A DECOMPOSITION ALGORITHM COUPLED WITH OPERATIONAL MATRICES APPROACH WITH APPLICATIONS TO FRACTIONAL DIFFERENTIAL EQUATIONS

by

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In this article, we solve numerically the linear and non-linear fractional initial value problems of multiple orders by developing a numerical method that is based on the decomposition algorithm coupled with the operational matrices approach. By means of this, the fractional initial value problems of multiple orders are decomposed into a system of fractional initial value problems which are then solved by using the operational matrices approach. The efficiency and advantage of the developed numerical method are highlighted by comparing the results obtained otherwise in the literature. The construction of the new derivative operational matrix of fractional legendre function vectors in the Caputo sense is also a part of this research. As applications, we solve several fractional initial value problems of multiple orders. The numerical results are displayed in tables and plots.

Key words: *fractional legendre function vectors, caputo derivative, multi order fractional differential equations, spectral tau method, spectral collocation method*

Introduction

Our motivation in this study is to solve the following generalized linear and nonlinear fractional differential equations of multiple orders (FDEMO):

$$ {}_C\mathcal{D}^{\delta} y(u) = f[u, y(u), {}_C\mathcal{D}^{\vartheta_0} y(u), {}_C\mathcal{D}^{\vartheta_1} y(u), \ldots, {}_C\mathcal{D}^{\vartheta_n} y(u)] $$

$$ y^{(k)}(0) = h_k, \quad k = 0,1,\ldots,m \tag{1} $$

where $m-1<\delta\le m,\ 0<\vartheta_0<\vartheta_1<\ldots<\vartheta_n<\delta$, f in general is a non-linear function, and ${}_C\mathcal{D}^{\delta}$ is a fractional derivative of order $\delta>0$ defined in Caputo sense

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Properties of fractional legendre function vectors

The fractional legendre function vectors (LFV) can be expressed as, see [1].

$$FP_k^{\beta}(u) = \frac{(2k+1)(2u^{\beta}-1)}{k+1} FP_k^{\beta}(u) - \frac{k}{k+1} FP_{k-1}^{\beta}(u), \quad k = 1, 2, \cdots \tag{2}$$

where

$$FP_0^{\beta}(u) = 1, \quad FP_1^{\beta}(u) = 2u^{\beta} - 1$$

Equation (2) can also be written:

$$FP_k^{\beta}(u) = \sum_{s=0}^{k} A_{(s,k)} u^{s\beta} \tag{3}$$

The orthogonality conditions are:

$$\int_0^1 FP_k^{\beta}(u) FP_{k'}^{\beta}(u) w(u) \mathrm{d}u = \begin{cases} \dfrac{1}{\beta(2k+1)}, & \text{for} \quad k = k', \\ 0, \quad \text{for} & k \neq k' \end{cases} \tag{4}$$

where $w(u) = u^{\beta-1}$ is a weight function.

Functions approximation using FLFV

Any function $y(u) \in L(0,1)$, can be expanded in the form of fractional LFV:

$$y(u) = \sum_{k=0}^{\infty} a_k FP_k^{\beta}(u) \tag{5}$$

Using (4), the series coefficients can be computed:

$$a_k = \beta(2k+1) \int_0^1 y(u) FP_k^{\beta}(u) w(u) \mathrm{d}u, \quad k = 0, 1, 2, \cdots \tag{6}$$

By truncating (5) to $M + 1$ terms, we have:

$$\begin{aligned} y(u) &\simeq \sum_{k=0}^{M} a_k FP_k^{\beta}(u) \\ &= \Upsilon^T \Theta(u) \end{aligned} \tag{7}$$

where

$$\Upsilon^T = (a_0, a_1, \cdots, a_M)$$

and

$$\Theta(u) = [FP_0^{\beta}(u), FP_1^{\beta}(u), FP_2^{\beta}(u), \cdots, FP_M^{\beta}(u)]^T \tag{8}$$

Operational matrices

Operational matrices together with spectra tau and spectral collocation methods have been frequently used to solve ordinary and partial FDE, [1-5] and references therein. In this study, we develop a generalized derivative operational matrix in Caputo sense of fractional LFV. The operational matrix developed in [5] is a special case of our developed operational matrix for $\beta = 1$.

