, \varpi(\varrho)) \int_0^1 v(\varrho, \eta, \varpi(\eta)) d\eta, \varpi(\varrho) \int_0^\varrho u(\varrho, \eta, \varpi(\eta)) d\eta\right) - f(\varrho, 0, 0, 0)\right| + |f(\varrho, 0, 0, 0)| \\
\leq &\psi(|\varpi(\varrho)|) + \left|g(\varrho, \varpi(\varrho))\right| \int_0^1 \left|v(\varrho, \eta, \varpi(\eta)) d\eta\right| + |\varpi(\varrho)| \int_0^\varrho \left|u(\varrho, \eta, \varpi(\eta)) d\eta\right| + |f(\varrho, 0, 0, 0)| \\
\leq &\psi(|\varpi(\varrho)|) + (|g(\varrho, \varpi(\varrho))| + |\varpi(\varrho)|)l + |f(\varrho, 0, 0, 0)| \\
\leq &\psi(|\varpi(\varrho)|) + (|g(\varrho, \varpi(\varrho)) - g(\varrho, 0)| + |g(\varrho, 0)| + |\varpi(\varrho)|)l + |f(\varrho, 0, 0, 0)| \\
\leq &\psi(|\varpi(\varrho)|) + (\kappa|\varpi(\varrho)| + \kappa' + |\varpi(\varrho)|)l + |f(\varrho, 0, 0, 0)| \\
\leq &\psi(\|\varpi\|) + (\|\varpi\| (\kappa + 1) + \kappa')l + N.
\end{aligned}$$

Therefore,

$$\|T\varpi\| \leq \psi(\|\varpi\|) + (\|\varpi\| (\kappa + 1) + \kappa')l + N.$$

For $r_0 > 0$ such that $\|\varpi\| \leq r_0$, by assumption (A4), $\|T\varpi\| \leq r_0$, i.e. T maps B_{r_0} into itself. Now, we prove T is continuous. Let $\{\varpi_n\}_{n=1}^\infty$ be a sequence in B_{r_0} such that $\varpi_n \to \varpi$, we obtain

$$
\begin{aligned}
&\left|(T\varpi_n)(\varrho)-(T\varpi)(\varrho)\right|\\
=&\left|f\left(\varrho,\ \varpi_n(\varrho),\ g(\varrho,\varpi_n(\varrho))\int_0^1 v(\varrho,\eta,\varpi_n(\eta))d\eta,\ \varpi_n(\varrho)\int_0^\varrho u(\varrho,\eta,\varpi_n(\eta))d\eta\right)\right.\\
&\left.-f\left(\varrho,\ \varpi(\varrho),\ g(\varrho,\varpi(\varrho))\int_0^1 v(\varrho,\eta,\varpi(\eta))d\eta,\ \varpi(\varrho)\int_0^\varrho u(\varrho,\eta,\varpi(\eta))d\eta\right)\right|\\
\leq&\ \psi(|\varpi_n(\varrho)-\varpi(\varrho)|)+\left|g(\varrho,\varpi_n(\varrho))\int_0^1 v(\varrho,\eta,\varpi_n(\eta))d\eta-g(\varrho,\varpi(\varrho))\int_0^1 v(\varrho,\eta,\varpi(\eta))d\eta\right|\\
&+\left|\varpi_n(\varrho)\int_0^\varrho u(\varrho,\eta,\varpi_n(\eta))d\eta-\varpi(\varrho)\int_0^\varrho u(\varrho,\eta,\varpi(\eta))d\eta\right|.
\end{aligned}
$$

Let us now consider

$$
\begin{aligned}
&\left|g(\varrho,\varpi_n(\varrho))\int_0^1 v(\varrho,\eta,\varpi_n(\eta))d\eta-g(\varrho,\varpi(\varrho))\int_0^1 v(\varrho,\eta,\varpi(\eta))d\eta\right|\\
\leq&\left|g(\varrho,\varpi_n(\varrho))\right|\int_0^1\left|v(\varrho,\eta,\varpi_n(\eta))-v(\varrho,\eta,\varpi(\eta))\right|d\eta\\
&+\left|g(\varrho,\varpi_n(\varrho))-g(\varrho,\varpi(\varrho))\right|\int_0^1\left|v(\varrho,\eta,\varpi(\eta))\right|d\eta\\
\leq&\left|g(\varrho,\varpi_n(\varrho))\right|V_{r_0}(\epsilon)+\kappa l\left|\varpi_n(\varrho)-\varpi(\varrho)\right|\\
\leq&(\kappa\left\|\varpi_n\right\|+\kappa')V_{r_0}(\epsilon)+\kappa l\left\|\varpi_n-\varpi\right\|,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\varpi_n(\varrho)\int_0^\varrho u(\varrho,\eta,\varpi_n(\eta))d\eta-\varpi(\varrho)\int_0^\varrho u(\varrho,\eta,\varpi(\eta))d\eta\right|\\
\leq&\left|\varpi_n(\varrho)\int_0^\varrho u(\varrho,\eta,\varpi_n(\eta))d\eta-\varpi_n(\varrho)\int_0^\varrho u(\varrho,\eta,\varpi(\eta))d\eta\right|\\
&+\left|\varpi_n(\varrho)\int_0^\varrho u(\varrho,\eta,\varpi(\eta))d\eta-\varpi(\varrho)\int_0^\varrho u(\varrho,\eta,\varpi(\eta))d\eta\right|\\
\leq&\left\|\varpi_n\right\|U_{r_0}(\epsilon)+l\left\|\varpi_n-\varpi\right\|,
\end{aligned}
$$

where

$$V_{r_0}(\epsilon)=\sup_{\varrho,\eta\in I}\{\left|v(\varrho,\eta,\varpi(\eta))-v(\varrho,\eta,y(\eta))\right|:\varpi,y\in B_{r_0}\&|\varpi-y|\leq\epsilon\},$$

and

$$U_{r_0}(\epsilon)=\sup_{\varrho,\eta\in I}\{\left|u(\varrho,\eta,\varpi(\eta))-u(\varrho,\eta,y(\eta))\right|:\varpi,y\in B_{r_0}\&|\varpi-y|\leq\epsilon\}.$$

As $\epsilon\to 0$, $V_{r_0}\to 0$ and $U_{r_0}\to 0$. We are enabled to obtain

$$\left\|T\varpi_n-T\varpi\right\|\leq\psi(\left\|\varpi_n-\varpi\right\|)+l(\kappa+1)\left\|\varpi_n-\varpi\right\|,$$

i.e., $T\varpi_n \to T\varpi$. Hence, we have proved that T is continuous in B_{r_0}. Clearly $T(B_{r_0}^+) \subseteq B_{r_0}^+$, where $B_{r_0}^+ = \{\varpi \in B_{r_0} : \varpi(\varrho) \geq 0, \quad \forall \varrho \in I\}$.

Further, we suppose $A \subseteq B_{r_0}$ such that A is nonempty and $\varpi \in A$. Also, let $\epsilon > 0$ and $\varrho_1, \varrho_2 \in I$ such that $|\varrho_1 - \varrho_2| \leq \epsilon$. Without loss of generality, let us also assume that $\varrho_2 \geq \varrho_1$, we estimate

$$
\begin{aligned}
&\Big|(T\varpi)(\varrho_2) - (T\varpi)(\varrho_1)\Big| \\
\leq &\Big| f\Big(\varrho_2,\ \varpi(\varrho_2),\ g(\varrho_2, \varpi(\varrho_2)) \int_0^1 v(\varrho_2, \eta, \varpi(\eta))d\eta,\ \varpi(\varrho_2) \int_0^{\varrho_2} u(\varrho_2, \eta, \varpi(\eta))d\eta\Big) \\
&- f\Big(\varrho_2,\ \varpi(\varrho_1),\ g(\varrho_2, \varpi(\varrho_2)) \int_0^1 v(\varrho_2, \eta, \varpi(\eta))d\eta,\ \varpi(\varrho_2) \int_0^{\varrho_2} u(\varrho_2, \eta, \varpi(\eta))d\eta\Big)\Big| \\
&+ \Big| f\Big(\varrho_2,\ \varpi(\varrho_1),\ g(\varrho_2, \varpi(\varrho_2)) \int_0^1 v(\varrho_2, \eta, \varpi(\eta))d\eta,\ \varpi(\varrho_2) \int_0^{\varrho_2} u(\varrho_2, \eta, \varpi(\eta))d\eta\Big) \\
&- f\Big(\varrho_1,\ \varpi(\varrho_1),\ g(\varrho_2, \varpi(\varrho_2)) \int_0^1 v(\varrho_2, \eta, \varpi(\eta))d\eta,\ \varpi(\varrho_2) \int_0^{\varrho_2} u(\varrho_2, \eta, \varpi(\eta))d\eta\Big)\Big| \\
&+ \Big| f\Big(\varrho_1,\ \varpi(\varrho_1),\ g(\varrho_2, \varpi(\varrho_2)) \int_0^1 v(\varrho_2, \eta, \varpi(\eta))d\eta,\ \varpi(\varrho_2) \int_0^{\varrho_2} u(\varrho_2, \vartheta, \varpi(\eta))d\eta\Big) \\
&- f\Big(\varrho_1,\ \varpi(\varrho_1),\ g(\varrho_2, \varpi(\varrho_2)) \int_0^1 v(\varrho_1, \eta, \varpi(\eta))d\eta,\ \varpi(\varrho_2) \int_0^{\varrho_1} u(\varrho_2, \eta, \varpi(\eta))d\eta\Big)\Big| \\
&+ \Big| f\Big(\varrho_1,\ \varpi(\varrho_1),\ g(\varrho_2, \varpi(\varrho_2)) \int_0^1 v(\varrho_1, \eta, \varpi(\eta))d\eta,\ \varpi(\varrho_2) \int_0^{\varrho_1} u(\varrho_2, \eta, \varpi(\eta))d\eta\Big) \\
&- f\Big(\varrho_1,\ \varpi(\varrho_1),\ g(\varrho_1, \varpi(\varrho_1)) \int_0^1 v(\varrho_1, \eta, \varpi(\eta))d\eta,\ \varpi(\varrho_1) \int_0^{\varrho_1} u(\varrho_1, \eta, \varpi(\eta))d\eta\Big)\Big| \\
\leq\ & \psi(|\varpi(\varrho_2) - \varpi(\varrho_1)|) + \omega(f, \epsilon) + \Big|g(\varrho_2, \varpi(\varrho_2))\Big| \int_0^1 \Big|v(\varrho_2, \eta, \varpi(\eta)) - v(\varrho_1, \eta, \varpi(\eta))\Big| d\eta \\
&+ |\varpi(\varrho_2)| \int_{\varrho_1}^{\varrho_2} \Big|u(\varrho_2, \eta, \varpi(\eta))\Big| d\eta + \Big|g(\varrho_2, \varpi(\varrho_2)) - g(\varrho_1, \varpi(\varrho_1))\Big| \int_0^1 \Big|v(\varrho_1, \eta, \varpi(\eta))\Big| d\eta \\
&+ \Big|\varpi(\varrho_2) \int_0^{\varrho_1} u(\varrho_2, \eta, \varpi(\eta))d\eta - \varpi(\varrho_1) \int_0^{\varrho_1} u(\varrho_1, \eta, \varpi(\eta))d\eta\Big| \\
\leq\ & \psi(|\varpi(\varrho_2) - \varpi(\varrho_1)|) + \omega(f, \epsilon) + \Big|g(\varrho_2, \varpi(\varrho_2))\Big| \omega(v, \epsilon) + |\varpi(\varrho_2)| l \epsilon \\
&+ \Big|g(\varrho_2, \varpi(\varrho_2)) - g(\varrho_1, \varpi(\varrho_1))\Big| \int_0^1 \Big|v(\varrho_1, \eta, \varpi(\eta))d\eta\Big| \\
&+ \Big|\varpi(\varrho_2) \int_0^{\varrho_1} u(\varrho_2, \eta, \varpi(\eta))d\eta - \varpi(\varrho_1) \int_0^{\varrho_1} u(\varrho_1, \eta, \varpi(\eta))d\eta\Big|,
\end{aligned}
$$

wherein,

$$
\begin{aligned}
&\Big|g(\varrho_2, \varpi(\varrho_2)) - g(\varrho_1, \varpi(\varrho_1))\Big| \int_0^1 \Big|v(\varrho_1, \eta, \varpi(\eta))\Big| d\eta \\
\leq &\Big\{\Big|g(\varrho_2, \varpi(\varrho_2)) - g(\varrho_1, \varpi(\varrho_2))\Big| + \Big|g(\varrho_1, \varpi(\varrho_2)) - g(\varrho_1, \varpi(\varrho_1))\Big|\Big\} l \\
\leq & l\omega(g, \epsilon) + \kappa l \Big|\varpi(\varrho_2) - \varpi(\varrho_1)\Big|,
\end{aligned}
$$

and

$$\begin{aligned}
&\left|\varpi(\varrho_2)\int_0^{\varrho_1} u(\varrho_2,\eta,\varpi(\eta))d\eta - \varpi(\varrho_1)\int_0^{\varrho_1} u(\varrho_1,\eta,\varpi(\eta))d\eta\right| \\
\leq&\left|\varpi(\varrho_2)\int_0^{\varrho_1} u(\varrho_2,\eta,\varpi(\eta))d\eta - \varpi(\varrho_2)\int_0^{\varrho_1} u(\varrho_1,\eta,\varpi(\eta))d\eta\right| \\
&+\left|\varpi(\varrho_2)\int_0^{\varrho_1} u(\varrho_1,\eta,\varpi(\eta))d\eta - \varpi(\varrho_1)\int_0^{\varrho_1} u(\varrho_1,\eta,\varpi(\eta))d\eta\right| \\
\leq&\left|\varpi(\varrho_2)\right|\int_0^{\varrho_1}\left|u(\varrho_2,\eta,\varpi(\eta)) - u(\varrho_1,\eta,\varpi(\eta))\right|d\eta + \left|\varpi(\varrho_2)-\varpi(\varrho_1)\right|\int_0^{\varrho_1}\left|u(\varrho_1,\eta,\varpi(\eta))d\eta\right| \\
\leq&|\varpi(\varrho_2)|\omega(u,\epsilon) + l\left|\varpi(\varrho_2)-\varpi(\varrho_1)\right|,
\end{aligned}$$

together with

$$\begin{aligned}
\omega(f,\epsilon) &= \sup_{\varrho_1,\varrho_2\in I}\{\left|f(\varrho_1,u,v,w)-f(\varrho_2,u,v,w)\right| : |\varrho_1-\varrho_2|\leq\epsilon\},\\
\omega(u,\epsilon) &= \sup_{\varrho_1,\varrho_2\in I}\{\left|u(\varrho_1,\eta,\varpi)-u(\varrho_2,\eta,\varpi)\right| : |\varrho_1-\varrho_2|\leq\epsilon\},\\
\omega(v,\epsilon) &= \sup_{\varrho_1,\varrho_2\in I}\{\left|v(\varrho_1,\eta,\varpi)-v(\varrho_2,\eta,\varpi)\right| : |\varrho_1-\varrho_2|\leq\epsilon\},\\
\omega(g,\epsilon) &= \sup_{\varrho_1,\varrho_2\in I}\{|g(\varrho_2,x)-g(\varrho_1,x)| : |\varrho_2-\varrho_1|\leq\epsilon\}.
\end{aligned}$$