Lemma 1 The fractional derivative of order $\delta > 0$ of fractional LFV in Caputo sense can be computed by using the following, see [1]:

$$ {}_{C}\mathcal{D}^{\delta} FP_{k}^{\beta}(u) = \sum_{s=0}^{k} A'_{(s,k)} \frac{\Gamma(s\beta+1)}{\Gamma(s\beta-\delta+1)} u^{s\beta-\delta} \tag{9} $$

where $A'_{(s,k)} = 0$ when $s\beta \in \mathbb{R}_{+}$ and $s\beta < \delta$. For other cases, $A'_{(s,k)} = A_{(s,k)}$.

The fractional derivative operational matrix of FLFV

In this section, we develop the generalized derivative operational matrix of fractional LFV in Caputo sense.

Theorem 1 Suppose $\Theta(u)$ be the fractional LFV as defined in (8), and also suppose $\delta > 0$, then:

$$ {}_{C}\mathcal{D}^{\delta}\Theta(u) \simeq \mathrm{D}^{\delta}_{(M+1,M+1)}\Theta(u) \tag{10} $$

where D^{δ} is the derivative operational matrix of fractional LFV which elements can be computed using:

$$ \mathrm{D}^{\delta}_{(M+1,M+1)} = \sum_{s=\frac{\lceil\delta\rceil}{\beta}}^{k} \Lambda_{(k,j,s)}, \quad k = \frac{\lceil\delta\rceil}{\beta},\cdots,M, \quad j = 0,1,\cdots,M \tag{11} $$

and

$$ \Lambda_{(k,j,s)} = \sum_{r=0}^{j} (-1)^{k+s} \frac{(k+s)!}{(k-s)!(s!)^2} \frac{\Gamma(s\beta+1)}{\Gamma(s\beta-\delta+1)} \times $$
$$ \times (-1)^{j+r} \frac{(j+r)!}{(j-r)!(r!)^2} \frac{\beta(2j+1)}{\beta(s+r+1)-\delta} $$

Proof Applying linearity of Caputo derivative on (8), and *Lemma 1* we have:

$$ \begin{gathered} {}_{C}\mathcal{D}^{\delta} FP_{k}^{\beta}(u) = \sum_{s=\frac{\lceil\delta\rceil}{\beta}}^{k} (-1)^{k+s} \frac{(k+s)!}{(k-s)!(s!)^2} \frac{\Gamma(s\beta+1)}{\Gamma(s\beta-\delta+1)} u^{s\beta-\delta} \\ k = \frac{\lceil\delta\rceil}{\beta}, \frac{\lceil\delta\rceil}{\beta}+1,\cdots,M \end{gathered} \tag{12} $$

Now, the term $u^{s\beta-\delta}$ can be approximated:

$$u^{s\beta-\delta} \simeq \sum_{j=0}^{M} B_{(r,j)} FP_j^{\beta}(u) \tag{13}$$

where

$$\begin{aligned} B_{(r,j)} &= \beta(2j+1)\int_0^1 FP_j^{\beta}(u)u^{s\beta-\delta}u^{\beta-1}\mathrm{d}u \\ &= \beta(2j+1)\sum_{r=0}^{j}(-1)^{j+r}\frac{(j+r)!}{(j-r)!(r!)^2}\frac{1}{\beta(s+r+1)-\delta} \end{aligned} \tag{14}$$

Employing eq. (14) and eq. (13), into eq. (12), we have the required result.

Decomposition technique

Set $y_1 = y$ in (1), and suppose:

$${}_C\mathcal{D}^{\vartheta_1} y_1 = y_2 \tag{15}$$

Case 1 If $m-1 \le \vartheta_1 < \vartheta_2 \le m$, then we define:

$${}_C\mathcal{D}^{\vartheta_2-\vartheta_1} y_2 = y_3 \tag{16}$$

Claim: $y_3 = {}_C\mathcal{D}^{\vartheta_2} y$.
If $\vartheta_1 = m-1$, then:

$${}_C\mathcal{D}^{\vartheta_2-\vartheta 1} y_2 = {}_C\mathcal{D}^{\vartheta_2-(m-1)}\,{}_C\mathcal{D}^{(m-1)} y_1 = {}_C\mathcal{D}^{\vartheta_2} y_1 = {}_C\mathcal{D}^{\vartheta_2} y$$