Then, we can find

$$\begin{aligned}
&\left|(T\varpi)(\varrho_2)-(T\varpi)(\varrho_1)\right| \\
\leq&\psi(|\varpi(\varrho_2)-\varpi(\varrho_1)|) + \omega(f,\epsilon) + \left|g(\varrho_2,\varpi(\varrho_2))\right|\omega(v,\epsilon) + l\epsilon\left|\varpi(\varrho_2)\right| + l\omega(g,\epsilon) \\
&+\kappa l\left|\varpi(\varrho_2)-\varpi(\varrho_1)\right| + |\varpi(\varrho_2)|\omega(u,\epsilon) + l\left|\varpi(\varrho_2)-\varpi(\varrho_1)\right| \\
\leq&\psi(\omega(\varpi,\epsilon)) + \omega(f,\epsilon) + (\kappa\|\varpi\|+\kappa')\omega(v,\epsilon) + l\epsilon\|\varpi\| + l\omega(g,\epsilon) + (\kappa l + l)\omega(\varpi,\epsilon) + \|\varpi\|\,\omega(u,\epsilon).
\end{aligned}$$

Now, applying uniform continuity of the function $f(\varrho,u,v,w)$, $u(\varrho,\eta,\varpi)$, $v(\varrho,\eta,\varpi)$ and $g(\varrho,\varpi)$ on the set $I\times[-r_0,r_0]\times[-r_0,r_0]\times[-r_0,r_0]$, $I\times I\times[-r_0,r_0]$, $I\times I\times[-r_0,r_0]$, $I\times[-r_0,r_0]$, respectively. We are enabled to deduce that $\omega(f,\epsilon)\to 0$, $\omega(u,\epsilon)\to 0$, $\omega(v,\epsilon)\to 0$, and $\omega(g,\epsilon)\to 0$, when $\epsilon\to 0$. Thus, we can find

$$\begin{aligned}
\omega_0(TA) &\leq \psi(\omega_0(A)) + (\kappa l + l)\omega_0(A)\\
&\leq (\psi + (\kappa+1)l)(\omega_0(A)).
\end{aligned}\tag{3.5}$$

Suppose $\varpi\in A$, and $\varrho_1,\varrho_2\in I$, together with $\varrho_1<\varrho_2$, we estimate that

$$\begin{aligned}
&\left|(T\varpi)(\varrho_2)-(T\varpi)(\varrho_1)\right| - [|(T\varpi)(\varrho_2)-(T\varpi)(\varrho_1)|] \\
=&\left|f\Big(\varrho_2,\ \varpi(\varrho_2),\ g(\varrho_2,\varpi(\varrho_2)))\int_0^1 v(\varrho_2,\eta,\varpi(\eta))d\eta,\ \varpi(\varrho_2)\int_0^{\varrho_2} u(\varrho_2,\eta,\varpi(\eta))d\eta\Big)\right. \\
&\left. - f\Big(\varrho_1,\ \varpi(\varrho_1),\ g(\varrho_1,\varpi(\varrho_1))\int_0^1 v(\varrho_1,\eta,\varpi(\eta))d\eta,\ \varpi(\varrho_1)\int_0^{\varrho_1} u(\varrho_1,\eta,\varpi(\eta))d\eta\Big)\right|
\end{aligned}$$

$$
\begin{aligned}
&-\Big[\Big|f\Big(\varrho_2,\ \varpi(\varrho_2),\ g(\varrho_2,\varpi(\varrho_2))\int_0^1 v(\varrho_2,\eta,\varpi(\eta))d\eta,\ \varpi(\varrho_2)\int_0^{\varrho_2} u(\varrho_2,\eta,\varpi(\eta))d\eta\Big)\\
&\quad-f\Big(\varrho_1,\ \varpi(\varrho_1),\ g(\varrho_1,\varpi(\varrho_1))\int_0^1 v(\varrho_1,\eta,\varpi(\eta))d\eta,\ \varpi(\varrho_1)\int_0^{\varrho_1} u(\varrho_1,\eta,\varpi(\eta))d\eta\Big)\Big|\Big]\\
&\leq\Big|f\Big(\varrho_2,\ \varpi(\varrho_2),\ g(\varrho_2,\varpi(\varrho_2))\int_0^1 v(\varrho_2,\eta,\varpi(\eta))d\eta,\ \varpi(\varrho_2)\int_0^{\varrho_2} u(\varrho_2,\eta,\varpi(\eta))d\eta\Big)\\
&\quad-f\Big(\varrho_2,\ \varpi(\varrho_1),\ g(\varrho_2,\varpi(\varrho_2))\int_0^1 v(\varrho_2,\eta,\varpi(\eta))d\eta,\ \varpi(\varrho_2)\int_0^{\varrho_2} u(\varrho_2,\eta,\varpi(\eta))d\eta\Big)\Big|\\
&\quad+\Big|f\Big(\varrho_2,\ \varpi(\varrho_1),\ g(\varrho_2,\varpi(\varrho_2))\int_0^1 v(\varrho_2,\eta,\varpi(\eta))d\eta,\ \varpi(\varrho_2)\int_0^{\varrho_2} u(\varrho_2,\eta,\varpi(\eta))d\eta\Big)\\
&\quad-f\Big(\varrho_1,\ \varpi(\varrho_1),\ g(\varrho_2,\varpi(\varrho_2))\int_0^1 v(\varrho_2,\eta,\varpi(\eta))d\eta,\ \varpi(\varrho_2)\int_0^{\varrho_2} u(\varrho_2,\eta,\varpi(\eta))d\eta\Big)\Big|\\
&\quad+\Big|f\Big(\varrho_1,\ \varpi(\varrho_1),\ g(\varrho_2,\varpi(\varrho_2))\int_0^1 v(\varrho_2,\eta,\varpi(\eta))d\eta,\ \varpi(\varrho_2)\int_0^{\varrho_2} u(\varrho_2,\eta,\varpi(\eta))d\eta\Big)\\
&\quad-f\Big(\varrho_1,\ \varpi(\varrho_1),\ g(\varrho_2,\varpi(\varrho_2))\int_0^1 v(\varrho_1,\eta,\varpi(\eta))d\eta,\ \varpi(\varrho_2)\int_0^{\varrho_1} u(\varrho_2,\eta,\varpi(\eta))d\eta\Big)\Big|\\
&\quad+\Big|f(\varrho_1,\ \varpi(\varrho_1),\ g(\varrho_2,\varpi(\varrho_2))\int_0^1 v(\varrho_1,\eta,\varpi(\eta))d\eta,\ \varpi(\varrho_2)\int_0^{\varrho_1} u(\varrho_2,\eta,\varpi(\eta))d\eta\Big)\\
&\quad-f\Big(\varrho_1,\ \varpi(\varrho_1),\ g(\varrho_1,\varpi(\varrho_1))\int_0^1 v(\varrho_1,\eta,\varpi(\eta))d\eta,\ \varpi(\varrho_1)\int_0^{\varrho_1} u(\varrho_1,\eta,\varpi(\eta))d\eta\Big)\Big|\\
&\quad-\Big[\Big|f\Big(\varrho_2,\ \varpi(\varrho_2),\ g(\varrho_2,\varpi(\varrho_2))\int_0^1 v(\varrho_2,\eta,\varpi(\eta))d\eta,\ \varpi(\varrho_2)\int_0^{\varrho_2} u(\varrho_2,\eta,\varpi(\eta))d\eta\Big)\\
&\quad-f\Big(\varrho_2,\ \varpi(\varrho_1),\ g(\varrho_2,\varpi(\varrho_2))\int_0^1 v(\varrho_2,\eta,\varpi(\eta))d\eta,\ \varpi(\varrho_2)\int_0^{\varrho_2} u(\varrho_2,\eta,\varpi(\eta))d\eta\Big)\Big|\Big]\\
&\quad-\Big[\Big|f\Big(\varrho_2,\ \varpi(\varrho_1),\ g(\varrho_2,\varpi(\varrho_2))\int_0^1 v(\varrho_2,\eta,\varpi(\eta))d\eta,\ \varpi(\varrho_2)\int_0^{\varrho_2} u(\varrho_2,\eta,\varpi(\eta))d\eta\Big)\\
&\quad-f\Big(\varrho_1,\ \varpi(\varrho_1),\ g(\varrho_2,\varpi(\varrho_2))\int_0^1 v(\varrho_2,\eta,\varpi(\eta))d\eta,\ \varpi(\varrho_2)\int_0^{\varrho_2} u(\varrho_2,\eta,\varpi(\eta))d\eta\Big)\Big|\Big]\\
&\quad-\Big[\Big|f\Big(\varrho_1,\ \varpi(\varrho_1),\ g(\varrho_2,\varpi(\varrho_2))\int_0^1 v(\varrho_2,\eta,\varpi(\eta))d\eta,\ \varpi(\varrho_2)\int_0^{\varrho_2} u(\varrho_2,\eta,\varpi(\eta))d\eta\Big)\\
&\quad-f\Big(\varrho_1,\ \varpi(\varrho_1),\ g(\varrho_2,\varpi(\varrho_2))\int_0^1 v(\varrho_1,\eta,\varpi(\eta))d\eta,\ \varpi(\varrho_2)\int_0^{\varrho_1} u(\varrho_2,\eta,\varpi(\eta))d\eta\Big)\Big|\Big]\\
&\quad-\Big[\Big|f\Big(\varrho_1,\ \varpi(\varrho_1),\ g(\varrho_2,\varpi(\varrho_2))\int_0^1 v(\varrho_1,\eta,\varpi(\eta))d\eta,\ \varpi(\varrho_2)\int_0^{\varrho_1} u(\varrho_2,\eta,\varpi(\eta))d\eta\Big)\\
&\quad-f\Big(\varrho_1,\ \varpi(\varrho_1),\ g(\varrho_1,\varpi(\varrho_1))\int_0^1 v(\varrho_1,\eta,\varpi(\eta))d\eta,\ \varpi(\varrho_1)\int_0^{\varrho_1} u(\varrho_1,\eta,\varpi(\eta))d\eta\Big)\Big|\Big]\\
&\leq\psi(|\varpi(\varrho_2)-\varpi(\varrho_1)|)+\kappa l|\varpi(\varrho_2)-\varpi(\varrho_1)|+l|\varpi(\varrho_2)-\varpi(\varrho_1)|\\
&\quad-\psi([|\varpi(\varrho_2)-\varpi(\varrho_1)|])-\kappa l([|\varpi(\varrho_2)-\varpi(\varrho_1)|])-l[|\varpi(\varrho_2)-\varpi(\varrho_1)|]\\
&\leq(\psi+(\kappa+1)l)(|\varpi(\varrho_2)-\varpi(\varrho_1)|-[|\varpi(\varrho_2)-\varpi(\varrho_1)|]).
\end{aligned}
$$

Now, we are enabled to write

$$i(T\varpi) \leq (\psi + (\kappa + 1)l)(i(\varpi)).$$

Thus, we can find

$$i(TA) \leq (\psi + (\kappa + 1)l)(i(A)). \tag{3.6}$$

From inequalities (3.5) and (3.6), together with the definition of MNC μ, we can write

$$\mu(TA) = \omega_0(TA) + i(TA) \leq \psi'(\omega_0(A) + i(A)) \leq \psi'(\mu(A)),$$

where $\psi' = \psi + (\kappa + 1)l,\ \psi' \in \Psi$.

Hence, by the generalized Darbo fixed point theorem, T has a fixed point in $C(I)$, i.e., the nonlinear equation (1.1) has at least one solution in $C[0, 1]$. □

Further, we give two examples with a verification of all the five assumptions of our main theorem.

Example 3.1. In the first example, we consider the following nonlinear integral equation:

$$\varpi(\varrho) = \frac{\varrho}{4(1 + \varpi(\varrho)^2)}\left(1 + \int_0^1 \frac{\varrho\eta}{16(1 + \varpi(\eta)^2)}d\eta\right) + \varpi(\varrho)\int_0^\varrho \frac{\varrho\eta}{16(1 + \varpi(\eta)^2)^2}d\eta. \tag{3.7}$$

Herein, we have $f(\varrho, u, v, w) = \dfrac{\varrho}{4(1 + u^2)} + v + w$ and $g(\varrho, \varpi(\varrho)) = \dfrac{\varrho}{4(1 + \varpi^2)}$ satisfying the assumptions (A1) and (A2) respectively such that

$$\begin{aligned} |f(\varrho, u_1, v_1, w_1) - f(\varrho, u_2, v_2, w_2)| &\leq \left|\frac{\varrho}{4(1 + u_1^2)} - \frac{\varrho}{4(1 + u_2^2)}\right| + |v_1 - v_2| + |w_1 - w_2| \\ &\leq \psi(|u_1 - u_2|) + |v_1 - v_2| + |w_1 - w_2|, \end{aligned}$$

where $\psi(\varrho) = \frac{\varrho}{4}$ and

$$\begin{aligned} g(\varrho, 0) &= \frac{\varrho}{4} \leq \frac{1}{4} = \kappa', \\ |g(\varrho, y_1) - g(\varrho, y_2)| &\leq \frac{\varrho}{4}|y_1 - y_2| \leq \frac{1}{4}|y_1 - y_2| = \kappa|y_1 - y_2|. \end{aligned}$$

Also, the functions $u(\varrho, \eta, \varpi) = \dfrac{\varrho\eta}{16(1 + \varpi^2)}$ and $v(\varrho, \eta, \varpi) = \dfrac{\varrho\eta}{16(1 + \varpi^2)^2}$ satisfy the assumption (A3) such that $l = \dfrac{1}{16}$. Thus $\kappa l < \dfrac{1}{4}$ and for $r_0 = 1$, the assumption (A4) is fulfilled as

$$\psi(r_0) + (r_0(\kappa + 1) + \kappa')l + N = \frac{r_0}{4} + \left(\frac{5r_0}{4} + \frac{1}{4}\right)\frac{1}{16} + \frac{1}{4} \leq r_0.$$

Now, we are enabled to observe that the last inequality admit a positive solution, for $r_0 = 1$. Hence, all assumptions (A1)–(A5) of Theorem 3.1 are fulfilled. Thus, we are enabled to conclude that the nonlinear integral equation (3.7) admits at least one solution in the space $C(I)$.