Hence the claim is verified.
If $\vartheta_1 \notin \mathbb{N}$, then, ${}_C\mathcal{D}^{\vartheta_1} y_1(0) = 0$, and as $\vartheta_2 - \vartheta_1 < 1$:

$$\begin{aligned} {}_C\mathcal{D}^{\vartheta_2-\vartheta_1}({}_C\mathcal{D}^{\vartheta_1} y_1) &= D_{\mathrm{RL}} I^{1+\vartheta_1-\vartheta_2}\,{}_{\mathrm{RL}} I^{m-\vartheta_1} y_1^{(m)} \\ &= D_{\mathrm{RL}} I^{1+m-\vartheta_2} y_1^{(m)} = {}_{\mathrm{RL}}\, I^{m-\vartheta_2} y_1^{(m)} = {}_C\mathcal{D}^{\vartheta_2} y_1 = {}_C\mathcal{D}^{\vartheta_2} y \end{aligned}$$

Therefore $y_3 = {}_C\mathcal{D}^{\vartheta_2-\vartheta_1} y_2 = {}_C\mathcal{D}^{\vartheta_2} y$.
Case 2 Consider $m-1 \le \vartheta_1 < m \le \vartheta_2$. If $\vartheta_1 = m-1$, then define:

$${}_C\mathcal{D}^{\vartheta_2-\vartheta 1} y_2 = y_3 \tag{17}$$

As ${}_C\mathcal{D}^{\vartheta_2-\vartheta_1} y_2 = {}_C\mathcal{D}^{\vartheta_2-m+1} \cdot {}_C\mathcal{D}^{m-1} y_1 = {}_C\mathcal{D}^{\vartheta_2} y_1$.
If $m-1 < \vartheta_1 < m \le \vartheta_2$, then define:

$${}_C\mathcal{D}^{m-\vartheta_1} y_2 = y_3 \tag{18}$$

Claim: $y_3 = y^{(m)}$. As $\vartheta_1 \notin \mathbb{N}$, then ${}_C\mathcal{D}^{\vartheta_1} y_1(0) = y_2(0) = 0$ and $0 < m - \vartheta_1 < 1$,

$${}_C\mathcal{D}^{m-\vartheta_1} y_2 = {}_C\,\mathcal{D}^{m-\vartheta_1}\,{}_C\mathcal{D}^{\vartheta_1} y_1 = D_{\mathrm{RL}} I^{1+\vartheta_1-m}\,{}_{\mathrm{RL}} I^{m-\vartheta_1} y_1^{(m)} = D_{\mathrm{RL}} I y_1^{(m)} = y_1^{(m)} = y^{(m)}$$

Hence $y_3 = y^{(m)}$. Further we define:

$$ {}_C\mathcal{D}^{\vartheta_2 - m} y_3 = y_4 \tag{19}$$

Claim: $y_4 = {}_C\mathcal{D}^{\vartheta_2} y$. As $y_4 = {}_C\mathcal{D}^{\vartheta_2 - m} y_3 = {}_C\mathcal{D}^{\vartheta_2 - m} y^{(m)} = {}_C\mathcal{D}^{\vartheta_2} y$.

The process will be continued until the decomposition of the problem (1) into a system of fractional initial value problems (FIVP).

Test examples

In this section, the applicability of the method is analyzed by solving various problems and comparing their analytical solutions with their approximate solutions obtained using our method. In addition to that, the approximate results obtained by using our method are compared with the results obtained otherwise in the literature.

Example 1 Consider the following linear FIVP of multiple orders [6]:

$$ \begin{gathered} {}_C\mathcal{D}^{\delta} y(u) = a\, {}_C\mathcal{D}^{\vartheta_0} y(u) - y(u) + x(u), \quad u \in [0,1], \quad 0 < \vartheta_0 < \delta \le 1, \\ y(0) = 0 \end{gathered} \tag{20}$$

The source term is given:

$$ x(u) = \frac{5u^{\frac{3}{2}}}{2} + u^{\frac{5}{2}} + \frac{15\sqrt{\pi}\, u^{\frac{9}{4}}}{8\Gamma\left(\frac{13}{4}\right)} $$