Example 3.2. Consider the following nonlinear integral equation:

$$\varpi(\varrho) = \frac{\varrho}{1+\varrho^2} + \frac{\ln\varpi(\varrho)}{1+\varrho} + \frac{\cos(\varrho\varpi(\varrho))}{2(1+\varrho)}\int_0^1 \frac{\varrho+\eta}{16(1+\exp(-\varpi(\eta)))}d\eta + \varpi(\varrho)\int_0^\varrho \frac{\arctan\varpi(\eta)}{4(1+\eta)}d\eta. \quad (3.8)$$

We have $f(\varrho,u,v,w) = \dfrac{\varrho}{1+\varrho^2} + \ln u + v + w$ and $g(\varrho,\varpi(\varrho)) = \dfrac{\cos(\varrho\varpi(\varrho))}{2(1+\varrho)}$ satisfying the assumptions (A1) and (A2) respectively such that $\psi(\varrho) = \dfrac{\varrho}{16}$, $\kappa' = \dfrac{1}{2}$ and $\kappa = \dfrac{1}{4}$. Also, the functions $u(\varrho,\eta,\varpi) = \dfrac{\varrho+\eta}{16(1+\exp(-\varpi(\eta)))}$ and $v(\varrho,\eta,\varpi) = \dfrac{\arctan\varpi(\eta)}{4(1+\eta)}$ fulfill the assumption (A3) such that $l = \dfrac{1}{8}$. Thus, $\kappa l < \dfrac{1}{4}$ and for $r_0 = 1$, the assumption (A4) is fulfilled. Thus, our second example also fulfills all the five assumptions (A1)–(A5) of Theorem 3.1 and hence by the same theorem, Eq (3.8) has at least one solution in the space $C(I)$.

4. Algorithm to find the solution

In this section, we rephrase and solve Eq (3.7) by using MHP and Adomian decomposition method. The Homotopy perturbation method is a coupling of the idea of homotopy and perturbation methods to eliminate the limitation of traditional perturbation methods. It is also a powerful concept in perturbations theory and topology. In the proposed method, we alter problems of the nonlinear functional equation to some simpler problems and to be free of non-linearity, we apply a linear combination of Adomian polynomials [42]. Thus, we present an iterative algorithm to solve Eq (3.7).

Let us consider the general form of Eq (3.7) as follows:

$$\mathcal{D}(\varrho,\varpi(\varrho)) - f(\varrho,\varpi(\varrho)) = 0,\ \varrho\in[0,1]. \quad (4.1)$$

Here, $\mathcal{D}$ is a nonlinear operator and f is some known function. We divide the operator $\mathcal{D}$ according to [43–45] into operators $\mathcal{M}$ and $\mathcal{N}$ to some linear or nonlinear operators. We also divide the function f into f_1 and f_2. Then Eq (4.1) can be rewritten as $\mathcal{M}(\varrho,\varpi) - f_1(\varrho,\varpi) + \mathcal{N}(\varrho,\varpi) - f_2(\varrho,\varpi) = 0$.

Now, we are enabled to define a MHP as follows:

$$H(p,v) = \mathcal{M}(\varrho,v) - f_1(\varrho,v) + p(\mathcal{N}(\varrho,v) - f_2(\varrho,v)) = 0, \quad (4.2)$$

$$\text{and} \quad v(\varrho) = \sum_{i=0}^{\infty} p^i v_i(\varrho) \ \text{ together with } \ \lim_{p\to1} v(\varrho) \simeq \varpi(\varrho). \quad (4.3)$$

Herein, $p = [0,1]$ and is called an embedding parameter. Letting $p = 0$ to $p = 1$, we are enabled to find $\mathcal{M}(\varrho,v) = f_1(\varrho,v)$ to $\mathcal{N}(\varrho,v) = f_2(\varrho,v)$. Further, in Eq (3.7), we choose the operators as follows:

$$\mathcal{M}(\varrho,\varpi(\varrho)) = \varpi(\varrho),$$

$$\mathcal{N}(\varrho,\varpi(\varrho)) = -\frac{\varrho}{4(1+\varpi(\varrho)^2)}\int_0^1 \frac{\varrho\eta}{16(1+\varpi(\eta)^2)}d\eta - \varpi(\varrho)\int_0^\varrho \frac{\varrho\eta}{16(1+\varpi(\eta)^2)^2}d\eta,$$

$$f_1(\varrho,(\varpi(\varrho)) = \varrho \ \text{ and } \ f_2(\varrho,\varpi(\varrho)) = \frac{\varrho}{4(1+\varpi(\varrho)^2)} - \varrho.$$

For our ease in calculation, we further divide the operator $\mathcal{N}$ into $\mathcal{N}'$ and $\mathcal{N}''$ and approximate them by using Adomian decomposition as

$$\begin{aligned}
\mathcal{N}'(\varrho, v(\varrho)) &= -\frac{\varrho}{4(1+v(\varrho)^2)}\int_0^1 \frac{\varrho\eta}{16(1+v(\eta)^2)}d\eta = -\sum_{i=0}^{\infty} p^i \mathcal{A}'_i(\varrho),\\
\mathcal{N}''(\varrho, v(\varrho)) &= -v(\varrho)\int_0^{\varrho} \frac{\varrho\eta}{16(1+v(\eta)^2)^2}d\eta = -\sum_{i=0}^{\infty} p^i \mathcal{A}''_i(\varrho),\\
f_2(\varrho, v(\varrho)) &= \sum_{i=0}^{\infty} p^i \mathcal{A}_i(\varrho),
\end{aligned} \tag{4.4}$$

wherein the Adomian polynomials are given as follows:

$$\begin{aligned}
\mathcal{A}_i(\varrho) &= \frac{1}{i!}\Big[\frac{d^i}{dp^i}\Big(\frac{\varrho}{4(1+(\sum_{j=0}^{i} p^j v_j(\varrho))^2)} - \varrho\Big)\Big]_{p=0},\\
\mathcal{A}'_i(\varrho) &= \frac{1}{i!}\Big[\frac{d^i}{dp^i}\Big(\frac{\varrho}{4(1+(\sum_{j=0}^{i} p^j v_j(\varrho))^2)}\int_0^1 \frac{\varrho\eta}{16(1+(\sum_{j=0}^{i} p^j v_j(\eta))^2)}d\eta\Big)\Big]_{p=0},\\
\mathcal{A}''_i(\varrho) &= \frac{1}{i!}\Big[\frac{d^i}{dp^i}\Big((\sum_{j=0}^{i} p^j v_j(\varrho))\int_0^{\varrho} \frac{\varrho\eta}{16(1+(\sum_{j=0}^{i} p^j v_j(\eta))^2)^2}d\eta\Big)\Big]_{p=0}.
\end{aligned}$$

Putting Eqs (4.3) and (4.4) in Eq (4.2), we get

$$\mathcal{M}\Big(\sum_{i=0}^{\infty} p^i v_i(\varrho)\Big) - f_1(\varrho) + p\Big(-\sum_{i=0}^{\infty} p^i\mathcal{A}'_i(\varrho) - \sum_{i=0}^{\infty} p^i\mathcal{A}''_i(\varrho) - \sum_{i=0}^{\infty} p^i\mathcal{A}_i(\varrho)\Big) = 0.$$

Equating the coefficients of powers of p to zero and solving for v_i, $i = 0, 1, 2, ...,$ we are now enabled to find an iterative algorithm for the numerical solution of Eq (3.7) as follows:

$$\begin{aligned}
v_0(\varrho) &= \mathcal{M}^{-1}(f_1(\varrho)),\\
v_j(\varrho) &= \mathcal{M}^{-1}(A_{j-1}(\varrho) + A'_{j-1}(\varrho) + A''_{j-1}(\varrho)) \quad j = 1, 2, \ldots .
\end{aligned} \tag{4.5}$$

Applying the algorithm to the same example, we get $v_0(\varrho) = \varrho$. For the case $j = 1$, we have

$$A_0(\varrho) = \frac{\varrho}{4}\Big(\frac{1}{1+\varrho^2}\Big) - \varrho, \quad A'_0(\varrho) = \frac{\ln 2}{128}\Big(\frac{\varrho^2}{1+\varrho^2}\Big), \quad A''_0(\varrho) = \frac{\varrho^4}{32(1+\varrho^2)},$$

$$\text{and thus}\quad v_1(\varrho) = \mathcal{M}^{-1}(A_0(\varrho) + A'_0(\varrho) + A''_0(\varrho)) = \frac{1}{1+\varrho^2}\Big(\frac{\ln 2}{128}\varrho^2 + \frac{\varrho}{4} + \frac{\varrho^4}{32}\Big) - \varrho.$$

Now, let us consider the case $j = 2$, we have

$$\begin{aligned}
A_1(\varrho) &= \frac{1}{(1+\varrho^2)^3}\Big(-\frac{\ln 2}{256}\varrho^4 - \frac{\varrho^3}{8} - \frac{\varrho^6}{64}\Big) + \frac{\varrho^3}{2(1+\varrho^2)^2},\\
A'_1(\varrho) &= \frac{1}{(1+\varrho^2)^3}\Big(-\frac{\ln 2}{2048}\varrho^7 - \frac{\ln^2 2}{8192}\varrho^5 - \frac{\ln 2}{256}\varrho^4\Big) + \frac{\ln 2}{64(1+\varrho^2)^2}\varrho^4 + \frac{0.07985190531633895\varrho^2}{32(1+\varrho^2)},
\end{aligned}$$

and

$$A_1''(\varrho) = \frac{\varrho^4}{32(1+\varrho^2)} - \frac{((3\ln 2+60)\varrho^8 + (9\ln 2+180)\varrho^6 + (9\ln 2+180)\varrho^4 + 3\ln 2\varrho^2 + 60\varrho^2)\arctan\varrho}{24576\varrho^6 + 73728\varrho^4 + 73728\varrho^2 + 24576}$$
$$- \frac{(-1408\varrho^8 + (3\ln 2-132)\varrho^7 - 1152\varrho^6 - (8\ln 2+160)\varrho^5 - (3\ln 2+60)\varrho^3)}{+24576\varrho^6 + 73728\varrho^4 + 73728\varrho^2 + 24576}.$$

Thus, we have

$$\begin{aligned}
v_2(\varrho) = {} & \frac{1}{(1+\varrho^2)^3}\Big(-\frac{\ln 2}{2048}\varrho^7 - \frac{\varrho^6}{64} - \frac{\varrho^3}{8} - \frac{\ln 2}{128}\varrho^4 - \frac{\ln^2 2}{8192})\varrho^5\Big) + \frac{1}{(1+\varrho^2)^2}\Big(-\frac{\ln 2}{64}\varrho^4 - \frac{\varrho^3}{2}\Big) \\
& + \frac{1}{1+\varrho^2}\Big(0.03125\varrho^4 + 0.0024953720411\varrho^2\Big) \\
& - \frac{((3\ln 2+60)\varrho^8 + (9\ln 2+180)\varrho^6 + (9\ln 2+180)\varrho^4 + 3\ln 2\varrho^2 + 60\varrho^2)\arctan\varrho}{24576\varrho^6 + 73728\varrho^4 + 73728\varrho^2 + 24576} \\
& - \frac{(-1408\varrho^8 + (3\ln 2-132)\varrho^7 - 1152\varrho^6 - (8\ln 2+160)\varrho^5 - (3\ln 2+60)\varrho^3)}{+24576\varrho^6 + 73728\varrho^4 + 73728\varrho^2 + 24576}.
\end{aligned}$$

By Eq (4.3), we get an approximated solution,

$$\begin{aligned}
\varpi_1(\varrho) = {} & \frac{1}{1+\varrho^2}\Big(\frac{\ln 2}{128}\varrho^2 + \frac{\varrho}{4} + \frac{\varrho^4}{32}\Big). \\
\varpi_2(\varrho) = {} & \frac{1}{(1+\varrho^2)^3}\Big(-\frac{\ln 2}{2048}\varrho^7 - \frac{\varrho^6}{64} - \frac{\ln^2 2}{8192}\varrho^5 - \frac{\ln 2}{128}\varrho^4 - \frac{\varrho^3}{8})\Big) + \frac{1}{(1+\varrho^2)^2}\Big(\frac{\ln 2}{64}\varrho^4 + \frac{\varrho^3}{2}\Big) \\
& + \frac{1}{1+\varrho^2}\Big(\frac{\varrho^4}{32} + \frac{\ln 2}{128}\varrho^2 + \frac{\varrho}{4} + 0.03125\varrho^4 + 0.0024953720411\varrho^2\Big) \\
& - \frac{((3\ln 2+60)\varrho^8 + (9\ln 2+180)\varrho^6 + (9\ln 2+180)\varrho^4 + 3\ln 2\varrho^2 + 60\varrho^2)\arctan\varrho}{24576\varrho^6 + 73728\varrho^4 + 73728\varrho^2 + 24576} \\
& - \frac{(-1408\varrho^8 + (3\ln 2-132)\varrho^7 - 1152\varrho^6 - (8\ln 2+160)\varrho^5 - (3\ln 2+60)\varrho^3)}{+24576\varrho^6 + 73728\varrho^4 + 73728\varrho^2 + 24576}.
\end{aligned}$$

Thus, by putting $\varpi_1(\varrho)$ together with $\varpi_2(\varrho)$ in Eq (3.7) and equating the both sides, we are enabled to obtain the absolute errors of $\varpi_1(\varrho)$ together with $\varpi_2(\varrho)$ as shown in Table 1. In this way, we are now enabled to observe that increasing the number of cases in the algorithm (4.5), we get better approximations of $\varpi(\varrho)$. The graphs of $v(\varrho)$ and $\varpi(\varrho)$ are shown in Figure 1.

Table 1. Absolute errors.

ϱ	Absolute error of $\varpi_1(\varrho)$	Absolute error of $\varpi_2(\varrho)$
0	0	1.61×10^{-2}
0.1	3×10^{-4}	1.59×10^{-2}
0.2	1.9×10^{-3}	1.42×10^{-2}
0.3	5.9×10^{-3}	1.01×10^{-2}
0.4	1.3×10^{-2}	3.8×10^{-3}
0.5	2.34×10^{-2}	4.2×10^{-3}
0.6	3.68×10^{-2}	1.29×10^{-2}
0.7	5.3×10^{-2}	2.12×10^{-2}
0.8	7.15×10^{-2}	2.82×10^{-2}
0.9	9.18×10^{-2}	3.42×10^{-2}
1.0	1.135×10^{-1}	3.91×10^{-2}

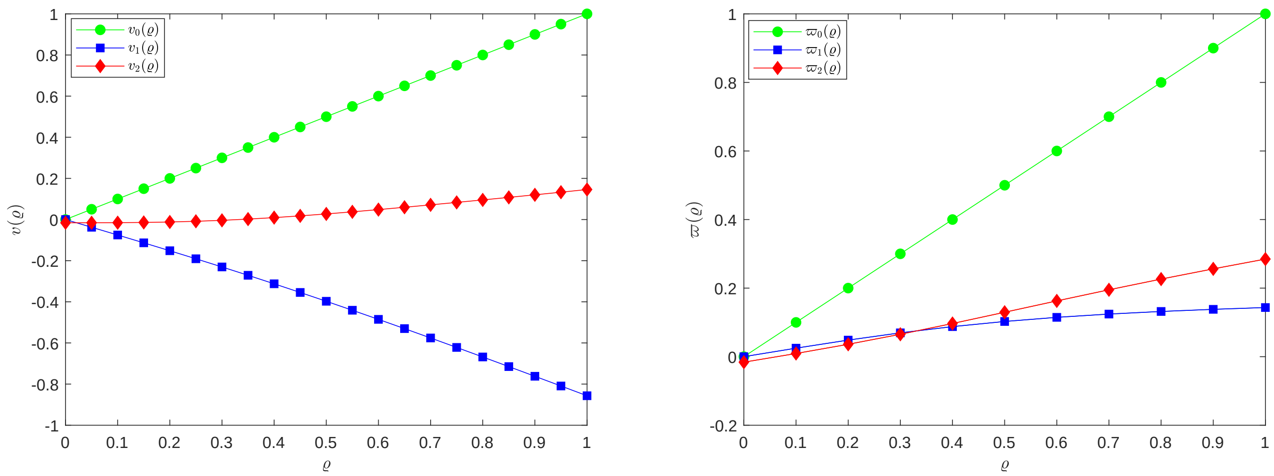

Figure 1. Plots of $v(\varrho)$ and $\varpi(\varrho)$.