The exact solution of (20) at $\delta = 1, a = -1,$ and $\vartheta_0 = 1/4$ is:

$$ y(u) = u^2 \sqrt{u} $$

The problem (20) can be decomposed into a system of FIVP by applying the algorithm studied in section *Decomposition technique*:

$$ {}_C\mathcal{D}^{\vartheta_0} y_1(u) = y_2(u), \quad y_1(0) = 0 $$

$$ {}_C\mathcal{D}^{\delta - \vartheta_0} y_2(u) = -y_2(u) - y_1(u) + x(u), \quad y_2(0) = 0 \tag{21}$$

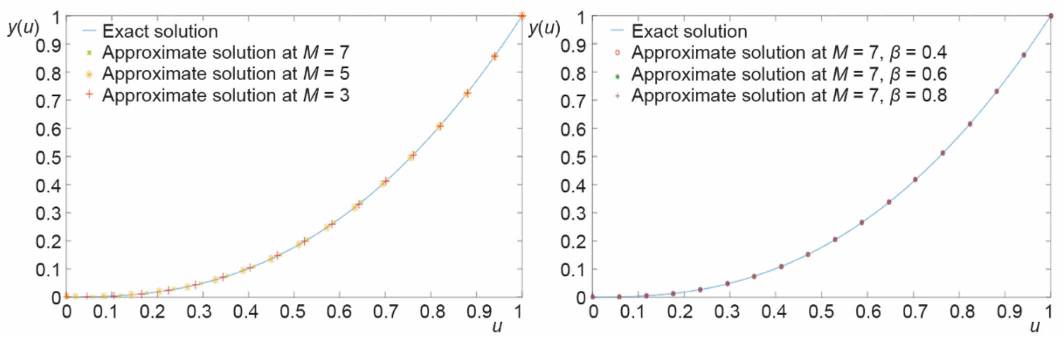

Figure 1. Plots of approximate solution and exact solution of *Example 1* at various values of *M*, and *β*

Table 1. Approximate results of *Example 1* at various values of *M*, and *β*

Our method				Method in [6], *Example 3*	
M	β	L^∞	L^2	L^∞	L^2
3	1	$1.6 \cdot 10^{-3}$	$3.7 \cdot 10^{-3}$	–	–
4	1	$3.36 \cdot 10^{-4}$	$7.40 \cdot 10^{-4}$	$1.21 \cdot 10^{-3}$	$5.92 \cdot 10^{-4}$
6	0.5	$4.81 \cdot 10^{-6}$	$5.71 \cdot 10^{-6}$	–	–
8	0.5	$2.35 \cdot 10^{-7}$	$2.87 \cdot 10^{-7}$	$5.80 \cdot 10^{-5}$	$2.50 \cdot 10^{-5}$
9	0.5	$7.01 \cdot 10^{-8}$	$8.81 \cdot 10^{-8}$	–	–
16	1	$2.13 \cdot 10^{-12}$	$3.15 \cdot 10^{-12}$	$2.45 \cdot 10^{-6}$	$9.89 \cdot 10^{-7}$

Example 2 Consider the following nonlinear FIVP of multiple orders [3]:

$$\begin{gathered} {}_C\mathcal{D}^{\delta} y(u) = a\,{}_C\mathcal{D}^{\vartheta_0} y(u) - y^2(u) + x(u), \quad u \in [0,1], \quad 2 < \vartheta_0 < \delta \le 3 \\ y(0) = 0, \;\; y'(0) = 0, \quad y'(0) = 2 \end{gathered} \tag{22}$$

The source term is given:

$$x(u) = u^4$$

The exact solution of (22) at $\delta = 3$, $a = 1$, and $\vartheta_0 = 2.5$ is:

$$y(u) = u^2$$

Table 2. Approximate results of *Example 2* are compared with the results obtained in, [3] *Example 4*, at *M* = 3

u	$y(u)$	Our method	Method in [3]
0	0	0	0
0.1	0.01	0.01	0.01
0.2	0.04	0.04	0.04
0.3	0.09	0.09	0.09
0.4	0.16	0.16	0.16
0.5	0.25	0.25	0.25
0.6	0.36	0.36	0.36
0.7	0.49	0.49	0.49
0.8	0.64	0.64	0.64
0.9	0.81	0.81	0.81
1.0	1.0	1.0	1.0

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