5. Error analysis

In this section, we give the following two theorems and prove that they also concur with our functional integral equation (1.1).

Theorem 5.1. *Let* $\varpi_k(\varrho) = \sum_{i=0}^{k} v_i(\varrho)$, $k \in \mathbb{N}$ *and* $\mathcal{M}^{-1}$ *be a linear operator. Then the following equation is equivalent to the algorithm* (4.5)*:*

$$\varpi_{n+1} = \mathcal{M}_1^{-1}(f_1) + \mathcal{M}^{-1}(f_2(\varrho, \varpi_n) - \mathcal{N}(\varpi_n)). \tag{5.1}$$

Proof. Given that

$$\varpi_{n+1}(\varrho) = v_0(\varrho) + v_1(\varrho) + v_2(\varrho) + ...v_{n+1}(\varrho).$$

By using algorithm (4.5) , we are enabled to find

$$\begin{aligned}
\varpi_{n+1}(\varrho) &= \mathcal{M}^{-1}(f_1(\varrho)) + \mathcal{M}^{-1}(A_0(\varrho) + A_0'(\varrho) + A_0''(\varrho)) + \mathcal{M}^{-1}(A_1(\varrho) + A_1'(\varrho) + A_1''(\varrho)) \\
&\quad + ... + \mathcal{M}^{-1}(A_n(\varrho) + A_n'(\varrho) + A_n''(\varrho)) \\
&= \mathcal{M}^{-1}(f_1(\varrho)) + \mathcal{M}^{-1}\Big(\sum_{i=0}^{n} \mathcal{A}_i(\varrho) + \sum_{i=0}^{n} \mathcal{A}_i'(\varrho) + \sum_{i=0}^{n} \mathcal{A}_i''(\varrho)\Big) \\
&= \mathcal{M}^{-1}(f_1(\varrho)) + \mathcal{M}^{-1}\Big(f_2\Big(\varrho, \sum_{i=0}^{n} v_i(\varrho)\Big) - \mathcal{N}'\Big(\varrho, \sum_{i=0}^{n} v_i(\varrho)\Big) - \mathcal{N}''\Big(\varrho, \sum_{i=0}^{n} v_i(\varrho)\Big)\Big) \\
&= \mathcal{M}^{-1}(f_1(\varrho)) + \mathcal{M}^{-1}\Big(f_2\Big(\varrho, \sum_{i=0}^{n} v_i(\varrho)\Big) - \mathcal{N}\Big(\varrho, \sum_{i=0}^{n} v_i(\varrho)\Big)\Big) \\
&= \mathcal{M}^{-1}(f_1(\varrho)) + \mathcal{M}^{-1}(f_2(\varrho, \varpi_n(\varrho)) - \mathcal{N}(\varrho, \varpi_n(\varrho))).
\end{aligned}$$

Now, we prove the equivalence by using induction. By using assumption, we know that $\varpi_0 = v_0$.

$$\begin{aligned}
\varpi_1 &= \mathcal{M}^{-1}(f_1) + \mathcal{M}^{-1}(- \mathcal{N}(\varpi_0) + f_2(\varpi_0)) \\
&= \mathcal{M}^{-1}(f_1) + \mathcal{M}^{-1}(- \mathcal{N}(v_0) + f_2(v_0)) \\
&= \mathcal{M}^{-1}(f_1) + \mathcal{M}^{-1}(- \mathcal{N}'(v_0) - \mathcal{N}''(v_0) + f_2(v_0)), \\
\text{i.e., } \varpi_1(\varrho) &= \mathcal{M}^{-1}(f_1(\varrho)) + \mathcal{M}^{-1}(A_0'(\varrho) + A_0''(\varrho) + A_0(\varrho)) = v_0(\varrho) + v_1(\varrho).
\end{aligned}$$

Therefore,

$$v_1(\varrho) = \mathcal{M}^{-1}(A_0'(\varrho) + A_0''(\varrho) + A_0(\varrho)).$$

Hence, the equivalence is satisfied for v_1. We now verify for v_2.

$$\begin{aligned}
\varpi_2 &= \mathcal{M}^{-1}(f_1) + \mathcal{M}^{-1}(- \mathcal{N}(\varpi_1) + f_2(\varpi_1)) \\
&= \mathcal{M}^{-1}(f_1) + \mathcal{M}^{-1}(- \mathcal{N}'(v_0 + v_1) - \mathcal{N}''(v_0 + v_1) + f_2(v_0 + v_1)), \\
\text{i.e., } \varpi_2(\varrho) &= \mathcal{M}^{-1}(f_1(\varrho)) + \mathcal{M}^{-1}(A_0'(\varrho) + A_1'(\varrho) + A_0''(\varrho) + A_1''(\varrho) + A_0(\varrho) + A_1(\varrho)) \\
&= \mathcal{M}^{-1}(f_1(\varrho)) + \mathcal{M}^{-1}(A_0'(\varrho) + A_0''(\varrho) + A_0(\varrho)) + \mathcal{M}^{-1}(A_1'(\varrho) + A_1''(\varrho) + A_1(\varrho)) \\
&= v_0(\varrho) + v_1(\varrho) + v_2(\varrho).
\end{aligned}$$

Therefore,

$$v_2(\varrho) = \mathcal{M}^{-1}(A_1'(\varrho) + A_1''(\varrho) + A_1(\varrho)).$$

By using induction hypothesis, let us assume that the statement is true for v_n and we prove that Eq (5.1) gives rise to v_{n+1} as in the algorithm (4.5).

$$\begin{aligned}
\varpi_{n+1}(\varrho) &= \mathcal{M}^{-1}(f_1(\varrho)) + \mathcal{M}^{-1}(f_2(\varrho, \varpi_n(\varrho)) - \mathcal{N}(\varrho, \varpi_n(\varrho))) \\
&= \mathcal{M}^{-1}(f_1(\varrho)) + \mathcal{M}^{-1}\Big(f_2(\varrho, \sum_{i=0}^{n} v_i(\varrho)) - \mathcal{N}(\varrho, \sum_{i=0}^{n} v_i(\varrho))\Big) \\
&= \mathcal{M}^{-1}(f_1(\varrho)) + \mathcal{M}^{-1}\Big(\sum_{i=0}^{n} \mathcal{A}_i(\varrho) + \sum_{i=0}^{n} \mathcal{A}_i'(\varrho) + \sum_{i=0}^{n} \mathcal{A}_i''(\varrho)\Big)
\end{aligned}$$

$$= \mathcal{M}^{-1}(f_1(\varrho)) + \mathcal{M}^{-1}\Big(\sum_{i=0}^{n}\mathcal{A}_i(\varrho) + \sum_{i=0}^{n}\mathcal{A}'_i(\varrho) + \sum_{i=0}^{n}\mathcal{A}''_i(\varrho)\Big)$$
$$= \mathcal{M}^{-1}(f_1(\varrho)) + \sum_{i=0}^{n}\Big(\mathcal{M}^{-1}(\mathcal{A}_i(\varrho) + \mathcal{A}'_i(\varrho) + \mathcal{A}''_i(\varrho))\Big)$$
$$= v_0(\varrho) + \sum_{i=1}^{n} v_i(\varrho) + \mathcal{M}^{-1}(A_n(\varrho) + A'_n(\varrho) + A''_n(\varrho)) = \sum_{i=0}^{n+1} v_i(\varrho).$$

Therefore,

$$v_{n+1}(\varrho) = \mathcal{M}^{-1}(A_n(\varrho) + A'_n(\varrho) + A''_n(\varrho)).$$

Hence, the algorithm is proved. □

Theorem 5.2. *Let* $(C[0,1], \|\varpi\|_\infty)$ *be a Banach space and let* $\varpi_n(\varrho) = \sum_{i=0}^{n} v_i(\varrho)$, $n \in \mathbb{N}$ *such that* $\|v_i\|_\infty \le \alpha\, \|v_{i-1}\|_\infty$, $\forall i \in \mathbb{N}$, *where* $0 \le \alpha < 1$, *then:*

(i) The sequence $\{\varpi_n\}_{n=0}^{\infty}$ *is convergent.*

(ii) The limit of the sequence say $\lim_{n\to\infty} \varpi_n = \varpi^*$ *fulfills the algorithm* (4.5) *and Eq* (4.1).

Proof. (i) It is enough to show the sequence is Cauchy.

Let $m, n \in \mathbb{N}$ with $m > n$ and let $\epsilon > 0$. Then, for $\varrho \in I$, with the above assumptions, we estimate

$$\begin{aligned}\Big|\varpi_m(\varrho) - \varpi_n(\varrho)\Big| &= \Big|\sum_{i=n+1}^{m} v_i(\varrho)\Big| = |v_{n+1}(\varrho) + v_{n+2}(\varrho) + ... + v_m(\varrho)|\\ &\le \|v_{n+1}\|_\infty + ... + \|v_m\|_\infty\\ &\le \alpha^{n+1}\|v_0\|_\infty + \alpha^{n+2}\|v_0\|_\infty + ... + \alpha^{m}\|v_0\|_\infty\\ &= \Big(\frac{1-\alpha^{m-n}}{1-\alpha}\Big)\alpha^{n+1}\|v_0\|_\infty,\\ \text{i.e., } \|\varpi_m - \varpi_n\|_\infty &\le \Big(\frac{\alpha^{n+1}}{1-\alpha}\Big)\|v_0\|_\infty < \epsilon.\end{aligned}$$

Hence, the sequence $\{\varpi_n\}_{n=0}^{\infty}$ is Cauchy in $(C[0,1], \|\varpi\|_\infty)$, i.e., $\lim_{n\to\infty}\varpi_n(\varrho) = \lim_{n\to\infty}\sum_{i=0}^{n} v_i(\varrho) = \sum_{i=0}^{\infty} v_i(\varrho) = \varpi^*(\varrho)$.

(ii) We first prove that the limit of the sequence fulfills the algorithm (4.5). Let $\{\varpi_n\} \to \varpi^*$, we estimate

$$\begin{aligned}\lim_{n\to\infty}\varpi_n(\varrho) &= \lim_{n\to\infty}\Big(\mathcal{M}^{-1}(f_1(\varrho)) + \mathcal{M}^{-1}(-\mathcal{N}(\varrho, \varpi_{n-1}(\varrho)) + f_2(\varrho, \varpi_{n-1}(\varrho)))\Big)\\ &= \lim_{n\to\infty}\Big(\mathcal{M}^{-1}(f_1(\varrho)) + \mathcal{M}^{-1}\Big(-\mathcal{N}\Big(\varrho, \sum_{i=0}^{n-1} v_i(\varrho)\Big) + f_2\Big(\varrho, \sum_{i=0}^{n-1} v_i(\varrho)\Big)\Big)\Big)\\ &= \mathcal{M}^{-1}(f_1(\varrho)) + \mathcal{M}^{-1}\Big(-\lim_{n\to\infty}\mathcal{N}'\Big(\varrho, \sum_{i=0}^{n-1} v_i(\varrho)\Big) - \lim_{n\to\infty}\mathcal{N}''\Big(\varrho, \sum_{i=0}^{n-1} v_i(\varrho)\Big)\end{aligned}$$

$$
\begin{aligned}
&+\lim_{n\to\infty} f_2\Big(\varrho, \sum_{i=0}^{n-1} v_i(\varrho)\Big)\Big) \\
&= \mathcal{M}^{-1}(f_1(\varrho)) + \mathcal{M}^{-1}\Big(\lim_{n\to\infty}\sum_{i=0}^{n-1}\mathcal{A}'_i(\varrho) + \lim_{n\to\infty}\sum_{i=0}^{n-1}A''_i(\varrho) + \lim_{n\to\infty}\sum_{i=0}^{n-1}\mathcal{A}_i(\varrho)\Big) \\
&= \mathcal{M}^{-1}(f_1(\varrho)) + \mathcal{M}^{-1}\Big(\sum_{i=0}^{\infty}\mathcal{A}'_i(\varrho) + \sum_{i=0}^{\infty}A''_i(\varrho) + \sum_{i=0}^{\infty}\mathcal{A}_i(\varrho)\Big),
\end{aligned}
$$

$$
\text{i.e., } \lim_{n\to\infty}\varpi_n(\varrho) = \mathcal{M}^{-1}(f_1(\varrho)) + \mathcal{M}^{-1}\Big(-\mathcal{N}'\Big(\varrho, \sum_{i=0}^{\infty}v_i(\varrho)\Big) - \mathcal{N}''\Big(\varrho, \sum_{i=0}^{\infty}v_i(\varrho)\Big) + f_2\Big(\varrho, \sum_{i=0}^{\infty}v_i(\varrho)\Big)\Big).
$$

In other words,

$$
\varpi^*(\varrho) = \mathcal{M}^{-1}(f_1(\varrho)) + \mathcal{M}^{-1}(-\mathcal{N}(\varrho, \varpi^*(\varrho)) + f_2(\varrho, \varpi^*(\varrho))). \tag{5.2}
$$

Hence, the algorithm is proved. Now for proving the Eq (4.1), we apply the operator $\mathcal{M}$ to the above equation as

$$
\mathcal{M}(\varpi^*) = f_1(\varrho) - \mathcal{N}(\varpi^*) + f_2(\varpi^*) \Rightarrow \mathcal{M}(\varpi^*) + \mathcal{N}(\varpi^*) = f_1(\varrho) + f_2(\varpi^*) \Rightarrow A(\varpi^*) = f(\varpi^*),
$$

and the proof is completed.

□

The following corollary gives upper bound of the errors.

Corollary 5.1. *Under the assumptions of the above theorem,* $\|\varpi^* - \varpi_n\|_\infty \leq \left(\frac{\alpha^{n+1}}{1-\alpha}\right)\|f_1\|_\infty$, $\forall n \in \mathbb{N}$.

6. Conclusions

We have thus verified and proved the existence result of the considered nonlinear functional integral equation on the Banach space $C[0,1]$. The result is obtained by the applications of the generalized Darbo fixed point theorem associated with the MNC in the Banach space. Our result is demonstrated with the two examples. Also, we have introduced a numerical algorithm by using the MHP approach along with the Adomian decomposition method to find the approximate solution with relevant accuracy. Moreover, an error analysis with the upper bound of errors and the condition of convergence are presented. In this paper, the MATLAB program has been used for computations and programming.

Conflict of interest

The authors declare no conflict of interest.

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